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A top-down approach to identifying bull and bear market states

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Abstract

Bull and bear markets receive considerable media and academic attention. It is widely believed that such states are important determinants of wider market dynamics, yet no agreed definition exists. This paper investigates frameworks for ex post classification of asset prices in two-state (bull and bear) markets. An emphasis is placed on identifying state transition points that might achieve consensus. A number of potential difficulties with existing methodologies are highlighted. A principle-based approach is adopted from which a new, flexible, hierarchical methodology is proposed that addresses these issues and permits varying degrees of resolution allowing secondary trends such as bear rallies to be incorporated. The methodology is shown to be optimal under one measure of performance.

Keywords: bull market, bear market, financial cycles, dating rules

JEL classification: G10, C18, C63.

1 Introduction

The difficulties experienced by many major economies over the past decade have prompted renewed interest in economic cycles, financial cycles, and the global interactions between them. A particular focus is synchronization and the degree to which this magnifies outcomes. Such work is underpinned by the model used to identify the start and end of each cycle. Business cycles, comprising of expansion and recession phases, have widely accepted definitions and authorities to whom most observers are willing to defer. Equipped with a standard set of dates, economists can

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analyse relationships with economic activity. Economic phases are clearly influenced by the practical realities of companies operating in a market-based financial system (Woodford, 2010) yet, by contrast, the definition of a financial cycle is open to interpretation. Market prices should reflect future profit expectations and economic forecasts but, unlike economic output, are immediately observable and continuously updated. In this paper we specifically consider the financial cycle, comprising of bull and bear phases, in relation to equity prices. A new approach is proposed that formalises the dating of these cycles in a manner that can attract broad support. The value of the proposed model lies in the importance of the systematic study of bull and bear markets as they relate to wider market and economic dynamics. To obtain reliable results, great care must be taken in defining and measuring the underlying cycles.

The terms ‘bull’ and ‘bear’ and their derivatives are frequently used during market commentary. When applied to entire markets, they are generally taken to mean extended intervals of time over which prices have broadly increased or broadly decreased respectively (Chauvet and Potter, 2000). Such intervals are bounded by peaks and troughs, and naturally form alternating phases. When applied to individual participants, bull and bear terminology reflects sentiment. A bullish investor has expectations of returns higher than some average value (Brown and Cliff, 2004). This expectation is potentially both influenced by, and an influence on, other investors (Shiller, 1995). Bull markets naturally arise when such investors dominate trading activity. Thus bull and bear labels characterise the prevailing sentiment in a market, a concept that is difficult to formalise. The Adaptive Markets Hypothesis posits that markets evolve and their participants adapt. On this basis, bear markets may impose natural selection. Lo (2004) notes that around the turn of the century, a significant group of investors had never experienced a bear market. If market dynamics do evolve then alternating bull and bear phases are part of this evolution.

Google’s Ngram Viewer reveals that the terms ‘bullish’ and ‘bearish’ have been used for at least three centuries, and have been explicitly applied to markets since before 1900. While prevalent in the financial lexicon, these terms have no formally agreed definition. Commonly cited, but otherwise arbitrary, price change thresholds are 20% for bull/bear phases and 10% for corrections. They are also considered to have minimum durations spanning several months (Pagan and Sossounov, 2003). Despite this, these market states receive considerable attention from investors and researchers. Examples of such work include Kaminsky and Schmukler (2003) who analyse the impact of financial liberalization, Jansen and Tsai (2010) who examine the impact of surprise monetary policy, Candelon et al. (2008) and Claessens et al.

(2010) who both investigate synchronization of markets across countries, [Ntantamis and Zhou \(2015\)](#) who look for relationships between commodity prices and commodity stocks, and [Albuquerque et al. \(2015\)](#) who examine the relationship between returns and fundamentals during bull and bear periods.

Given that much research is conducted on market dynamics during bull and bear states, there is a clear need for a more formal definition, and a means with which to determine the state of a market, at least on a historical ex post basis. By contrast, economic cycles have attracted considerably more attention and there is now a widely accepted consensus on their definition. In the US, early attempts to address issues relating to the measurement of economic output, and timing of cycles, were led by the National Bureau of Economic Research (NBER). This work has clearly influenced the dating of financial cycles. In their book on ‘Cyclical Analysis of Time Series’ [Bry and Boschan \(1971\)](#) presented techniques adapted from NBER to identify such phases in economic data. A key motivation was to split cycles into homogeneous phases, yet the authors recognise the inherent difficulty in settling on a particular algorithm which ‘cannot be regarded as objective in the sense that all reasonable and conscientious investigators would agree on the answers’.

The lack of an agreed definition means that some flexibility is desirable in any classification model. As market trends increase in magnitude or duration, one might expect greater consensus on applying a bull or bear label. A classification model that maximises the level of agreement between different observers, even if they differ in their definitions, is also appealing. Methodologies that lack broad consensus, are subject to future revision, or those which permit anomalies, have the potential to undermine the results of derivative work. A widely accepted ex post classification is also critical for the validation of future bull and bear predictions in such studies as [Chen \(2009\)](#) or [Wu and Lee \(2015\)](#). As with business cycles, a dating algorithm that is ‘simple, robust, as transparent as possible and replicable’ is desirable ([Harding and Pagan, 2003](#)).

This paper highlights potential deficiencies in existing ex post rule-based methodologies and proposes adjustments to address such issues. While early work was inspired by the treatment of business cycles, here a principle-based approach is adopted specifically for the treatment of bull and bear market phases. Through this, an alternative approach is developed, free from such deficiencies, while retaining the flexibility to adopt elements from earlier models. It can be applied to any time series regardless of frequency or prior filtering. The approach is top-down in nature and begins by first identifying low-frequency, long-term trends where wide agreement between market observers would be expected. The different approaches

are applied to various time series and the results compared. The top-down methodology is shown to be optimal under one measure of performance, and therefore has the potential to act as a benchmark against which actual performance could be compared. In contrast to other studies, daily rather than monthly price series are considered. Examples are provided to show that filtering of the data may lead to anomalies. This paper proceeds as follows. In section 2 existing ex post classification models are reviewed. Section 3 outlines the new top-down approach which is then applied in section 4 and contrasted with existing models. Section 5 concludes the paper.

2 Review of existing methodologies

Any methodology \mathcal{M} for identifying market phases within a discrete time series $P = (t_i, P(t_i))$ for $i = 0, \dots, n$ can be thought of as an operator $\mathcal{M}(P, \theta)$ which identifies a subset of P that partitions the series into distinct phases for a given parameterization θ . The partitioning subset is defined by the set of points which mark the transition between phases. This creates a partial ordering for phase classifications based on set inclusion where $\mathcal{M}(P, \theta) \preceq \mathcal{M}(P, \theta^*) \Leftrightarrow \mathcal{M}(P, \theta) \subseteq \mathcal{M}(P, \theta^*)$. This provides a convenient framework in which to compare the partitions generated by different parameterizations and to formalise their behaviour.

Assuming the methodology respects the accepted bull/bear essence, the fewer points that occur in $\mathcal{M}(P, \theta)$, the more we would expect market participants to agree on the trend exhibited between points, and the selection of points themselves. While one market participant may have *stricter* criteria for defining a bull market than another, if the parameters themselves can be compared in these terms then a ‘well behaved’ methodology would be monotonically decreasing in the sense that $\theta \preceq \theta^* \Leftrightarrow \mathcal{M}(P, \theta^*) \preceq \mathcal{M}(P, \theta)$.

To determine bull and bear cycles, one must depart from traditional economic approaches which typically use (monthly) smoothed data series, often without a long term trend. Ex post methodologies for identifying such cycles are largely based on the identification of turning points (i.e. peaks and troughs). Such local extrema are candidates for the start and end of phases within the cycle. In determining which of these candidates to retain in the final classification, one must impose further rules based on considerations such as the length of the phase or magnitude of change. The following principles are adopted for guidance:

P1. Bull and bear phases alternate.

P2. Bull (bear) markets exhibit a significant rise (fall) in prices between the start and end.

P3. The prices over each phase should be bounded by the values achieved at the phase end points.

Principle P1 follows logically in a two-state model where we insist every point must be classified as belonging to either a bull or bear phase. Principle P2 is necessary to align with common usage of the terms. This allows short-term reversals (secondary trends) to occur mid-phase, but insists each phase exhibit a clear primary trend distinct from irregular fluctuations. Principle P3 means that the maximum and minimum values achieved over any phase coincide precisely with the phase start and end dates. We will refer to phases satisfying this property as being *end-bounded*. This prohibits bulls (bears) reaching new phase lows (highs) or ending at a price lower (higher) than an earlier phase high (low). A violation of this principle implies the presence of an intermediate point representing a phase maximum or minimum value. This raises the question why the prior phase was not extended to this point.

In what follows, unless stated otherwise, we incorporate this into our definition of bull and bear. The use of daily price series reduces the risk of this principle being violated. Conversely, the use of monthly data increases the risk of anomalies¹. The principles require phase end dates to be turning points. To these we add:

P4. A small change in parameterization should not fundamentally alter the phase dates.

P5. Extending the time series should not fundamentally alter the phase dates.

Principle P4 seeks to impose a form of stability so that methodologies are well behaved in the sense that a small change in the model parameterization θ to θ^* should have a small, incremental and order-preserving change in the model outputs. Such a change might add or remove phase partitioning points, but would be expected to leave $\mathcal{M}(P, \theta) \subseteq \mathcal{M}(P, \theta^*)$ or $\mathcal{M}(P, \theta^*) \subseteq \mathcal{M}(P, \theta)$.

Principle P5 requires that model results should not be subject to continuous revision. Thus, extending the time series should not lead to a reevaluation of historical partition points (Claessens et al., 2010). Clearly this would not be acceptable in the analogous case of business cycles.

Akin to Kole and van Dijk (2016), the principles are inspired by a trader who is required to be fully invested (long or short) at all times (P1) acting with the benefit

¹Of course it would be possible to claim that an intraday high, occurring on some otherwise unremarkable day, marked the true market peak

of hindsight. Principle P2 ensures sufficient profit can be made to cover trading costs while principle P3 would allow such a trader to time the market perfectly. With principle P4 operating in tandem with P2, we would expect increased profits follow from more frequent trading activity. Principle P5 is desirable when investigating market dynamics linked to market state. This property mitigates the risk of having results undermined when future data becomes available.

While other methodologies exist (see for example the naive moving average in [Chen \(2009\)](#) or the ‘CC Turning Point Detection’ method in [Gonzalez et al. \(2006\)](#)) we focus on two frequently cited rule-based approaches and the more general class of Markov Switching models.

2.1 Pagan and Sossounov (PS)

The approach of [Pagan and Sossounov \(2003\)](#) is influenced [Bry and Boschan \(1971\)](#) but adapted to stock prices in accordance with Dow Theory. Duration rather than magnitude is used as a filtering mechanism. They argue against imposing lower bounds on the size of the market move, by noting that extreme movements are more akin to booms and busts. *Phases* refer to intervals between adjacent opposite extrema, *cycles* refer to intervals between adjacent similar extrema. The algorithm they define, with their suggested parameterization, is as follows:

Phase 1 - determine initial turning points using raw data.

- (a) Use a **8 month** window extending in both directions to determine local extrema.
- (b) Enforce alternation of peaks and troughs; for two adjacent peaks (troughs) choose the highest (lowest).

Phase 2 - censoring operations (ensure alternation after each).

- (a) Elimination of turns within **6 months** of the start/end of the series
- (b) Elimination of peaks (or troughs) at both ends of the series which are lower (or higher) than points closer to the end.
- (c) Remove cycles of length less than **16 months**
- (d) Remove phases of length less than **4 months** unless the market move exceeds **20%**.

They demonstrate that this parameterization identifies turning points closely aligned to periods thought of as (US) bull and bear markets. To implement the

approach several passes across the price series are required. Their algorithm could be considered to be a ‘bottom up’ approach in the sense that it begins by identifying all phase partition candidates before selectively paring back to the final selection. No procedure is explicitly defined for determining which local extrema to exclude to ensure rules are satisfied. In the case of two peaks in close proximity, in violation of cycle rule 1(b), it is natural to exclude the lower peak. Figure 1a depicts a stylised situation where the phase between peak $P(t_2)$ and trough $P(t_3)$ is deemed to be too short. Eliminating this phase removes the lowest trough (or highest peak) from the visible landscape, and potentially violates principle P3.

