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Safety and Invariance for Constrained Switching Systems

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Abstract— **We study discrete time linear switching systems subject to additive disturbances. We consider two types of constraints, namely on the states and on the switching signal. A switching sequence is admissible if it is accepted by an automaton. Contrary to the arbitrary switching case, stability does not imply the existence of an invariant[1] set. In this article, we propose a generalization of a bounded invariant set, namely, the notion of an invariant multi-set and show its significance in terms of dynamical systems. Under standard assumptions, we provide an iterative algorithm to approximate the minimal invariant multi-set with a guarantee of accuracy and an algorithm to compute the maximal invariant multi-set. Application of the established framework to switching systems with minimum dwell time reveals potential computational benefits and allows formulations of more refined notions.**

I. Introduction

Switching systems are being extensively studied in the context of stability analysis and control [1], [2], [3]. Apart from the theoretical challenges they pose, these systems appear often in practice since they accurately model real-world systems from different fields and provide close approximations of complex hybrid or non-linear systems. In several settings in control applications, the switching signal is not arbitrary. For example, constrained switching appears when there are different controllers to choose from, each one achieving a different performance (with a different cost), or, when a fault occurs in feedback control and the process is in open-loop for a small period of time. Additionally, several theoretical challenges require more refined tools for studying their stability properties than the arbitrary case [4], [5], [6]. In this article, we express the switching constraints with a labelled, strongly connected directed graph. In specific, a switching sequence is an admissible switching sequence if there exists a path such that it can be realized by the labels of the edges involved.

The notion of invariance used here, namely positive invariance, is important as it implies that all trajectories starting from a set will remain there forever. Since almost every system in practice is subject to some type of constraints on its states or outputs, the notion of invariance becomes extremely relevant in control applications [7]. Specifically, problems related to safety and viability [8] can be addressed by computing sets which possess the invariance property or a variant of it. Although the stability and stabilizability are currently addressed in the literature, see e.g., [6], there is little work available on safety and invariance properties of switching systems, mainly concerning important subclasses [9], [10], [11].

In this article, to study invariance, we first provide an appropriate generalization, namely the invariance of a multi-set. By multi-set we refer to a collection of sets in one-to-one correspondence with the nodes of the graph that defines the admissible switching sequences. Roughly, a *multi-set* is invariant if the system trajectory visits at each time instant the member set which corresponds to the node reached in order the admissible switching signal to be realized. By extending standard results in the literature that concern systems under arbitrary switching, we show that the forward reachability multi-set sequence converges to the minimal invariant multi-set. This sequence might not converge to the minimal invariant multi-set in finite-time, thus, we approximate it with a guaranteed accuracy. Moreover, we propose a constructive approach for computing the maximal admissible invariant multi-set with respect to a state constraint set and consequently for computing the maximal safe set. Finally, to show the relevance of the results, we focus on the case of systems under minimum dwell-time requirement, see e.g., [10], [12], [13]. In specific, we show that our proposed technique offers computational benefits and allows for a better understanding of the behavior of these dynamical systems.

Notation: We write vectors x, y with small letters and sets $\mathcal{S}, \mathcal{X}, \mathcal{V}$ with capital letters in italics. The ball of radius α of an arbitrary norm in $\mathbb{R}^n$ is denoted by $\mathbb{B}(\alpha)$. The distance between a vector $x \in \mathbb{R}^n$ and a compact set $\mathcal{S} \subset \mathbb{R}^n$ is denoted by $\mathrm{d}(x, \mathcal{S})$, while the Hausdorff distance between two compact sets $\mathcal{S}_1 \subset \mathbb{R}^n$, $\mathcal{S}_2 \subset \mathbb{R}^n$ is denoted by $\mathrm{haus}(\mathcal{S}_1, \mathcal{S}_2)$ The Minkowski sum between two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is denoted by $\mathcal{S}_1 \oplus \mathcal{S}_2$, the set difference is denoted by $\mathcal{S}_1 \setminus \mathcal{S}_2$ and the interior of a set $\mathcal{S}$ is denoted by $\mathrm{int}(\mathcal{S})$.

II. Preliminaries

A. System description and basic assumptions

We are interested in studying invariance and safety for systems whose switching patterns are constrained by a set of rules. In our case, these rules are induced by a connected labelled directed graph. To this purpose, we consider a set of matrices $\mathcal{A} := \{A_1, ..., A_N\} \subset \mathbb{R}^{n \times n}$ and a set of disturbance sets $\mathbb{W} = \{\mathcal{W}_1, ..., \mathcal{W}_N\}$, where $\mathcal{W}_i \subset \mathbb{R}^n$,

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[1] Throughout the paper and for simplicity of exposition, by stability we mean asymptotic stability and by invariance we mean robust positive invariance.

$i \in [1,N]$. Moreover, we consider a set of nodes $\mathcal{V} := \{1,2,...,M\}$ and a set of edges $\mathcal{E} = \{(s,d,\sigma) : s \in \mathcal{V}, d \in \mathcal{V}, \sigma \in [1,N]\}$, where s is the source node, d is the destination node and σ the label of the edge. We denote the graph with a set of nodes $\mathcal{V}$ and a set of edges $\mathcal{E}$ as $\mathcal{G}(\mathcal{V},\mathcal{E})$. We also consider a state constraint set $\mathcal{X} \subset \mathbb{R}^n$. The set of outgoing edges of a node $s \in \mathcal{V}$ in $\mathcal{G}(\mathcal{V},\mathcal{E})$ is denoted by $\mathrm{out}(s) := \{d \in \mathcal{V} : (\exists \sigma \in [1,N] : (s,d,\sigma) \in \mathcal{E})\}$. We consider the System

$$x(t+1) = A_{\sigma(t)}x(t) + w(t) \tag{II.1}$$
$$y(t+1) \in \mathrm{out}(y(t)) \tag{II.2}$$
$$(x(0), y(0)) \in \mathbb{R}^n \times \mathcal{V} \tag{II.3}$$

with $w(t) \in \mathcal{W}_{\sigma(t)}$, subject to the constraints

$$\sigma(t) \in \{\sigma : (y(t), y(t+1), \sigma) \in \mathcal{E}\}, \quad \forall t \geq 0, \tag{II.4}$$
$$x(t) \in \mathcal{X}, \quad \forall t \geq 0. \tag{II.5}$$

We take into account the following assumptions throughout the paper.

