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HOMOLOGICAL LOCALISATION OF MODEL CATEGORIES

DAVID BARNES AND CONSTANZE ROITZHEIM

ABSTRACT. One of the most useful methods for studying the stable homotopy category is localising at some spectrum E . For an arbitrary stable model category we introduce a candidate for the E -localisation of this model category. We study the properties of this new construction and relate it to some well-known categories.

1. INTRODUCTION

The stable homotopy category is spectacularly complicated and yet of fundamental importance to homotopy theorists. A standard and highly successful method of dealing with this complexity is to “filter out” some of this information via a Bousfield localisation. In return we obtain a more structured category with useful and interesting patterns.

More precisely, we choose some homology theory E_* and replace the stable homotopy category $\mathrm{Ho}(\mathcal{S})$ with $\mathrm{Ho}(L_E\mathcal{S})$, the full subcategory of $\mathrm{Ho}(\mathcal{S})$ with objects the E -local spectra. This means that in the passage from $\mathrm{Ho}(\mathcal{S})$ to $\mathrm{Ho}(L_E\mathcal{S})$, the E_* -isomorphisms are formally inverted. Bousfield’s paper [Bou79] is the original source of this idea.

There are a number of other model categories whose homotopy categories share many of the properties of $\mathrm{Ho}(\mathcal{S})$, namely stable model categories. It would be advantageous if we could generalise the notion of E -localisation to this class of categories. Thus we are interested in the construction of a *homological localisation* of a *stable* model category, one that is the analogue of forming $\mathrm{Ho}(L_E\mathcal{S})$ from $\mathrm{Ho}(\mathcal{S})$.

The main motivation comes again from the study of the stable homotopy category. In order to understand spectra, $\mathrm{Ho}(\mathcal{S})$ and its various E -localisations it is necessary to relate $\mathcal{S}$ and $L_E\mathcal{S}$ to other stable model categories $\mathcal{C}$. For example, one can study to what extent there is a stable model category $\mathcal{C}$ whose homotopy category “models” $\mathrm{Ho}(L_E\mathcal{S})$ and how similar $\mathcal{C}$ is to $L_E\mathcal{S}$ in terms of higher homotopy behaviour. To make those links it would be a desirable tool to have the corresponding E -localisations of $\mathcal{C}$ in order to compare E -local spectra to other counterparts related to $\mathcal{C}$.

A stable model category $\mathcal{C}$ is a model category whose associated homotopy category $\mathrm{Ho}(\mathcal{C})$ is triangulated via the construction of [Hov99, Section 7]. Lenhardt proved in [Len12] that $\mathrm{Ho}(\mathcal{C})$ is a module over $\mathrm{Ho}(\mathcal{S})$ whenever $\mathcal{C}$ is a stable model category. Hence we have a tensor product

$$- \wedge^L -: \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

and an enrichment of $\mathrm{Ho}(\mathcal{C})$ in $\mathrm{Ho}(\mathcal{S})$. This technique is called *stable frames*.

Using this action on the homotopy category of a stable model category one could try to make a new model structure on $\mathcal{C}$ such that the weak equivalences are the “ E_* -isomorphisms”: those maps $f: X \rightarrow Y$ in $\mathcal{C}$ such that

$$f \wedge^L E: X \wedge^L E \rightarrow Y \wedge^L E$$

is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. Such a model structure would deserve the name $L_E\mathcal{C}$. The machinery that allows one to create new model structures with larger collections of weak

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equivalences is Bousfield localisation, see [Hir03, Part I]. But it seems particularly difficult to check that $L_E\mathcal{C}$ exists for general $\mathcal{C}$. For spectra, the argument appears in [EKMM97, Section VIII.1] and requires numerous unpleasant cardinality arguments.

For well-behaved stable model categories $\mathcal{C}$ we are going to produce a new model structure $\mathcal{C}_E$ that avoids such set-theoretic awkwardness. This $\mathcal{C}_E$ is a good candidate for the E -localisation of $\mathcal{C}$ because of the following universal property: $\mathcal{C}_E$ is the “closest” model category to $\mathcal{C}$ such that any Quillen adjunction from spectra to $\mathcal{C}$

$$\mathcal{S} \rightleftarrows \mathcal{C}$$

gives rise to a Quillen adjunction

$$L_E\mathcal{S} \rightleftarrows \mathcal{C}_E$$

from E -local spectra to $\mathcal{C}_E$. We are also able to give another description of $\mathcal{C}_E$ in terms of pushouts of model categories, which shows how strong the universal property of this new model structure is.

We are also able to give an improvement of [BR11, Theorem 9.5]: we can show that for all E , the homotopy information of E -local spectra is entirely encoded in the $\mathrm{Ho}(\mathcal{S})$ -module structure on the E -local stable homotopy category. This was previously only possible with the strong restriction that E is smashing. Hence we have the following, which appears as Theorem 9.1.

Theorem 1. *Let $\mathcal{C}$ be a stable model category. Assume we have an equivalence of triangulated categories*

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

then $L_E\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent if and only if Φ is an equivalence of $\mathrm{Ho}(\mathcal{S})$ -module categories.

2. ORGANISATION

Firstly, we recall some definitions and conventions regarding Bousfield localisation and stable frames. We also re-introduce the concept of *stably E -familiar model categories*: in [BR11] we studied those $\mathcal{C}$ such that the action of $\mathrm{Ho}(\mathcal{S})$ factors over the functor $\mathrm{Ho}(\mathcal{S}) \rightarrow \mathrm{Ho}(L_E\mathcal{S})$. In particular the homotopy category of such a model category has an enrichment in the more structured category $\mathrm{Ho}(L_E\mathcal{S})$. We called such categories *stably E -familiar*.

We then turn to the question of altering a model structure on a given category so as to obtain a stably E -familiar model category. In Section 5 we consider the simpler case of spectral model categories: such a model category is defined in a similar way to a simplicial model category, but with simplicial sets replaced by the model category of symmetric spectra. We construct the *stable E -familiarisation* of a spectral model category in this section.

In Section 6 we extend our results to more general stable model categories. We prove that the *stable E -familiarisation* of a model category $\mathcal{C}$ is the closest stably E -familiar model category to $\mathcal{C}$ in the following sense. The result below also implies that our construction has the universal property we described earlier.

Theorem 2. *Let $\mathcal{C}$ be a stable, proper and cellular model category such that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. Then there is a model structure $\mathcal{C}_E$ on $\mathcal{C}$ such that*

- (1) $\mathcal{C}_E$ is stably E -familiar,
- (2) if $F : \mathcal{C} \rightarrow \mathcal{E}$ is a left Quillen functor and $\mathcal{E}$ is stably E -familiar, then F factors over $\mathcal{C} \rightarrow \mathcal{C}_E$.

Section 7 consists of several *examples* of $\mathcal{C}_E$ for some E and $\mathcal{C}$ involving algebraic model categories, chromatic localisations and module categories over a ringoid spectrum.

In Section 8 we rephrase the universal property of $\mathcal{C}_E$ in terms of *homotopy pushouts* of model categories.

Finally, we prove a full version of the *modular rigidity* theorem that all homotopy information of E -local spectra is governed by the $\mathrm{Ho}(\mathcal{S})$ -action on $\mathrm{Ho}(L_E \mathcal{S})$ given by framings.

3. BOUSFIELD LOCALISATION

We begin with an introduction to Bousfield localisation at a homology theory E . Throughout the paper when we refer to spectra, we mean symmetric spectra equipped with the stable model structure [HSS00] unless stated otherwise.

Let E be a spectrum and let $[-, -]_*$ denote maps in the stable homotopy category. Then E corepresents a homology functor E_* on the category of spectra via

$$E_*(X) = [S^0, E \wedge X]_*$$

where S^0 denotes the sphere spectrum. Bousfield used this to construct a homotopy category of spectra where maps which induce isomorphisms on E_* -homology are isomorphisms [Bou79]. We recap some of the definitions from this work.