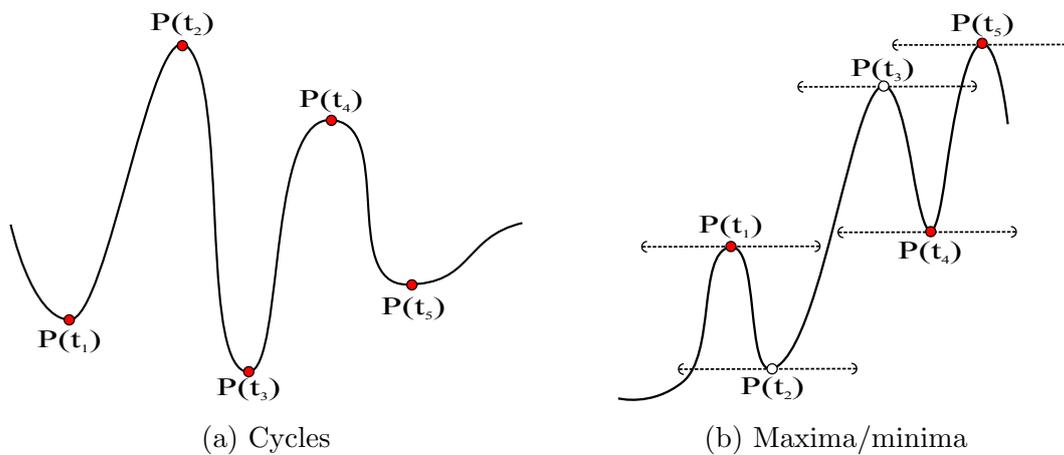


Figure 1: PS approach. Figure (a) represents a situation where the phase from peak $P(t_2)$ to trough $P(t_3)$ is deemed too short. Eliminating this phase removes the lowest trough (or highest peak) from the visible landscape. A bull market extending from $P(t_1)$ to $P(t_4)$ would also achieve its peak long before the bull ends. Figure (b) illustrates an issue when the initial window size used to locate turning points, eliminates turns $P(t_2)$ and $P(t_3)$. Peak $P(t_1)$ therefore marks the start of a bear market that ends at trough $P(t_4)$, and represents a rising market!

A second issue is illustrated in Figure 1b where a trough is found between two local peaks. Given the proximity of the local peaks, only one is considered to be a local maxima under rule 1(a). This results in a fall between the prior peak and the following trough to give a rising bear market in violation of principle P2. Rapid oscillations therefore pose potential problems. Rules 2(a) and 2(b) exercise caution when dealing with the series ends, recognising that the market state may only be revealed when the series is extended. This leaves some points, at least temporarily, unclassified. A third related issue is that extending the price series into the future can cause major historical revisions conflicting with principle P5. Thus, as new price data becomes available, a re-application of the methodology may result in some previously designated phase end points being discarded in preference for others. These revisions need not be limited to the recent past. As an extreme example, consider a price series that alternatives between peaks and troughs of increasing magnitude at precisely 5 month intervals. Due to phase constraints, only every other peak and every other trough can be retained. Extending the series to

include one additional peak and trough, could lead to all previous turning points being discarded.

In rule 2(a) there is the potential to overlook short phases that most market commentators would not hesitate to call. Steady-but-slow trends should be captured, but may be too small in magnitude for consensus to be reached that they represent complete market phases². Starting with a given configuration, decreasing any of the parameters will allow new turning points to enter contention for designation as phase end points. However, there is no guarantee that doing so will only add fresh points without removing others thereby endangering principle P4.

2.2 Lunde and Timmermann (LT)

This approach is based on the magnitude of the market move without any duration constraints. The model is configured by choosing the market moves $\lambda_{bull}, \lambda_{bear} > 0$ (λ_{bear} is also bounded by 1) required to trigger a change in state to a bull or bear market respectively. An initial starting point and state must also be chosen. While in a bull phase, the market peak is allowed to ratchet up to new highs $P^{max}(t_i)$ extending the current bull run. A bear phase is triggered once the market falls by an amount λ_{bear} from the previous high (i.e. $P(t_{i+k}) \leq (1 - \lambda_{bear})P^{max}(t_i)$). All intermediate points are assigned to the bear state. While in a bear phase, the market trough is allowed to ratchet down to new lows $P^{min}(t_i)$ extending the current bear run. A bull phase is triggered once the market rises by an amount λ_{bull} from the previous low (i.e. $P(t_{i+k}) \geq (1 + \lambda_{bull})P^{min}(t_i)$). All intermediate points are assigned to the bull state. Note that the point at which a bull (bear) is declared will normally precede the peak (trough) by some distance. A precise mathematical definition can be found in [Lunde and Timmermann \(2004\)](#).

The LT approach is simpler than PS, requiring a single ‘left to right’ pass along the time series. Unlike PS, the state changes as events unfold, demonstrating a pragmatic “*we are in a bull market until we’re not*” attitude. While there will be a delay determining the state ([Nyberg \(2013\)](#) refers to this as the ‘real-time information lag’), once the state has been set, no subsequent revisions would occur regardless of future market movements. This has the attractive feature that future information will not result in historic revisions. In their paper, Lunde and Timmermann consider both symmetric (for example, $\lambda_{bull} = \lambda_{bear} = 15\%$) and non-symmetric filters (for example, $\lambda_{bull} = 20\% > 10\% = \lambda_{bear}$).

As [Pagan and Sossounov \(2003\)](#) note, one drawback to the LT approach is the

²This could easily be addressed with an extra parameter and simple rule change but is perhaps not in keeping with the spirit of PS.

requirement to choose a starting point. Another issue is that long, steady-but-slow trends may not register. Varying the values of λ_{bull} , λ_{bear} could cause the timing of bull and bear markets to change in violation of principle P4. For example, as λ_{bear} decreases, smaller market falls could be designated as bears. A difficulty here is the potential ‘knock-on’ effect as previously important turning points are removed from the landscape; when viewed at increased resolution one would not expect the start and end of major phases to change. A further related difficulty is that it is possible to achieve all-time market highs during a ‘bear phase’ in violation of principle P3. This is illustrated in Figure 2.

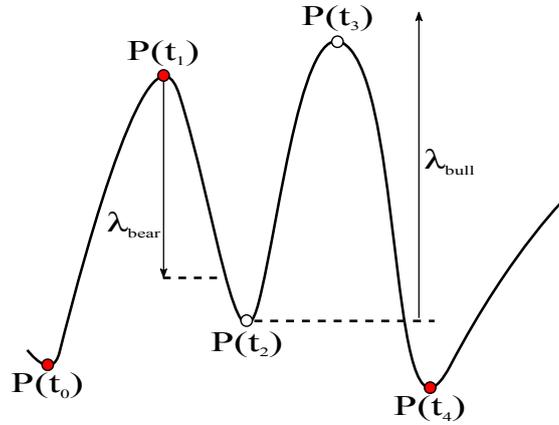


Figure 2: LT approach. A bear phase is entered when the drop from peak $P(t_1)$ to $P(t_2)$ exceeds threshold λ_{bear} . The subsequent rise from trough $P(t_2)$ to $P(t_3)$ is not sufficient to breach threshold λ_{bull} and trigger a new bull market, but a new low $P(t_4)$ is found. The bear market therefore extends from $P(t_1)$ to $P(t_4)$ but includes a new market high in $P(t_3)$.

One way to avoid this would be to adapt the algorithm to extend the previous bull phase to the intermediate peak. Alternatively, under their specification, such difficulties can be avoided by insisting upon an alternative form of symmetry defined by:

$$\lambda_{bull} = \frac{\lambda_{bear}}{(1 - \lambda_{bear})} \quad (1)$$

This effectively requires the same absolute change in log prices to trigger a change in state in either direction. Under this condition LT bull and bear market phases must be end-bound (see [Appendix](#), Lemma 1). An implication of this is that an LT bull market (under equation 1) cannot peak mid-cycle, nor can a bear market bottom out mid-cycle. This precludes fresh new highs (lows) being reached during bear (bull) phases, and leads to the first result:

R1. Under equation 1 LT satisfies principle P3

Technically a bull (bear) phase could reach the same peak (trough) more than

once³, but with asset prices, such an occurrence would be exceptional. Should the peaks be separated by a significant trough, both would ultimately be incorporated as phase ends. If there is no significant intermediate trough, the choice of market peak becomes arbitrary. The LT method would allow the bull (bear) to continue to the last such peak (trough). For convenience we assume that any price series has already been filtered in some way to remove adjacent points of equal value.

It can be shown (see [Appendix](#), Lemma 2) that, under certain assumptions, the same bull/bear sequence can be generated moving either forwards or backwards in time. This provides a mechanism for removing the need to choose an arbitrary starting point: simply start with the global maximum or minimum and move forward and backwards from there. We refer to the LT method applied in this manner under equation 1 as the LT2 method.

2.3 Markov Switching (MS)

Markov switching models popularised by [Hamilton \(1989\)](#) provide an alternative approach to rule-based methodologies. A survey of the use of MS models in empirical finance can be found in [Guidolin \(2011\)](#). Under these models, the state of the market is treated as an unobserved discrete variable upon which stock return dynamics depend. One simple such model is where the stock return r_t (i.e. change in log price) follows a different regime depending on the unobserved market state s_t .

$$r_t = \mu_{s_t} + \sigma_{s_t}\varepsilon_t \tag{2}$$

Here ε_t is identically and independently distributed (IID) with zero mean and unit variance. Examples of attempts to identify regimes within stock market returns in this manner include [Turner et al. \(1989\)](#), [Maheu and McCurdy \(2000\)](#) who incorporate duration dependence, [Kim et al. \(2004\)](#) who incorporate volatility feedback, and [Guidolin and Timmermann \(2005\)](#) who find evidence for a 3-state model.

While rule-based approaches are subject to arbitrary parameterizations, the results of a MS approach depend upon the validity of the underlying statistical model ([Harding and Pagan, 2003](#)). Rule based models are generally easy to understand and implement but suffer from delayed state identification due to the information lag. By contrast MS models are perhaps less intuitive, but produce immediate results ([Berge and Jordà, 2013](#)). Under such fully specified models the current state can be judged on the basis of probability, and future states can be forecast. [Kole and van Dijk \(2016\)](#) make the observation that the PS and LT rule-based models essentially

³[Bry and Boschan \(1971, p12\)](#) refer to this difficulty as ‘double turns’.

rely on mean return while parametric MS models incorporate return variance. This is used to explain their finding that rule-based models are superior for in-sample identification, but inferior for out-of-sample forecasting.

Clearly any 2-state MS model will satisfy principle P1 but not necessarily principle P2 or P3. The parameterization of a MS model may include such choices as the autoregressive lag, or the threshold probability required to indicate a state transition. It can be demonstrated that principles P4 and P5 are easily violated under such parameterization changes. Indeed, extending the time series could lead to a MS model detecting fewer state changes, while an increase in the frequency of the observations could lead to many more state changes. Thus while MS models offer many insights and benefits, they may struggle to achieve broad consensus on the phase partitions that they produce.

3 Methodology

The proposed top-down approach first seeks to identify major bull and bear trends in line with the guiding principles. Recursive application then identifies minor trends which must necessarily include some reversal of the major trend. Before outlining the full methodology, we consider how to identify these reversals.

3.1 Reversals

Consider a time interval $[t_a, t_b]$ over which corresponding price series $P(t)$ achieves minimum and maximum values at the start and end points respectively. Such an interval could be described as having a primary bull market trend. We define:

- Left maximums as points t_i such that $P(t_j) \leq P(t_i) \forall j < i$.
- Right minimums as points t_i such that $P(t_i) \leq P(t_j) \forall i < j$.

Moving chronologically over the price series, left maximums represent new highs, and moving in a reverse chronological direction, right minimums represent new lows. Right maximums and left minimums are defined analogously. The maximum reversal achieved over a bull (bear) period must occur between left maximum (minimum) and right minimum (maximum) turning points (see [Appendix](#), Lemma 3). This result is illustrated in [Figure 3](#).

To find candidates for the maximum reversal a simple algorithm can be applied. For a bull phase, sweep from left to right, starting with the assumption that the last point to be added was a left maximum.

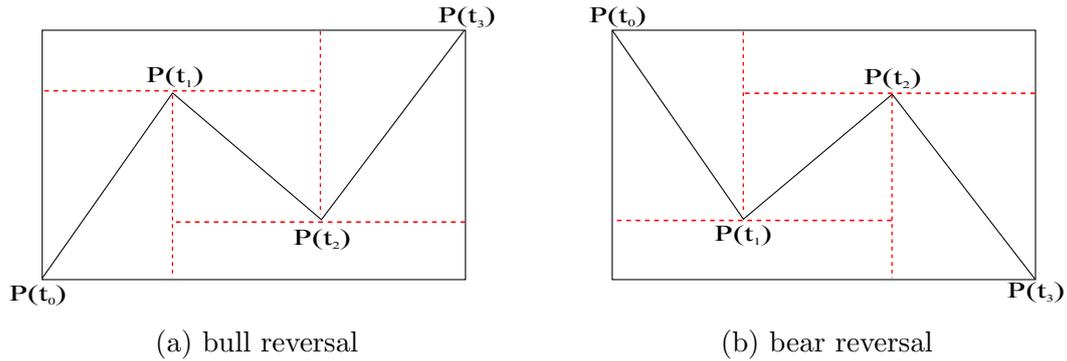


Figure 3: Reversals: within any primary trend we seek to find the maximum reversal. For bulls, we therefore seek the largest fall in prices. If this occurs between $P(t_1)$ and $P(t_2)$ then $P(t_1)$ must be a left maximum, and $P(t_2)$ a right minimum. The largest reversal can be used to further partition a market phase into distinct sub-phases, each of which attains its maximum and minimum values at its end points.