Assumption 1: The constraint set $\mathcal{X} \subset \mathbb{R}^n$ is compact, convex and contains the origin in its interior.

Assumption 2: The disturbance sets $\mathcal{W}_i$, $i \in [1,N]$, are compact, convex and contain the origin in their interior.

The stability of the disturbance-free system has been studied and characterized by the introduction of the constrained joint spectral radius [4], which is a generalization of the joint spectral radius [3].

Definition 1 ([4]): The constrained joint spectral radius (CJSR) of the disturbance-free System is

$$\rho(\mathcal{A},\mathcal{G}) := \lim_{k\to\infty} \rho_k(\mathcal{A},\mathcal{G}),$$

where $\rho_k(\mathcal{A},\mathcal{G}) := \max\{|\prod_{j=1}^{k} A_{i_j}|^{1/k} : (\exists s_j \in [1,M], j \in [0,k] : (s_j, s_{j+1}, i_j) \in \mathcal{E})\}$ is the maximum growth rate up to time k.

It has been shown [4, Corollary 2.8], that the nominal system (II.1)-(II.3) under constraints (II.4) is asymptotically stable if and only if $\rho(\mathcal{A},\mathcal{G}) < 1$. Moreover, asymptotic stability is equivalent to exponential stability.

Assumption 3: $\rho(\mathcal{A},\mathcal{G}) < 1$.

Assumption 3 is necessary in the context of this article since $\rho(\mathcal{A},\mathcal{G}) > 1$ excludes existence of invariant multi-sets or safe sets. The study of the limiting case $\rho(\mathcal{A},\mathcal{G}) = 1$, although interesting, is outside the scope of this study. See [5] for techniques allowing to guarantee that Assumption 3 holds. The following assumption concerns the structure of the constraints in the switching signal and holds true for many interesting cases in stability analysis of control systems.

Assumption 4: The graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is strongly connected, i.e., there is a path connecting any node $s \in \mathcal{V}$ to any node $d \in \mathcal{V}$.

B. Invariant multi-sets

We first recall the notion of invariant set, and then generalize it to *multi-sets*.

Definition 2 (Invariance): A set $\mathcal{S} \subset \mathbb{R}^n$ is called *invariant* with respect to the System (II.1)-(II.3) if $x(0) \in \mathcal{S}$ implies $x(t) \in \mathcal{S}$, for any initial condition $y(0) \in \mathcal{V}$ and any switching signal $\sigma(t), t \geq 0$, satisfying (II.4). If additionally $\mathcal{S} \subseteq \mathcal{X}$, the set $\mathcal{S}$ is called *admissible invariant* with respect to the System (II.1)-(II.3) and the constraints (II.5). Moreover, if for any admissible invariant set $\mathcal{M} \subseteq \mathcal{X}$ it holds that $\mathcal{M} \subseteq \mathcal{S}$, $\mathcal{S}$ is called the *maximal admissible invariant set*.

Definition 3 (Multi-set invariance): The collection of sets $\{\mathcal{S}^i\}_{i\in[1,M]}$ is called an *invariant multi-set* with respect to the System (II.1)-(II.3) if $x(0) \in \mathcal{S}^{y(0)}$ implies $x(t) \in \mathcal{S}^{y(t)}$, for all $t \geq 0$, for any initial condition $y(0) \in \mathcal{V}$ and for any switching signal $\sigma(t), t \geq 0$, satisfying (II.4). If additionally $\mathcal{S}^i \subseteq \mathcal{X}$, for all $i \in [1,M]$, the multi-set $\{\mathcal{S}^i\}_{i\in[1,M]}$ is called an *admissible invariant multi-set* with respect to the System (II.1)-(II.2) and the constraints (II.5). Moreover, an admissible invariant multi-set $\{\mathcal{S}^i_M\}_{i\in[1,M]}$ is called the *maximal admissible invariant multi-set* if for any admissible invariant multi-set $\{\mathcal{S}^i\}_{i\in[1,M]}$ it holds that $\mathcal{S}^i \subseteq \mathcal{S}^i_M$, for all $i \in [1,M]$. Last, an invariant multi-set $\{\mathcal{S}^i_m\}_{i\in[1,M]}$ is called the *minimal invariant multi-set* if for any invariant multi-set $\{\mathcal{S}^i\}_{i\in[1,M]}$ it holds that $\mathcal{S}^i_m \subseteq \mathcal{S}^i$, for all $i \in [1,M]$.

Definition 4 (Safety): A set $\mathcal{S} \subset \mathbb{R}^n$ is called *safe* with respect to the System (II.1)-(II.3), the constraints (II.4), (II.5) and with respect to a set of nodes $\mathcal{Y} \subseteq \mathcal{V}$ if $(x(0), y(0)) \in \mathcal{S} \times \mathcal{Y}$, implies $x(t) \in \mathcal{X}$, for any initial condition $y(0) \in \mathcal{Y}$ and for any switching signal $\sigma(t), t \geq 0$, satisfying (II.4). A safe set $\mathcal{S}^\star$ is called the *maximal safe set* if for any other safe set $\mathcal{M} \subset \mathcal{X}$ it holds that $\mathcal{S} \subset \mathcal{S}^\star$.