Definition 3.1. *A map $f: X \rightarrow Y$ of spectra is an E -equivalence if $E_*(f)$ is an isomorphism. A spectrum Z is E -local if $f^*: [Y, Z] \rightarrow [X, Z]$ is an isomorphism for all E -equivalences $f: X \rightarrow Y$. A spectrum A is E -acyclic if $[A, Z] = 0$ for all E -acyclic Z . An E -equivalence from X to an E -local object Z is called an E -localisation.*

Bousfield localisation of spectra gives a homotopy theory that is particularly sensitive towards E_* and E -local phenomena. The E -local homotopy theory is obtained from the category of spectra by formally inverting the E -equivalences.

This can be seen as a special case of a more general result by Hirschhorn. Let $\mathcal{C}$ be a model category. For $X, Y \in \mathcal{C}$, we let $\mathrm{map}_{\mathcal{C}}(X, Y) \in \mathbf{sSet}$ denote the homotopy function object, see [Hir03, Chapter 17] and Section 4.

Definition 3.2. *Let S be a class of maps in $\mathcal{C}$. Then an object $Z \in \mathcal{C}$ is S -local if*

$$\mathrm{map}_{\mathcal{C}}(s, Z) : \mathrm{map}_{\mathcal{C}}(B, Z) \longrightarrow \mathrm{map}_{\mathcal{C}}(A, Z)$$

is a weak equivalence in simplicial sets for any $s: A \rightarrow B$ in S . A map $f: X \rightarrow Y \in \mathcal{C}$ is an S -equivalence if

$$\mathrm{map}_{\mathcal{C}}(f, Z) : \mathrm{map}_{\mathcal{C}}(Y, Z) \longrightarrow \mathrm{map}_{\mathcal{C}}(X, Z)$$

is a weak equivalence for any S -local $Z \in \mathcal{C}$. An object $W \in \mathcal{C}$ is S -acyclic if

$$\mathrm{map}_{\mathcal{C}}(W, Z) \simeq *$$

for all S -local $Z \in \mathcal{C}$.

A *left Bousfield localisation* of a model category $\mathcal{C}$ with respect to a class of maps S is a new model structure $L_S \mathcal{C}$ on $\mathcal{C}$ such that

- the weak equivalences of $L_S \mathcal{C}$ are the S -equivalences,
- the cofibrations of $L_S \mathcal{C}$ are the cofibrations of $\mathcal{C}$,
- the fibrations of $L_S \mathcal{C}$ are those maps that have the right lifting property with respect to cofibrations that are also S -equivalences.

Hirschhorn proves that if S is a set and $\mathcal{C}$ is left proper and cellular then $L_S\mathcal{C}$ exists. (We will give rough definitions of these two terms below.) Note that an object is fibrant in $L_S\mathcal{C}$ if and only if it is fibrant in $\mathcal{C}$ and S -local.

In the case of localising spectra at a homology theory one wants to invert the class of E_* -isomorphisms, i.e. those maps of spectra that induce isomorphisms in the homology theory E_* . Since this is not a set, one cannot use Hirschhorn's result directly. In [EKMM97, Section VIII.1] it is shown that there is a set S whose S -equivalences are exactly the E_* -isomorphisms. Hence, the key to proving the existence of homological localisations is to find a set giving the correct notion of equivalence. We shall encounter this idea again when constructing $\mathcal{C}_E$.

A model category is *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. A model category is *right proper* if the pullback of a weak equivalence along a fibration is a weak equivalence. If a model category is both left and right proper, we say that it is *proper*.

We also need a stronger version of “cofibrantly generation”, one which forces cell complexes to be better behaved. The actual definition is technical and not particularly illuminating, so we shall simply say that a model category is *cellular* if it is cofibrantly generated by sets I and J , and the domains and codomains of I and J satisfy some nice cardinality conditions. We leave the details to [Hir03, Definition 12.1.1].

4. STABLE FRAMINGS

Framings are a powerful tool that describe and classify Quillen functors from simplicial sets or spectra to arbitrary model categories. They were first developed by Hovey in [Hov99, Section 5.2]. For a model category $\mathcal{C}$, he investigates cosimplicial and simplicial resolutions of objects in $\mathcal{C}$. These are called “frames”. In more detail, a frame of an object $A \in \mathcal{C}$ is a cofibrant replacement of the constant cosimplicial object $A \in \mathcal{C}^\Delta$ in the Reedy model category of cosimplicial objects in $\mathcal{C}$. From these notions one obtains bifunctors

$$\begin{aligned} - \otimes - &: \mathcal{C} \times \mathbf{sSet} \longrightarrow \mathcal{C}, \\ \mathrm{map}_l(-, -) &: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathbf{sSet}, \\ (-)^{(-)} &: \mathcal{C} \times \mathbf{sSet}^{op} \longrightarrow \mathcal{C}, \\ \mathrm{map}_r(-, -) &: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathbf{sSet} \end{aligned}$$

satisfying certain adjunction properties. The notation $\otimes$ stems from the fact that

$$A \otimes \Delta[0] \simeq A.$$

However, this set-up does not equip $\mathcal{C}$ with the structure of a simplicial model category because the “mapping spaces” $\mathrm{map}_l(X, Y)$ and $\mathrm{map}_r(X, Y)$ only agree up to a zig-zag of weak equivalences for cofibrant $X \in \mathcal{C}$ and fibrant $Y \in \mathcal{C}$ [Hov99, Proposition 5.4.7]. But their derived functors agree, leaving us with the following [Hov99, Theorem 5.5.3].

Theorem 4.1 (Hovey). *Let $\mathcal{C}$ be any model category. Then its homotopy category $\mathrm{Ho}(\mathcal{C})$ is a closed $\mathrm{Ho}(\mathbf{sSet})$ -module category.*

In particular, this equips any model category with the notion of a homotopy mapping space. Moreover, framings satisfy the following important properties.

- If $\mathcal{C}$ carries the structure of a simplicial model category [Hov99, Definition 4.2.18], then the two $\mathrm{Ho}(\mathbf{sSet})$ -module structures coming from either framings or the simplicial structure agree [Hov99, Theorem 5.6.2].
- If $F : \mathbf{sSet} \longrightarrow \mathcal{C}$ is a left Quillen functor with $F(\Delta[0]) = A$, then the left derived functors of F and of the framing functor $A \otimes - : \mathbf{sSet} \longrightarrow \mathcal{C}$ agree. Thus, every left Quillen functor from simplicial sets to any model category can be described, up to homotopy, by a frame.

The second property follows from the fact that the category of cosimplicial objects $\mathcal{C}^\Delta$ is equivalent to the category of adjunctions $\mathbf{sSet} \rightleftarrows \mathcal{C}$ [Hov99, Proposition 3.1.5]. A cosimplicial object $A^\bullet$ corresponds to a Quillen adjunction under this equivalence if and only if it is a frame, that is $A^\bullet$ is cofibrant and homotopically constant, [BR11, Proposition 3.2].

In [Len12] Fabian Lenhardt described an analogous set-up for spectra and stable model categories. Now let $\mathcal{C}$ be a stable model category. First, Lenhardt shows that the category of adjunctions between spectra and a stable model category $\mathcal{C}$ is equivalent to the category of “ Σ -cospectra” $\mathcal{C}^\Delta(\Sigma)$. An object in $\mathcal{C}^\Delta(\Sigma)$ consists of a sequence of cosimplicial objects $X_n \in \mathcal{C}^\Delta$ together with structure maps

$$\Sigma X_{n+1} \longrightarrow X_n.$$

The suspension of cosimplicial objects is described in [Len12, Section 3.3]. He then characterises those Σ -cospectra that give rise to Quillen adjunctions under this equivalence, calling them stable frames. These give rise to bifunctors $-\wedge-$ and $\mathrm{Map}(-, -)$ satisfying the expected adjunction properties.