1. If the last point added was a left maximum
 - (a) if new left maximum replace last left maximum
 - (b) otherwise, if new right minimum found, add new right minimum
2. If the last point added was a right minimum
 - (a) if new right minimum replace last right minimum
 - (b) otherwise, if new left maximum found, add new left maximum

These two sets of minimums and maximums represent alternating local extrema. Any candidate for a maximum reversal must occur between two such points. By calculating all possible reversals (respecting time), the largest one that satisfies required criteria can be determined. These criteria can be based on either a minimum price move or duration considerations. However, in line with earlier statements, we require reversals to have maximums or minimums bounding each sub-interval (i.e. intermediate extrema are prohibited in line with principle P3)⁴.

3.2 Top-down approach (TD)

The essence of the proposed new method is to first identify major bull and bear trends in line with the guiding principles, but taking a high level view seeking long term trends over years or even decades. It then uses a recursive approach to detect the most significant sub-trends which must necessarily include some reversal of the major trend. For this reason, the method is referred to as the top-down approach. While the approach is flexible enough to accommodate elements from both PS and

⁴This is to avoid the problem described in Figure 1a if, say, duration considerations prohibit the maximum reversal being selected.

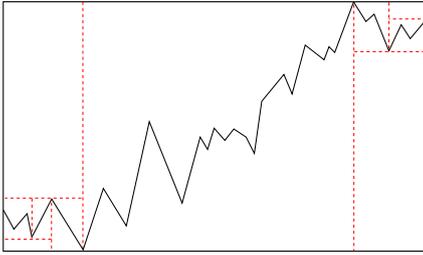
LT, here only minimum bull λ_{bull} and bear λ_{bear} movements are specified. We start with a single interval $[t_0, t_n]$ and proceed as follows:

Phase 1 - recursively partition the interval into sub-intervals until no new partitions are found

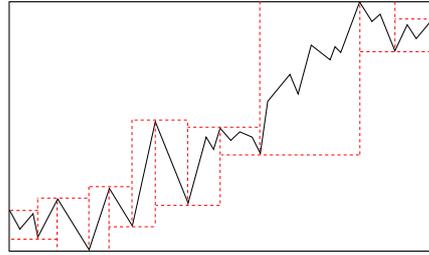
- (a) Over each (sub-)interval $[t_a, t_d]$ find the minimum and maximum values, which we assume (without loss of generality) occur at times $t_b < t_c$ respectively.
- (b) In the event of a tie choose the value that occurs chronologically last.
- (c) If $[t_b, t_c] \subsetneq [t_a, t_d]$ and, either $P_{t_c}/P_{t_b} < 1 - \lambda_{bear}$ or $P_{t_c}/P_{t_b} > 1 + \lambda_{bull}$, then partition the interval at t_b and t_c .

Phase 2 - recursively partition each sub-interval, until no new partitions are found

- (a) Over each (sub-)interval $[t_a, t_d]$ find the maximum valid reversal over $[t_b, t_c]$ and decompose the interval into 3 sub-intervals if one is found.



(a) Phase 1 partition



(b) Phase 2 recursive partitioning

Figure 4: TD approach. Figure (a) shows the partition after phase 1. Figure (b) shows the partition after recursively identifying the maximum valid reversal over each sub-interval. Sub-partitioning stops once no suitable reversal can be found.

The process recursively splits market phases into sub-phases as shown in Figure 4. End points occurring at t_0 and t_n should be discarded⁵. By construction, each new interval created by the partitioning process has end points which coincide with both the maximum and minimum values obtained over the interval, and therefore represents a bull or bear phase. If rule (1b) must be applied because of equal peaks (say), then either the peaks are separated by a trough sufficient to trigger a reversal, or the price must remain within a relatively small band between the peaks. In the former case, the interval between the peaks will be partitioned in phase (2). In the latter case, we follow the LT approach and continue in a bull phase until a sufficient

⁵End points mark the current classification horizon; only additional data can reveal if they truly represent turning points and phase end dates. On an temporary basis one could retain those end points which represent maximum or minimum values over the first and final intervals.

price fall is experienced to trigger a bear. By construction the TD method abides by principles P1-4.

The TD approach requires more computational effort than the LT approach but naturally reveals a hierarchy of phases and sub-phases. Each successful search for a reversal, decomposes a bull (bear) parent phase into two bull (bear) child phases separated by a bear (bull) child phase (see Figure 3). The hierarchy developed under phase 2 can therefore be viewed as a tree with trinomial branching. Note the resemblance to the ‘fractal zig-zag generators’ described by Mandelbrot and Hudson (2008). One can first identify long term, low-frequency bull and bear trends, and then by decreasing the size of the minimum magnitude of bull and bear phases λ_{bull} , λ_{bear} , one increases the number of phases and the resolution of the price series viewed through its peaks and troughs.

While the TD methodology has been described here based on price movements in line with LT, elements from PS can readily be adopted. More generally constraints can easily be added based on the points that define any reversal. One potential difficulty would be the imposition of a cycle-based constraint such as PS Rule 2(c). This would prohibit reversals where the interval between t_0 and t_2 or between t_1 and t_3 (Figure 3) was deemed too short. However this rule could be violated across boundaries established in TD phase 1. To remedy this, one could simply eliminate the lower (higher) peak (trough) or either side of the local minimum (maximum) that must exist at such a boundary. We now present the second result:

R2. If $\lambda_{bull}^* \leq \lambda_{bull}$ and $\lambda_{bear}^* \leq \lambda_{bear}$ then $TD(P, \lambda_{bull}, \lambda_{bear}) \preceq TD(P, \lambda_{bull}^*, \lambda_{bear}^*)$.

This result follows by construction since partitioning of the TD algorithm is guaranteed to stop earlier for higher values of λ_{bull} and λ_{bear} . This confirms that the TD method abides by principle P4. Note that the TD approach can equally be applied to a subset of the original price series, and could therefore be applied to $PS(P, \theta)$. In identifying the hierarchy of market phases of the initial turning points generated by phase 1 application of $PS(P, \theta)$, one could formulate rules for appropriate elimination of invalid PS phases. Post-application of TD to either PS or LT will reveal their embedded hierarchy.

Within the TD framework, it is possible to relax the requirement for equation 1 to hold without creating anomalies such as new highs being reached during bear phases. However, one must be willing to accept that each reversal serves to split the primary phase into three sub-phases of smaller magnitude. Referring to figure 3a,

suppose $P(t_2) = P(t_1)(1 - \lambda_{bear})$ (representing the minimum bull reversal size) then

$$P(t_2) > P(t_0) \Leftrightarrow P(t_0) \left(1 + \frac{P(t_1)}{P(t_0)} - 1 \right) (1 - \lambda_{bear}) > P(t_0) \quad (3)$$

$$\Leftrightarrow \frac{P(t_1)}{P(t_0)} - 1 > \frac{\lambda_{bear}}{1 - \lambda_{bear}} \quad (4)$$

and

$$P(t_3) > P(t_1) \Leftrightarrow P(t_1)(1 - \lambda_{bear}) \left(1 + \frac{P(t_3)}{P(t_2)} - 1 \right) > P(t_1) \quad (5)$$

$$\Leftrightarrow \frac{P(t_3)}{P(t_2)} - 1 > \frac{\lambda_{bear}}{1 - \lambda_{bear}} \quad (6)$$

thus allowing bull reversals of size λ_{bear} potentially creates bull sub-phases of size $\lambda_{bear}/(1 - \lambda_{bear})$. Similarly, referring to figure 3b, suppose $P(t_2) = P(t_1)(1 + \lambda_{bull})$ then

$$P(t_2) < P(t_0) \Leftrightarrow P(t_0) \left(1 - \left(1 - \frac{P(t_1)}{P(t_0)} \right) \right) (1 + \lambda_{bull}) < P(t_0) \quad (7)$$

$$\Leftrightarrow \frac{\lambda_{bull}}{1 + \lambda_{bull}} < 1 - \frac{P(t_1)}{P(t_0)} \quad (8)$$

and

$$P(t_3) < P(t_1) \Leftrightarrow P(t_1)(1 + \lambda_{bull}) \left(1 - \left(1 - \frac{P(t_3)}{P(t_2)} \right) \right) < P(t_1) \quad (9)$$

$$\Leftrightarrow \frac{\lambda_{bull}}{1 + \lambda_{bull}} < 1 - \frac{P(t_3)}{P(t_2)} \quad (10)$$

thus allowing bear reversals of size λ_{bull} potentially creates bear sub-phases of size $\lambda_{bull}/(1 + \lambda_{bull})$. We refer to TD applied without duration constraints, but under the restriction imposed by equation 1, as TD2.

3.3 TD2 Properties

We first note that since TD2 is effectively a single parameter model, it follows from result R2 that TD2 classifications for a given price series form a total order. In particular therefore the model is well behaved in the sense of principle P4. By decreasing λ_{bull} in sufficiently small increments any change in the partition is likely to consist of either one additional point (added during phase 1 partitioning) or two additional points (added during phase 2 partitioning). For any bull/bear partition $\mathcal{M}(P, \theta) = T' = \{t'_j : j = 1, \dots, m\}$ with $m > 1$ we define:

$$R(T') = \sum_{i=2}^m |\log(P(t'_i)) - \log(P(t'_{i-1}))| \quad (11)$$

This measures the sum of absolute log price differences. We will refer to any m -point partition with the highest possible R -score as being *optimal*. From the perspective of a trader with the benefit of hindsight, identifying such a partition would allow them to obtain maximum benefit from price swings, while minimising trading activity and transaction costs. We present two further results:

R3. If T' is a TD2 m -point partition of the time series P , then T' is optimal.

R4. The LT2 and TD2 methods are equivalent.

The somewhat technical proof of result R3 is provided in [Appendix](#) (Theorem 10). Essentially the R -score measure captures the total magnitude of swings between peaks and troughs as the market alternates between bull and bear phases. This result demonstrates that the TD2 algorithm mechanically generates a partition that, in at least one sense, cannot be bettered. Such a property is conducive for maximising consensus on bull/bear phases. Note that while any TD2 m -point partition will be optimal it is not the case that the TD2 model can be suitably parameterized to find the optimal m -point partition for any value of m ⁶. Nevertheless, in practical terms, the performance of any ex ante strategy relying on turning point prediction could be judged against equivalent optimal TD2 partitions.

The proof of result R4 is provided in [Appendix](#) (Theorem 12). It follows that TD2 must abide by all five principles. Clearly TD2 prohibits historical revisions given that, by construction, LT2 has this property, but what of the more general TD methodology if we relax the requirement for equation 1 to hold? Theorem 14 confirms the final result:

R5. The TD model does not permit historical revisions.

The equivalence of the TD2 and LT2 methodologies simplifies the implementation of TD2 when a hierarchy of major and minor phases is not required. While both methodologies will identify the same set of phase-partitioning turning points, the top-down approach will also reveal the natural hierarchy of major and minor market trends. The more general TD method can also be used to relax the bull/bear symmetry requirement, and impose additional constraints. We conclude this section by presenting a summary of the models as shown in Table 1.

⁶To construct a suitable counter example consider partitioning an end-bounded interval with an odd number of points m .

	MS	PS	LT	LT2	TD	TD2
Alternate phases (P1)	✓	✓	✓	✓	✓	✓
Significant rise/fall (P2)	×	×	✓	✓	✓	✓
End-bounded phases (P3)	×	×	×	✓	✓	✓
Parameter Stable (P4)	×	×	×	✓	✓	✓
Prohibits historical revisions (P5)	×	×	✓	✓	✓	✓
Free from end points difficulties	×	×	×	✓	✓	✓
Hierarchical	×	×	×	×	✓	✓
Implementation	optimization	multi-pass	single-pass		recursive	

Table 1: A comparison of the different classification models

Note that while we can safely classify the state of points lying between the penultimate and final partition points, we cannot be sure the final partition will delimit the end of the current phase. For this reason we exclude it from the set of turning points. Finally, we note that the imposition of additional duration based constraints could undermine this result and violate principle P4.

