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Athanasopoulos, N., Lazar, M., & Bitsoris, G. (2012). On the construction of invariant proper C-polytopic sets for continuous-time linear systems. In *Proceedings of the 16th International Conference on System Theory, Control and Computing* Institute of Electrical and Electronics Engineers Inc.. http://ieeexplore.ieee.org/document/6379313/#full-text-section

Published in:
Proceedings of the 16th International Conference on System Theory, Control and Computing

Document Version:
Peer reviewed version

On the Construction of Invariant Proper $\mathcal{C}$–polytopic Sets for Continuous-time Linear Systems

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***Abstract*— This paper presents a method for iteratively enlarging a given invariant (contractive) proper $\mathcal{C}$–polytopic set for *continuous-time* linear systems. It is proven that the proposed algorithm generates a monotonic sequence of invariant (contractive) proper $\mathcal{C}$–polytopic sets. The distinguishing feature of the algorithm is a novel, geometric approach to the set expansion problem. Several examples demonstrate the effectiveness of the proposed method.**

I. Introduction

A problem of interest in stability analysis of linear systems in the presence of hard state constraints is the computation of the admissible, with respect to constraints, domain of attraction. This problem is relevant in several applications, which include determining whether a desired set of initial conditions belongs to the admissible domain of attraction.

It is well known [1] that invariant (contractive) subsets of the admissible state–space provide an approximation of the admissible domain of attraction. As such, construction of an invariant set is the typical approach to solving the above-mentioned problem. In numerous real–life applications, state constraints are specified by bounded polyhedral sets, which, in the linear case, can be equivalently formulated *via* proper $\mathcal{C}$–polytopic sets, i.e., bounded and closed polyhedral sets that contain the origin in their interior. Several methods are available in this case to construct an admissible invariant proper $\mathcal{C}$–polytopic set for both continuous–time and discrete–time linear systems [1]. Most of these methods can be related to the algebraic necessary and sufficient conditions originally proposed by the authors of [2] in 1986, see, for example, [3]–[8]. Recently, alternative algebraic necessary and sufficient conditions for construction of invariant proper $\mathcal{C}$–polytopic sets *via* proper conic partitions [9] and polyhedral Lyapunov functions [10] were proposed.

However, these methods typically generate invariant proper $\mathcal{C}$–polytopic sets of an arbitrary shape, which may result in a conservative approximation of the admissible domain of attraction, with respect to a given proper $\mathcal{C}$–polytopic set of initial conditions of interest. That is why, in light of the considered problem, it is of further interest to monotonically enlarge a given, invariant (contractive) proper $\mathcal{C}$–polytopic set. This is possible for *discrete–time* linear systems by means of iterative computation of (forward) reachability sets [11], [12], or controllability (backward reachability) sets [13], [14]. Moreover, this approach can be translated to *continuous–time* linear systems *via* discrete approximations such as the exponential discretized system [12] or the Euler approximating system [6]. To the best of the authors' knowledge, a method for enlarging a given invariant (contractive) proper $\mathcal{C}$–polytopic set, which does not involve discrete approximations, is not available for continuous–time linear systems.

Therefore, this paper proposes a novel, geometric approach to the set expansion problem for continuous–time linear dynamics. Given an invariant (contractive) proper $\mathcal{C}$–polytopic subset of a desired set of initial conditions, we provide an iterative algorithm that generates a monotonically increasing (with respect to set inclusion) sequence of admissible invariant (contractive) proper $\mathcal{C}$–polytopic sets. Essentially, for each hyperplane of a previously computed invariant proper $\mathcal{C}$–polytopic set, the algorithm returns a vector which is not in the interior of the set. Moreover, the convex hull of the new vector and the vertices of the set is preserving the invariance (contractivity) property. All involved operations come down to solving a sequence of low–complexity linear programs, which makes the developed algorithm computationally efficient.

Remark 1: The proposed geometric approach to the set expansion problem is also valid for discrete–time linear systems. Improved results in terms of the expansion rate, compared to the classical controllable sets approach, were reported in [15], [16].