Definition 5 (m-returnability): Given an integer $m \geq 1$, a set $\mathcal{S} \subset \mathbb{R}^n$ is called m-returnable with respect to (i) a set $\mathcal{S}_0$, (ii) a set of nodes $\mathcal{Y} \subseteq \mathcal{V}$, (iii) the system (II.1)–(II.3) and (iv) the constraints (II.4), (II.5) if $(x(0), y(0)) \in \mathcal{S}_0 \times \mathcal{Y}$ implies that for any two time instants $t_1 \geq 0$, $t_2 \geq m-1$ such that $t_2 - t_1 \geq m$ there exists at least one time instant $t^\star \in [t_1, t_2]$ such that $x(t^\star) \in \mathcal{S}$.

While it is not necessary for the System (II.1)-(II.2) to possess an invariant set, we show that under the Assumptions 1-3 there always exist minimal and maximal invariant multi-sets. These particularities are highlighted below.

Running Example Part 1: We consider a scalar system (II.1)-(II.3). The graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ where the switching constraints are defined is shown in Figure 1. In specific, $\mathcal{V} = \{1,2\}$, $\mathcal{E} = \{(1,1,2),(1,2,1),(2,1,2)\}$. There is one un-

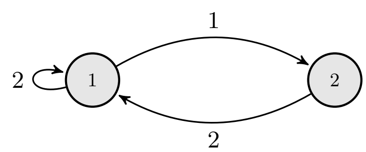

Fig. 1. The graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ for the Running Example.

stable mode, while we do not consider any disturbances, i.e., $\mathcal{A} := \{A_1, A_2\} = \{-2, \frac{1}{4}\}$ and $\mathcal{W}_1 = \mathcal{W}_2 = \{0\}$. We observe that apart from the trivial set $\mathcal{S} = \{0\}$, no invariant set exists. Indeed, for any set $\mathcal{S}$, picking $x(0) = \text{argmax}_{x\in\mathcal{S}}|x|$, $y(0) = 1$ and $\sigma(0) = 1$, it holds that $x(1) \notin \mathcal{S}$. Nevertheless, it is possible to find an invariant multi-set. For example, an invariant multi-set is $\{\mathcal{S}^1, \mathcal{S}^2\}$, with $\mathcal{S}^1 = [-0.5, 0.5]$, $\mathcal{S}^2 = [-1, 1]$.

III. Main results

We define the sequence of multi sets $\{\mathcal{N}_l^j\}_{j\in[1,M]}$, $i \geq 0$ generated by the following set of initial conditions and iterations

$$\mathcal{N}_0^j := \bigcup_{(s,j,\sigma)\in\mathcal{E}} \mathcal{W}_\sigma, \quad j \in [1, M], \tag{III.1}$$

$$\mathcal{N}_{l+1}^j := \bigcup_{(s,j,\sigma)\in\mathcal{E}} A_\sigma \mathcal{N}_l^s, \quad j \in [1, M] \tag{III.2}$$

The elements of the multi-set sequence $\{\mathcal{N}_l^j\}_{j\in[1,M]}$ can be seen as the *l-step forward reachability multi-sets* of the disturbance-free System (II.1)-(II.3), i.e., when $w(t) = 0$, starting from $\mathcal{N}_0^j$ and for all $t \geq 0$. Lemma 1 is a consequence of [5, Theorem 1] and stems from Assumption 3, i.e., the exponential stability of the System (II.1)–(II.3), expressed with set inclusions. The metrics ε, Γ can be computed in practice, e.g., using reachability-based methods [5] or Lyapunov functions [6].

Lemma 1: Consider the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, the sets $\mathcal{W}_i \subset \mathbb{R}^n$, $i \in [1, N]$ and the set of matrices $\mathcal{A} \subset \mathbb{R}^{n\times n}$. Under Assumptions 2, 3, there exist scalars $\varepsilon \in (0, 1)$, $\Gamma \geq 1$ such that

$$\mathcal{N}_t^j \subseteq \Gamma\varepsilon^t \mathcal{N}_0^j, \tag{III.3}$$

for all $t \geq 0$ and for all $j \in [1, M]$.

A. The minimal invariant multi-set

Let us consider the sequence of multi-sets $\{\mathcal{F}_l^j\}_{j\in[1,M]}$, $l \geq 0$, with

$$\mathcal{F}_0^j := \{0\}, \quad j \in [1, M], \tag{III.4}$$

$$\mathcal{F}_{l+1}^j := \bigcup_{(s,j,\sigma)\in\mathcal{E}} \left(A_\sigma \mathcal{F}_l^s \oplus \mathcal{W}_\sigma\right), \quad j \in [1, M]. \tag{III.5}$$

The multi-set sequence $\{\mathcal{F}_l^j\}_{j\in[1,M]}$ has as elements the *l-step forward reachability multi-sets* of the System (II.1)-(II.3), starting from the zero singleton. Next, we show that the set sequence (III.4)-(III.5) converges to the minimal invariant multi-set. First, a few technical results are required.