As in the unstable case, this is not rigid enough to equip any stable model category $\mathcal{C}$ with the structure of a spectral model category. However, the above bifunctors give rise to the following [Len12, Theorem 6.3].

Theorem 4.2 (Lenhardt). *Let $\mathcal{C}$ be a stable model category. Then $\mathrm{Ho}(\mathcal{C})$ is a closed $\mathrm{Ho}(\mathcal{S})$ -module category.*

As expected, this satisfies the following key properties.

- If $\mathcal{C}$ is already a spectral model category, then the $\mathrm{Ho}(\mathcal{S})$ -module structure derived from the spectral structure agrees with the $\mathrm{Ho}(\mathcal{S})$ -module structure coming from stable frames [BR11, Example 6.7].
- By construction, every left Quillen functor $F : \mathcal{S} \longrightarrow \mathcal{C}$ is, up to homotopy, of the form $X \wedge - : \mathcal{S} \longrightarrow \mathcal{C}$ for some fibrant-cofibrant $X \in \mathcal{C}$.
- In particular, for any fibrant-cofibrant $X \in \mathcal{C}$ there is a left Quillen functor $\mathcal{S} \longrightarrow \mathcal{C}$ that sends the sphere spectrum to X . We denote this functor by $X \wedge -$ and its right adjoint by $\mathrm{Map}_{\mathcal{C}}(X, -)$.
- Any stable frame and thus any Quillen functor $\mathcal{S} \longrightarrow \mathcal{C}$ is, up to homotopy, entirely determined by its image on the sphere.

As we have already mentioned, the homotopy theory of $L_E \mathcal{S}$ is often much better understood than $\mathcal{S}$. So it is worth asking if some stable model categories have more in common with $L_E \mathcal{S}$ than $\mathcal{S}$. We answer this question and obtain several useful results using this idea in [BR11]. We give the fundamental definitions below.

Definition 4.3. *We say that a stable frame $X \in \mathcal{C}^\Delta(\Sigma)$ is an E -local frame if it gives rise to a Quillen functor pair*

$$X \wedge - : L_E \mathcal{S} \rightleftarrows \mathcal{C} : \mathrm{Map}_{\mathcal{C}}(X, -).$$

A stable model category $\mathcal{C}$ is stably E -familiar if every stable frame is an E -local frame.

This is [BR11, Definition 7.1]. This generalises the notion of an $L_E \mathcal{S}$ -model category in the following sense: if $\mathcal{C}$ is already an $L_E \mathcal{S}$ -model category, then the $\mathrm{Ho}(\mathcal{C})$ -module structure on $\mathrm{Ho}(\mathcal{C})$ agrees with the $\mathrm{Ho}(L_E \mathcal{S})$ -module structure given by E -local frames [BR11, Proposition 7.6]. We can further characterise stably E -familiar model categories as follows [BR11, Theorem 7.8].

Theorem 4.4. *Let $\mathcal{C}$ be a stable model category. Then $\mathcal{C}$ is stably E -familiar if and only if the homotopy mapping spectrum $\mathbb{R} \mathrm{Map}_{\mathcal{C}}(X, Y)$ is an E -local spectrum for all $X, Y \in \mathcal{C}$.*

We can use the theory of E -local framings to study *algebraic model categories*. An algebraic model category is a $\text{Ch}(\mathbb{Z})$ -model category in the sense of [Hov99, Definition 4.2.18]. Thus a $\text{Ch}(\mathbb{Z})$ -model category is enriched, tensored and cotensored over chain complexes and satisfies the $\text{Ch}(\mathbb{Z})$ -analogue of the compatibility axiom (SM7). This implies that the homomorphism spectra obtained via framings are products of Eilenberg–Mac Lane spectra [GJ99, Proposition III.2.20], [DS07, Proposition 1.6]. Using the computations of Gutiérrez in [Gut10] one can draw the following conclusions [BR11, Section 9].

- For $n \geq 1$ there are no algebraic stably $K(n)$ -familiar model categories, where n denotes the n^{th} Morava– K -theory.
- Let $E(n)$ denote the n^{th} chromatic Johnson–Wilson spectrum. An algebraic model category is stably $E(n)$ -familiar if and only if it is rational.

Now we turn to the question of whether any model category can be made stably E -familiar in some natural way.

5. E -FAMILIARISATION OF SPECTRAL MODEL CATEGORIES

For any homology theory E we can consider the category of spectra with the E -local model structure, $L_E\mathcal{S}$. Hence we would like to know if a reasonable notion of E -localisation exists for an arbitrary stable model category $\mathcal{C}$.

Intuitively, a promising definition would be a Bousfield localisation $L_E\mathcal{C}$ of $\mathcal{C}$ where one localises at the class of “ E -equivalences” given by

$$\{f : X \longrightarrow Y \in \mathcal{C} \mid f \wedge^L E : X \wedge^L E \longrightarrow Y \wedge^L E \text{ is an isomorphism in } \text{Ho}(\mathcal{C})\},$$

where the action $\wedge$ of a spectrum on an element of $\mathcal{C}$ is defined via stable frames. However, showing the existence of Bousfield localisations at a class of maps is set-theoretically awkward. The standard method to circumvent this difficulty is to find a set of maps S such that the S -equivalences are precisely the E -equivalences. This is an extremely difficult task, see [EKMM97, Section VIII.1], so it is not clear if a good notion of E -localisation exists for general model categories.

Instead, we will construct the *stable E -familiarisation* $\mathcal{C}_E$ of $\mathcal{C}$ which is the “closest” stably E -familiar model category to $\mathcal{C}$. We will then draw some conclusions about its properties which will show that this construction is the right choice for an analogue of E -localisation for general stable $\mathcal{C}$. For example, the first theorem will show that every Quillen adjunction

$$\mathcal{S} \rightleftarrows \mathcal{C}$$

will give rise to a Quillen adjunction

$$L_E\mathcal{S} \rightleftarrows \mathcal{C}_E.$$

The first question to answer is: what kind of maps do we want to invert in order to construct $\mathcal{C}_E$? In a stably E -familiar model category $\mathcal{D}$ any map of the form

$$X \wedge^L j : X \wedge^L A \rightarrow X \wedge^L B$$

for $j : A \rightarrow B$ an E -equivalence of spectra and $X \in \mathcal{D}$ is a weak equivalence. Hence we could try to localise $\mathcal{C}$ at this class of maps. So we must find some set of maps S such that the S -equivalences equals this class.

We need a couple of technical results first. For this section we shall work with $\mathcal{S}$ -model categories in the sense of [Hov99, Definition 4.2.18], where $\mathcal{S}$ again denotes the model category of symmetric spectra. Such a model category $\mathcal{D}$ is enriched, tensored and cotensored over symmetric spectra in simplicial sets and satisfies the appropriate analogue of Quillen’s (SM7) axiom for simplicial model categories. We shall refer to $\mathcal{D}$ as being a *spectral model category*. We may also talk about $L_E\mathcal{S}$ -model categories, where we use

the E -local model structure on $\mathcal{S}$. A spectral model category is in particular stable and simplicial, see [SS03, Lemma 3.5.2]. We will see later that the restriction to spectral model categories is not as big a restriction as it might seem at first.

We denote the pushout-product of two maps by $\square$, so for $f : X \rightarrow Y$ and $g : A \rightarrow B$ the pushout-product of f and g is

$$f \square g : X \wedge B \coprod_{X \wedge A} Y \wedge A \rightarrow Y \wedge B.$$

Recall that a set of maps S in a stable model category $\mathcal{D}$ is said to be *stable* if the class of S -local objects is closed under suspension. By [BR14, Proposition 4.6] if $\mathcal{D}$ and S are stable then so is $L_S \mathcal{D}$.