3.4 Corrections and Rallies

Once one has arrived at a full bull/bear classification for a particular price series (according to one’s preferred approach and configuration) a natural next step is to examine, so called, bull corrections and bear rallies. These represent periods in the market where a clear secondary trend is observable that runs contrary to a primary trend. Viewed with hindsight, these secondary trends are not significant enough to warrant their own ‘bull’ or ‘bear’ designation. Of course at the time, such hindsight is unavailable; the new trend may yet develop into a full bull or bear, or may simply be a temporary interruption of the primary trend. Being able to distinguish between such primary and secondary trends *ex tempore* would be invaluable. One attempt to make such a determination is the four-state Markov-switching model used by [Maheu et al. \(2012\)](#).

Such secondary trends can be captured by a further modified (but non-recursive) application of the TD algorithm. First, criteria must be set to determine what constitutes a secondary trend. This could be in the form of thresholds λ_{bc} and λ_{br} defining minimum bull correction and bear rally magnitudes. Second, reversals of sufficient magnitude within in phase would be identified. In terms of the hierarchy of phases, such counter-trend sub-phases would all be considered children of the same parent phase so that, for example, the end of a bear rally would mark the continuation of the primary bear phase.

4 Data and Results

We now compare the different approaches by applying them to various data sets. This exercise is performed using daily data for the S&P500 from 1950-2015⁷ and FT30 from 1930-2015⁸.

4.1 Comparison with PS

Here we apply the PS algorithm with suggested parameterization and the TD algorithm applied using PS rules 2(b), 2(c) and 2(d) to stop recursive splitting of phases. While this can be viewed as setting $\lambda_{bull} = \lambda_{bear} = 0$, recursive partitioning continues to seek the largest (valid) reversal over each interval. Table 2 compares the results generated by the two models for daily S&P500 data, and contrasts these with the peaks and troughs identified by Pagan and Sossounov using monthly data (for completeness we include their baseline results for stock market growth cycles from Niemira and Klein (1994)) and NBER US business cycles⁹. Unsurprisingly, the stock market cycle models identify market peaks that precede NBER recessions. Of course they can only do so with the benefit of hindsight due to the information lag, and bear phases may not be followed by a recession. Note that while the recession of 1980 experienced a fall in S&P500 prices, the duration and magnitude of the fall failed to invoke PS rule 2(d).

NBER	Peak				Trough				
	Pagan Sossounov	Niermira Klein	PS	TD	NBER	Pagan Sossounov	Niermira Klein	PS	TD
1953/07	1952/12	1953/01	1953/01	1953/01	1954/05	1953/08	1953/09	1953/09	1953/09
1957/08	1956/07	1956/07	1956/08	1956/08	1958/04	1957/12	1957/12	1957/10	1957/10
1960/04	1959/07	1959/07	1959/08	1959/08	1961/02	1960/10	1960/10	1960/10	1960/10
	1961/12	1961/12	1961/12	1961/12		1962/06	1962/06	1962/06	1962/06
	1966/01	1966/01	1966/02	1966/02		1966/09	1966/10	1966/10	1966/10
1969/12	1968/11	1968/12	1968/11	1968/11	1970/11	1970/06	1970/06	1970/05	1970/05
	1971/04	1971/04	1971/04	1971/04		1971/11	1971/11	1971/11	1971/11
1973/11	1972/12	1973/01	1973/01	1973/01	1975/03	1974/09	1974/12	1974/10	1974/10
	1976/12	1976/09	1976/09	1976/09		1978/02	1978/03	1978/03	1978/03
1980/01			1978/09		1980/07			1980/03	
1981/07	1980/11	1980/11	1980/11	1980/11	1982/11	1982/07	1982/07	1982/08	1982/08
	1983/06	1983/10	1983/10	1983/10		1984/05	1984/07	1984/07	1984/07
	1987/08	1987/09	1987/08	1987/08		1987/11	1987/12	1987/12	1987/12
1990/07	1990/05	1990/06			1991/03	1990/10	1990/10		
	1994/01	1994/01				1994/06	1994/06		
2001/03			2000/03	2000/03	2001/11			2002/10	2002/10
			2004/02	2004/02				2004/08	2004/08
2007/12			2007/10	2007/10	2009/06			2009/03	2009/03
			2011/04	2011/04				2011/10	2011/10

Table 2: S&P500 bull bear cycles. The first column represents NBER business cycle reference dates. Columns two and three are those reported in earlier work, finishing at the dotted line. The fourth and fifth columns are those identified by the author's implementation of PS and TD using daily price series.

⁷sourced from finance.yahoo.com

⁸sourced from www.globalfinancialdata.com

⁹see www.nber.org/cycles.html

A visual comparison of the PS and TD approaches is also given in Figure 5a. Both approaches broadly identify the same turning points and hence the same bull and bear phases. The TD approach has two extra points which lie at either end of the series because PS rule 2(a) has not been enforced.

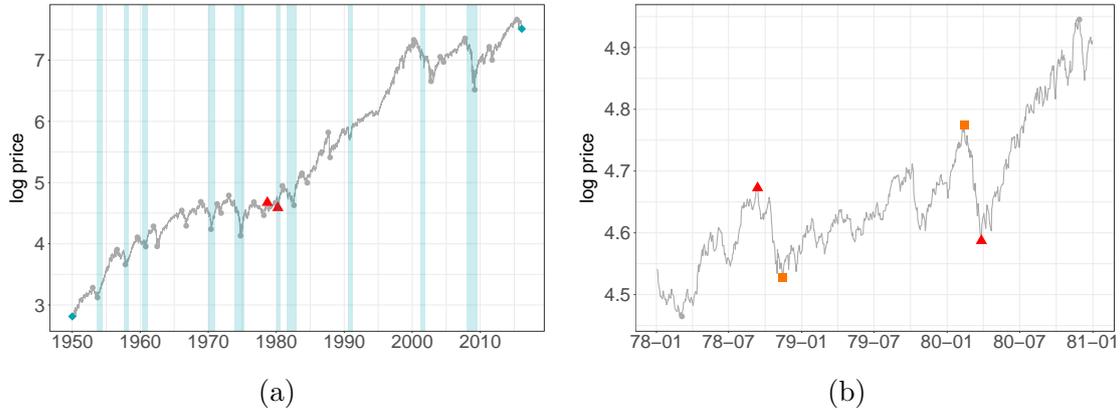


Figure 5: Comparison of the PS and TD models applied to S&P500. Grey circles indicate common turning points, red triangles are unique to PS, green diamonds are unique to TD (here only points at the ends of the time series). Figure (a) shows close agreement between the two models. Shading indicates NBER designed recessions. Figure (b) highlights one interval (1978-81) where the models disagree. Orange squares indicate the lowest and highest intermediate points over this interval.

The PS approach has two extra turning points in September 1978 and March 1980. These points were rejected by TD in compliance with principle P3. Figure 5b highlights that over the course of this PS bear phase, a lower trough is achieved in November 1978 and a higher peak is achieved in February 1980. These intermediate turning points were rejected by TD based on PS rule 1(a). It is not immediately obvious why these peaks and troughs were not identified by Pagan and Sossounov. This appears to be linked to their use of monthly observations. Tests using month end data confirm this. Interestingly, in adapting the work of Bry and Boschan, Pagan and Sossounov point out that the process of data filtering (particularly to remove outliers) commonly applied to identify economic cycles, may actually suppress key events. Their original work also identifies bears between May and October 1990, and between January and June 1994. Using daily price series the former is eliminated due to PS rule 1(a), while the latter is eliminated due to PS rule 2(d).

Figure 6a shows a comparison of the results when applied to FT30 data. While still broadly similar, more differences appear than with the S&P500. Numerous turning points are rejected by PS because of rule 1(a) which are still otherwise PS compliant. Figure 6b reveals a case where the PS and TD models dispute the timing of a turning point peak. Again this is the result of PS filtering rule 1(a) which rejects the later peak due to its proximity to future higher prices. That these higher prices occur after a trough (and end of a bear market) is not accounted for. Again, mark

peaks precede official UK recessions (defined as two consecutive quarters of negative economic growth¹⁰).

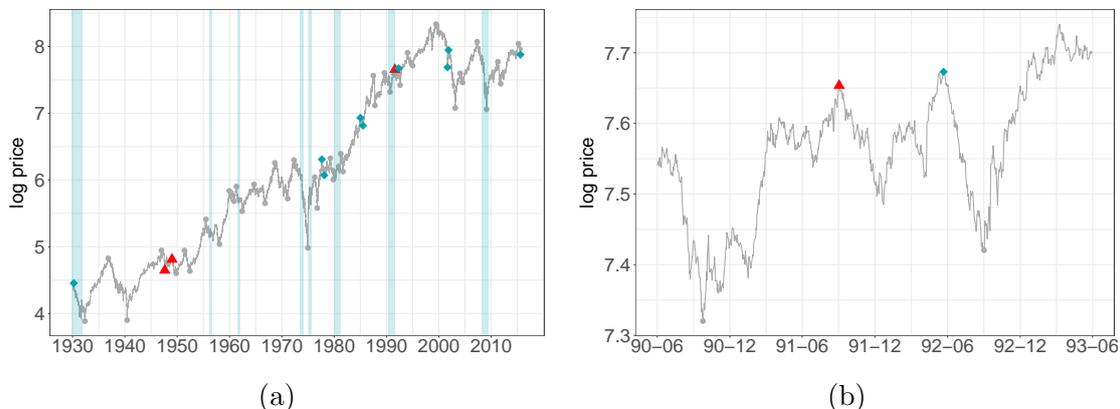


Figure 6: Comparison of the PS and TD models applied to FT30. Grey circles indicate common turning points, red triangles are unique to PS, green diamonds are unique to TD. Figure (a) shows close agreement between the two models. Shading indicates official recessions. Figure (b) highlights one interval (1990-93) where the models disagree. Here the precise timing of a peak is disputed. PS has rejected the higher turning point because of its filtering rule 1(a).

Table 3 provides statistics which summarise the bull and bear markets identified by the two approaches. We see that bull markets have larger absolute returns, longer durations, and therefore more time is spent in the bull market state. This is consistent with Pagan and Sossounov (2003) who find that US equity markets spend approximately 60-70% of time in a bull state. Positive returns are achieved in approximately 11 (9) out of every 20 days in a bull (bear) state. We note that bears also exhibit higher volatility, a defining feature of the bear definition used by Maheu and McCurdy (2000) who report that the market spends on average 90% of time in a bull state under their 2-state model. Box plots of the distribution of daily returns are shown in Figure 7.

	S&P500				FT30			
	PS		TD		PS		TD	
	bear	bull	bear	bull	bear	bull	bear	bull
State count	17	17	16	16	21	22	23	24
Average duration (months)	13.25	30.78	12.92	33.86	16.64	29.31	15.94	26.15
Average return (%)	-31.67	57.44	-33.12	60.50	-46.27	63.05	-44.28	59.75
Cumulative return (%)	-538.40	976.51	-529.84	967.95	-971.59	1387.13	-1018.40	1433.94
Positive return days (%)	45.79	56.43	45.19	56.29	45.89	57.11	46.26	57.19
Volatility (%)	19.18	13.61	19.62	13.60	19.43	14.78	19.09	14.89
Median daily return (%)	-0.08	0.09	-0.09	0.09	-0.10	0.11	-0.09	0.12
Time in state (%)	30.09	69.91	27.62	72.38	35.15	64.85	36.88	63.12

Table 3: Descriptive statistics for PS and TD models applied with suggested parameters. Here the end points from the TD classification have been excluded so both approaches span the same period. Following Pagan and Sossounov the return over each phase is calculated as the change in log price. Cumulative return should therefore increase with a rise in cycles.

