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# HOMOLOGICAL LOCALISATION OF MODEL CATEGORIES

DAVID BARNES AND CONSTANZE ROITZHEIM

ABSTRACT. One of the most useful methods for studying the stable homotopy category is localising at some spectrum  $E$ . For an arbitrary stable model category we introduce a candidate for the  $E$ -localisation of this model category. We study the properties of this new construction and relate it to some well-known categories.

## 1. INTRODUCTION

The stable homotopy category is spectacularly complicated and yet of fundamental importance to homotopy theorists. A standard and highly successful method of dealing with this complexity is to “filter out” some of this information via a Bousfield localisation. In return we obtain a more structured category with useful and interesting patterns.

More precisely, we choose some homology theory  $E_*$  and replace the stable homotopy category  $\mathrm{Ho}(\mathcal{S})$  with  $\mathrm{Ho}(L_E\mathcal{S})$ , the full subcategory of  $\mathrm{Ho}(\mathcal{S})$  with objects the  $E$ -local spectra. This means that in the passage from  $\mathrm{Ho}(\mathcal{S})$  to  $\mathrm{Ho}(L_E\mathcal{S})$ , the  $E_*$ -isomorphisms are formally inverted. Bousfield’s paper [Bou79] is the original source of this idea.

There are a number of other model categories whose homotopy categories share many of the properties of  $\mathrm{Ho}(\mathcal{S})$ , namely stable model categories. It would be advantageous if we could generalise the notion of  $E$ -localisation to this class of categories. Thus we are interested in the construction of a *homological localisation* of a *stable* model category, one that is the analogue of forming  $\mathrm{Ho}(L_E\mathcal{S})$  from  $\mathrm{Ho}(\mathcal{S})$ .

The main motivation comes again from the study of the stable homotopy category. In order to understand spectra,  $\mathrm{Ho}(\mathcal{S})$  and its various  $E$ -localisations it is necessary to relate  $\mathcal{S}$  and  $L_E\mathcal{S}$  to other stable model categories  $\mathcal{C}$ . For example, one can study to what extent there is a stable model category  $\mathcal{C}$  whose homotopy category “models”  $\mathrm{Ho}(L_E\mathcal{S})$  and how similar  $\mathcal{C}$  is to  $L_E\mathcal{S}$  in terms of higher homotopy behaviour. To make those links it would be a desirable tool to have the corresponding  $E$ -localisations of  $\mathcal{C}$  in order to compare  $E$ -local spectra to other counterparts related to  $\mathcal{C}$ .

A stable model category  $\mathcal{C}$  is a model category whose associated homotopy category  $\mathrm{Ho}(\mathcal{C})$  is triangulated via the construction of [Hov99, Section 7]. Lenhardt proved in [Len12] that  $\mathrm{Ho}(\mathcal{C})$  is a module over  $\mathrm{Ho}(\mathcal{S})$  whenever  $\mathcal{C}$  is a stable model category. Hence we have a tensor product

$$- \wedge^L -: \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

and an enrichment of  $\mathrm{Ho}(\mathcal{C})$  in  $\mathrm{Ho}(\mathcal{S})$ . This technique is called *stable frames*.

Using this action on the homotopy category of a stable model category one could try to make a new model structure on  $\mathcal{C}$  such that the weak equivalences are the “ $E_*$ -isomorphisms”: those maps  $f: X \rightarrow Y$  in  $\mathcal{C}$  such that

$$f \wedge^L E: X \wedge^L E \rightarrow Y \wedge^L E$$

is an isomorphism in  $\mathrm{Ho}(\mathcal{C})$ . Such a model structure would deserve the name  $L_E\mathcal{C}$ . The machinery that allows one to create new model structures with larger collections of weak

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equivalences is Bousfield localisation, see [Hir03, Part I]. But it seems particularly difficult to check that  $L_E\mathcal{C}$  exists for general  $\mathcal{C}$ . For spectra, the argument appears in [EKMM97, Section VIII.1] and requires numerous unpleasant cardinality arguments.

For well-behaved stable model categories  $\mathcal{C}$  we are going to produce a new model structure  $\mathcal{C}_E$  that avoids such set-theoretic awkwardness. This  $\mathcal{C}_E$  is a good candidate for the  $E$ -localisation of  $\mathcal{C}$  because of the following universal property:  $\mathcal{C}_E$  is the “closest” model category to  $\mathcal{C}$  such that any Quillen adjunction from spectra to  $\mathcal{C}$

$$\mathcal{S} \rightleftarrows \mathcal{C}$$

gives rise to a Quillen adjunction

$$L_E\mathcal{S} \rightleftarrows \mathcal{C}_E$$

from  $E$ -local spectra to  $\mathcal{C}_E$ . We are also able to give another description of  $\mathcal{C}_E$  in terms of pushouts of model categories, which shows how strong the universal property of this new model structure is.

We are also able to give an improvement of [BR11, Theorem 9.5]: we can show that for all  $E$ , the homotopy information of  $E$ -local spectra is entirely encoded in the  $\mathrm{Ho}(\mathcal{S})$ -module structure on the  $E$ -local stable homotopy category. This was previously only possible with the strong restriction that  $E$  is smashing. Hence we have the following, which appears as Theorem 9.1.

**Theorem 1.** *Let  $\mathcal{C}$  be a stable model category. Assume we have an equivalence of triangulated categories*

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

*then  $L_E\mathcal{S}$  and  $\mathcal{C}$  are Quillen equivalent if and only if  $\Phi$  is an equivalence of  $\mathrm{Ho}(\mathcal{S})$ -module categories.*

## 2. ORGANISATION

Firstly, we recall some definitions and conventions regarding Bousfield localisation and stable frames. We also re-introduce the concept of *stably  $E$ -familiar model categories*: in [BR11] we studied those  $\mathcal{C}$  such that the action of  $\mathrm{Ho}(\mathcal{S})$  factors over the functor  $\mathrm{Ho}(\mathcal{S}) \rightarrow \mathrm{Ho}(L_E\mathcal{S})$ . In particular the homotopy category of such a model category has an enrichment in the more structured category  $\mathrm{Ho}(L_E\mathcal{S})$ . We called such categories *stably  $E$ -familiar*.

We then turn to the question of altering a model structure on a given category so as to obtain a stably  $E$ -familiar model category. In Section 5 we consider the simpler case of spectral model categories: such a model category is defined in a similar way to a simplicial model category, but with simplicial sets replaced by the model category of symmetric spectra. We construct the *stable  $E$ -familiarisation* of a spectral model category in this section.

In Section 6 we extend our results to more general stable model categories. We prove that the *stable  $E$ -familiarisation* of a model category  $\mathcal{C}$  is the closest stably  $E$ -familiar model category to  $\mathcal{C}$  in the following sense. The result below also implies that our construction has the universal property we described earlier.

**Theorem 2.** *Let  $\mathcal{C}$  be a stable, proper and cellular model category such that the domains of the generating cofibrations of  $\mathcal{C}$  are cofibrant. Then there is a model structure  $\mathcal{C}_E$  on  $\mathcal{C}$  such that*

- (1)  $\mathcal{C}_E$  is stably  $E$ -familiar,
- (2) if  $F : \mathcal{C} \rightarrow \mathcal{E}$  is a left Quillen functor and  $\mathcal{E}$  is stably  $E$ -familiar, then  $F$  factors over  $\mathcal{C} \rightarrow \mathcal{C}_E$ .

Section 7 consists of several *examples* of  $\mathcal{C}_E$  for some  $E$  and  $\mathcal{C}$  involving algebraic model categories, chromatic localisations and module categories over a ringoid spectrum.

In Section 8 we rephrase the universal property of  $\mathcal{C}_E$  in terms of *homotopy pushouts* of model categories.

Finally, we prove a full version of the *modular rigidity* theorem that all homotopy information of  $E$ -local spectra is governed by the  $\mathrm{Ho}(\mathcal{S})$ -action on  $\mathrm{Ho}(L_E \mathcal{S})$  given by framings.

### 3. BOUSFIELD LOCALISATION

We begin with an introduction to Bousfield localisation at a homology theory  $E$ . Throughout the paper when we refer to spectra, we mean symmetric spectra equipped with the stable model structure [HSS00] unless stated otherwise.

Let  $E$  be a spectrum and let  $[-, -]_*$  denote maps in the stable homotopy category. Then  $E$  corepresents a homology functor  $E_*$  on the category of spectra via

$$E_*(X) = [S^0, E \wedge X]_*$$

where  $S^0$  denotes the sphere spectrum. Bousfield used this to construct a homotopy category of spectra where maps which induce isomorphisms on  $E_*$ -homology are isomorphisms [Bou79]. We recap some of the definitions from this work.