The remainder of the paper is structured as follows. Notation and basic definitions are introduced in Section II along with the problem formulation. Section III presents preliminary technical results that will be instrumental in establishing the properties of the proposed method. The main algorithm along with the properties of the resulting sequence of sets are described in Section IV. Several examples illustrate the effectiveness of the proposed approach in Section V, while conclusions are summarized in Section VI.

II. Notation, basic definitions and problem formulation

- The field of reals and the sets of non–negative reals and non–negative integers are denoted by $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ respectively.
- For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ we define $\Pi_{\geq c} := \{k \in \Pi : k \geq c\}$ and similarly $\Pi_{\leq c}$. Given two arbitrary sets $\mathcal{P}, \mathcal{S} \subseteq \mathbb{R}$, let $\mathcal{P}_{\mathcal{S}} := \mathcal{P} \cap \mathcal{S}$.
- For a matrix $A \in \mathbb{R}^{n\times m}$, $[A]_{ij}$ denotes the element in the i–th row and j–th column, $[A]_{i\bullet} \in \mathbb{R}^m$ denotes

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the i–th row and $[A]_{\bullet j} \in \mathbb{R}^n$ denotes the j–th column. For matrices $A, B \in \mathbb{R}^{n\times m}$, by $A \leq B$ we denote the corresponding set of componentwise scalar inequalities $[A]_{ij} \leq [B]_{ij}$ for all $(i,j) \in \mathbb{N}_{[1:n]} \times \mathbb{N}_{[1:m]}$.

- The vector with all its elements equal to one is denoted by $\mathbf{1}_n \in \mathbb{R}^n$, the $n \times m$ real matrix with all its elements equal to zero is denoted by $\mathbf{0}_{n\times m}$ and the $n\times n$ identity matrix is denoted by I_n.
- A matrix $A \in \mathbb{R}^{n\times n}$ is called a *Metzler* matrix if its non-diagonal elements are nonnegative, i.e., $[A]_{ij} \geq 0$, for all $i \neq j$.
- Given a set $\mathcal{S} \subset \mathbb{R}^n$ and a real matrix A of compatible dimensions (possibly a scalar), the image of $\mathcal{S}$ under A is denoted by $A\mathcal{S} := \{Ax \ : x \in \mathcal{S}\}$.
- A set $\mathcal{S} \subset \mathbb{R}^n$ is called a proper $\mathcal{C}$–set if it is convex, compact and contains the origin in its interior. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces and a *polytope* is a closed and bounded polyhedron.

Let $\mathbb{P} \subseteq \mathbb{R}^n$ denote the set of all proper $\mathcal{C}$–polytopic sets. We consider both the half–space and the vertex representation of proper $\mathcal{C}$–polytopic sets. More specifically:

- The half–space representation of an arbitrary proper $\mathcal{C}$–polytopic set is given by
$$\mathcal{S} := \{x \in \mathbb{R}^n : Gx \leq \mathbf{1}_p\},$$
where $G \in \mathbb{R}^{p\times n}$ is a full column–rank matrix, $p \in \mathbb{N}_{\geq n+1}$.
- Given a proper $\mathcal{C}$–polytopic set $\mathcal{S}$, the mapping $\mathcal{V} : \mathbb{P} \rightrightarrows \mathbb{R}^n$, with $\mathcal{V}(\mathcal{S}) = \{v_i\}_{i\in\mathbb{N}_{[1:q]}}$, gives the vertices of $\mathcal{S}$. The vertex–representation of $\mathcal{S}$ is given by
$$\mathcal{S} := \mathrm{convh}(\{v_i\}_{i\in\mathbb{N}_{[1:q]}}),$$
for some $q \in \mathbb{N}_{\geq n+1}$. Let $V := [v_1, v_2, \ldots, v_q] \in \mathbb{R}^{n\times q}$ and note that V has full row–rank.
- Given any two arbitrary sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^n$, the relation $\mathcal{S}_2 \subset \mathcal{S}_1$ holds if $x \in \mathcal{S}_2$ implies $x \in \mathcal{S}_1$ and there exists at least one $\xi \in \mathcal{S}_1$ such that $\xi \notin \mathcal{S}_2$.
- The convex hull of a proper $\mathcal{C}$–polytopic set $\mathcal{S}$ and a vector $\xi \in \mathbb{R}^n$ is denoted by $\mathcal{S}^\xi := \mathrm{convh}(\{\mathcal{V}(\mathcal{S}), \xi\})$. Observe that $\mathcal{S}^\xi$ is also a proper $\mathcal{C}$–polytopic set and, moreover, $\mathcal{S} \subset \mathcal{S}^\xi$ for all $\xi \notin \mathcal{S}$.
- Given an arbitrary set $\mathcal{S} \subset \mathbb{R}^n$, its interior is denoted by $\mathrm{interior}(S)$ and its closure is denoted by $\mathrm{closure}(\mathcal{S})$.