Proposition 5.1. *Let $\mathcal{D}$ be a left proper, cellular and spectral model category. Let S be a stable set of maps in $\mathcal{D}$. Then $L_S \mathcal{D}$ is also a spectral model category.*

Proof. Since $\mathcal{D}$ is left proper and cellular, $L_S \mathcal{D}$ exists by [Hir03, Theorem 4.1.1]. We must prove that if i is a cofibration of $L_S \mathcal{D}$ and j is a cofibration of $\mathcal{S}$ then $i \square j$ is a cofibration of $L_S \mathcal{D}$ that is a weak equivalence (in $L_S \mathcal{D}$) if either of i or j is. Since $\mathcal{D}$ is spectral and the cofibrations are unchanged by left Bousfield localisation, we know that $i \square j$ is a cofibration whenever i and j are. Furthermore if j is an acyclic cofibration of symmetric spectra, then $i \square j$ is a weak equivalence in $\mathcal{D}$ and hence it is also an S -equivalence.

The third case is where i is an acyclic cofibration of $L_S \mathcal{D}$ and j is a cofibration of symmetric spectra. We must show that $i \square j$ is an S -equivalence. By [Hov99, Lemma 4.2.4] it suffices to prove this for j a generating cofibration of symmetric spectra and i a generating acyclic cofibration of $L_S \mathcal{D}$. By [HSS00, Proposition 3.4.2] we may assume that j is of the form

$$F_n K \rightarrow F_n L$$

where F_n is the left adjoint to evaluation at level n , and K and L are simplicial sets. By [Hir03, Proposition 4.5.1] the domain of i is cofibrant, so it follows that both the domain and codomain of $i \square j$ are cofibrant. The set S is stable, so the class of S -equivalences in $\text{Ho}(\mathcal{D})$ is closed under suspension and desuspension. Thus $i \square j$ is an S -equivalence if and only if

$$\Sigma^n(i \square j) \cong i \square \Sigma^n j$$

is an S -equivalence for all n .

We know that $\Sigma^n F_n K$ is weakly equivalent to $F_0 K$ in $\mathcal{S}$. Hence for any cofibrant $X \in \mathcal{D}$,

$$X \wedge \Sigma^n F_n K \rightarrow X \wedge F_0 K$$

is a weak equivalence of $\mathcal{D}$. We also know that the domains of the maps $i \square \Sigma^n j$ and $i \square (F_0 K \rightarrow F_0 L)$ are pushouts of cofibrations between cofibrant objects. It follows that $i \square \Sigma^n j$ is weakly equivalent to the map $i \square (F_0 K \rightarrow F_0 L)$. The bifunctor

$$- \wedge F_0 - : \mathcal{D} \times \text{sSet} \rightarrow \mathcal{D}$$

gives $\mathcal{D}$ the structure of a simplicial model category. We may now use [Hir03, Theorem 4.1.1], which states that since $\mathcal{D}$ is simplicial, so is $L_S \mathcal{D}$. Consequently we see that $i \square (F_0 K \rightarrow F_0 L)$ is an S -equivalence. Hence $i \square j$ is also an S -equivalence and $L_S \mathcal{D}$ is a spectral model category. $\square$

Proposition 5.2. *Let $\mathcal{D}$ be a left proper, cellular and spectral model category with generating cofibrations $I_{\mathcal{D}}$ and generating acyclic cofibrations $J_{\mathcal{D}}$. Let J_E be the set of generating acyclic cofibrations for $L_E \mathcal{S}$. Define*

$$S = I_{\mathcal{D}} \square J_E = \{i \square j \mid i \in I_{\mathcal{D}}, j \in J_E\}.$$

Then $L_S \mathcal{D}$ is an $L_E \mathcal{S}$ -model category and hence is stably E -familiar.

Proof. The set J_E is closed under desuspension in the sense that for any element $j \in J_E$ there is an element j' with $\Sigma j' \simeq j$. It follows that the same holds for S , so it is stable in the sense of [BR14, Definition 3.2]. Thus $L_S \mathcal{D}$ is also a stable model category. By Lemma 5.1 it is also an $\mathcal{S}$ -model category.

To see that it is an $L_E \mathcal{S}$ -model category we only need to check that if i is a cofibration of $L_S \mathcal{D}$ and j is an acyclic cofibration of $L_E \mathcal{S}$ then $i \square j$ is an S -equivalence. By [Hov99, Lemma 4.2.4] it suffices to prove this for $i \in I_{\mathcal{D}}$ and $j \in J_E$. But then $i \square j$ is an element of S and hence is an S -equivalence. $\square$

Proposition 5.3. *Let $\mathcal{D}$ be a left proper, cellular and spectral model category and S as in Proposition 5.2. Assume that the domains of the generating cofibrations of $\mathcal{D}$ are cofibrant. Then if $\mathcal{D}$ is a monoidal model category so is $L_S \mathcal{D}$.*

Proof. Since $\mathcal{D}$ is spectral, the maps in S are cofibrations between cofibrant objects. Thus by [BR14, Lemma 6.1] $L_S \mathcal{D}$ is monoidal if and only if

$$I_{\mathcal{D}} \square S = I_{\mathcal{D}} \square (I_{\mathcal{D}} \square J_E) \cong (I_{\mathcal{D}} \square I_{\mathcal{D}}) \square J_E$$

lies in the class of S -equivalences. As $\mathcal{D}$ is monoidal, $I_{\mathcal{D}} \square I_{\mathcal{D}}$ consists of cofibrations. By Proposition 5.2, $L_S \mathcal{D}$ is an $L_E \mathcal{S}$ -model category. Hence the pushout product of a cofibration of $\mathcal{D}$ and an acyclic cofibration of $L_E \mathcal{S}$ is an S -equivalence as required. $\square$

We now show that this set S has the correct homotopical behaviour in terms of E -familiarity by giving another description of the weak equivalences of $L_S \mathcal{D}$.

Proposition 5.4. *Let $\mathcal{D}$ be a left proper, cellular spectral model category, such that the domains of the generating cofibrations of $\mathcal{D}$ are cofibrant. Let T be the class of maps*

$$T = \{X \wedge^L f \mid X \in \mathcal{D}, f \text{ is an } E\text{-equivalence of spectra}\}.$$

Then the class of T -equivalences is equal to the class of S -equivalences.

Proof. Take some cofibrant $X \in \mathcal{D}$. Then the functor

$$X \wedge - : L_E \mathcal{S} \rightarrow L_S \mathcal{D}$$

is a left Quillen functor by Proposition 5.2. Hence $X \wedge -$ takes E -equivalences between cofibrant spectra to S -equivalences. Thus every element of T is a weak equivalence in $L_S \mathcal{D}$.

Now we will show that every element of S is also a T -equivalence. Consider $i \square j \in S$ for $i: X \rightarrow Y$ a generating cofibration of $\mathcal{D}$ and $j: A \rightarrow B$ a generating acyclic cofibration for $L_E \mathcal{S}$. Since X, Y, A and B are all cofibrant, the maps $X \wedge j$ and $Y \wedge j$ are in the class T . Let P be the domain of $i \square j$, then by [Hir03, Lemma 3.4.2], the map from $Y \wedge A \rightarrow P$ is also a T -equivalence. It follows by the two-out-of-three property that $i \square j$ is a T -equivalence. $\square$

If the category $\mathcal{D}$ is already stably E -familiar then the class T is already contained in the category of weak equivalences. Hence so is the set S , and $\mathcal{D}$ is in fact an $L_E \mathcal{S}$ -model category.