**Definition 3.1.** *A map  $f: X \rightarrow Y$  of spectra is an  $E$ -equivalence if  $E_*(f)$  is an isomorphism. A spectrum  $Z$  is  $E$ -local if  $f^*: [Y, Z] \rightarrow [X, Z]$  is an isomorphism for all  $E$ -equivalences  $f: X \rightarrow Y$ . A spectrum  $A$  is  $E$ -acyclic if  $[A, Z] = 0$  for all  $E$ -acyclic  $Z$ . An  $E$ -equivalence from  $X$  to an  $E$ -local object  $Z$  is called an  $E$ -localisation.*

Bousfield localisation of spectra gives a homotopy theory that is particularly sensitive towards  $E_*$  and  $E$ -local phenomena. The  $E$ -local homotopy theory is obtained from the category of spectra by formally inverting the  $E$ -equivalences.

This can be seen as a special case of a more general result by Hirschhorn. Let  $\mathcal{C}$  be a model category. For  $X, Y \in \mathcal{C}$ , we let  $\mathrm{map}_{\mathcal{C}}(X, Y) \in \mathbf{sSet}$  denote the homotopy function object, see [Hir03, Chapter 17] and Section 4.

**Definition 3.2.** *Let  $S$  be a class of maps in  $\mathcal{C}$ . Then an object  $Z \in \mathcal{C}$  is  $S$ -local if*

$$\mathrm{map}_{\mathcal{C}}(s, Z) : \mathrm{map}_{\mathcal{C}}(B, Z) \longrightarrow \mathrm{map}_{\mathcal{C}}(A, Z)$$

*is a weak equivalence in simplicial sets for any  $s: A \rightarrow B$  in  $S$ . A map  $f: X \rightarrow Y \in \mathcal{C}$  is an  $S$ -equivalence if*

$$\mathrm{map}_{\mathcal{C}}(f, Z) : \mathrm{map}_{\mathcal{C}}(Y, Z) \longrightarrow \mathrm{map}_{\mathcal{C}}(X, Z)$$

*is a weak equivalence for any  $S$ -local  $Z \in \mathcal{C}$ . An object  $W \in \mathcal{C}$  is  $S$ -acyclic if*

$$\mathrm{map}_{\mathcal{C}}(W, Z) \simeq *$$

*for all  $S$ -local  $Z \in \mathcal{C}$ .*

A *left Bousfield localisation* of a model category  $\mathcal{C}$  with respect to a class of maps  $S$  is a new model structure  $L_S \mathcal{C}$  on  $\mathcal{C}$  such that

- the weak equivalences of  $L_S \mathcal{C}$  are the  $S$ -equivalences,
- the cofibrations of  $L_S \mathcal{C}$  are the cofibrations of  $\mathcal{C}$ ,
- the fibrations of  $L_S \mathcal{C}$  are those maps that have the right lifting property with respect to cofibrations that are also  $S$ -equivalences.

Hirschhorn proves that if  $S$  is a set and  $\mathcal{C}$  is left proper and cellular then  $L_S\mathcal{C}$  exists. (We will give rough definitions of these two terms below.) Note that an object is fibrant in  $L_S\mathcal{C}$  if and only if it is fibrant in  $\mathcal{C}$  and  $S$ -local.

In the case of localising spectra at a homology theory one wants to invert the class of  $E_*$ -isomorphisms, i.e. those maps of spectra that induce isomorphisms in the homology theory  $E_*$ . Since this is not a set, one cannot use Hirschhorn's result directly. In [EKMM97, Section VIII.1] it is shown that there is a set  $S$  whose  $S$ -equivalences are exactly the  $E_*$ -isomorphisms. Hence, the key to proving the existence of homological localisations is to find a set giving the correct notion of equivalence. We shall encounter this idea again when constructing  $\mathcal{C}_E$ .

A model category is *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. A model category is *right proper* if the pullback of a weak equivalence along a fibration is a weak equivalence. If a model category is both left and right proper, we say that it is *proper*.

We also need a stronger version of “cofibrantly generation”, one which forces cell complexes to be better behaved. The actual definition is technical and not particularly illuminating, so we shall simply say that a model category is *cellular* if it is cofibrantly generated by sets  $I$  and  $J$ , and the domains and codomains of  $I$  and  $J$  satisfy some nice cardinality conditions. We leave the details to [Hir03, Definition 12.1.1].

#### 4. STABLE FRAMINGS

Framings are a powerful tool that describe and classify Quillen functors from simplicial sets or spectra to arbitrary model categories. They were first developed by Hovey in [Hov99, Section 5.2]. For a model category  $\mathcal{C}$ , he investigates cosimplicial and simplicial resolutions of objects in  $\mathcal{C}$ . These are called “frames”. In more detail, a frame of an object  $A \in \mathcal{C}$  is a cofibrant replacement of the constant cosimplicial object  $A \in \mathcal{C}^\Delta$  in the Reedy model category of cosimplicial objects in  $\mathcal{C}$ . From these notions one obtains bifunctors

$$\begin{aligned} - \otimes - &: \mathcal{C} \times \mathbf{sSet} \longrightarrow \mathcal{C}, \\ \mathrm{map}_l(-, -) &: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathbf{sSet}, \\ (-)^{(-)} &: \mathcal{C} \times \mathbf{sSet}^{op} \longrightarrow \mathcal{C}, \\ \mathrm{map}_r(-, -) &: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathbf{sSet} \end{aligned}$$

satisfying certain adjunction properties. The notation  $\otimes$  stems from the fact that

$$A \otimes \Delta[0] \simeq A.$$

However, this set-up does not equip  $\mathcal{C}$  with the structure of a simplicial model category because the “mapping spaces”  $\mathrm{map}_l(X, Y)$  and  $\mathrm{map}_r(X, Y)$  only agree up to a zig-zag of weak equivalences for cofibrant  $X \in \mathcal{C}$  and fibrant  $Y \in \mathcal{C}$  [Hov99, Proposition 5.4.7]. But their derived functors agree, leaving us with the following [Hov99, Theorem 5.5.3].

**Theorem 4.1** (Hovey). *Let  $\mathcal{C}$  be any model category. Then its homotopy category  $\mathrm{Ho}(\mathcal{C})$  is a closed  $\mathrm{Ho}(\mathbf{sSet})$ -module category.*

In particular, this equips any model category with the notion of a homotopy mapping space. Moreover, framings satisfy the following important properties.

- If  $\mathcal{C}$  carries the structure of a simplicial model category [Hov99, Definition 4.2.18], then the two  $\mathrm{Ho}(\mathbf{sSet})$ -module structures coming from either framings or the simplicial structure agree [Hov99, Theorem 5.6.2].
- If  $F : \mathbf{sSet} \longrightarrow \mathcal{C}$  is a left Quillen functor with  $F(\Delta[0]) = A$ , then the left derived functors of  $F$  and of the framing functor  $A \otimes - : \mathbf{sSet} \longrightarrow \mathcal{C}$  agree. Thus, every left Quillen functor from simplicial sets to any model category can be described, up to homotopy, by a frame.

The second property follows from the fact that the category of cosimplicial objects  $\mathcal{C}^\Delta$  is equivalent to the category of adjunctions  $\mathbf{sSet} \rightleftarrows \mathcal{C}$  [Hov99, Proposition 3.1.5]. A cosimplicial object  $A^\bullet$  corresponds to a Quillen adjunction under this equivalence if and only if it is a frame, that is  $A^\bullet$  is cofibrant and homotopically constant, [BR11, Proposition 3.2].

In [Len12] Fabian Lenhardt described an analogous set-up for spectra and stable model categories. Now let  $\mathcal{C}$  be a stable model category. First, Lenhardt shows that the category of adjunctions between spectra and a stable model category  $\mathcal{C}$  is equivalent to the category of “ $\Sigma$ -cospectra”  $\mathcal{C}^\Delta(\Sigma)$ . An object in  $\mathcal{C}^\Delta(\Sigma)$  consists of a sequence of cosimplicial objects  $X_n \in \mathcal{C}^\Delta$  together with structure maps

$$\Sigma X_{n+1} \longrightarrow X_n.$$

The suspension of cosimplicial objects is described in [Len12, Section 3.3]. He then characterises those  $\Sigma$ -cospectra that give rise to Quillen adjunctions under this equivalence, calling them stable frames. These give rise to bifunctors  $-\wedge-$  and  $\mathrm{Map}(-, -)$  satisfying the expected adjunction properties.