For the S&P500, the TD approach has one fewer cycles. This leads to an increase in average bull duration, but a decrease in average bear duration since an extra

¹⁰see www.ons.gov.uk

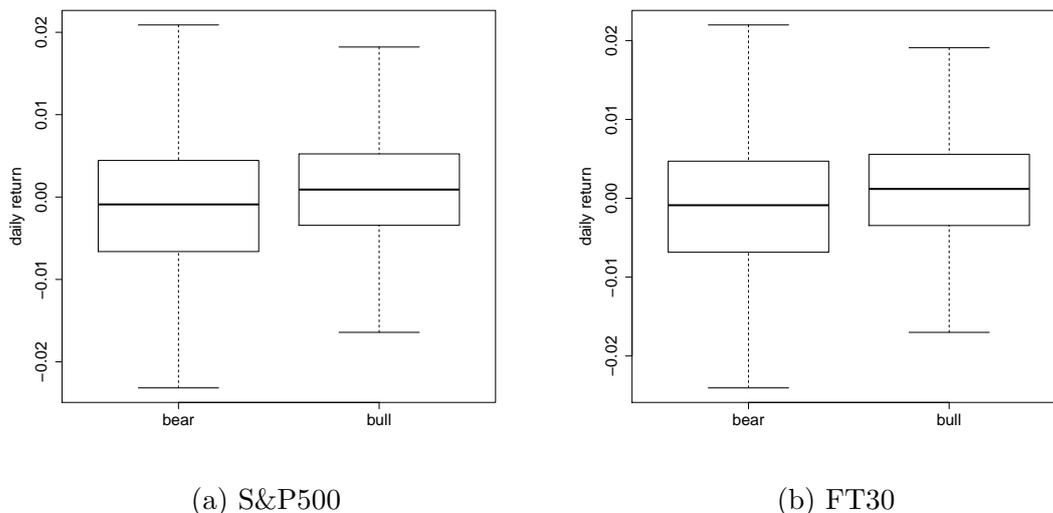


Figure 7: Box plots of daily return for TD model using a PS-style minimum duration approach.

above-average ‘PS bear’ is identified. For both models the shortest bear is only 101 days in duration (because of PS rule 2(d)) due to a 33.5% fall between August and December 1987. The TD partition ends at the series all time high on 21st May 2015. Suppose the partition is extended to include the series start point and all time series low. This adds an additional initial bull phase and causes the full partition to be end-bounded. The R -score becomes 15.62, which as lemma 5 shows, comprises the change in log price between the start and end points, plus twice the sum of the reversals 5.38. As a comparison the partition generated by TD2 with $\lambda_{bull} = 20\%$ comprises of an equal number phases but achieves an R -score of 16.49 with reversals totalling 5.82.

For the FT30, the TD approach identifies more cycles resulting in a slight decrease in duration and amplitude (i.e. absolute return). For TD the shortest bull market is only 76 days due to a 29.5% rise between September and December 2001. Despite satisfying PS rule 2(d) this phase is eliminated based on PS rule 1(a).

An interesting question to consider is how dates identified by any model relate to those of market analysts, and further, whether analyst forecasts provide a means to identify turning points ex-ante. If analysts can successfully recognise market turning points, and distinguish between short-term reversals and long-term trends, this should be reflected in their forecasts. Figure 8a plots one such forecast¹¹ from 1999-2015. Notice that the turning points in the forecast series lag those of the actual price series. Furthermore, Figure 8b shows that during bear phases, forecasts tend to be overly optimistic, with actual year-end prices considerably lower than the

¹¹Bloomberg’s SPXSFRC Index, an average year-end S&P500 forecasts compiled from Wall Street strategists

forecasts. Conversely, forecasts during bull phases are conservative. Interestingly, it appears that forecasters are quick to ‘call the bottom’ of the market in 2002/3 and 2008/9 but quickly lose the courage of their conviction, as the bear market re-emerges from a short lived reversal. While forecast price series are less volatile, and have more readily identifiable turning points, they ultimately suffer the same information lag problem.

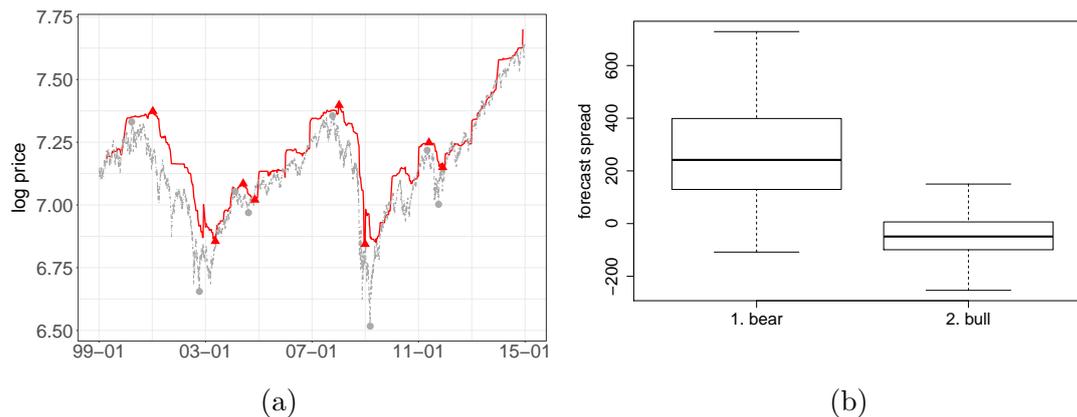


Figure 8: S&P500 Forecasts. Figure (a) shows the S&P500 (grey dashed line) versus Bloomberg’s average year-end forecast (red solid line). Markers identify turning points identified using the TD model configured to match PS. Figure (b) is a boxplot of the forecast spread (i.e. year-end forecast minus the actual year-end price).

4.2 Comparison with LT

The equivalence of LT2 and TD2 has already been demonstrated. Instead here we compare the LT and TD models for different parameterizations not consistent with equation 1. Figure 9a shows a comparison when applied to S&P500 with non-symmetric parameterization $\lambda_{bull} = 0.20$ and $\lambda_{bear} = 0.10$, a configuration considered by [Lunde and Timmermann](#). Here the TD model identifies more turning points.

Figure 9b examines one such difference. Both models concur that a bull market ends in October 1989. However the LT bear market continues until October 1990 and therefore spans a period containing, what was at the time, an all-time market high in late January 1990. Either LT called the bull market high too soon, or fails to recognise a prerequisite bear phase required to reach such a high.

Table 4 provides statistics which summarise the bull and bear markets identified for different parameterizations under TD2. Since increasing λ_{bull} from 20.00% to 30.00% increases λ_{bear} from 16.67% to 23.08%, it is not surprising that the number of cycles under the latter parameterization decreases significantly, with a dramatic impact on the length of bull durations. Given the prescriptive nature of the LT model, it is possible to calculate [Nyberg’s](#) ‘real-time information lag’, that is, the

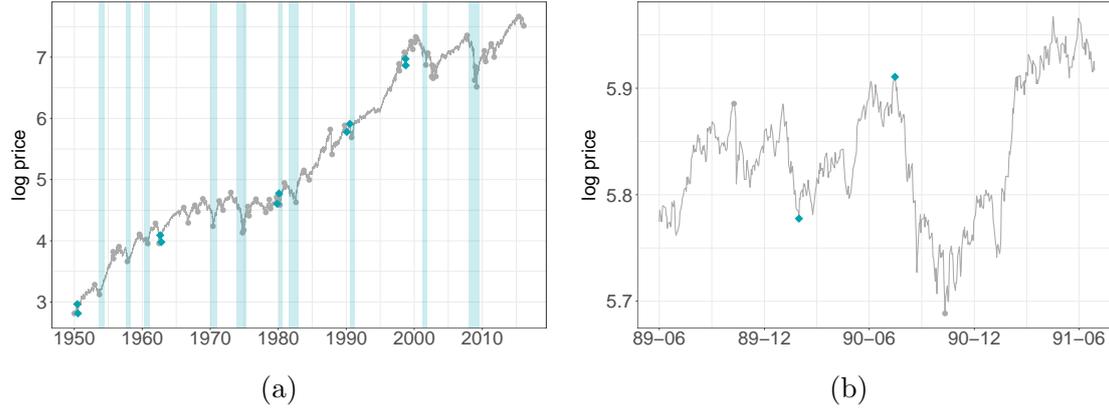


Figure 9: Comparison of the LT and TD models applied to S&P500 with $\lambda_{bull} = 0.20$ and $\lambda_{bear} = 0.10$. Grey circles indicate common turning points, red triangles are unique to LT, green diamonds are unique to TD. Figure (a) shows close agreement between the two models. Shading indicates official recessions. Figure (b) highlights one interval where the models disagree. Here the LT bear market between October 1989 and October 1990 contains a (then) all-time market high achieved in late January 1990.

time taken to trigger a change in state. We see that, on average, this lag is smaller for bulls in absolute terms, and significantly smaller in relative terms. Once again we see that returns in the bull state are less volatile.

	S&P500				FT30			
	TD2 $\lambda_{bull}=20\%$		TD2 $\lambda_{bull}=30\%$		TD2 $\lambda_{bull}=20\%$		TD2 $\lambda_{bull}=30\%$	
	bear	bull	bear	bull	bear	bull	bear	bull
State count	17	18	7	8	31	31	20	20
Average duration (months)	9.48	34.63	16.61	83.54	11.76	21.13	17.73	33.26
Average return (%)	-34.23	59.28	-52.47	106.55	-41.01	52.58	-51.57	69.51
Cumulative return (%)	-581.95	1067.07	-367.26	852.39	-1271.30	1629.97	-1031.45	1390.12
Positive return days (%)	44.12	55.74	44.72	54.86	44.41	57.52	45.66	56.61
Volatility (%)	20.88	13.53	23.70	13.40	19.48	14.67	19.63	14.74
Median daily return (%)	-0.11	0.09	-0.12	0.07	-0.13	0.12	-0.10	0.11
Time in state (%)	20.54	79.46	14.82	85.18	35.76	64.24	34.77	65.23
Average time to call state (months)	5.95	3.24	10.43	7.36	4.84	3.60	8.96	7.59

Table 4: Descriptive statistics for TD2 model applied with different parameters

By comparing Tables 3 and 4 we see that results for S&P500 are similar, but that results for FT30 differ significantly, thus while PS and LT can be expected to produce similar results, this is sensitive to their parameterizations.

4.3 Comparison with MS

The MS model from equation 2 is applied to a monthly S&P500 price series. The results generated by MS models are markedly different to those of the rule-based approaches so no direct comparison of partition points is attempted. The market states found correspond to a positive return, low volatility bull state ($\mu_1 = 1.04\%$, $\sigma_1 = 3.31\%$) and a negative return, high volatility bear state ($\mu_2 = -1.32\%$, $\sigma_2 = 6.40\%$). Similar to Maheu and McCurdy (2000) the market is found to spend only 10% of time in a bear state. Figure 10a illustrates the filtered probabilities of being

in a bear state.

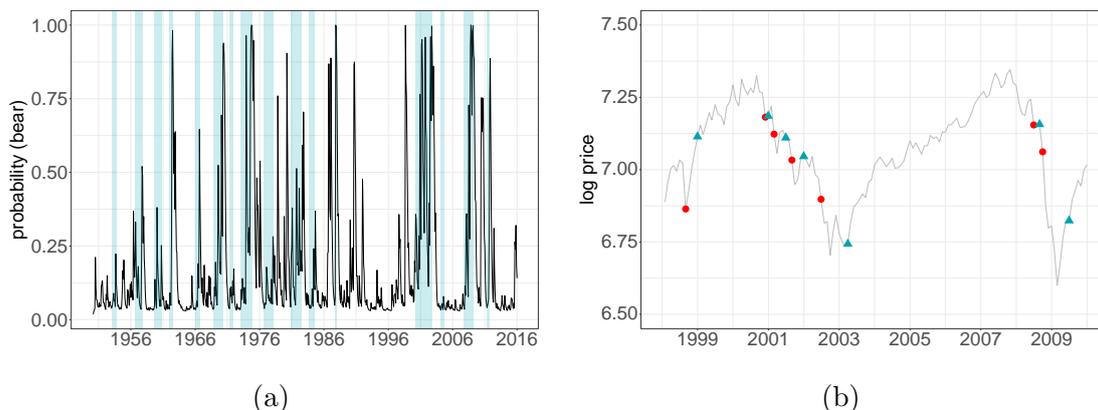


Figure 10: A simple MS model applied to S&P500. The filtered probability of a bear state is plotted. In figure (a) shaded areas indicate bear phases identified using the PS approach, but implemented within the TD framework. Figure (b) plots log price. Red circles indicate points where the probability of a bear state has risen above a 50% threshold, green triangles where the probability has fallen below a 50% threshold.

The probability spikes tend to coincide with periods identified as bear phases by rule-based approaches. A real-time information lag is again evident. This is further illustrated by Figure 10b where precise partition points are highlighted. These points rarely coincide with local extrema, and are therefore unlikely to find universal support as definitive phase transition dates.

4.4 Extension to reversals

We consider now extending the classification to four states to admit the secondary counter-trends of bull corrections and bear reversals. Here the TD2 model is first used to identify primary trends. Then, as described in section 3.4, reversals within phases are sought. A threshold for bear rallies is set at precisely half that required for bull phases. Results are shown in Table 5 and Figure 11. The median daily return is higher in bear reversals than in bulls, but lower in bears than in bull corrections. The average impact of secondary trends is approximately the same, although the bull corrections are much longer and their cumulative impact is much larger given their more frequent occurrence. Volatility remains relatively high in bear reversals suggesting that a simple 2-state model based on stable positive returns and volatile negative returns might not capture bear rallies.