Corollary 5.5. *Let $\mathcal{D}$ be a left proper cellular spectral model category that is stably E -familiar. Assume that the domains of the generating cofibrations of $\mathcal{D}$ are cofibrant. Then $\mathcal{D}$ is an $L_E \mathcal{S}$ -model category.* $\square$

6. E -FAMILIARISATION OF STABLE MODEL CATEGORIES

We now want to consider model categories that are not necessarily spectral. Consider a proper and cellular stable model category $\mathcal{C}$. By [BR14, Theorem 8.2] $\mathcal{C}$ is Quillen equivalent to a spectral model category, namely the category $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$ of symmetric spectra in simplicial objects in $\mathcal{C}$ equipped with a non-standard model structure. Hence there is a Quillen equivalence which by abuse of notation we call

$$\Sigma^\infty : \mathcal{C} \rightleftarrows \mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C}) : \Omega^\infty$$

This model category $\mathcal{D}$ is also proper and cellular. Furthermore, if the generating cofibrations for $\mathcal{C}$ have cofibrant domains, then so do the generating cofibrations for $\mathcal{D}$.

Theorem 6.1. *Let $\mathcal{C}$ be a stable, proper and cellular model category, such that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. Let $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$, with generating cofibrations $I_{\mathcal{D}}$ and fibrant replacement $\hat{f}$. We set $S = I_{\mathcal{D}} \square J_E$ as in Proposition 5.2.*

Define $\mathcal{C}_E$ to be the left Bousfield localisation of $\mathcal{C}$ at the set of maps $\Omega^\infty \hat{f} S$. Then

- (1) $\mathcal{C}_E$ is stably E -familiar,
- (2) the weak equivalences of $\mathcal{C}_E$ are the T' -equivalences, for T' the class below

$$T' = \{X \wedge^L f \mid X \in \mathcal{C}, f \text{ is an } E\text{-equivalence of spectra}\}$$

- (3) if $F : \mathcal{C} \rightarrow \mathcal{E}$ is a left Quillen functor and $\mathcal{E}$ is stably E -familiar, then F factors over $\mathcal{C} \rightarrow \mathcal{C}_E$, i.e. $F : \mathcal{C}_E \rightarrow \mathcal{E}$ is also a left Quillen functor.

Proof. The model categories $\mathcal{C}$ and $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$ are Quillen equivalent. Hence [Hir03, Theorem 3.3.20] tell us that the adjunction

$$\Sigma^\infty : \mathcal{C} \rightleftarrows \mathcal{D} : \Omega^\infty.$$

induces a Quillen equivalence between $L_{\Omega^\infty \hat{f} S} \mathcal{C}$ and $L_{\Sigma^\infty \hat{c} \Omega^\infty \hat{f} S} \mathcal{D}$. (Here, $\hat{c}$ denotes the cofibrant replacement in $\mathcal{C}$.) The model category $L_{\Sigma^\infty \hat{c} \Omega^\infty \hat{f} S} \mathcal{D}$ is equal to $L_S \mathcal{D}$ since $(\Sigma^\infty, \Omega^\infty)$ is a Quillen equivalence. Thus we have a Quillen equivalence between $\mathcal{C}_E = L_{\Omega^\infty \hat{f} S} \mathcal{C}$ and $L_S \mathcal{D}$. The second category is stably E -familiar by Proposition 5.2. Hence so is $\mathcal{C}_E$ by [BR11, Lemma 7.10].

We may also conclude that the left derived functor of Σ^∞ induces a bijection between the weak equivalences of $\mathcal{C}_E$ (considered as a class in $\text{Ho } \mathcal{C}$) and the S -equivalences of $\text{Ho } \mathcal{D}$. Proposition 5.4 tells us that the class of S -equivalences in $\mathcal{D}$ is equal to the class of T -equivalences where

$$T = \{X \wedge^L f \mid X \in \mathcal{D}, f \text{ is an } E\text{-equivalence of spectra}\}.$$

Consider the class of maps

$$T' = \{X \wedge^L f \mid X \in \mathcal{C}, f \text{ is an } E\text{-equivalence of spectra}\}.$$

Let $\mathbb{L}\Sigma^\infty$ and $\mathbb{R}\Omega^\infty$ denote the left and right derived functors of Σ^∞ and Ω^∞ respectively. By [Len12, Theorem 6.3]

$$\mathbb{L}\Sigma^\infty(X \wedge^L f) = (\mathbb{L}\Sigma^\infty X) \wedge^L f.$$

Hence $\mathbb{L}\Sigma^\infty$ takes elements of T' to elements of T . Consider some element $Y \wedge^L f$ of T . This is weakly equivalent to

$$(\mathbb{L}\Sigma^\infty \mathbb{R}\Omega^\infty Y) \wedge^L f$$

and hence is in $\mathbb{L}\Sigma^\infty T'$. Thus the derived functor of Σ^∞ induces a bijection between the class T' and the class T up to weak equivalence. As a consequence the derived functor of Σ^∞ induces a bijection between the class of T' -equivalences and the class of T -equivalences. It follows that the T' -equivalences must be the class of weak equivalences of $\mathcal{C}_E$.

For the final point, let $F: \mathcal{C} \rightarrow \mathcal{E}$ be a left Quillen functor. If $\mathcal{E}$ is stably E -familiar, then the left derived functor of F takes the T' -equivalences to weak equivalences of $\mathcal{E}$. Hence $F: \mathcal{C}_E \rightarrow \mathcal{E}$ is also a left Quillen functor. $\square$

Remark 6.2. *Let $\mathcal{C}$ be a stable, proper and cellular model category, such that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. Then the above result says that $\mathcal{C}_E$ is the “closest” stably E -familiar model category to $\mathcal{C}$.*

In particular a model category $\mathcal{C}$ is stably E -familiar if and only if $\mathcal{C}_E = \mathcal{C}$.

Remark 6.3. *The assumptions on $\mathcal{C}$ are more reasonable than they might seem in practice. Since we want to perform a left Bousfield localisation, we will have to assume that $\mathcal{C}$ is left proper and cellular. To assume that $\mathcal{C}$ is also right proper is not too much of a restriction.*

We also need another assumption: that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. This is a subtle assumption that occurs elsewhere in the literature, for example in [Hov01]. We note that this assumption holds for almost all of the cofibrantly generated model categories that arise naturally.

It is easy to check that the homotopy mapping spectra for $\mathcal{C}_E$ are given by the formula below, where Y_E is the fibrant replacement of Y in $\mathcal{C}_E$.

$$\mathbb{R} \operatorname{Map}_{\mathcal{C}_E}(X, Y) = \mathbb{R} \operatorname{Map}_{\mathcal{C}}(X, Y_E)$$

In particular, this mapping spectrum is E -local. We can use this to draw some immediate consequences of E -familiarisation.

For example, the chromatic Johnson–Wilson theories $E(n)$ satisfy

$$L_{E(n-1)} L_{E(n)} = L_{E(n-1)}$$

[Rav92, 7.5.3]. Thus,

Corollary 6.4. *For a proper and cellular stable model category $\mathcal{C}$ we have*

$$(\mathcal{C}_{E(n)})_{E(n-1)} = \mathcal{C}_{E(n-1)}.$$

$\square$

We can further use our knowledge of stably E -familiar algebraic model categories described at the end of Section 4 to read off the following corollaries.

Corollary 6.5. *Let $\mathcal{C}$ be an algebraic model category and $K(n)$ the n^{th} Morava– K -theory for $n \geq 1$. Then $\mathcal{C}_{K(n)}$ is trivial.* $\square$

Corollary 6.6. *Let $\mathcal{C}$ be an algebraic model category and let $E(n)$ denote the n^{th} chromatic Johnson–Wilson spectrum. Then $\mathcal{C}_{E(n)} = \mathcal{C}_{H\mathbb{Q}}$.* $\square$

If we assume that localisation at E is smashing, we can obtain a nicer description of the weak equivalences of $\mathcal{C}_E$: in the smashing case $\mathcal{C}_E$ is precisely the “naive” localisation of $\mathcal{C}$ at $L_E S^0$ as described in the introduction of Section 5. That is, the left Bousfield localisation of $\mathcal{C}$ at the class of $L_E S^0$ -equivalences (which we denote as $L_{L_E S^0} \mathcal{C}$) exists and is equal to $\mathcal{C}_E$. With this extra assumption we also see that

$$\mathcal{C}_E = \mathcal{C}_{L_E S^0}.$$

However, for a general model category $\mathcal{C}$ and smashing E it is unclear whether the model category $L_{L_E S^0} \mathcal{C}$ exists and if it would be Quillen equivalent to $\mathcal{C}_E$.