Fact 1: Consider the multi-set sequence (III.4), (III.5). Under Assumption 2, for all $l \geq 0$ it holds

$$\mathcal{F}_l^j = \bigcup_{i\in[0,l]} \mathcal{F}_i^j, \quad j \in [1, M]. \tag{III.6}$$

Proposition 1: Consider the multi-set sequence (III.4), (III.5). Under Assumptions 2 and 3, there exist scalars $\varepsilon \in (0, 1)$, $\Gamma \geq 1$ such that for any $l \geq 0$, it holds

$$\mathcal{F}_l^j \subseteq \mathcal{F}_{l+1}^j \subseteq \mathcal{F}_l^j \oplus \left(\Gamma\varepsilon^l \bigcup_{i\in[1,N]} \mathcal{W}_i\right). \tag{III.7}$$

Proof: The left inclusion holds from Fact 1. To prove the right inclusion, we first recall that from Lemma 1 there exist scalars Γ, $\varepsilon \in (0, 1)$ such that (III.3) holds. Setting

$$\mathcal{Z}_l := \varepsilon^l\Gamma \left(\bigcup_{i\in[1,N]} \mathcal{W}_i\right),$$

it follows that

$$\begin{aligned}
\mathcal{F}_{l+1}^j = &\bigcup_{(s_l,j,\sigma_l)\in\mathcal{E}} A_{\sigma_l}\left(\ldots\left(\bigcup_{(s_1,s_2,\sigma_1)\in\mathcal{E}} A_{\sigma_1}\right(\right.\right.\\
&\left.\left.\bigcup_{(s_0,s_1,\sigma_0)\in\mathcal{E}} (A_{\sigma_0}\mathcal{F}_0^{s_0} \oplus \mathcal{W}_{\sigma_0})\right) \oplus \mathcal{W}_{\sigma_1}\right)\ldots\Bigg) \oplus \mathcal{W}_{\sigma_l}\\
&= \bigcup_{(s_l,j,\sigma_l)\in\mathcal{E}} \left(\ldots\left(\bigcup_{(s_1,s_2,\sigma_1)\in\mathcal{E}} A_{\sigma_l}\ldots A_{\sigma_1}\mathcal{N}_0^{s_1} \oplus\right.\right.\\
&\left.\left. A_{\sigma_l}\ldots A_2\mathcal{W}_{\sigma_1}\right)\ldots\right) \oplus A_{\sigma_l}\mathcal{W}_{\sigma_{l-1}}\big) \oplus \mathcal{W}_{\sigma_l}\\
&\subseteq \bigcup_{(s_l,j,\sigma_l)\in\mathcal{E}} \left(\ldots\left(\bigcup_{(s_1,s_2,\sigma_1)\in\mathcal{E}} \mathcal{Z}_l \oplus A_{\sigma_l}\ldots A_2\mathcal{W}_{\sigma_1}\right)\right.\\
&\left.\ldots\right) \oplus A_{\sigma_l}\mathcal{W}_{\sigma_{l-1}}\big) \oplus \mathcal{W}_{\sigma_l}\\
&= \mathcal{Z}_l \oplus \bigcup_{(s_l,j,\sigma_l)\in\mathcal{E}} \left(\ldots\left(\bigcup_{(s_1,s_2,\sigma_1)\in\mathcal{E}} A_{\sigma_l}\ldots A_2\mathcal{W}_{\sigma_1}\right)\right.\\
&= \mathcal{F}_l^j \oplus \mathcal{Z}_l,
\end{aligned}$$

thus, the right inclusion in (III.7) holds. ∎

Theorem 1: Consider the multi-set sequence (III.4), (III.5). Under Assumptions 1-4, the following hold.

(i) The multi-set sequence is convergent, in the metric space of compact sets having as metric the Hausdorff distance, i.e., there are sets $\mathcal{F}_\infty^j$, $j \in [1, M]$, such that $\lim_{l\to\infty} \mathcal{F}_l^j = \mathcal{F}_\infty^j$.

(ii) Let $\alpha := \min\left\{\alpha : \bigcup_{i\in[1,N]} \mathcal{W}_i \subseteq \alpha\mathbb{B}(1)\right\}$ and let $\Gamma \geq 1$, $\varepsilon \in (0, 1)$ be scalars satisfying (III.3). For any $\epsilon > 0$ and $l \geq \left\lceil \log_\varepsilon\left(\frac{\epsilon(1-\varepsilon)}{\alpha\Gamma}\right)\right\rceil$ the relation

$$\mathcal{F}_\infty^j \subseteq \mathcal{F}_l^j \oplus \mathbb{B}(\epsilon) \tag{III.8}$$

holds, for all $j \in [1, M]$.

(iii) The multi-set sequence converges to the minimal compact invariant multi-set with respect to the System (II.1)-(II.3) and constraints (II.5), i.e., $\mathcal{S}_m^j = \mathcal{F}_\infty^j$, $j \in [1, M]$.

Proof: (i) By Proposition 1, the set sequence $\{\mathcal{F}_i^j\}_{i\geq0}$, for each $j \in [1, M]$, is monotonically non-decreasing and is a Cauchy sequence. Thus, a limit $\mathcal{F}_\infty^j$ exists, for all $j \in [1, M]$. (ii) From Proposition 1, it holds that $\text{haus}(\mathcal{F}_l^j, \mathcal{F}_{l+1}^j) \leq \Gamma\alpha\varepsilon^l$, for any $l \geq 0$, $j \in [1, M]$. Consequently, for any $j \in [1, M]$, $m \geq 1$, $l \geq 0$ we have $\mathcal{F}_{l+m}^j \subseteq \Gamma\alpha\frac{1-\varepsilon^m}{1-\varepsilon}\mathbb{B}(1) \oplus \mathcal{F}_l^j$. Taking the limit as $m \to \infty$, it follows that $\mathcal{F}_\infty^j \subseteq \frac{\Gamma\alpha\varepsilon^k}{1-\varepsilon}\mathbb{B}(1) \oplus \mathcal{F}_l$. Thus, relation (III.8) is satisfied for any $l \geq \left\lceil \log_\varepsilon\left(\frac{\epsilon(1-\varepsilon)}{\alpha\Gamma}\right)\right\rceil$.