As in the unstable case, this is not rigid enough to equip any stable model category  $\mathcal{C}$  with the structure of a spectral model category. However, the above bifunctors give rise to the following [Len12, Theorem 6.3].

**Theorem 4.2** (Lenhardt). *Let  $\mathcal{C}$  be a stable model category. Then  $\mathrm{Ho}(\mathcal{C})$  is a closed  $\mathrm{Ho}(\mathcal{S})$ -module category.*

As expected, this satisfies the following key properties.

- If  $\mathcal{C}$  is already a spectral model category, then the  $\mathrm{Ho}(\mathcal{S})$ -module structure derived from the spectral structure agrees with the  $\mathrm{Ho}(\mathcal{S})$ -module structure coming from stable frames [BR11, Example 6.7].
- By construction, every left Quillen functor  $F : \mathcal{S} \longrightarrow \mathcal{C}$  is, up to homotopy, of the form  $X \wedge - : \mathcal{S} \longrightarrow \mathcal{C}$  for some fibrant-cofibrant  $X \in \mathcal{C}$ .
- In particular, for any fibrant-cofibrant  $X \in \mathcal{C}$  there is a left Quillen functor  $\mathcal{S} \longrightarrow \mathcal{C}$  that sends the sphere spectrum to  $X$ . We denote this functor by  $X \wedge -$  and its right adjoint by  $\mathrm{Map}_{\mathcal{C}}(X, -)$ .
- Any stable frame and thus any Quillen functor  $\mathcal{S} \longrightarrow \mathcal{C}$  is, up to homotopy, entirely determined by its image on the sphere.

As we have already mentioned, the homotopy theory of  $L_E \mathcal{S}$  is often much better understood than  $\mathcal{S}$ . So it is worth asking if some stable model categories have more in common with  $L_E \mathcal{S}$  than  $\mathcal{S}$ . We answer this question and obtain several useful results using this idea in [BR11]. We give the fundamental definitions below.

**Definition 4.3.** *We say that a stable frame  $X \in \mathcal{C}^\Delta(\Sigma)$  is an  $E$ -local frame if it gives rise to a Quillen functor pair*

$$X \wedge - : L_E \mathcal{S} \rightleftarrows \mathcal{C} : \mathrm{Map}_{\mathcal{C}}(X, -).$$

*A stable model category  $\mathcal{C}$  is stably  $E$ -familiar if every stable frame is an  $E$ -local frame.*

This is [BR11, Definition 7.1]. This generalises the notion of an  $L_E \mathcal{S}$ -model category in the following sense: if  $\mathcal{C}$  is already an  $L_E \mathcal{S}$ -model category, then the  $\mathrm{Ho}(\mathcal{C})$ -module structure on  $\mathrm{Ho}(\mathcal{C})$  agrees with the  $\mathrm{Ho}(L_E \mathcal{S})$ -module structure given by  $E$ -local frames [BR11, Proposition 7.6]. We can further characterise stably  $E$ -familiar model categories as follows [BR11, Theorem 7.8].

**Theorem 4.4.** *Let  $\mathcal{C}$  be a stable model category. Then  $\mathcal{C}$  is stably  $E$ -familiar if and only if the homotopy mapping spectrum  $\mathbb{R} \mathrm{Map}_{\mathcal{C}}(X, Y)$  is an  $E$ -local spectrum for all  $X, Y \in \mathcal{C}$ .*

We can use the theory of  $E$ -local framings to study *algebraic model categories*. An algebraic model category is a  $\text{Ch}(\mathbb{Z})$ -model category in the sense of [Hov99, Definition 4.2.18]. Thus a  $\text{Ch}(\mathbb{Z})$ -model category is enriched, tensored and cotensored over chain complexes and satisfies the  $\text{Ch}(\mathbb{Z})$ -analogue of the compatibility axiom (SM7). This implies that the homomorphism spectra obtained via framings are products of Eilenberg–Mac Lane spectra [GJ99, Proposition III.2.20], [DS07, Proposition 1.6]. Using the computations of Gutiérrez in [Gut10] one can draw the following conclusions [BR11, Section 9].

- For  $n \geq 1$  there are no algebraic stably  $K(n)$ -familiar model categories, where  $n$  denotes the  $n^{\text{th}}$  Morava– $K$ -theory.
- Let  $E(n)$  denote the  $n^{\text{th}}$  chromatic Johnson–Wilson spectrum. An algebraic model category is stably  $E(n)$ -familiar if and only if it is rational.

Now we turn to the question of whether any model category can be made stably  $E$ -familiar in some natural way.

## 5. $E$ -FAMILIARISATION OF SPECTRAL MODEL CATEGORIES

For any homology theory  $E$  we can consider the category of spectra with the  $E$ -local model structure,  $L_E\mathcal{S}$ . Hence we would like to know if a reasonable notion of  $E$ -localisation exists for an arbitrary stable model category  $\mathcal{C}$ .

Intuitively, a promising definition would be a Bousfield localisation  $L_E\mathcal{C}$  of  $\mathcal{C}$  where one localises at the class of “ $E$ -equivalences” given by

$$\{f : X \longrightarrow Y \in \mathcal{C} \mid f \wedge^L E : X \wedge^L E \longrightarrow Y \wedge^L E \text{ is an isomorphism in } \text{Ho}(\mathcal{C})\},$$

where the action  $\wedge$  of a spectrum on an element of  $\mathcal{C}$  is defined via stable frames. However, showing the existence of Bousfield localisations at a class of maps is set-theoretically awkward. The standard method to circumvent this difficulty is to find a set of maps  $S$  such that the  $S$ -equivalences are precisely the  $E$ -equivalences. This is an extremely difficult task, see [EKMM97, Section VIII.1], so it is not clear if a good notion of  $E$ -localisation exists for general model categories.

Instead, we will construct the *stable  $E$ -familiarisation*  $\mathcal{C}_E$  of  $\mathcal{C}$  which is the “closest” stably  $E$ -familiar model category to  $\mathcal{C}$ . We will then draw some conclusions about its properties which will show that this construction is the right choice for an analogue of  $E$ -localisation for general stable  $\mathcal{C}$ . For example, the first theorem will show that every Quillen adjunction

$$\mathcal{S} \rightleftarrows \mathcal{C}$$

will give rise to a Quillen adjunction

$$L_E\mathcal{S} \rightleftarrows \mathcal{C}_E.$$

The first question to answer is: what kind of maps do we want to invert in order to construct  $\mathcal{C}_E$ ? In a stably  $E$ -familiar model category  $\mathcal{D}$  any map of the form

$$X \wedge^L j : X \wedge^L A \rightarrow X \wedge^L B$$

for  $j : A \rightarrow B$  an  $E$ -equivalence of spectra and  $X \in \mathcal{D}$  is a weak equivalence. Hence we could try to localise  $\mathcal{C}$  at this class of maps. So we must find some set of maps  $S$  such that the  $S$ -equivalences equals this class.

We need a couple of technical results first. For this section we shall work with  $\mathcal{S}$ -model categories in the sense of [Hov99, Definition 4.2.18], where  $\mathcal{S}$  again denotes the model category of symmetric spectra. Such a model category  $\mathcal{D}$  is enriched, tensored and cotensored over symmetric spectra in simplicial sets and satisfies the appropriate analogue of Quillen’s (SM7) axiom for simplicial model categories. We shall refer to  $\mathcal{D}$  as being a *spectral model category*. We may also talk about  $L_E\mathcal{S}$ -model categories, where we use

the  $E$ -local model structure on  $\mathcal{S}$ . A spectral model category is in particular stable and simplicial, see [SS03, Lemma 3.5.2]. We will see later that the restriction to spectral model categories is not as big a restriction as it might seem at first.

We denote the pushout-product of two maps by  $\square$ , so for  $f : X \rightarrow Y$  and  $g : A \rightarrow B$  the pushout-product of  $f$  and  $g$  is

$$f \square g : X \wedge B \coprod_{X \wedge A} Y \wedge A \rightarrow Y \wedge B.$$

Recall that a set of maps  $S$  in a stable model category  $\mathcal{D}$  is said to be *stable* if the class of  $S$ -local objects is closed under suspension. By [BR14, Proposition 4.6] if  $\mathcal{D}$  and  $S$  are stable then so is  $L_S \mathcal{D}$ .