Lemma 6.7. *In addition to the assumptions of Theorem 6.1, assume that localisation at E is smashing. Then a map f in $\mathcal{C}_E$ is a weak equivalence if and only if $f \wedge^L L_E S^0$ is a weak equivalence in $\mathcal{C}$. Hence the weak equivalences of $\mathcal{C}_E$ are precisely the $L_E S^0$ -equivalences.*

Proof. We first show the statement for a spectral model category $\mathcal{D}$. Recall the model category $L_S\mathcal{D}$ for S the set $I_{\mathcal{D}} \sqcup J_E$ from the previous section. We will show that the S -equivalences are precisely the L_ES^0 -equivalences of $\mathcal{D}$.

Every map in the set S is an L_ES^0 -equivalence, hence every S -equivalence is a L_ES^0 -equivalence. Now take some L_ES^0 -equivalence $f: X \rightarrow Y$ in $\mathcal{D}$. The map

$$X \rightarrow X \wedge^L L_ES^0$$

is a T -equivalence with

$$T = \{X \wedge^L f \mid X \in \mathcal{D}, f \text{ is an } E\text{-equivalence of spectra}\}$$

defined earlier in this section. Hence it is an S -equivalence. Thus the commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X \wedge^L L_ES^0 & \xrightarrow{f \wedge^L L_ES^0} & Y \wedge^L L_ES^0 \end{array}$$

shows that f is S -equivalent to a weak equivalence in $\mathcal{D}$. Weak equivalences in $\mathcal{D}$ are in particular S -equivalences, so by the 2-out-of-3 axiom of model categories, f must be an S -equivalence.

To move this result from a spectral $\mathcal{D}$ to a general $\mathcal{C}$ we use a similar argument to that of the second point of Theorem 6.1. The Quillen equivalence $(\Sigma^\infty, \Omega^\infty)$ takes the L_ES^0 -equivalences of $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$ bijectively to the L_ES^0 -equivalences of $\mathcal{C}$. It follows that the L_ES^0 -equivalences of $\mathcal{C}$ are precisely the weak equivalences of $\mathcal{C}_E$. $\square$

The following corollary shows that stable E -familiarisation restricts to E -localisation in the case of spectra. This shows that the notion of $\mathcal{C}_E$ is indeed a good candidate for an analogue of E -localisation of a general $\mathcal{C}$.

Corollary 6.8. *Consider the category of modules over a ring spectrum R . Then*

$$(R\text{-mod})_E = L_E(R\text{-mod})$$

where the right hand side is the naive localisation of $R\text{-mod}$. It has weak equivalences those maps of R -modules which forget to E -equivalences of spectra and the same cofibrations as $R\text{-mod}$. In particular

$$\mathcal{S}_E = L_ES.$$

Proof. We start by noting that $L_E(R\text{-mod})$ is equal to the model structure of R -modules in L_ES from [SS00, Theorem 4.1]. Hence this model structure has generating sets of cofibrations and acyclic cofibrations $R \wedge I_S$ and $R \wedge J_E$ [SS00, Lemma 2.3].

We claim that every map in $R \wedge J_E$ is a T' -equivalence, where T' is from Theorem 6.1. We know that the domains of J_E are cofibrant, hence $R \wedge j$ is weakly equivalent to $R \wedge^L j$ for any $j \in J_E$. Thus the claim holds. It follows that every acyclic cofibration of $L_E(R\text{-mod})$ is a weak equivalence (and also a cofibration) of $(R\text{-mod})_E$.

We must now show the converse. Every acyclic cofibration of $(R\text{-mod})_E$ is an E -equivalence of underlying spectra and hence is an acyclic cofibration of $L_E(R\text{-mod})$. Thus the two model structures agree. $\square$

We can now give a simple proof that stable E -familiarisation preserves Quillen equivalences.

Proposition 6.9. *Let $\mathcal{C}$ and $\mathcal{E}$ be proper, cellular and stable model categories such that the domains of their generating cofibrations are cofibrant. Let*

$$F: \mathcal{C} \rightleftarrows \mathcal{E}: G$$

be a Quillen equivalence. Then there is a Quillen equivalence between the E -familiarised model categories

$$F : \mathcal{C}_E \rightleftarrows \mathcal{E}_E : G.$$

Proof. Composing F with the identity on $\mathcal{E}$ gives us a left Quillen functor

$$F : \mathcal{C} \rightarrow \mathcal{E}_E$$

and $\mathcal{E}_E$ is of course stably E -familiar. Hence by the universal property of $\mathcal{C}_E$ proved in Theorem 6.1 we have a left Quillen functor $F : \mathcal{C}_E \rightarrow \mathcal{E}_E$. We now need to show that gives us a Quillen equivalence. We do so using Proposition 5.4 and the method of the second part of the proof of Theorem 6.1.

Let T be the class of maps

$$T = \{A \wedge^L f \mid A \in \mathcal{C}, f \text{ is an } E\text{-equivalence of spectra}\}.$$

Similarly, let T' be the class of maps

$$T' = \{B \wedge^L f \mid B \in \mathcal{E}, f \text{ is an } E\text{-equivalence of spectra}\}.$$

Then $\mathcal{C}_E = L_T \mathcal{C}$ and $\mathcal{E}_E = L_{T'} \mathcal{E}$. Let $\mathbb{L}F$ and $\mathbb{R}G$ denote the left and right derived functors of F and G respectively. By [Hir03, Theorem 3.3.20], the adjunction (F, G) induces a Quillen equivalence between $L_T \mathcal{C}$ and $L_{\mathbb{L}F(T)} \mathcal{E}$. But the set $\mathbb{L}F(T)$ is isomorphic in $\text{Ho } \mathcal{E}$ to the set T' because Quillen equivalences induce equivalences of $\text{Ho}(\mathcal{S})$ -module categories [Len12, Theorem 6.3]. $\square$

Remark 6.10. *One could try to prove an analogue of Proposition 5.3 and show that if $\mathcal{C}$ is monoidal then so is $\mathcal{C}_E$. This would require the adjunction $(\Sigma^\infty, \Omega^\infty)$ at the beginning of this section to be monoidal. However we do not know if this is the case.*

7. EXAMPLES

Let $\mathcal{C}$ be a spectral model category, such that the domains of its generating cofibrations are cofibrant. (Recall from [BR14, Theorem 7.2] that any stable, proper and cellular model category is Quillen equivalent to a spectral one.) Assume that $\mathcal{C}$ has a set of compact generators for its homotopy category, [SS03, Definition 2.1.2]. Schwede and Shipley prove in the above-mentioned paper that any such $\mathcal{C}$ is Quillen equivalent to a category $\text{mod-}\mathcal{E}$ where $\mathcal{E}$ can be thought of as a “ring spectrum with several objects”. In the case of $\mathcal{C}$ having a single compact generator, $\mathcal{E}$ is simply a ring spectrum.