(iii) Invariance of the multi-set $\{\mathcal{F}_\infty^j\}_{j\in[1,M]}$ follows from Fact 1. To show minimality, we follow a similar reasoning as in [14, Lemma 3.1]. To this end, let us assume there exists a compact invariant multi-set $\{\mathcal{S}^j\}_{j\in[1,M]}$ for which there exists an index $j^\star \in [1,M]$ such that $\mathcal{F}_\infty^{j^\star} \nsubseteq \mathcal{S}^{j^\star}$. Then, for any $x(0) \in \mathbb{R}^n$, for any $y(0) \in \mathcal{V}$ and under Assumptions 2, 4, we pick $w(t) = 0$, for all $t \geq 0$. Additionally, we study the solution $(x(t), y(t))$, $t \geq 0$, for which there exists a time sequence $\{t_i\}_{i\geq 0}$ such that $y(t_i) = j^\star$, $i \geq 0$. From Assumption 1, $x(t) \to 0$ as $t \to \infty$. Since $\mathcal{S}^{j^\star}$ is a compact set, $x(t_i) \in \mathcal{S}^{j^\star}$ and $\{x(t_i)\}_{i\geq 0}$ converges to 0, it necessarily holds that $0 \in \mathcal{S}^{j^\star}$. However, since $\mathcal{S}^{j^\star}$ is a member of the invariant multi-set $\{\mathcal{S}^j\}_{j\in[1,M]}$, it necessarily holds from Fact 1 that $\mathcal{F}_\infty^{j^\star} \subseteq \mathcal{S}^{j^\star}$, which is a contradiction. Thus, $\mathcal{F}_\infty^{j^\star} \subseteq \mathcal{S}^{j^\star}$. ■

As it can be observed from Theorem 1 and Proposition 1, if there exists an integer $\overline{k}$ such that $\mathcal{F}_{\overline{k}}^j = \mathcal{F}_{\overline{k}+1}^j$, for all $j \in [1,M]$, then the multi-set series converges in finite time with $\mathcal{S}_m^j = \mathcal{F}_{\overline{k}}^j$, $j \in [1,M]$. However, since only asymptotic convergence can be guaranteed, we can use Theorem 1(ii) to provide ϵ-inner approximations. Nevertheless, it is still hard to compute the set sequence (III.4), (III.5) as the members of the multi-set might not be convex. To alleviate this computational burden and in the same spirit as in [15], [16] we propose to compute the minimal *convex* invariant multi-set. We consider the multi-set sequence

$$\overline{\mathcal{F}}_0^j := \{0\}, \quad j \in [1,M] \tag{III.9}$$

$$\overline{\mathcal{F}}_{l+1}^j := \bigcup_{(s,j,\sigma)\in\mathcal{E}} \mathrm{conv}(A_\sigma \overline{\mathcal{F}}_l^s \oplus \mathcal{W}_\sigma), j \in [1,M]. \tag{III.10}$$

Theorem 2: Consider the multi-set sequences (III.4), (III.5) and (III.4), (III.5). Under Assumptions 1-3, the following hold.

(i) $\mathrm{conv}(\mathcal{F}_l^j) = \mathrm{conv}(\overline{\mathcal{F}}_l^j)$, $\quad \forall l \geq 0, \quad \forall j \in [1,M]$.

(ii) Let $\alpha := \min\{\alpha : \mathcal{W}^\star \subseteq \alpha\mathbb{B}(1)\}$. Then, For any $\epsilon > 0$ and $l \geq \left\lceil \log_\varepsilon \left(\frac{\epsilon(1-\varepsilon)}{\alpha\Gamma} \right) \right\rceil$ the relation $\mathrm{conv}(\overline{\mathcal{F}}_l^j) \subseteq \mathrm{conv}(\overline{\mathcal{F}}_\infty^j) \subseteq \mathrm{conv}(\overline{\mathcal{F}}_l^j) \oplus \mathbb{B}(\epsilon)$ holds, for all $j \in [1,M]$.

(iii) The multi-set $\{\mathrm{conv}(\overline{\mathcal{F}}_\infty^j)\}_{j\in[1,M]}$ is the minimal convex invariant multi-set with respect to the System (II.1)-(II.3) and the constraints (II.5).

The proofs are omitted for brevity: Theorem 2(i) follows similar steps as in [16, Section 3], [5, Proposition 1], while Theorem 2(ii), (iii) can be shown using the same arguments as in Theorem 1(ii), (iii) and the fact that the Minkowski sum and convex hull operations commute.

Running Example Part 2: We compute an ϵ– approximation of the minimal invariant multi-set $\{\overline{\mathcal{F}}_m^1, \overline{\mathcal{F}}_m^2\}$, when $\mathcal{W}_1 = [-0.1, 0.5]$ and $\mathcal{W}_2 = [-0.5, 0.1]$ and by setting $\epsilon = 10^{-3}$. Assumptions 2 and 3 hold and relation (III.3) of Lemma 1 is satisfied with $\Gamma = 32.49$, $\varepsilon = 0.8123$. From Theorem 1(ii), we obtain $l \geq 55$, thus, we compute the 10^{-3}–approximation $\{\mathcal{F}_1^{55}, \mathcal{F}_2^{55}\} = \{[-0.85, 0.65], [-1.4, 2.2]\}$.

B. The maximal invariant multi-set

First, we show that all trajectories of the System (II.1)-(II.3) converge exponentially to the minimal invariant multi-set $\{\mathcal{S}_m^j\}_{j\in[1,M]}$

Lemma 2: Let $(x(\cdot), y(\cdot))$ be any solution of the System (II.1)-(II.3) subject to the constraints (II.5). Under Assumptions 2, 3, for any initial condition $(x(0), y(0))$ and any given $\epsilon > 0$, there exists an integer $l^\star$ such that

$$d(x(t), \mathcal{S}_m^{y(t)}) \leq \epsilon, \tag{III.11}$$

for any $t \geq l^\star$, where $\{\mathcal{S}_m^j\}_{j\in[1,M]}$ is the minimal invariant multi-set.