**Proposition 5.1.** *Let  $\mathcal{D}$  be a left proper, cellular and spectral model category. Let  $S$  be a stable set of maps in  $\mathcal{D}$ . Then  $L_S \mathcal{D}$  is also a spectral model category.*

*Proof.* Since  $\mathcal{D}$  is left proper and cellular,  $L_S \mathcal{D}$  exists by [Hir03, Theorem 4.1.1]. We must prove that if  $i$  is a cofibration of  $L_S \mathcal{D}$  and  $j$  is a cofibration of  $\mathcal{S}$  then  $i \square j$  is a cofibration of  $L_S \mathcal{D}$  that is a weak equivalence (in  $L_S \mathcal{D}$ ) if either of  $i$  or  $j$  is. Since  $\mathcal{D}$  is spectral and the cofibrations are unchanged by left Bousfield localisation, we know that  $i \square j$  is a cofibration whenever  $i$  and  $j$  are. Furthermore if  $j$  is an acyclic cofibration of symmetric spectra, then  $i \square j$  is a weak equivalence in  $\mathcal{D}$  and hence it is also an  $S$ -equivalence.

The third case is where  $i$  is an acyclic cofibration of  $L_S \mathcal{D}$  and  $j$  is a cofibration of symmetric spectra. We must show that  $i \square j$  is an  $S$ -equivalence. By [Hov99, Lemma 4.2.4] it suffices to prove this for  $j$  a generating cofibration of symmetric spectra and  $i$  a generating acyclic cofibration of  $L_S \mathcal{D}$ . By [HSS00, Proposition 3.4.2] we may assume that  $j$  is of the form

$$F_n K \rightarrow F_n L$$

where  $F_n$  is the left adjoint to evaluation at level  $n$ , and  $K$  and  $L$  are simplicial sets. By [Hir03, Proposition 4.5.1] the domain of  $i$  is cofibrant, so it follows that both the domain and codomain of  $i \square j$  are cofibrant. The set  $S$  is stable, so the class of  $S$ -equivalences in  $\text{Ho}(\mathcal{D})$  is closed under suspension and desuspension. Thus  $i \square j$  is an  $S$ -equivalence if and only if

$$\Sigma^n(i \square j) \cong i \square \Sigma^n j$$

is an  $S$ -equivalence for all  $n$ .

We know that  $\Sigma^n F_n K$  is weakly equivalent to  $F_0 K$  in  $\mathcal{S}$ . Hence for any cofibrant  $X \in \mathcal{D}$ ,

$$X \wedge \Sigma^n F_n K \rightarrow X \wedge F_0 K$$

is a weak equivalence of  $\mathcal{D}$ . We also know that the domains of the maps  $i \square \Sigma^n j$  and  $i \square (F_0 K \rightarrow F_0 L)$  are pushouts of cofibrations between cofibrant objects. It follows that  $i \square \Sigma^n j$  is weakly equivalent to the map  $i \square (F_0 K \rightarrow F_0 L)$ . The bifunctor

$$- \wedge F_0 - : \mathcal{D} \times \text{sSet} \rightarrow \mathcal{D}$$

gives  $\mathcal{D}$  the structure of a simplicial model category. We may now use [Hir03, Theorem 4.1.1], which states that since  $\mathcal{D}$  is simplicial, so is  $L_S \mathcal{D}$ . Consequently we see that  $i \square (F_0 K \rightarrow F_0 L)$  is an  $S$ -equivalence. Hence  $i \square j$  is also an  $S$ -equivalence and  $L_S \mathcal{D}$  is a spectral model category.  $\square$

**Proposition 5.2.** *Let  $\mathcal{D}$  be a left proper, cellular and spectral model category with generating cofibrations  $I_{\mathcal{D}}$  and generating acyclic cofibrations  $J_{\mathcal{D}}$ . Let  $J_E$  be the set of generating acyclic cofibrations for  $L_E \mathcal{S}$ . Define*

$$S = I_{\mathcal{D}} \square J_E = \{i \square j \mid i \in I_{\mathcal{D}}, j \in J_E\}.$$

*Then  $L_S \mathcal{D}$  is an  $L_E \mathcal{S}$ -model category and hence is stably  $E$ -familiar.*



*Proof.* The set  $J_E$  is closed under desuspension in the sense that for any element  $j \in J_E$  there is an element  $j'$  with  $\Sigma j' \simeq j$ . It follows that the same holds for  $S$ , so it is stable in the sense of [BR14, Definition 3.2]. Thus  $L_S \mathcal{D}$  is also a stable model category. By Lemma 5.1 it is also an  $\mathcal{S}$ -model category.

To see that it is an  $L_E \mathcal{S}$ -model category we only need to check that if  $i$  is a cofibration of  $L_S \mathcal{D}$  and  $j$  is an acyclic cofibration of  $L_E \mathcal{S}$  then  $i \square j$  is an  $S$ -equivalence. By [Hov99, Lemma 4.2.4] it suffices to prove this for  $i \in I_{\mathcal{D}}$  and  $j \in J_E$ . But then  $i \square j$  is an element of  $S$  and hence is an  $S$ -equivalence.  $\square$

**Proposition 5.3.** *Let  $\mathcal{D}$  be a left proper, cellular and spectral model category and  $S$  as in Proposition 5.2. Assume that the domains of the generating cofibrations of  $\mathcal{D}$  are cofibrant. Then if  $\mathcal{D}$  is a monoidal model category so is  $L_S \mathcal{D}$ .*

*Proof.* Since  $\mathcal{D}$  is spectral, the maps in  $S$  are cofibrations between cofibrant objects. Thus by [BR14, Lemma 6.1]  $L_S \mathcal{D}$  is monoidal if and only if

$$I_{\mathcal{D}} \square S = I_{\mathcal{D}} \square (I_{\mathcal{D}} \square J_E) \cong (I_{\mathcal{D}} \square I_{\mathcal{D}}) \square J_E$$

lies in the class of  $S$ -equivalences. As  $\mathcal{D}$  is monoidal,  $I_{\mathcal{D}} \square I_{\mathcal{D}}$  consists of cofibrations. By Proposition 5.2,  $L_S \mathcal{D}$  is an  $L_E \mathcal{S}$ -model category. Hence the pushout product of a cofibration of  $\mathcal{D}$  and an acyclic cofibration of  $L_E \mathcal{S}$  is an  $S$ -equivalence as required.  $\square$

We now show that this set  $S$  has the correct homotopical behaviour in terms of  $E$ -familiarity by giving another description of the weak equivalences of  $L_S \mathcal{D}$ .

**Proposition 5.4.** *Let  $\mathcal{D}$  be a left proper, cellular spectral model category, such that the domains of the generating cofibrations of  $\mathcal{D}$  are cofibrant. Let  $T$  be the class of maps*

$$T = \{X \wedge^L f \mid X \in \mathcal{D}, f \text{ is an } E\text{-equivalence of spectra}\}.$$

*Then the class of  $T$ -equivalences is equal to the class of  $S$ -equivalences.*

*Proof.* Take some cofibrant  $X \in \mathcal{D}$ . Then the functor

$$X \wedge - : L_E \mathcal{S} \rightarrow L_S \mathcal{D}$$

is a left Quillen functor by Proposition 5.2. Hence  $X \wedge -$  takes  $E$ -equivalences between cofibrant spectra to  $S$ -equivalences. Thus every element of  $T$  is a weak equivalence in  $L_S \mathcal{D}$ .

Now we will show that every element of  $S$  is also a  $T$ -equivalence. Consider  $i \square j \in S$  for  $i: X \rightarrow Y$  a generating cofibration of  $\mathcal{D}$  and  $j: A \rightarrow B$  a generating acyclic cofibration for  $L_E \mathcal{S}$ . Since  $X, Y, A$  and  $B$  are all cofibrant, the maps  $X \wedge j$  and  $Y \wedge j$  are in the class  $T$ . Let  $P$  be the domain of  $i \square j$ , then by [Hir03, Lemma 3.4.2], the map from  $Y \wedge A \rightarrow P$  is also a  $T$ -equivalence. It follows by the two-out-of-three property that  $i \square j$  is a  $T$ -equivalence.  $\square$

If the category  $\mathcal{D}$  is already stably  $E$ -familiar then the class  $T$  is already contained in the category of weak equivalences. Hence so is the set  $S$ , and  $\mathcal{D}$  is in fact an  $L_E \mathcal{S}$ -model category.