	S&P500				FT30			
	bear	bear ral.	bull cor.	bull	bear	bear ral.	bull cor.	bull
State count	29	12	27	45	71	40	25	56
Average duration (months)	4.83	1.74	3.01	12.05	3.97	2.08	2.92	10.39
Average return (%)	-25.25	12.53	-12.39	31.15	-25.27	13.07	-12.47	34.68
Cumulative return (%)	-732.26	150.31	-334.49	1401.57	-1794.30	523.00	-311.85	1941.82
Positive return days (%)	41.83	59.36	42.16	57.79	38.87	63.21	43.92	59.27
Volatility (%)	20.48	21.66	15.34	13.11	19.36	17.92	16.05	14.34
Median daily return (%)	-0.17	0.16	-0.15	0.11	-0.24	0.20	-0.13	0.14
Time in state (%)	17.87	2.67	10.36	69.11	27.61	8.15	7.16	57.08

Table 5: Descriptive statistics for TD2 model applied with second pass to identify reversals. Here $\lambda_{bull} = 20\%$ and bear rallies are defined by $\frac{1}{2}\lambda_{bull}$, with λ_{bear} and bull corrections being defined in accordance with equation 1.

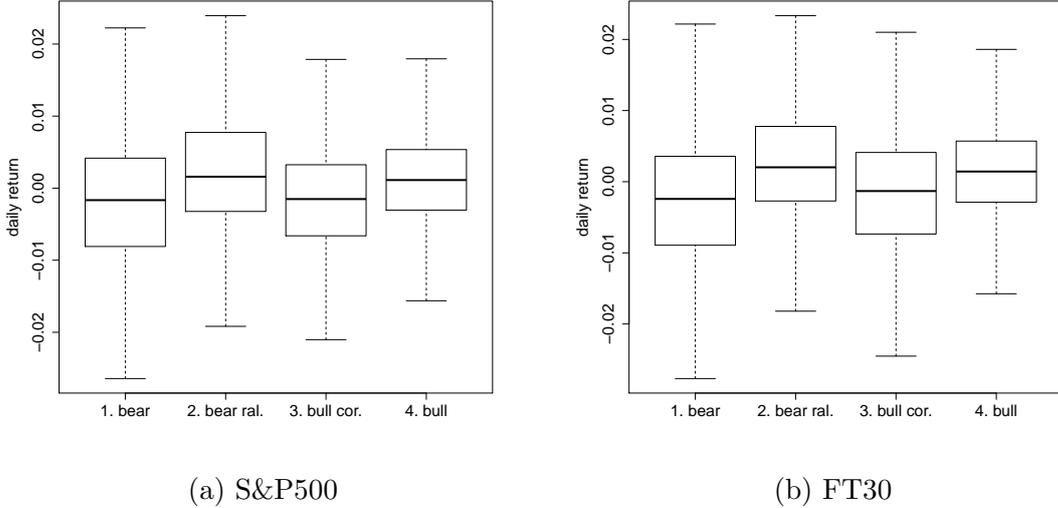


Figure 11: Box plots of daily return for TD model using a LT-style minimum magnitude approach with reversals.

In their 4-state approach [Maheu et al.](#) report unconditional state probabilities of 0.070 (bear), 0.157 (bear rally), 0.304 (bull correction) and 0.469 (bull), which differ significantly from those reported here¹². They also report the probabilities of positive returns in each of the same four states as 0.44, 0.53, 0.48, and 0.59. The results here are more pronounced with positive returns achieved in approximately 12 out of every 20 days in either a bull or bear rally state, and in approximately 8 out of every 20 days in either a bear or bull correction state.

4.5 Parameterization

One difficulty with any ex post classification is the parametrization choice, and how this impacts the resulting classification. While the TD approach outlined here ensures the stability of turning points (principle P4), individual days are subject to classification change as the resolution of the classification is changed. For example,

¹²Note that under their model it is possible to transition from a bear rally into a bull, and from a bull correction to a bear.

a point classified in a bull state for a low resolution classification, could move to a bear state in a higher resolution classification, and then a bear reversal state when reversals are considered. We examine now how this issue of resolution might impact the dynamics of each state, and how the personal preference for a particular parameterization would influence how one might experience them.

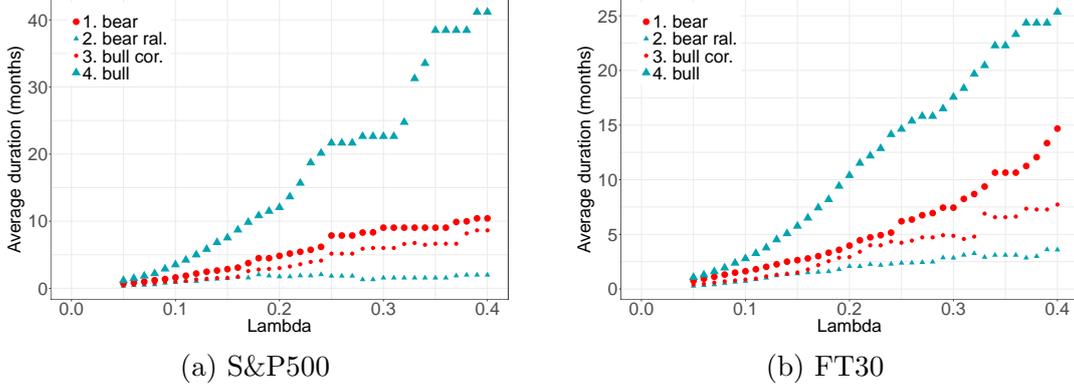


Figure 12: Average duration by state based on TD2 with varying λ_{bull} .

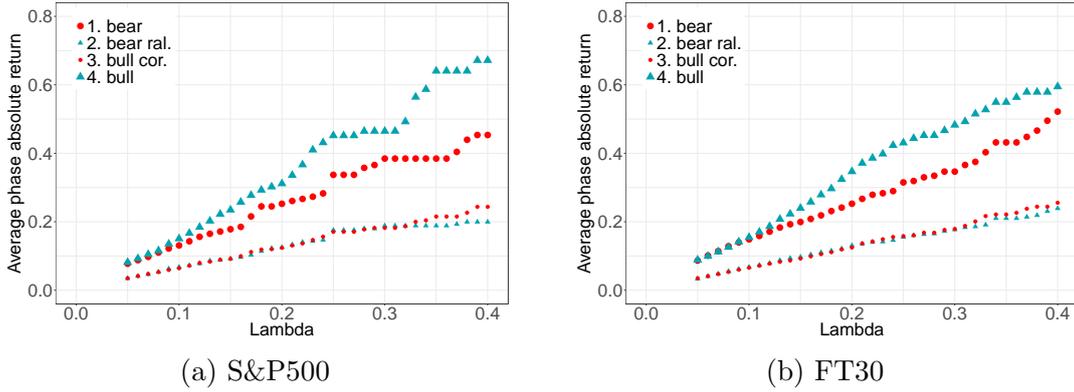


Figure 13: Average phase return (absolute log price change) by state based on TD2 with varying λ_{bull} .

Clearly decreasing the minimum threshold required to qualify as a phase will reduce both the average duration (Figure 12) and magnitude (Figure 13) of phases. However in each case, bull phases are on average significantly longer than bear phases, but while bear rallies are generally short-lived, bull corrections are closer to bear in average duration. We see that, under the TD2 configuration, bull corrections and bear rallies have on average the same absolute return, regardless of resolution. We see in Figure 14 that as the minimum threshold decreases, there is a general increase in the magnitude of the absolute median daily return across all states. This in part reflects a purification process: as resolution increases the number of positive return days increases (Figure 15). The absolute median return in the bear state is consistently larger than that in a bull correction state, yet the bear reversal absolute

median return dominates that of the bull. This latter observation runs contrary to [Maheu et al. \(2012\)](#) who estimate lower returns in their bear rally relative to their bull state.

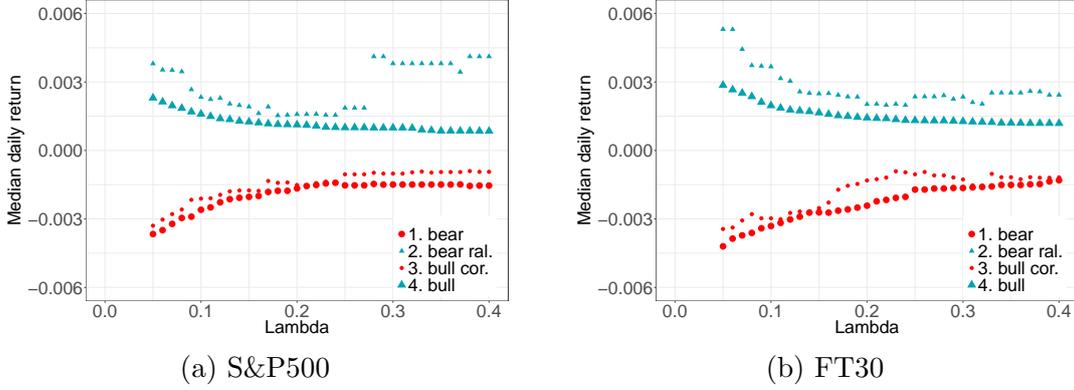


Figure 14: Daily median return by state based on TD2 with varying λ_{bull} .

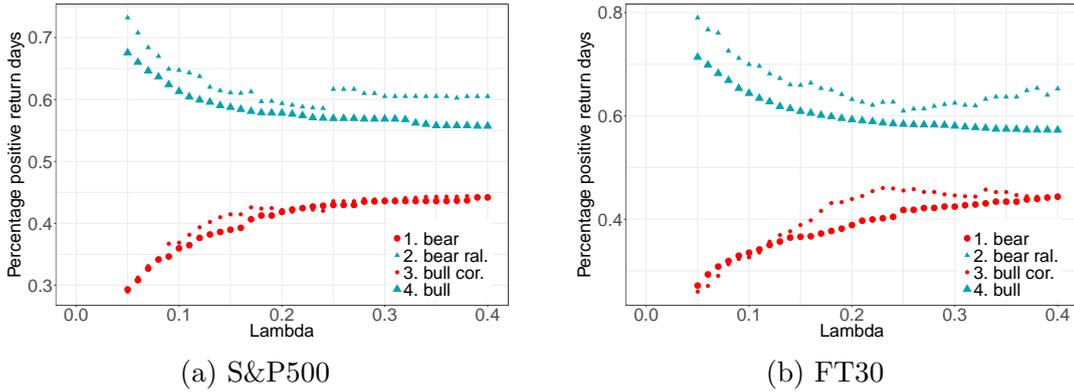


Figure 15: Percentage of days with positive return by state based on TD2 with varying λ_{bull} .

Figure 16 shows the percentage time spent in each state. While bull is the dominant market state, as one would expect of any random process with positive drift, increasing resolution sees a redistribution into the bear state. The time spent in secondary trend states remains consistently low. Figure 17 presents the equivalent results for state volatility. We note here that volatility in the bull state is both stable and consistently low regardless of the choice of resolution. For other states volatility falls as the resolution is increased. Bear volatility remains consistently higher than the volatility in both bull and bull correction states.

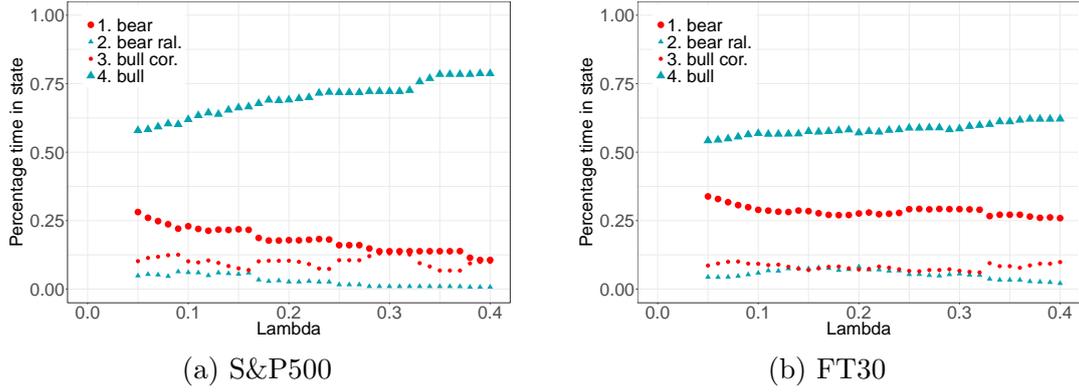


Figure 16: Percentage time in state based on TD2 with varying λ_{bull} .