The proof of Lemma 2 is based on decomposing the trajectory of the system in two parts, one of which is vanishing exponentially and the other is included in the minimal invariant multi-set. It is omitted due to space limitations.

Given an integer $\sigma \in [1,N]$ and a set $\mathcal{S} \subset \mathbb{R}^n$, we define the mapping

$$\mathcal{C}(\sigma, \mathcal{S}) := \{x : A_\sigma x + w \in \mathcal{S}, w \in \mathcal{W}_\sigma\}. \tag{III.12}$$

Consider the state constraint set $\mathcal{X} \subset \mathbb{R}^n$. We define the sequence of multi-sets $\{\mathcal{S}_l^j\}_{j\in[1,M]}$, $l \geq 0$ as follows

$$\mathcal{S}_0^j = \mathcal{X}, \quad j \in [1,M] \tag{III.13}$$

$$\mathcal{S}_{l+1}^j = \left(\bigcap_{(j,d,\sigma)\in\mathcal{E}} \mathcal{C}(\sigma, \mathcal{S}_l^d) \right) \bigcap \mathcal{S}_0^j, \quad j \in [1,M] \tag{III.14}$$

Theorem 3: Consider the System (II.1)-(II.3) subject to the constraints (II.4), (II.5). Suppose that Assumptions 1-3 hold, and moreover, $\mathcal{S}_m^j \subseteq \mathrm{int}(\mathcal{X})$, $j \in [1,M]$, where $\{\mathcal{S}_m^j\}_{j\in[1,M]}$ is the minimal invariant multi-set. Consider the sequence of multi-sets (III.13), (III.14). Then, there exists an integer $\overline{k} \geq 0$ such that

(i) $\mathcal{S}_{\overline{k}+1}^j = \mathcal{S}_{\overline{k}}^j$, for all $j \in [1,M]$.

(ii) The multi-set $\{\mathcal{S}_{\overline{k}}^j\}_{j\in[1,M]}$ is the maximal admissible invariant multi-set with respect to the System (II.1)-(II.3) subject to the constraints (II.4), (II.5).

Proof: (i) The proof uses Lemma 2. In specific, under Assumption 3 and [4, Corollary 2.8], there exist scalars $\Gamma \geq 1$, $\varepsilon \in [0,1)$ such that $|x(t)| \leq \Gamma\varepsilon^t |x(0)|$. Under Assumption 1, there exist scalars R, r such that $R := \min\{R : \mathcal{X} \subseteq \mathbb{B}(R)\}$, $r := \max\{r : \mathbb{B}(r) \subseteq \mathcal{X}\}$. Then, setting $\overline{k} = \lceil \log_\varepsilon \frac{r}{R\Gamma} \rceil$, it follows that $x(0) \in \mathcal{X}$ implies $x(t) \in \mathcal{X}$, for all $t \geq \overline{k}$, for all $y(0) \in \mathcal{V}$. Let us assume that $x(0) \in \mathcal{S}_{\overline{k}}^{y(0)}$ but $x(0) \notin \mathcal{S}_{\overline{k}+1}^{y(0)}$. Then, $x(\overline{k}+1) \notin \mathcal{X}$ which is a contradiction. Thus, $\mathcal{S}_{\overline{k}+1}^{y(0)} \supseteq \mathcal{S}_{\hat{k}}^{y(0)}$. Taking into account that $\mathcal{S}_{l+1}^j \subseteq \mathcal{S}_l^j$ holds by construction for all $j \in [1,M]$, $l \geq 0$, the result follows.

(ii) We can take similar steps as in the proofs of standard results concerning the linear case or the case of arbitrary switching, e.g., [17]: From (i), it follows that $\{\mathcal{S}_{\overline{k}}^j\}_{j\in[1,M]}$ is an admissible invariant multi-set. Let us suppose that there exists an admissible invariant multi-set $\{\mathcal{M}^j\}_{j\in[1,M]}$ and an index $j^\star$ for which $\mathcal{M}^{j^\star} \nsubseteq \mathcal{S}_{\overline{k}}^{j^\star}$. Then, for all $x(0) \in \mathcal{M}^{j^\star} \setminus \mathcal{S}_{\overline{k}}^{j^\star}$, $y(0) = j^\star$, it follows that $x(\overline{k}) \notin \mathcal{X}$ and $\{\mathcal{M}^{j^\star}\}_{j\in[1,M]}$ is not admissible, which is a contradiction. Thus, $\mathcal{M}^{j^\star} \subseteq \mathcal{S}_{\overline{k}}^{j^\star}$ and $\{\mathcal{S}_{\overline{k}}^j\}_{j\in[1,M]}$ is the maximal admissible invariant multi-set with respect to the System (II.1)-(II.3) subject to the constraints (II.4), (II.5). ■

It is worth noting for Theorem 3 that for the disturbance-free linear system under constrained switching, Assumption 1 can be weakened by replacing convexity with basic semi-alegbraicity. For more details we refer to [18]. The following is a direct consequence of Theorem 3.

Corollary 1: Consider the System (II.1)-(II.3) subject to the constraints (II.4), (II.5). Let $\{\mathcal{S}_M^j\}_{j\in[1,M]}$ be the maximal admissible invariant multi-set and $\mathcal{Y} \subseteq \mathcal{V}$ be a set of nodes of the graph $\mathcal{G}(\mathcal{V},\mathcal{E})$. Then, the maximal safe set $\mathcal{S}_\mathcal{Y}$ with respect to the System (II.1)-(II.3), the constraints (II.4), (II.5) and with respect to $\mathcal{Y} \subseteq \mathcal{V}$ is $\mathcal{S}_\mathcal{Y} = \bigcap_{j\in\mathcal{Y}} \mathcal{S}_M^j$.