**Corollary 5.5.** *Let  $\mathcal{D}$  be a left proper cellular spectral model category that is stably  $E$ -familiar. Assume that the domains of the generating cofibrations of  $\mathcal{D}$  are cofibrant. Then  $\mathcal{D}$  is an  $L_E \mathcal{S}$ -model category.*  $\square$

6.  $E$ -FAMILIARISATION OF STABLE MODEL CATEGORIES

We now want to consider model categories that are not necessarily spectral. Consider a proper and cellular stable model category  $\mathcal{C}$ . By [BR14, Theorem 8.2]  $\mathcal{C}$  is Quillen equivalent to a spectral model category, namely the category  $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$  of symmetric spectra in simplicial objects in  $\mathcal{C}$  equipped with a non-standard model structure. Hence there is a Quillen equivalence which by abuse of notation we call

$$\Sigma^\infty : \mathcal{C} \rightleftarrows \mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C}) : \Omega^\infty$$

This model category  $\mathcal{D}$  is also proper and cellular. Furthermore, if the generating cofibrations for  $\mathcal{C}$  have cofibrant domains, then so do the generating cofibrations for  $\mathcal{D}$ .

**Theorem 6.1.** *Let  $\mathcal{C}$  be a stable, proper and cellular model category, such that the domains of the generating cofibrations of  $\mathcal{C}$  are cofibrant. Let  $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$ , with generating cofibrations  $I_{\mathcal{D}}$  and fibrant replacement  $\hat{f}$ . We set  $S = I_{\mathcal{D}} \square J_E$  as in Proposition 5.2.*

Define  $\mathcal{C}_E$  to be the left Bousfield localisation of  $\mathcal{C}$  at the set of maps  $\Omega^\infty \hat{f} S$ . Then

- (1)  $\mathcal{C}_E$  is stably  $E$ -familiar,
- (2) the weak equivalences of  $\mathcal{C}_E$  are the  $T'$ -equivalences, for  $T'$  the class below

$$T' = \{X \wedge^L f \mid X \in \mathcal{C}, f \text{ is an } E\text{-equivalence of spectra}\}$$

- (3) if  $F : \mathcal{C} \rightarrow \mathcal{E}$  is a left Quillen functor and  $\mathcal{E}$  is stably  $E$ -familiar, then  $F$  factors over  $\mathcal{C} \rightarrow \mathcal{C}_E$ , i.e.  $F : \mathcal{C}_E \rightarrow \mathcal{E}$  is also a left Quillen functor.

*Proof.* The model categories  $\mathcal{C}$  and  $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$  are Quillen equivalent. Hence [Hir03, Theorem 3.3.20] tell us that the adjunction

$$\Sigma^\infty : \mathcal{C} \rightleftarrows \mathcal{D} : \Omega^\infty.$$

induces a Quillen equivalence between  $L_{\Omega^\infty \hat{f} S} \mathcal{C}$  and  $L_{\Sigma^\infty \hat{c} \Omega^\infty \hat{f} S} \mathcal{D}$ . (Here,  $\hat{c}$  denotes the cofibrant replacement in  $\mathcal{C}$ .) The model category  $L_{\Sigma^\infty \hat{c} \Omega^\infty \hat{f} S} \mathcal{D}$  is equal to  $L_S \mathcal{D}$  since  $(\Sigma^\infty, \Omega^\infty)$  is a Quillen equivalence. Thus we have a Quillen equivalence between  $\mathcal{C}_E = L_{\Omega^\infty \hat{f} S} \mathcal{C}$  and  $L_S \mathcal{D}$ . The second category is stably  $E$ -familiar by Proposition 5.2. Hence so is  $\mathcal{C}_E$  by [BR11, Lemma 7.10].

We may also conclude that the left derived functor of  $\Sigma^\infty$  induces a bijection between the weak equivalences of  $\mathcal{C}_E$  (considered as a class in  $\text{Ho } \mathcal{C}$ ) and the  $S$ -equivalences of  $\text{Ho } \mathcal{D}$ . Proposition 5.4 tells us that the class of  $S$ -equivalences in  $\mathcal{D}$  is equal to the class of  $T$ -equivalences where

$$T = \{X \wedge^L f \mid X \in \mathcal{D}, f \text{ is an } E\text{-equivalence of spectra}\}.$$

Consider the class of maps

$$T' = \{X \wedge^L f \mid X \in \mathcal{C}, f \text{ is an } E\text{-equivalence of spectra}\}.$$

Let  $\mathbb{L}\Sigma^\infty$  and  $\mathbb{R}\Omega^\infty$  denote the left and right derived functors of  $\Sigma^\infty$  and  $\Omega^\infty$  respectively. By [Len12, Theorem 6.3]

$$\mathbb{L}\Sigma^\infty(X \wedge^L f) = (\mathbb{L}\Sigma^\infty X) \wedge^L f.$$

Hence  $\mathbb{L}\Sigma^\infty$  takes elements of  $T'$  to elements of  $T$ . Consider some element  $Y \wedge^L f$  of  $T$ . This is weakly equivalent to

$$(\mathbb{L}\Sigma^\infty \mathbb{R}\Omega^\infty Y) \wedge^L f$$

and hence is in  $\mathbb{L}\Sigma^\infty T'$ . Thus the derived functor of  $\Sigma^\infty$  induces a bijection between the class  $T'$  and the class  $T$  up to weak equivalence. As a consequence the derived functor of  $\Sigma^\infty$  induces a bijection between the class of  $T'$ -equivalences and the class of  $T$ -equivalences. It follows that the  $T'$ -equivalences must be the class of weak equivalences of  $\mathcal{C}_E$ .

For the final point, let  $F: \mathcal{C} \rightarrow \mathcal{E}$  be a left Quillen functor. If  $\mathcal{E}$  is stably  $E$ -familiar, then the left derived functor of  $F$  takes the  $T'$ -equivalences to weak equivalences of  $\mathcal{E}$ . Hence  $F: \mathcal{C}_E \rightarrow \mathcal{E}$  is also a left Quillen functor.  $\square$

**Remark 6.2.** *Let  $\mathcal{C}$  be a stable, proper and cellular model category, such that the domains of the generating cofibrations of  $\mathcal{C}$  are cofibrant. Then the above result says that  $\mathcal{C}_E$  is the “closest” stably  $E$ -familiar model category to  $\mathcal{C}$ .*

In particular a model category  $\mathcal{C}$  is stably  $E$ -familiar if and only if  $\mathcal{C}_E = \mathcal{C}$ .

**Remark 6.3.** *The assumptions on  $\mathcal{C}$  are more reasonable than they might seem in practice. Since we want to perform a left Bousfield localisation, we will have to assume that  $\mathcal{C}$  is left proper and cellular. To assume that  $\mathcal{C}$  is also right proper is not too much of a restriction.*

*We also need another assumption: that the domains of the generating cofibrations of  $\mathcal{C}$  are cofibrant. This is a subtle assumption that occurs elsewhere in the literature, for example in [Hov01]. We note that this assumption holds for almost all of the cofibrantly generated model categories that arise naturally.*

It is easy to check that the homotopy mapping spectra for  $\mathcal{C}_E$  are given by the formula below, where  $Y_E$  is the fibrant replacement of  $Y$  in  $\mathcal{C}_E$ .

$$\mathbb{R} \operatorname{Map}_{\mathcal{C}_E}(X, Y) = \mathbb{R} \operatorname{Map}_{\mathcal{C}}(X, Y_E)$$

In particular, this mapping spectrum is  $E$ -local. We can use this to draw some immediate consequences of  $E$ -familiarisation.

For example, the chromatic Johnson–Wilson theories  $E(n)$  satisfy

$$L_{E(n-1)} L_{E(n)} = L_{E(n-1)}$$

[Rav92, 7.5.3]. Thus,

**Corollary 6.4.** *For a proper and cellular stable model category  $\mathcal{C}$  we have*

$$(\mathcal{C}_{E(n)})_{E(n-1)} = \mathcal{C}_{E(n-1)}.$$

$\square$

We can further use our knowledge of stably  $E$ -familiar algebraic model categories described at the end of Section 4 to read off the following corollaries.

**Corollary 6.5.** *Let  $\mathcal{C}$  be an algebraic model category and  $K(n)$  the  $n^{\text{th}}$  Morava– $K$ -theory for  $n \geq 1$ . Then  $\mathcal{C}_{K(n)}$  is trivial.*  $\square$

**Corollary 6.6.** *Let  $\mathcal{C}$  be an algebraic model category and let  $E(n)$  denote the  $n^{\text{th}}$  chromatic Johnson–Wilson spectrum. Then  $\mathcal{C}_{E(n)} = \mathcal{C}_{H\mathbb{Q}}$ .*  $\square$

If we assume that localisation at  $E$  is smashing, we can obtain a nicer description of the weak equivalences of  $\mathcal{C}_E$ : in the smashing case  $\mathcal{C}_E$  is precisely the “naive” localisation of  $\mathcal{C}$  at  $L_E S^0$  as described in the introduction of Section 5. That is, the left Bousfield localisation of  $\mathcal{C}$  at the class of  $L_E S^0$ -equivalences (which we denote as  $L_{L_E S^0} \mathcal{C}$ ) exists and is equal to  $\mathcal{C}_E$ . With this extra assumption we also see that

$$\mathcal{C}_E = \mathcal{C}_{L_E S^0}.$$

However, for a general model category  $\mathcal{C}$  and smashing  $E$  it is unclear whether the model category  $L_{L_E S^0} \mathcal{C}$  exists and if it would be Quillen equivalent to  $\mathcal{C}_E$ .