Let us briefly recap some of the definitions and constructions of that result. Let $\mathcal{G}$ denote the set of generators of $\mathcal{C}$. Then the $\mathcal{S}$ -enriched category $\mathcal{E}$ is simply defined as the full $\mathcal{S}$ -enriched subcategory of $\mathcal{C}$ with objects $\mathcal{G}$. An object $M \in \text{mod-}\mathcal{E}$ consists of a spectrum $M(G)$ for each $G \in \mathcal{G}$ plus morphisms of spectra

$$\mathcal{E}(G', G) \wedge M(G) \longrightarrow M(G') \text{ for } G, G' \in \mathcal{G}$$

satisfying certain coherence conditions. By adjunction, such an M is the same as a contravariant spectral functor from $\mathcal{E}$ to $\mathcal{S}$. A standard example of an object of $\text{mod-}\mathcal{E}$ is given by the spectral functor $\mathcal{E}(-, G)$ for fixed $G \in \mathcal{G}$. The model structure on $\text{mod-}\mathcal{E}$ is has weak equivalences and fibrations defined objectwise [SS03, Theorem A.1.1]. This means that a natural transformation $f : M \longrightarrow N$ is a weak equivalence or a fibration if and only if

$$f_G : M(G) \longrightarrow N(G)$$

is so for each $G \in \mathcal{G}$. Theorem 3.9.3 of [SS03] then describes a Quillen equivalence

$$\text{Hom}(\mathcal{G}, -) : \mathcal{C} \rightleftarrows \text{mod-}\mathcal{E} : - \wedge_{\mathcal{E}} \mathcal{G}$$

for spectral $\mathcal{C}$.

This is a highly useful description of a stable model category and we would like to obtain a description of the E -familiarisation $\mathcal{C}_E$ of $\mathcal{C}$ in terms of $\text{mod-}\mathcal{E}$. We note that this is a rather special case as not every stable model category has a set of compact generators [HS99, Corollary B.13].

By Proposition 6.9 we know that $\mathcal{C}_E$ and $(\text{mod-}\mathcal{E})_E$ are Quillen equivalent, so we shall find another description of $(\text{mod-}\mathcal{E})_E$.

Since $\text{mod-}\mathcal{E}$ is a spectral model category, it is easily seen that $(\text{mod-}\mathcal{E})_E$ is given by $L_S \text{mod-}\mathcal{E}$ as in Proposition 5.2. Recall that $S = I \square J_E$ for I the set of generating cofibrations for $\text{mod-}\mathcal{E}$. Hence in $(\text{mod-}\mathcal{E})_E$ any map of the form below is a weak equivalence.

$$\mathcal{E}(-, G) \wedge (i \square j)$$

In the above, G is a cofibrant and fibrant replacement of one of the compact generators for $\mathcal{C}$, i is a generating cofibration for $\mathcal{S}$ and j is a generating acyclic cofibration for $L_E \mathcal{S}$.

We can make another model structure on $\text{mod-}\mathcal{E}$ by taking the same cofibrations as before, but taking the generating set of acyclic cofibrations to be those maps of the form

$$\mathcal{E}(-, G) \wedge j$$

for G a generator and j a generating acyclic cofibration for $L_E \mathcal{S}$. We shall call this set of maps K and let $\text{mod-}\mathcal{E}_K$ denote the corresponding model structure. One can either check directly that these sets give a model structure or one can alter [SS03, Theorem A.1.1] to use $L_E \mathcal{S}$ instead of $\mathcal{S}$.

We claim that this model structure equals the model structure of $(\text{mod-}\mathcal{E})_E$. An element of K can be described as

$$\mathcal{E}(-, G) \wedge ((* \rightarrow S^0) \square j).$$

Hence every element of K is an acyclic cofibration of $(\text{mod-}\mathcal{E})_E$. Conversely, $\text{mod-}\mathcal{E}$ equipped with this new model structure is stably E -familiar. Hence the identity functor $(\text{mod-}\mathcal{E})_E \rightarrow \text{mod-}\mathcal{E}_K$ is a left Quillen functor. Hence every acyclic cofibration of $(\text{mod-}\mathcal{E})_E$ is an acyclic cofibration of $\text{mod-}\mathcal{E}_K$. Thus these two model structures have the same cofibrations and acyclic cofibrations. We have therefore shown the following.

Proposition 7.1. *The model category $(\text{mod-}\mathcal{E})_E$ is the category of contravariant spectral functors from $\mathcal{E}$ to $L_E \mathcal{S}$, equipped with the model structure where fibrations and weak equivalences are defined objectwise. Thus the fibrant objects are those functors M such that $M(G)$ is fibrant in $\mathcal{S}$ and E -local for all $G \in \mathcal{E}$. $\square$*

Consider the case where $\mathcal{C}$ has a single compact generator. Following the above we can replace this by a category of functors to $\mathcal{S}$. Indeed, [SS03, Theorem 3.1.1] states that $\mathcal{C}$ is Quillen equivalent to the category of R -modules, $\text{mod-}R$, for some ring spectrum R . In this case, the above proposition recovers the result of Corollary 6.8.

8. E -FAMILIARISATION AND HOMOTOPY PUSHOUTS

We want to give another description of $\mathcal{C}_E$ via a universal property. We will relate $\mathcal{C}_E$ to a pushout of model categories. While the pullback of model categories is well-understood, [Ber11], the pushout is more complicated and is not often used. Roughly speaking, the homotopy pushout of a corner diagram of Quillen adjunctions

$$\mathcal{C} \rightleftarrows \mathcal{D} \rightleftarrows \mathcal{E}$$

is supposed to be a model category $\mathcal{P}$ that satisfies a universal property analogous to the pushout of a diagram within a category. Unfortunately, the homotopy-theoretic pushout construction is rather delicate and its existence and description not always clear.

However there is a special case where we can construct pushouts of model categories and verify that they have the correct universal property. By working in a particular context, we avoid the general question of whether homotopy pushouts of model categories exist in general.

Let $\mathcal{M}_2$ be a left Bousfield localisation of $\mathcal{M}_1$ at a class of maps W . Without loss of generality we assume that the maps in W are morphisms between cofibrant objects. (If the elements of W did not satisfy this, one can replace them with weakly equivalent morphisms between cofibrant objects. This would then give rise to the same Bousfield localisations.) In particular, this gives us a Quillen pair

$$\mathrm{Id} : \mathcal{M}_1 \rightleftarrows \mathcal{M}_2 = L_W \mathcal{M}_1 : \mathrm{Id}$$

Assume that we have a Quillen adjunction

$$F : \mathcal{M}_1 \rightleftarrows \mathcal{N}_1 : G.$$

We are now going to discuss the homotopy pushout of the corner diagram below for this special case

$$L_W \mathcal{M}_1 = \mathcal{M}_2 \xleftarrow{\sim} \mathcal{M}_1 \rightleftarrows \mathcal{N}_1.$$

Definition 8.1. *The homotopy pushout of the above diagram is defined, if it exists, as the Bousfield localisation $L_{\mathbb{L}FW} \mathcal{N}_1$ of $\mathcal{N}_1$. Here, $\mathbb{L}F$ denotes the left derived functor of F .*

To justify this definition we need to see that $\mathcal{N}_2 = L_{\mathbb{L}FW} \mathcal{N}_1$ (provided it exists) satisfies the desired properties that a homotopy pushout is supposed to have. First we note that by [Hir03, Theorem 3.3.20] F and G induce a Quillen adjunction

$$F : \mathcal{M}_2 \rightleftarrows \mathcal{N}_2 : G$$

Assume that there is a model category $\mathcal{D}$ with Quillen adjunctions

$$\begin{array}{ccc} \mathcal{M}_2 & \rightleftarrows & \mathcal{D} \\ F' : \mathcal{N}_1 & \rightleftarrows & \mathcal{D} : G' \end{array}$$

such that in the diagram below, the two different composites of left adjoints from $\mathcal{M}_1$ to $\mathcal{D}$ agree up to natural isomorphism.

$$\begin{array}{ccc} \mathcal{M}_1 & \rightleftarrows & \mathcal{N}_1 \\ \updownarrow & & \updownarrow \\ \mathcal{M}_2 & & \mathcal{D} \end{array}$$

Because the vertical functors in the square below are simply identity functors it follows immediately that we may add $\mathcal{N}_2$ and obtain a commutative diagram of adjoint pairs.

$$\begin{array}{ccc} \mathcal{M}_1 & \rightleftarrows & \mathcal{N}_1 \\ \updownarrow & & \updownarrow \\ \mathcal{M}_2 & \rightleftarrows & \mathcal{N}_2 \\ & \searrow & \downarrow \\ & & \mathcal{D} \end{array}$$

We must check that the adjunction below is a Quillen adjunction.

$$F' : \mathcal{N}_2 \rightleftarrows \mathcal{D} : G'$$

The model category $\mathcal{N}_2$ is the Bousfield localisation of $\mathcal{N}_1$ with respect to the class of maps Ff where f is a weak equivalence between cofibrant objects of $\mathcal{M}_2$. Thus $(F' \circ F)(f)$ is a weak equivalence in $\mathcal{D}$. This means that F' uniquely factors over $\mathcal{N}_2$. Furthermore, by construction, $\mathcal{N}_2$, if it exists, is unique up to Quillen equivalence.