Running Example Part 3: We compute the maximal invariant multi-set $\{\mathcal{S}_M^1, \mathcal{S}_M^2\}$ when $\mathcal{W}_1 = [-0.1, 0.5]$, $\mathcal{W}_2 = [-0.5, 1]$ and the constraint set is $\mathcal{X} = [-2.5, 2.5]$. Since all Assumptions 1-3 hold, and moreover, $\mathcal{S}_m^j \subseteq \mathrm{int}(\mathcal{X})$, $j = 1, 2$, from Theorem 3 the set sequence (III.13), (III.14) converges in finite time. Indeed, the maximal invariant multi-set is retrieved after three iterations (i.e., $\overline{k} = 3$), with $\mathcal{S}_M^1 = [-1, 0.95]$, $\mathcal{S}_M^2 := [-2, 2.5]$.

IV. Case study: Minimum Dwell Time

Let us consider the system in the first example of [12, Section 4] that concerns a linear system which switches between two modes with minimum dwell time $\tau = 15$. The corresponding graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ that captures such constraints is in Figure 2 and the corresponding constrained switching system is of the form (II.1)–(II.3) with the additional constraint $y(0) \in \{1, 16\}$. Although one can utilize directly Theorems 1 and 2 in Section III to compute the minimal and maximal invariant multi-sets, in this section we further

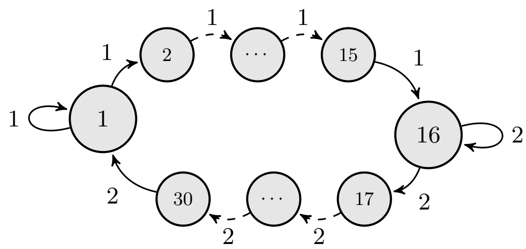

Fig. 2. Example from [12, Section 4], the corresponding graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ describing the admissible switching sequences constrained by the dwell time requirements.

refine the obtained results by observing that the computation of the aforementioned multi-sets can be done using a reduced graph. In detail, this graph consists only of a subset of the initial nodes, namely the set of unavoidable nodes.

Definition 6: Given a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ and an integer $m \geq 1$, the set of nodes $\mathcal{Y} \subseteq \mathcal{V}$ is called *m-unavoidable* if any path of length m passes through a node $v \in \mathcal{Y}$ at least once. If $m \geq V$, where V is the number of nodes of $\mathcal{G}(\mathcal{V},\mathcal{E})$, the set $\mathcal{Y}$ is simply called *unavoidable*.

In the setting studied here, the number of unavoidable nodes is equal to the number of modes. Thus, we can define the

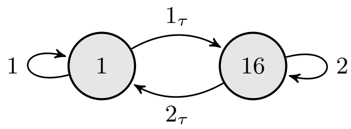

Fig. 3. Example from [12, Section 4], the reduced graph $\mathcal{G}(\mathcal{Y},\hat{\mathcal{E}})$, with $\mathcal{Y} = \{1, 16\}$ being the set of 15-unavoidable nodes of $\mathcal{G}(\mathcal{V},\mathcal{E})$. The labels 1_τ and 2_τ correspond to the dynamics produced by composition of the dynamics of mode 1 and 2 of the original system τ times consecutively.

Reduced System, which is a system of the form (II.1)–(II.3), with the set of matrices $\hat{\mathcal{A}} = \mathcal{A} \cup \{A_i^\tau\}_{i\in[1,N]}$ and the set of disturbance sets $\hat{\mathbb{W}} = \mathbb{W} \cup \{\oplus_{j=0}^{\tau-1} A_i^j \mathcal{W}_i\}_{i\in[1,N]}$. We define also the set of labels $\Sigma = \{1, ..., N\} \cup \{1_\tau, ..., N_\tau\}$, where i_τ corresponds to the mapping generated by the composition of the i-mode dynamics for τ times. Accordingly, the *Reduced Graph* related to the Reduced system is a fully connected graph $\mathcal{G}(\mathcal{Y},\hat{\mathcal{E}})$, where each node $y_i \in \mathcal{Y}$ corresponding to mode i of the original system has a self-loop with label i and is connected to all other nodes with the label i_τ.

When studying the nominal part of the System (II.1)-(II.3), it is known, see e.g., [6], that the stability properties of any two systems whose related graphs produce the same admissible switching sequences of infinite length coincide. Motivated by the above, we extend this reasoning by firstly computing the maximal and minimal invariant multi-set of the Reduced system and secondly by associating it with the corresponding notions of the original system via a simple transformation. We denote by $\mathcal{R}\left(\{\sigma_i\}_{i\in[0,p]}, \mathcal{S}\right)$ the $(p+1)$-step forward map from the set $\mathcal{S}$ under the switching sequence $\{\sigma_i\}_{i\in[0,p]}$, i.e., $\mathcal{R}\left(\{\sigma_i\}_{i\in[0,p]}, \mathcal{S}\right) = \bigoplus_{j=0}^{p-1} (\prod_{i=0}^{p-1-j} A_{\sigma_{p-1-i}} \mathcal{W}_{\sigma_j}) \oplus (\prod_{i=0}^{p} A_{\sigma_{p-i}} \mathcal{S}) \oplus \mathcal{W}_{\sigma_p}$. Analogously, we denote by $\mathcal{C}(\{\sigma_i\}_{i\in[0,p]})$ the $(p+1)$-step backward map to the set $\mathcal{S}$. To further simplify notation, given a source node $s \in \mathcal{V}$ and a destination node $d \in \mathcal{V}$ of the graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, we denote the ordered sequence of the labels of the path from s to d as $\sigma(s, d)$ and the ordered sequence of the nodes present in the path from s to d as $m(s, d)$.