**Lemma 6.7.** *In addition to the assumptions of Theorem 6.1, assume that localisation at  $E$  is smashing. Then a map  $f$  in  $\mathcal{C}_E$  is a weak equivalence if and only if  $f \wedge^L L_E S^0$  is a weak equivalence in  $\mathcal{C}$ . Hence the weak equivalences of  $\mathcal{C}_E$  are precisely the  $L_E S^0$ -equivalences.*

*Proof.* We first show the statement for a spectral model category  $\mathcal{D}$ . Recall the model category  $L_S\mathcal{D}$  for  $S$  the set  $I_{\mathcal{D}} \sqcup J_E$  from the previous section. We will show that the  $S$ -equivalences are precisely the  $L_ES^0$ -equivalences of  $\mathcal{D}$ .

Every map in the set  $S$  is an  $L_ES^0$ -equivalence, hence every  $S$ -equivalence is a  $L_ES^0$ -equivalence. Now take some  $L_ES^0$ -equivalence  $f: X \rightarrow Y$  in  $\mathcal{D}$ . The map

$$X \rightarrow X \wedge^L L_ES^0$$

is a  $T$ -equivalence with

$$T = \{X \wedge^L f \mid X \in \mathcal{D}, f \text{ is an } E\text{-equivalence of spectra}\}$$

defined earlier in this section. Hence it is an  $S$ -equivalence. Thus the commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X \wedge^L L_ES^0 & \xrightarrow{f \wedge^L L_ES^0} & Y \wedge^L L_ES^0 \end{array}$$

shows that  $f$  is  $S$ -equivalent to a weak equivalence in  $\mathcal{D}$ . Weak equivalences in  $\mathcal{D}$  are in particular  $S$ -equivalences, so by the 2-out-of-3 axiom of model categories,  $f$  must be an  $S$ -equivalence.

To move this result from a spectral  $\mathcal{D}$  to a general  $\mathcal{C}$  we use a similar argument to that of the second point of Theorem 6.1. The Quillen equivalence  $(\Sigma^\infty, \Omega^\infty)$  takes the  $L_ES^0$ -equivalences of  $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$  bijectively to the  $L_ES^0$ -equivalences of  $\mathcal{C}$ . It follows that the  $L_ES^0$ -equivalences of  $\mathcal{C}$  are precisely the weak equivalences of  $\mathcal{C}_E$ .  $\square$

The following corollary shows that stable  $E$ -familiarisation restricts to  $E$ -localisation in the case of spectra. This shows that the notion of  $\mathcal{C}_E$  is indeed a good candidate for an analogue of  $E$ -localisation of a general  $\mathcal{C}$ .

**Corollary 6.8.** *Consider the category of modules over a ring spectrum  $R$ . Then*

$$(R\text{-mod})_E = L_E(R\text{-mod})$$

*where the right hand side is the naive localisation of  $R\text{-mod}$ . It has weak equivalences those maps of  $R$ -modules which forget to  $E$ -equivalences of spectra and the same cofibrations as  $R\text{-mod}$ . In particular*

$$\mathcal{S}_E = L_ES.$$

*Proof.* We start by noting that  $L_E(R\text{-mod})$  is equal to the model structure of  $R$ -modules in  $L_ES$  from [SS00, Theorem 4.1]. Hence this model structure has generating sets of cofibrations and acyclic cofibrations  $R \wedge I_S$  and  $R \wedge J_E$  [SS00, Lemma 2.3].

We claim that every map in  $R \wedge J_E$  is a  $T'$ -equivalence, where  $T'$  is from Theorem 6.1. We know that the domains of  $J_E$  are cofibrant, hence  $R \wedge j$  is weakly equivalent to  $R \wedge^L j$  for any  $j \in J_E$ . Thus the claim holds. It follows that every acyclic cofibration of  $L_E(R\text{-mod})$  is a weak equivalence (and also a cofibration) of  $(R\text{-mod})_E$ .

We must now show the converse. Every acyclic cofibration of  $(R\text{-mod})_E$  is an  $E$ -equivalence of underlying spectra and hence is an acyclic cofibration of  $L_E(R\text{-mod})$ . Thus the two model structures agree.  $\square$

We can now give a simple proof that stable  $E$ -familiarisation preserves Quillen equivalences.

**Proposition 6.9.** *Let  $\mathcal{C}$  and  $\mathcal{E}$  be proper, cellular and stable model categories such that the domains of their generating cofibrations are cofibrant. Let*

$$F: \mathcal{C} \rightleftarrows \mathcal{E}: G$$

be a Quillen equivalence. Then there is a Quillen equivalence between the  $E$ -familiarised model categories

$$F : \mathcal{C}_E \rightleftarrows \mathcal{E}_E : G.$$

*Proof.* Composing  $F$  with the identity on  $\mathcal{E}$  gives us a left Quillen functor

$$F : \mathcal{C} \rightarrow \mathcal{E}_E$$

and  $\mathcal{E}_E$  is of course stably  $E$ -familiar. Hence by the universal property of  $\mathcal{C}_E$  proved in Theorem 6.1 we have a left Quillen functor  $F : \mathcal{C}_E \rightarrow \mathcal{E}_E$ . We now need to show that gives us a Quillen equivalence. We do so using Proposition 5.4 and the method of the second part of the proof of Theorem 6.1.

Let  $T$  be the class of maps

$$T = \{A \wedge^L f \mid A \in \mathcal{C}, f \text{ is an } E\text{-equivalence of spectra}\}.$$

Similarly, let  $T'$  be the class of maps

$$T' = \{B \wedge^L f \mid B \in \mathcal{E}, f \text{ is an } E\text{-equivalence of spectra}\}.$$

Then  $\mathcal{C}_E = L_T \mathcal{C}$  and  $\mathcal{E}_E = L_{T'} \mathcal{E}$ . Let  $\mathbb{L}F$  and  $\mathbb{R}G$  denote the left and right derived functors of  $F$  and  $G$  respectively. By [Hir03, Theorem 3.3.20], the adjunction  $(F, G)$  induces a Quillen equivalence between  $L_T \mathcal{C}$  and  $L_{\mathbb{L}F(T)} \mathcal{E}$ . But the set  $\mathbb{L}F(T)$  is isomorphic in  $\text{Ho } \mathcal{E}$  to the set  $T'$  because Quillen equivalences induce equivalences of  $\text{Ho}(\mathcal{S})$ -module categories [Len12, Theorem 6.3].  $\square$

**Remark 6.10.** *One could try to prove an analogue of Proposition 5.3 and show that if  $\mathcal{C}$  is monoidal then so is  $\mathcal{C}_E$ . This would require the adjunction  $(\Sigma^\infty, \Omega^\infty)$  at the beginning of this section to be monoidal. However we do not know if this is the case.*

## 7. EXAMPLES

Let  $\mathcal{C}$  be a spectral model category, such that the domains of its generating cofibrations are cofibrant. (Recall from [BR14, Theorem 7.2] that any stable, proper and cellular model category is Quillen equivalent to a spectral one.) Assume that  $\mathcal{C}$  has a set of compact generators for its homotopy category, [SS03, Definition 2.1.2]. Schwede and Shipley prove in the above-mentioned paper that any such  $\mathcal{C}$  is Quillen equivalent to a category  $\text{mod-}\mathcal{E}$  where  $\mathcal{E}$  can be thought of as a “ring spectrum with several objects”. In the case of  $\mathcal{C}$  having a single compact generator,  $\mathcal{E}$  is simply a ring spectrum.