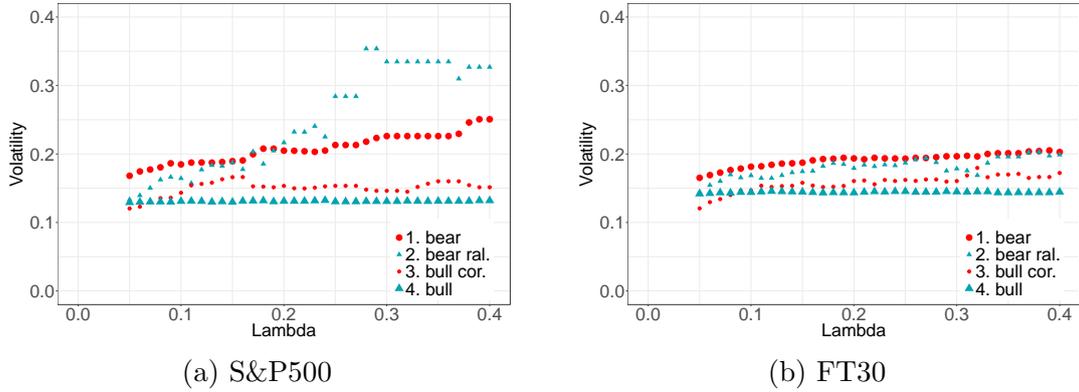


Figure 17: Volatility by state based on TD2 with varying λ_{bull} .

5 Conclusion

We have examined two commonly applied ex post rule-based methodologies to identify bull and bear states. Both approaches successfully identify phases and cycles that broadly align with consensus views of economic cycles and historic bulls and bear. However, the analysis presented demonstrates a number of deficiencies which have the potential to undermine their results.

The principle-based framework can be used as a basis to evaluate ex post methodologies. Its application has suggested modifications to existing classification methods, however, the new top-down approach provides several benefits that recommended its adoption. Firstly, it can be applied as an additional final step to existing methodologies, to create a hierarchy of market phases. Secondly, in doing so, it is possible to identify and remove phases which are unsuited to the application of bull or bear labels. Thirdly, the top-down methodology can be applied as a replacement for existing methodologies, but with their rules enforced to determine when the recursive top-down partitioning should cease. The approach avoids the ambiguity of first identifying turning points and then locally deciding which ones

to discard. Fourthly, the approach is robust and stable under changes of parameterization. Fifthly, a further application of the algorithm can be used to identify short-term secondary trends associated with bull corrections and bear rallies. Finally, we have demonstrated that under certain conditions, the top-down approach can be considered optimal in at least one sense. It therefore provides a benchmark against which other ex ante methodologies can be judged.

Given the potential sensitivity of any such model to its chosen parameterization, stability seems desirable. The top-down approach exhibits stability to changes in parameterization, and to extensions of the time series under analysis. It could therefore be used as an additional robustness check for any study relying on such ex post classification.

The approach is flexible enough to incorporate different formal definitions and could equally be applied to other time series and financial cycles. When implemented with magnitude based constraints the methodology does not permit historical revisions. Perhaps most importantly, the approach enforces principles which select key turning points more likely to receive consensus among market commentators.

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Appendix

5.1 Proofs for section 2.2

Lemma 1 *Under the LT algorithm, the prices achieved at the start and end of bull and bear market phases bound the values over the interval when:*

$$\lambda_{bull} = \frac{\lambda_{bear}}{(1 - \lambda_{bear})} \quad (12)$$

Proof. We consider a bull phase over an interval $[t_0, t_3]$. Let $t_2 \in (t_0, t_3)$. If $P(t_2) \leq P(t_0)$ we conclude that, since the prior bear phase did not extend to t_2 , a bull phase had been entered at $t_1 < t_2$. It follows that:

$$\frac{P(t_1)}{P(t_2)} \geq \frac{P(t_1)}{P(t_0)} \geq 1 + \lambda_{bull} = \frac{1}{1 - \lambda_{bear}} \Rightarrow \frac{P(t_2)}{P(t_1)} \leq 1 - \lambda_{bear} \quad (13)$$

meeting the criteria for entering a new bear phase, contradicting the fact that t_3 marks the end of the bull phase. Hence $P(t) > P(t_0) \forall t \in (t_0, t_3)$. Now let $t_1 \in (t_0, t_3)$. If $P(t_1) > P(t_3)$ we conclude that t_3 cannot coincide with the end of the bull market: either t_1 marks the highest peak achieved during the bull and t_3 is actually part of a bear, or there exists a point $t_4 > t_3$ such that $P(t_4) \geq P(t_1)$ and the bull extends beyond t_3 . A contradiction is reached so we must have $P(t) \leq P(t_3) \forall t \in (t_0, t_3)$. Bear market treatment is similar.

Lemma 2 *Let $LT(P, \lambda_{bull}, \lambda_{bear}, t_0^*) = \{t_j^* : j = 0, \dots, m\}$ delimit the phases generated by LT, satisfying equation 1 with designated starting point t_0^* . Now let $LT(P', \lambda_{bull}, \lambda_{bear}, t'_0) = \{t'_j : j = 0, \dots, k\}$ delimit the phases generated by LT with designated starting point t'_0 for reverse chronological series $P'(t_i) = P(t_{n-i})$. Further assume that:*

$$(P(t) - P(t_j))(P(t) - P(t_{j+1})) < 0 \forall t \in (t_j, t_{j+1}) \text{ for } j = 0, \dots, m - 1. \quad (14)$$

If $t'_0 = t_k^*$ ($k > 1$) then $t'_1 = t_{k-1}^*$.

Proof. Suppose t_k^* marks the end of a bear phase, then $[t_{k-1}^*, t_k^*]$ satisfies the minimum requirement for a bull phase under reverse chronological ordering since:

$$\frac{P(t_k^*)}{P(t_{k-1}^*)} < 1 - \lambda_{bear} \Rightarrow \frac{P(t_{k-1}^*)}{P(t_k^*)} > \frac{1}{1 - \lambda_{bear}} = 1 + \lambda_{bull}$$

Now $[t_{k-1}^*, t_k^*]$ is an uninterrupted bear so for any $[t_a, t_b] \subseteq [t_{k-1}^*, t_k^*]$ we have:

$$\frac{P(t_b)}{P(t_a)} < 1 + \lambda_{bull} \Rightarrow \frac{P(t_a)}{P(t_b)} > \frac{1}{1 + \lambda_{bull}} = 1 - \lambda_{bear}$$

so we must have $t'_1 \leq t_{k-1}^*$. If $t'_1 < t_{k-1}^*$, then the reverse chronological bull extends (without reversal) to a point $P(t'_1) > P(t_{k-1}^*)$ (by Lemma 1 and equation 14) but for any $[t_a, t_b] \subseteq [t'_1, t'_0]$ we have:

$$\frac{P(t_a)}{P(t_b)} > 1 - \lambda_{bear} \Rightarrow \frac{P(t_b)}{P(t_a)} < \frac{1}{1 - \lambda_{bear}} = 1 + \lambda_{bull}$$

Since bear phase $[t_{k-1}^*, t_k^*] \subset [t'_1, t'_0]$ we should not be able to find a point larger than $P(t_{k-1}^*)$ without also first finding a reversal (by implication of Lemma 1). We have thus arrived at a contradiction. Treatment for bull phases is similar.

Note that the imposition of equation 1 is critical here, without which Lemma 1 cannot be invoked. Figure 2 illustrates a counterexample to confirm this. Moving chronologically from time t_0 the LT methodology would select turning point t_1 and t_4 . Moving in reverse chronological order from time t_4 the LT methodology would select turning point t_3 and t_0 .

5.2 Proofs for section 3.1

Lemma 3 *The maximum reversal achieved over a bull (bear) period must occur between left maximum (minimum) and right minimum (maximum) turning points.*

Proof. Suppose that the maximum reversal within a bull phase occurs over sub interval (t_i, t_j) in violation of the Lemma. If t_i is not a left maximum then $\exists t_k < t_i$ such that $P(t_k) > P(t_i)$ meaning that (t_k, t_j) represents a larger reversal. Similarly if t_j is not a right minimum then $\exists t_k > t_j$ such that $P(t_j) > P(t_k)$ meaning that (t_i, t_k) represents a larger reversal. If t_i is not a turning point, then $P(t_{i+1}) > P(t_i)$ meaning that (t_{i+1}, t_j) represents a larger reversal. If t_j is not a turning point, then $P(t_{j-1}) < P(t_j)$ meaning that (t_i, t_{j-1}) represents a larger reversal.

Corollary 4 *Suppose that the maximum reversal within a bull (bear) phase occurs over sub interval (t_i, t_j) then $P(t_i) - P(t_k)$ is positive (negative) $\forall t_k \in (t_0, t_j)$ and $P(t_j) - P(t_k)$ is negative (positive) $\forall t_k \in (t_i, t_n)$.*

5.3 Proofs for section 3.3

Lemma 5 *Suppose time series P has its maximum and minimum values at times $\{t_1, t_m\}$. If $\{t_1, \dots, t_m\} \subseteq T^*$ is an m -point partition satisfying principals P1 and P2 then m is even and:*

$$R(T^*) = |\log(P(t_1)) - \log(P(t_m))| + 2 \sum_{j=1}^{m/2-1} |\log(P(t_{2j})) - \log(P(t_{2j+1}))| \quad (15)$$

Proof. We consider only the case where $P(t_1) < P(t_m)$. It follows that the first, last and odd numbered phases are bull phases. An odd number of phases requires an even number of partitioning points. Moreover:

$$R(T^*) = \sum_{i=2}^m |\log(P(t_i)) - \log(P(t_{i-1}))| \quad (16)$$

$$= (\log(P(t_2)) - \log(P(t_1))) + (\log(P(t_2)) - \log(P(t_3))) + \dots \\ + (\log(P(t_m)) - \log(P(t_{m-1}))) \quad (17)$$

$$= \log(P(t_m)) - \log(P(t_1)) + 2 \sum_{j=1}^{m/2-1} \log(P(t_{2j})) - \log(P(t_{2j+1})) \quad (18)$$

In considering what an optimal m -point partition might be, we first show that it is likely to share points in common with the TD approach.

Lemma 6 *Let T' be a TD2 partition of the time series P and T^* be any optimal m -point partition. Let $T'_1 \subseteq T'$ contain only points identified in phase 1 TD partitioning, then for each $t_i \in T'_1 \nexists j$ such that $t_j^* < t_i < t_{j+1}^*$.*

Proof. We consider only the case where t_i is a peak, $t_j^* < t_i < t_{j+1}^*$ and $P(t) < P(t_i) \forall t < t_i$, other cases being similar. If $P(t_j^*) > P(t_{j+1}^*)$ then moving t_j^* to t_i will create a partition with a greater R -score. If $P(t_j^*) < P(t_{j+1}^*)$ then $t_{i+1} < t_{j+1}^*$ since (t_i, t_{i+1}) is a TD bear phase. Moving t_j^* to t_{i+1} will therefore create a greater R -score since $t_{i+1} < P(t) \forall t < t_{i+1}$. Thus, in both cases we reach a contradiction.

Lemma 7 *Let T' be a TD2 partition of the time series P , and T^* be any optimal m -point partition. Suppose t_j^* is the smallest member of T^* in the open interval (t'_i, t'_{i+1}) then **either** the following properties hold:*

1. $t_{j+1}^* < t'_{i+1}$
2. $(P(t'_i) - P(t'_{i+1}))(P(t_{j+1}^*) - P(t_j^*)) < 0$
3. $\{t_k^* : t_k^* \in (t'_i, t'_{i+1})\}$ contains an even number of points.

or it is possible to set some t_k^* to one of $\{t'_i, t'_{i+1}\}$ without changing $R(T^*)$. Note that the latter case in this lemma is only necessary to deal with the treatment of double turns.

Proof. We consider only the case where $P(t'_i) < P(t'_{i+1})$. We first eliminate the case $j = m$. If (t_{j-1}^*, t_j^*) is a bull phase then moving t_j^* to t'_{i+1} would increase the size of the bull contradicting the optimal nature of T^* , or have no effect on $R(T^*)$

if $P(t_j^*) = P(t_{i+1})$. If (t_{j-1}^*, t_j^*) is a bear phase then we must have $t_{j-1}^* < t'_i$ since t_j^* was the smallest such point in the interval (t'_i, t'_{i+1}) . Here again moving t_j^* to t'_i would contradict the optimal nature of T^* , or have no effect on $R(T^*)$.

If (t_j^*, t_{j+1}^*) is a bull then moving t_j^* to t'_i would contradict the optimal nature of T^* unless $P(t_j^*) = P(t_i)$. If (t_j^*, t_{j+1}^*) is a bear then $t_{j+1}^* < t'_{i+1}$. If this were not the case then moving t_j^* to t'_{i+1} would contradict the optimal nature of T^* , or have no effect on $R(T^*)$.