Recall that the stable E -familiarisation $\mathcal{C}_E$ satisfies the following universal property. Given a left Quillen functor $F : \mathcal{C} \rightarrow \mathcal{D}$ with $\mathcal{D}$ stably E -familiar, F also gives rise to a left Quillen functor $\mathcal{C}_E \rightarrow \mathcal{D}$ via

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ id \downarrow & \nearrow & \\ \mathcal{C}_E & & \end{array}$$

This fact allows us to relate $\mathcal{C}_E$ and certain homotopy pushouts. Let $X \in \mathcal{C}$ be fibrant and cofibrant. Then we have a Quillen adjunction

$$X \wedge - : \mathcal{S} \rightleftarrows \mathcal{C} : \text{Map}_{\mathcal{C}}(X, -).$$

Using Definition 8.1 we can read off the following for a proper and cellular stable model category $\mathcal{C}$.

Lemma 8.2. *The homotopy pushout $\mathcal{P}_X$ of the diagram*

$$L_E \mathcal{S} \rightleftarrows \mathcal{S} \rightleftarrows \mathcal{C}$$

exists and is the Bousfield localisation of $\mathcal{C}$ with respect to the set of maps below, where J_E is the set of generating acyclic cofibrations of $L_E \mathcal{S}$.

$$X \wedge^L J_E = \{X \wedge^L j \mid j \in J_E\}$$

□

So in particular we know that this homotopy pushout exists. Because $\mathcal{C}_E$ is stably E -familiar we have a commutative square of Quillen adjunctions

$$\begin{array}{ccc} \mathcal{S} & \rightleftarrows & \mathcal{C} \\ \updownarrow & & \updownarrow \\ L_E \mathcal{S} & \rightleftarrows & \mathcal{C}_E \end{array}$$

By the universal property of $\mathcal{P}_X$, there is a Quillen adjunction $\mathcal{P}_X \rightleftarrows \mathcal{C}_E$ for each X . We can show that $\mathcal{C}_E$ is the “closest” model category to those pushouts in the following sense.

Theorem 8.3. *Let $\mathcal{C}$ be a stable, proper and cellular model category, such that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. The Quillen adjunction*

$$\mathcal{C} \rightleftarrows \mathcal{C}_E$$

factors over

$$\mathcal{P}_X \rightleftarrows \mathcal{C}_E$$

for all fibrant-cofibrant $X \in \mathcal{C}$. If there is any other stable $\mathcal{D}$ with a Quillen adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

that factors over

$$\mathcal{P}_X \rightleftarrows \mathcal{D}$$

for all fibrant-cofibrant X , then (F, G) also factors over $\mathcal{C}_E$.

Proof. The pushout $\mathcal{P}_X$ is defined as the Bousfield localisation of $\mathcal{C}$ at the set of maps $X \wedge^L j$ with $j \in J_E$. By Proposition 5.4 we know that $\mathcal{C}_E$ is the localisation of $\mathcal{C}$ at the class of maps of the form $X \wedge^L f$ for f an E -equivalence of spectra. Thus we see that for every $X \in \mathcal{C}$ the identity gives us a Quillen adjunction

$$\mathrm{Id} : \mathcal{P}_X \rightleftarrows \mathcal{C}_E : \mathrm{Id}$$

because every weak equivalence in $\mathcal{P}_X$ is also a weak equivalence in $\mathcal{C}_E$.

If the given Quillen adjunction (F, G) induces a Quillen adjunction

$$F : \mathcal{P}_X \rightleftarrows \mathcal{D} : G,$$

then F sends all morphisms of the form $X \wedge^L j$, for $j \in J_E$, to weak equivalences in $\mathcal{D}$. Hence F also sends all maps of the form $X \wedge^L f$, for f an E -equivalence of spectra, to weak equivalences in $\mathcal{D}$.

If (F, G) gives such Quillen adjunctions for all fibrant-cofibrant X then it must send any map of the form $X \wedge^L f$ with X fibrant-cofibrant and f an E -equivalence of spectra to a weak equivalence in $\mathcal{D}$. Thus it induces a Quillen adjunction

$$\mathcal{C}_E \rightleftarrows \mathcal{D}$$

by Theorem 6.1, which is what we wanted to prove. $\square$

9. MODULAR RIGIDITY FOR E -LOCAL SPECTRA

We can show that stable frames encode all homotopical information of the E -local stable homotopy category. The triangulated structure of $\mathrm{Ho}(L_E\mathcal{S})$ alone is not sufficient for this: given just a triangulated equivalence

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

for a stable model category $\mathcal{C}$ does not imply in general that $L_E\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent. In fact, Quillen equivalence can only be deduced from a triangulated equivalence of homotopy categories in some very special cases. To this date, the only nontrivial cases known of this ‘rigidity’ are the stable homotopy category itself [Sch07] and the case $E = K_{(2)}$ [Roi07]. However, if we do not only have a triangulated equivalence as above but also assume that this equivalence is a $\mathrm{Ho}(\mathcal{S})$ -module equivalence, we can show that $L_E\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent.

We now give a more general version of [BR11, Theorem 9.5], in particular the assumption that E is smashing is no longer required.

Theorem 9.1. *Let $\mathcal{C}$ be a stable model category. Assume we have an equivalence of triangulated categories*

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

then $L_E\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent if and only if Φ is an equivalence of $\mathrm{Ho}(\mathcal{S})$ -module categories.

Proof. The “only if” part is true by [Len12, Theorem 6.3]: a Quillen equivalence induces a $\mathrm{Ho}(\mathcal{S})$ -module equivalence.

Now let us assume that we have a $\mathrm{Ho}(\mathcal{S})$ -module equivalence

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

It follows that Φ^{-1} induces a weak equivalence of homotopy mapping spectra

$$\Phi^{-1} : \mathbb{R}\mathrm{Map}_{\mathcal{C}}(X, Y) \longrightarrow \mathbb{R}\mathrm{Map}_{L_E\mathcal{S}}(\Phi^{-1}X, \Phi^{-1}Y)$$

for $X, Y \in \mathcal{C}$. The right-hand-side is an E -local spectrum as $L_E\mathcal{S}$ is stably E -familiar. Hence every homotopy mapping spectrum of $\mathcal{C}$ is E -local, so $\mathcal{C}$ is stably E -familiar by [BR11, Theorem 7.8].

Thus for fibrant and cofibrant $X \in \mathcal{C}$, the Quillen functor

$$X \wedge - : \mathcal{S} \longrightarrow \mathcal{C}$$

factors over $L_E \mathcal{S}$ as a Quillen functor

$$X \wedge - : L_E \mathcal{S} \longrightarrow \mathcal{C}.$$

Now let X be a cofibrant–fibrant replacement of $\Phi(S^0)$. Because Φ is a $\mathrm{Ho}(\mathcal{S})$ –module equivalence we see that

$$X \wedge^L (-) = \Phi(S^0) \wedge^L (-) = \Phi(S^0 \wedge^L -) = \Phi(-).$$

This means that Φ is derived from a Quillen functor. This Quillen functor must therefore be a Quillen equivalence, which is what we wanted to prove. $\square$

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