Let us briefly recap some of the definitions and constructions of that result. Let  $\mathcal{G}$  denote the set of generators of  $\mathcal{C}$ . Then the  $\mathcal{S}$ -enriched category  $\mathcal{E}$  is simply defined as the full  $\mathcal{S}$ -enriched subcategory of  $\mathcal{C}$  with objects  $\mathcal{G}$ . An object  $M \in \text{mod-}\mathcal{E}$  consists of a spectrum  $M(G)$  for each  $G \in \mathcal{G}$  plus morphisms of spectra

$$\mathcal{E}(G', G) \wedge M(G) \longrightarrow M(G') \text{ for } G, G' \in \mathcal{G}$$

satisfying certain coherence conditions. By adjunction, such an  $M$  is the same as a contravariant spectral functor from  $\mathcal{E}$  to  $\mathcal{S}$ . A standard example of an object of  $\text{mod-}\mathcal{E}$  is given by the spectral functor  $\mathcal{E}(-, G)$  for fixed  $G \in \mathcal{G}$ . The model structure on  $\text{mod-}\mathcal{E}$  is has weak equivalences and fibrations defined objectwise [SS03, Theorem A.1.1]. This means that a natural transformation  $f : M \longrightarrow N$  is a weak equivalence or a fibration if and only if

$$f_G : M(G) \longrightarrow N(G)$$

is so for each  $G \in \mathcal{G}$ . Theorem 3.9.3 of [SS03] then describes a Quillen equivalence

$$\text{Hom}(\mathcal{G}, -) : \mathcal{C} \rightleftarrows \text{mod-}\mathcal{E} : - \wedge_{\mathcal{E}} \mathcal{G}$$

for spectral  $\mathcal{C}$ .

This is a highly useful description of a stable model category and we would like to obtain a description of the  $E$ -familiarisation  $\mathcal{C}_E$  of  $\mathcal{C}$  in terms of  $\text{mod-}\mathcal{E}$ . We note that this is a rather special case as not every stable model category has a set of compact generators [HS99, Corollary B.13].

By Proposition 6.9 we know that  $\mathcal{C}_E$  and  $(\text{mod-}\mathcal{E})_E$  are Quillen equivalent, so we shall find another description of  $(\text{mod-}\mathcal{E})_E$ .

Since  $\text{mod-}\mathcal{E}$  is a spectral model category, it is easily seen that  $(\text{mod-}\mathcal{E})_E$  is given by  $L_S \text{mod-}\mathcal{E}$  as in Proposition 5.2. Recall that  $S = I \square J_E$  for  $I$  the set of generating cofibrations for  $\text{mod-}\mathcal{E}$ . Hence in  $(\text{mod-}\mathcal{E})_E$  any map of the form below is a weak equivalence.

$$\mathcal{E}(-, G) \wedge (i \square j)$$

In the above,  $G$  is a cofibrant and fibrant replacement of one of the compact generators for  $\mathcal{C}$ ,  $i$  is a generating cofibration for  $\mathcal{S}$  and  $j$  is a generating acyclic cofibration for  $L_E \mathcal{S}$ .

We can make another model structure on  $\text{mod-}\mathcal{E}$  by taking the same cofibrations as before, but taking the generating set of acyclic cofibrations to be those maps of the form

$$\mathcal{E}(-, G) \wedge j$$

for  $G$  a generator and  $j$  a generating acyclic cofibration for  $L_E \mathcal{S}$ . We shall call this set of maps  $K$  and let  $\text{mod-}\mathcal{E}_K$  denote the corresponding model structure. One can either check directly that these sets give a model structure or one can alter [SS03, Theorem A.1.1] to use  $L_E \mathcal{S}$  instead of  $\mathcal{S}$ .

We claim that this model structure equals the model structure of  $(\text{mod-}\mathcal{E})_E$ . An element of  $K$  can be described as

$$\mathcal{E}(-, G) \wedge ((* \rightarrow S^0) \square j).$$

Hence every element of  $K$  is an acyclic cofibration of  $(\text{mod-}\mathcal{E})_E$ . Conversely,  $\text{mod-}\mathcal{E}$  equipped with this new model structure is stably  $E$ -familiar. Hence the identity functor  $(\text{mod-}\mathcal{E})_E \rightarrow \text{mod-}\mathcal{E}_K$  is a left Quillen functor. Hence every acyclic cofibration of  $(\text{mod-}\mathcal{E})_E$  is an acyclic cofibration of  $\text{mod-}\mathcal{E}_K$ . Thus these two model structures have the same cofibrations and acyclic cofibrations. We have therefore shown the following.

**Proposition 7.1.** *The model category  $(\text{mod-}\mathcal{E})_E$  is the category of contravariant spectral functors from  $\mathcal{E}$  to  $L_E \mathcal{S}$ , equipped with the model structure where fibrations and weak equivalences are defined objectwise. Thus the fibrant objects are those functors  $M$  such that  $M(G)$  is fibrant in  $\mathcal{S}$  and  $E$ -local for all  $G \in \mathcal{E}$ .  $\square$*

Consider the case where  $\mathcal{C}$  has a single compact generator. Following the above we can replace this by a category of functors to  $\mathcal{S}$ . Indeed, [SS03, Theorem 3.1.1] states that  $\mathcal{C}$  is Quillen equivalent to the category of  $R$ -modules,  $\text{mod-}R$ , for some ring spectrum  $R$ . In this case, the above proposition recovers the result of Corollary 6.8.

## 8. $E$ -FAMILIARISATION AND HOMOTOPY PUSHOUTS

We want to give another description of  $\mathcal{C}_E$  via a universal property. We will relate  $\mathcal{C}_E$  to a pushout of model categories. While the pullback of model categories is well-understood, [Ber11], the pushout is more complicated and is not often used. Roughly speaking, the homotopy pushout of a corner diagram of Quillen adjunctions

$$\mathcal{C} \rightleftarrows \mathcal{D} \rightleftarrows \mathcal{E}$$

is supposed to be a model category  $\mathcal{P}$  that satisfies a universal property analogous to the pushout of a diagram within a category. Unfortunately, the homotopy-theoretic pushout construction is rather delicate and its existence and description not always clear.

However there is a special case where we can construct pushouts of model categories and verify that they have the correct universal property. By working in a particular context, we avoid the general question of whether homotopy pushouts of model categories exist in general.

Let  $\mathcal{M}_2$  be a left Bousfield localisation of  $\mathcal{M}_1$  at a class of maps  $W$ . Without loss of generality we assume that the maps in  $W$  are morphisms between cofibrant objects. (If the elements of  $W$  did not satisfy this, one can replace them with weakly equivalent morphisms between cofibrant objects. This would then give rise to the same Bousfield localisations.) In particular, this gives us a Quillen pair

$$\text{Id} : \mathcal{M}_1 \rightleftarrows \mathcal{M}_2 = L_W \mathcal{M}_1 : \text{Id}$$

Assume that we have a Quillen adjunction

$$F : \mathcal{M}_1 \rightleftarrows \mathcal{N}_1 : G.$$

We are now going to discuss the homotopy pushout of the corner diagram below for this special case

$$L_W \mathcal{M}_1 = \mathcal{M}_2 \xleftarrow{\sim} \mathcal{M}_1 \rightleftarrows \mathcal{N}_1.$$

**Definition 8.1.** *The homotopy pushout of the above diagram is defined, if it exists, as the Bousfield localisation  $L_{\mathbb{L}FW} \mathcal{N}_1$  of  $\mathcal{N}_1$ . Here,  $\mathbb{L}F$  denotes the left derived functor of  $F$ .*

To justify this definition we need to see that  $\mathcal{N}_2 = L_{\mathbb{L}FW} \mathcal{N}_1$  (provided it exists) satisfies the desired properties that a homotopy pushout is supposed to have. First we note that by [Hir03, Theorem 3.3.20]  $F$  and  $G$  induce a Quillen adjunction

$$F : \mathcal{M}_2 \rightleftarrows \mathcal{N}_2 : G$$

Assume that there is a model category  $\mathcal{D}$  with Quillen adjunctions

$$\begin{array}{ccc} \mathcal{M}_2 & \rightleftarrows & \mathcal{D} \\ F' : \mathcal{N}_1 & \rightleftarrows & \mathcal{D} : G' \end{array}$$

such that in the diagram below, the two different composites of left adjoints from  $\mathcal{M}_1$  to  $\mathcal{D}$  agree up to natural isomorphism.

$$\begin{array}{ccc} \mathcal{M}_1 & \rightleftarrows & \mathcal{N}_1 \\ \updownarrow & & \updownarrow \\ \mathcal{M}_2 & & \mathcal{D} \end{array}$$

Because the vertical functors in the square below are simply identity functors it follows immediately that we may add  $\mathcal{N}_2$  and obtain a commutative diagram of adjoint pairs.

$$\begin{array}{ccc} \mathcal{M}_1 & \rightleftarrows & \mathcal{N}_1 \\ \updownarrow & & \updownarrow \\ \mathcal{M}_2 & \rightleftarrows & \mathcal{N}_2 \\ & \searrow & \downarrow \\ & & \mathcal{D} \end{array}$$

We must check that the adjunction below is a Quillen adjunction.

$$F' : \mathcal{N}_2 \rightleftarrows \mathcal{D} : G'$$

The model category  $\mathcal{N}_2$  is the Bousfield localisation of  $\mathcal{N}_1$  with respect to the class of maps  $Ff$  where  $f$  is a weak equivalence between cofibrant objects of  $\mathcal{M}_2$ . Thus  $(F' \circ F)(f)$  is a weak equivalence in  $\mathcal{D}$ . This means that  $F'$  uniquely factors over  $\mathcal{N}_2$ . Furthermore, by construction,  $\mathcal{N}_2$ , if it exists, is unique up to Quillen equivalence.