Finally, if (t_j^*, t_{j+1}^*) is a bear and $t_{j+1}^* < t_{i+1}$, if $j < m - 2$ then (t_{j+1}^*, t_{j+2}^*) is a bull. If $t_{j+2}^* < t'_{i+1}$ then $j < m - 3$ otherwise extending t_{j+2}^* to t'_{i+1} will create a contradiction, or have no effect on $R(T^*)$. Thus (t_{j+2}^*, t_{j+3}^*) is a bear but we must have $t_{j+3}^* < t'_{i+1}$ or another contradiction will be created. Induction completes the result.

Lemma 8 *Let T' be a TD2 partition of the time series P and T^* be any optimal partition. Suppose $t'_{i_1} = t_{j_1}^*$ and $t'_{i_2} = t_{j_2}^*$ form an interval such that $\nexists t \in (t'_{i_1}, t'_{i_2})$ where $t \in T' \cap T^*$, then the interval is end-bounded, and $|\log(P(t_{j_1}^*)) - \log(P(t_{j_2}^*))| \leq \alpha = \log(1 + \lambda_{\text{bull}})$ for any $(t_j^*, t_{j+1}^*) \subset (t'_{i_1}, t'_{i_2})$ with $(P(t_{j+1}^*) - P(t_j^*))(P(t'_{i_2}) - P(t'_{i_1})) < 0$.*

Proof. Suppose the interval is not end-bounded, then $\exists t \in (t'_{i_1}, t'_{i_2})$ with price greater than $\max(P(t'_{i_1}), P(t'_{i_2}))$ or less than $\min(P(t'_{i_1}), P(t'_{i_2}))$ ¹³. We consider the former (the latter case being similar). Find i such that $t'_i \leq t \leq t'_{i+1}$. If $t \notin T'$ then TD phase (t'_i, t'_{i+1}) is not end-bound, contradicting a property of TD partitions. Similarly, find j such that $t_j^* \leq t \leq t_{j+1}^*$. If $t \notin T^*$ then T^* phase (t_j^*, t_{j+1}^*) is not end-bound and cannot be optimal. Thus $t \in T' \cap T^*$ contradicting the original assumption.

We now consider the case where $P(t'_{i_2}) > P(t'_{i_1})$ and assume $P(t_{j+1}^*) < P(t_j^*)$ for some $(t_j^*, t_{j+1}^*) \subset (t'_{i_1}, t'_{i_2})$, other cases being similar. First find the largest $t'_i \in T'$ less than t_j^* and the smallest $t'_k \in T'$ greater than t_{j+1}^* . From Lemma 7 we know that $t'_i < t_j^* < t_{j+1}^* < t'_{i+1}$ thus $k = i + 1$. If (t'_i, t'_{i+1}) is a bear phase then moving t_j^* to t'_i would contradict the optimal nature of T^* . If (t'_i, t'_{i+1}) is a TD2 bull phase with no intermediate points, then we must have $|\log(P(t_{j+1}^*)) - \log(P(t_j^*))| < \alpha$.

Lemma 9 *Let T' be a TD2 m -point partition of the time series P , and T^* be any optimal m -point partition. Suppose t'_i is the smallest T' in the open interval (t_j^*, t_{j+1}^*) and $i < m$ then **either** the following properties hold:*

¹³The case where t exists as a double turn leads to only trivially different T' or T^* partitions and can therefore be neatly sidestepped.

1. $t'_{i+1} < t_{j+1}^*$
2. $(P(t_j^*) - P(t_{j+1}^*))(P(t'_{i+1}) - P(t'_i)) < 0$
3. $\{t'_k : t'_k \in (t_j^*, t_{j+1}^*)\}$ contains an even number of points.

or it is possible to set some t'_k to one of $\{t_j^*, t_{j+1}^*\}$ without fundamentally changing the partition T' .

Proof. We consider only the case where $P(t_j^*) < P(t_{j+1}^*)$. If (t'_i, t'_{i+1}) is a TD2 bull phase then we must have $P(t'_i) \leq P(t_j^*)$ since either $i = 1$ or $t'_{i-1} < t_j^*$. If $i = 1$ then $P(t'_i) \leq P(t) \forall t \leq t'_i$ by construction of T' . If $i > 1$ then (t'_{i-1}, t'_i) is a TD2 bear and $P(t'_i) \leq P(t) \forall t \in (t'_{i-1}, t'_i)$. If $P(t'_i) < P(t_j^*)$ this would contradict the optimal nature of T^* so we must have $P(t'_i) = P(t_j^*)$ and therefore setting $t_j^* = t'_i$ does not fundamentally alter either partition.

Now suppose (t'_i, t'_{i+1}) is a TD2 bear phase. If $t_{j+1}^* \leq t'_{i+1}$ then we must have $P(t_{j+1}^*) \leq P(t'_i)$ so moving t_{j+1}^* to t'_i would contradict the optimal nature of T^* unless $P(t_{j+1}^*) = P(t'_i)$. Finally, if $t'_{i+1} < t_{j+1}^*$, since (t'_i, t'_{i+1}) is a bear, if $j < m - 2$ then (t'_{i+1}, t'_{i+2}) is a bull. If $t'_{i+2} < t_{j+1}^*$ then $i < m - 3$ otherwise extend t'_{i+2} to t_{j+1}^* to create a contradiction. Thus (t'_{i+2}, t'_{i+3}) is a bear but we must have $t'_{i+3} < t_{j+1}^*$ or another contradiction will be created. Induction completes the result.

Theorem 10 *Let T' be a TD2 m -point partition of the time series P . If T^* is an optimal m -point partition then $R(T^*) = R(T')$.*

Proof. From Lemma 6 we can infer that $T' \cap T^* \neq \emptyset$ or a contradiction is quickly reached (for example, consider the global maximum and minimum points). Let $\{t'_{i_1}, \dots, t'_{i_p}\} = T' \cap T^*$ contain the p points common to both partitions. These points can be used to split the time series into $p + 1$ sub-intervals over which we define:

$$A_0^* = \{t_j^* : t_j^* \leq t'_{i_1}\} \quad (19)$$

$$A_k^* = \{t_j^* : t'_{i_k} \leq t_j^* \leq t'_{i_{k+1}}\} \quad (20)$$

$$A_p^* = \{t_j^* : t_j^* \geq t'_{i_p}\} \quad (21)$$

with A'_i being defined analogously. Using this partition we can write:

$$R(T^*) = R(A_0^*) + R(A_p^*) + \sum_{i=1}^{p-1} R(A_i^*) \quad (22)$$

We do not exclude the possibility that some $A_i^* = \emptyset$ with $R(A_i^*) = 0$. We now re-index the points of each A_i^* as $t_{i,1}^*, \dots, t_{i,m_i}^*$. From Lemmas 5 and 8, for $0 < k < p$ we now have:

$$R(A_k^*) = |\log(P(t'_{i_k})) - \log(P(t'_{i_{k+1}}))| + 2 \sum_{j=1}^{m_i/2-1} |\log(P(t_{i,2j}^*)) - \log(P(t_{i,2j+1}^*))| \quad (23)$$

and for $k \in \{0, p\}$:

$$R(A_k^*) = \sum_{j=2}^{m_k} |\log(P(t'_{i_j})) - \log(P(t'_{i_{j+1}}))| \quad (24)$$

However, we note that each T^* reversal in A_k^* will contribute less than $2\alpha = 2\log(1 + \lambda_{bull})$ by Lemmas 7 and 8. Also each phase in A_0^* and A_p^* will necessarily have an absolute difference of log prices below α by construction of TD2. Since both T' and T^* have equal number of points, each reversal unique to T^* can be mapped to two points unique to T' . Such points can either belong to some A'_i , creating TD2 reversals via Lemma 9 or points in either A'_0 or A'_p . Since the differences in $R(T^*)$ and $R(T')$ come from their unique points it follows that $R(T^*) \leq R(T')$, but since T^* is optimal, we must have $R(T^*) = R(T')$.

Lemma 11 *Let t_i^{LT} ($i = 0, \dots, m$) and t_j^{TD} ($j = 0, \dots, n$) delimit the phases generated by LT2 and TD2 respectively. If $t_i^{LT} = t_j^{TD}$ for some $i < m, j < n$ then $t_{i+1}^{LT} = t_{j+1}^{TD}$.*

Proof. Since $t_i^{LT} = t_j^{TD}$ represents a turning point under both schemas we begin by assuming it marks the start of a bull phase. If $t_{i+1}^{LT} < t_{j+1}^{TD}$ we reach a contradiction since $P(t_{i+1}^{LT}) < P(t_{j+1}^{TD})$ and (t_j^{TD}, t_{j+1}^{TD}) contains no valid reversals. If $t_{i+1}^{LT} \geq t_{j+1}^{TD}$ then either $j < m - 1$ and we enter an TD bear market (a contradiction, since this would then qualify as a reversal in LT) or $j = m - 1$ and $P(t_{j+1}^{TD})$ must coincide with $P(t_{i+1}^{LT})$ (the maximum value obtained over $[t_i^{LT}, t_{i+1}^{LT}]$). The argument can easily be extended to a bull phase.

Theorem 12 *The LT2 and TD2 methods are equivalent.*

Proof. This follows from Lemmas 2 and 11, and the choice of the global minimum (or maximum) as the starting point for the LT2 schema.

Corollary 13 *If $\lambda_{bull}^* \leq \lambda_{bull}$ then $LT2(P, \lambda_{bull}) \preceq LT2(P, \lambda_{bull}^*)$.*

Proof. This follows from Theorem 12 and result R2.

Theorem 14 *If price series P^* extends price series P then $TD(P, \lambda_{bull}, \lambda_{bear}) \preceq TD(P^*, \lambda_{bull}, \lambda_{bear})$.*

Proof. Suppose that this is not the case then there must be a price series $P^* = (t_i, P(t_i))$ for $i = 0, \dots, n + 1$ such that $TD(P, \lambda_{bull}, \lambda_{bear}) \not\preceq TD(P^*, \lambda_{bull}, \lambda_{bear})$ where $P = (t_i, P(t_i))$ for $i = 0, \dots, n$.

If $P(t_{n+1})$ does not represent a new global maximum or minimum, then there will no impact on the first iteration of phase 1 partitioning. Sub-partitioning of the first two intervals would therefore proceed unchanged. Sub-partitioning of the third interval returns us to the original problem. If $P(t_{n+1})$ does not represent a local maximum or minimum for any sub-partition, we can proceed as before.

We consider only the case when $P(t_{n+1})$ represents a new minimum over some phase 1 partition replacing the previous minimum $P(t_k)$. Phase 2 partitioning will seek reversals over some interval (t_j, t_{n+1}) containing t_k where $P(t_j)$ is a maximum. If no such intervals are found t_{n+1} simply extends the existing final phase. Suppose the largest such reversal occurs over (t_a, t_b) . If $t_b < t_k$ then this partition would also have been the largest reversal in (t_j, t_k) , and we can replace t_j with t_b and proceed. If $t_a \geq t_k$ then we must have $t_a = t_k$ since $P(t_k) < P(t_a)$ and (t_a, t_b) is the largest reversal. Any further partitioning of bull phase (t_k, t_b) and bear phase (t_b, t_{n+1}) would proceed as before since $P(t_n) > P(t_{n+1})$. Clearly $t_k \notin (t_a, t_b)$ since $P(t_k)$ is a minimum over the interval (t_j, t_n) .

5.4 Implementation

The process for determining reversals is described in section 3.1, moreover, it is shown that the maximum reversal over a bull phase must occur between left maximum and right minimum points. Explicit details are given for reversals during a bull market, that is, for a time interval $[t_a, t_b]$ over which corresponding price series $P(t)$ achieves minimum and maximum values at the start and end points respectively. Here we consider the practical implementation and demonstrate that dealing with this case is sufficient. The reader is referred to Figure 3.

Lemma 15 *The maximum reversal over bull interval $[t_a, t_b]$ occurs over $[t_1, t_2]$ and qualifies as a bear if and only if price series $1/P(t)$ achieves its maximum reversal over $[t_1, t_2]$ and qualifies as a bull where:*

$$1 + \lambda_{bull} = \frac{1}{1 - \lambda_{bear}}.$$

Proof. We first note that $P(t_2)/P(t_1)$ is a minimum if and only if $P(t_1)/P(t_2)$ is a maximum. The result follows since:

$$\frac{P(t_2)}{P(t_1)} < 1 - \lambda_{bear} \Leftrightarrow \frac{1/P(t_2)}{1/P(t_1)} > \frac{1}{1 - \lambda_{bear}} = 1 + \lambda_{bull}.$$

Note that to apply lemma 15 we do not require the TD method to impose equation 1.