Let $\{\hat{\mathcal{S}}_m^i\}_{i\in\mathcal{Y}}$ be the minimal invariant multi-set for the Reduced system. Then, the minimal invariant multi-set for the System is $\{\mathcal{S}_m^i\}_{i\in[M]}$, where $\mathcal{S}_m^j = \hat{\mathcal{S}}_m^j$, for all $j \in \mathcal{Y}$ and $\mathcal{S}_m^j = \bigcup_{s\in\mathcal{Y}} \mathcal{R}(\sigma(s, j), \hat{\mathcal{S}}_m^s)$, for all $j \in \mathcal{V} \setminus \mathcal{Y}$.

Similarly, let $\{\hat{\mathcal{S}}_M^i\}_{i\in\mathcal{Y}}$ be the maximal invariant multi-set for the Reduced system. Then, the maximal invariant multi-set for the System is $\{\mathcal{S}_M^i\}_{i\in[M]}$, where $\mathcal{S}_M^j = \bigcap_{d\in\mathcal{Y}} \left(\left(\bigcap_{\{i\in m(j,d)\}} \mathcal{C}(\sigma(j, i), \mathcal{X}) \right) \cap \mathcal{C}(\sigma(j, d), \mathcal{S}_M^d) \right) \cap \mathcal{S}_M^j$, for all $j \in \mathcal{Y}$ and

$\mathcal{S}_M^j = \bigcap_{d\in\mathcal{Y}} \left(\left(\bigcap_{\{i\in m(j,d)\}} \mathcal{C}(\sigma(j, i), \mathcal{X}) \right) \cap \mathcal{C}(\sigma(j, d), \mathcal{S}_M^d) \right) \cap \mathcal{X}$, for all $j \in \mathcal{V} \setminus \mathcal{Y}$.

It is worth observing that in the general case the reduced graph $\mathcal{G}(\mathcal{Y},\hat{\mathcal{E}})$ consists of N nodes and N^2 edges, which are significantly less than the $N(N-1)(\tau-1) + N$ nodes and

$N(N-1)\tau+N$ edges of the original graph $\mathcal{G}(\mathcal{V},\mathcal{E})$.

Example 1: We consider the first example in [12, Section 4] that concerns a switching system with two modes, i.e., $\mathcal{A}:=\{A_1,A_2\}$, with $A_1=\begin{bmatrix}1 & 0.1\\ -0.2 & 0.9\end{bmatrix}$, $A_2=\begin{bmatrix}1 & 0.1\\ -0.9 & 0.9\end{bmatrix}$, $\mathcal{W}_1=\mathcal{W}_2=\{0\}$, under minimum dwell time constraints with $\tau=15$ and state constraints the unit box $\mathcal{X}=\mathbb{B}_\infty(1)$. The reduced graph $\mathcal{G}(\mathcal{Y},\hat{\mathcal{E}})$ is shown in Figure 3, where $\mathcal{Y}=\{1,16\}$. Using the results of Section IV, we compute the maximal invariant multi-set $\{\mathcal{S}_M^i\}_{i\in\mathcal{V}}$ in 0.07 seconds. The

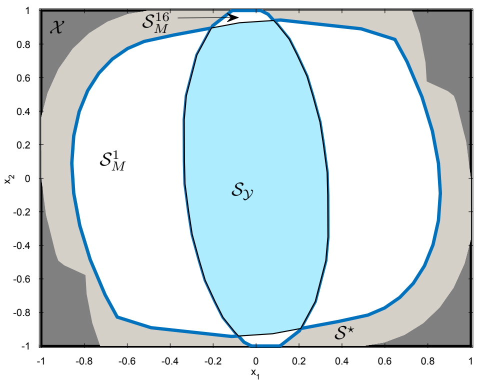

Fig. 4. Example 1, the sets $\mathcal{S}^\star=\cup_{i\in\mathcal{V}}\mathcal{S}_M^i$ (grey), the maximal safe w.r.t. the set of unavoidable nodes $\mathcal{Y}$ (blue), the sets $\mathcal{S}_M^1,\mathcal{S}_M^{16}$ (white) and the constraint set $\mathcal{X}$ (dark grey).

maximal safe set $\mathcal{S}_\mathcal{Y}=\cap_{i\in\mathcal{Y}}\mathcal{S}_M^i$ with respect to the nodes $\mathcal{Y}=\{1,16\}$ recovers the maximal Dwell Time invariant set of [12].

Additionally, in the proposed framework, more refined notions of invariance can be formulated; for example, the maximal safe set is a 15-returnable set with respect to itself and the set $\mathcal{Y}$, while $\mathcal{S}_M^1\cup\mathcal{S}_M^{16}$ is a 1-returnable set with respect to $\mathcal{S}_M^1\cap\mathcal{S}_M^{16}$ and the set $\mathcal{Y}$.

V. Conclusions and Future Work

Invariance and constraint satisfaction for switching systems have attracted much attention in the literature. One reason is that, even though these systems are extremely hard to study, classical results show that invariant sets and safe sets are algorithmically computable (under the assumption of stability for the nominal system). In this work, we have generalized these notions to constrained switching systems. We showed that the invariance notion must be replaced by a finer notion, namely an invariant multi-set, while the maximal safe set is given by the union of the individual sets in the maximal invariant multiset. As an application of our results, we showed that they can be translated into efficient algorithms for dwell time specifications.

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