Recall that the stable  $E$ -familiarisation  $\mathcal{C}_E$  satisfies the following universal property. Given a left Quillen functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  with  $\mathcal{D}$  stably  $E$ -familiar,  $F$  also gives rise to a left Quillen functor  $\mathcal{C}_E \rightarrow \mathcal{D}$  via

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ id \downarrow & \nearrow & \\ \mathcal{C}_E & & \end{array}$$

This fact allows us to relate  $\mathcal{C}_E$  and certain homotopy pushouts. Let  $X \in \mathcal{C}$  be fibrant and cofibrant. Then we have a Quillen adjunction

$$X \wedge - : \mathcal{S} \rightleftarrows \mathcal{C} : \text{Map}_{\mathcal{C}}(X, -).$$

Using Definition 8.1 we can read off the following for a proper and cellular stable model category  $\mathcal{C}$ .

**Lemma 8.2.** *The homotopy pushout  $\mathcal{P}_X$  of the diagram*

$$L_E \mathcal{S} \rightleftarrows \mathcal{S} \rightleftarrows \mathcal{C}$$

*exists and is the Bousfield localisation of  $\mathcal{C}$  with respect to the set of maps below, where  $J_E$  is the set of generating acyclic cofibrations of  $L_E \mathcal{S}$ .*

$$X \wedge^L J_E = \{X \wedge^L j \mid j \in J_E\}$$

□

So in particular we know that this homotopy pushout exists. Because  $\mathcal{C}_E$  is stably  $E$ -familiar we have a commutative square of Quillen adjunctions

$$\begin{array}{ccc} \mathcal{S} & \rightleftarrows & \mathcal{C} \\ \updownarrow & & \updownarrow \\ L_E \mathcal{S} & \rightleftarrows & \mathcal{C}_E \end{array}$$

By the universal property of  $\mathcal{P}_X$ , there is a Quillen adjunction  $\mathcal{P}_X \rightleftarrows \mathcal{C}_E$  for each  $X$ . We can show that  $\mathcal{C}_E$  is the “closest” model category to those pushouts in the following sense.

**Theorem 8.3.** *Let  $\mathcal{C}$  be a stable, proper and cellular model category, such that the domains of the generating cofibrations of  $\mathcal{C}$  are cofibrant. The Quillen adjunction*

$$\mathcal{C} \rightleftarrows \mathcal{C}_E$$

*factors over*

$$\mathcal{P}_X \rightleftarrows \mathcal{C}_E$$

*for all fibrant-cofibrant  $X \in \mathcal{C}$ . If there is any other stable  $\mathcal{D}$  with a Quillen adjunction*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

*that factors over*

$$\mathcal{P}_X \rightleftarrows \mathcal{D}$$

*for all fibrant-cofibrant  $X$ , then  $(F, G)$  also factors over  $\mathcal{C}_E$ .*



*Proof.* The pushout  $\mathcal{P}_X$  is defined as the Bousfield localisation of  $\mathcal{C}$  at the set of maps  $X \wedge^L j$  with  $j \in J_E$ . By Proposition 5.4 we know that  $\mathcal{C}_E$  is the localisation of  $\mathcal{C}$  at the class of maps of the form  $X \wedge^L f$  for  $f$  an  $E$ -equivalence of spectra. Thus we see that for every  $X \in \mathcal{C}$  the identity gives us a Quillen adjunction

$$\mathrm{Id} : \mathcal{P}_X \rightleftarrows \mathcal{C}_E : \mathrm{Id}$$

because every weak equivalence in  $\mathcal{P}_X$  is also a weak equivalence in  $\mathcal{C}_E$ .

If the given Quillen adjunction  $(F, G)$  induces a Quillen adjunction

$$F : \mathcal{P}_X \rightleftarrows \mathcal{D} : G,$$

then  $F$  sends all morphisms of the form  $X \wedge^L j$ , for  $j \in J_E$ , to weak equivalences in  $\mathcal{D}$ . Hence  $F$  also sends all maps of the form  $X \wedge^L f$ , for  $f$  an  $E$ -equivalence of spectra, to weak equivalences in  $\mathcal{D}$ .

If  $(F, G)$  gives such Quillen adjunctions for all fibrant-cofibrant  $X$  then it must send any map of the form  $X \wedge^L f$  with  $X$  fibrant-cofibrant and  $f$  an  $E$ -equivalence of spectra to a weak equivalence in  $\mathcal{D}$ . Thus it induces a Quillen adjunction

$$\mathcal{C}_E \rightleftarrows \mathcal{D}$$

by Theorem 6.1, which is what we wanted to prove.  $\square$

## 9. MODULAR RIGIDITY FOR $E$ -LOCAL SPECTRA

We can show that stable frames encode all homotopical information of the  $E$ -local stable homotopy category. The triangulated structure of  $\mathrm{Ho}(L_E\mathcal{S})$  alone is not sufficient for this: given just a triangulated equivalence

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

for a stable model category  $\mathcal{C}$  does not imply in general that  $L_E\mathcal{S}$  and  $\mathcal{C}$  are Quillen equivalent. In fact, Quillen equivalence can only be deduced from a triangulated equivalence of homotopy categories in some very special cases. To this date, the only nontrivial cases known of this ‘rigidity’ are the stable homotopy category itself [Sch07] and the case  $E = K_{(2)}$  [Roi07]. However, if we do not only have a triangulated equivalence as above but also assume that this equivalence is a  $\mathrm{Ho}(\mathcal{S})$ -module equivalence, we can show that  $L_E\mathcal{S}$  and  $\mathcal{C}$  are Quillen equivalent.

We now give a more general version of [BR11, Theorem 9.5], in particular the assumption that  $E$  is smashing is no longer required.

**Theorem 9.1.** *Let  $\mathcal{C}$  be a stable model category. Assume we have an equivalence of triangulated categories*

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

*then  $L_E\mathcal{S}$  and  $\mathcal{C}$  are Quillen equivalent if and only if  $\Phi$  is an equivalence of  $\mathrm{Ho}(\mathcal{S})$ -module categories.*

*Proof.* The “only if” part is true by [Len12, Theorem 6.3]: a Quillen equivalence induces a  $\mathrm{Ho}(\mathcal{S})$ -module equivalence.

Now let us assume that we have a  $\mathrm{Ho}(\mathcal{S})$ -module equivalence

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

It follows that  $\Phi^{-1}$  induces a weak equivalence of homotopy mapping spectra

$$\Phi^{-1} : \mathbb{R}\mathrm{Map}_{\mathcal{C}}(X, Y) \longrightarrow \mathbb{R}\mathrm{Map}_{L_E\mathcal{S}}(\Phi^{-1}X, \Phi^{-1}Y)$$

for  $X, Y \in \mathcal{C}$ . The right-hand-side is an  $E$ -local spectrum as  $L_E\mathcal{S}$  is stably  $E$ -familiar. Hence every homotopy mapping spectrum of  $\mathcal{C}$  is  $E$ -local, so  $\mathcal{C}$  is stably  $E$ -familiar by [BR11, Theorem 7.8].

Thus for fibrant and cofibrant  $X \in \mathcal{C}$ , the Quillen functor

$$X \wedge - : \mathcal{S} \longrightarrow \mathcal{C}$$

factors over  $L_E \mathcal{S}$  as a Quillen functor

$$X \wedge - : L_E \mathcal{S} \longrightarrow \mathcal{C}.$$

Now let  $X$  be a cofibrant–fibrant replacement of  $\Phi(S^0)$ . Because  $\Phi$  is a  $\mathrm{Ho}(\mathcal{S})$ –module equivalence we see that

$$X \wedge^L (-) = \Phi(S^0) \wedge^L (-) = \Phi(S^0 \wedge^L -) = \Phi(-).$$

This means that  $\Phi$  is derived from a Quillen functor. This Quillen functor must therefore be a Quillen equivalence, which is what we wanted to prove.  $\square$

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