We consider autonomous linear continuous-time systems described by linear differential equations
$$\dot{x}(t) = Ax(t), \tag{1}$$
where $x \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$, and $t \in \mathbb{R}_+$ denotes the continuous time. Given an initial condition $x(t_0) = x_0$, $t_0 \in \mathbb{R}_+$, let $x(t; x_0)$, $t \in \mathbb{R}_{[t_0:\infty)}$ denote the solution of system (1). The standing assumption throughout this paper is the following:

Assumption 1: System (1) is asymptotically stable.

Moreover, we study the case when the state vector is confined in a proper C–polytopic set $\mathcal{S}_x$:
$$\mathcal{S}_x = \{x \in \mathbb{R}^n : G_x x \leq \mathbf{1}_{p_x}\} = \mathrm{convh}(\{[V_x]_{\bullet i}\}_{i\in\mathbb{N}_{[1:q_x]}}) \tag{2}$$
for some suitable matrices $G_x \in \mathbb{R}^{p_x\times n}$ and $V_x \in \mathbb{R}^{n\times q_x}$.

Definition 1: Given system (1), a proper $\mathcal{C}$–polytopic constraint set $\mathcal{S}_x$ and a scalar $\varepsilon \in \mathbb{R}_{\geq 0}$, the set $\mathcal{S}$ is called *admissible contractive with contraction factor* ε or simply ε-*contractive* if and only if $x_0 \in \mathcal{S}$ implies $x(t; x_0) \in e^{-\varepsilon t}\mathcal{S}$, for all $t \in \mathbb{R}_{\geq t_0}$. When the previous relation holds with $\varepsilon = 0$, the set $\mathcal{S}$ is called *positively invariant*.

Next, we formally state the problem addressed in this paper.

Problem 1: Given system (1), a proper $\mathcal{C}$–polytopic constraint set $\mathcal{S}_x \subset \mathbb{R}^n$ and a proper $\mathcal{C}$–polytopic ε–contractive set $\mathcal{S}_0 \subset \mathcal{S}_x$, compute a sequence of proper $\mathcal{C}$–polytopic sets $\{\mathcal{S}_i\}_{i\in\mathbb{N}_{\geq 1}}$ such that the following relations hold[1] for all $i \in \mathbb{N}$:

1) $\mathcal{S}_i \subset \mathcal{S}_{i+1}$;
2) $\mathcal{S}_i \subseteq \mathcal{S}_x$;
3) $\mathcal{S}_i$ is a proper $\mathcal{C}$–polytopic set;
4) $\mathcal{S}_i$ is an ε–contractive set.

A sequence of sets $\{\mathcal{S}_i\}_{i\in\mathbb{N}_{\geq 1}}$ that satisfies properties *1)–4)* specified in Problem 1 is called an *admissible sequence of sets for system (1)*.

III. Preliminary results

In this section we present some results that are necessary for the development of the algorithmic procedure that solves Problem 1. The following theorem presents necessary and sufficient conditions for a proper $\mathcal{C}$–polytopic set to be ε–contractive with respect to (1) and it is important for the derivation of the main results. All relevant details can be found in [1].

Theorem 1: [1] Consider system (1) subject to the state constraints (2). Let $\mathcal{S} \subseteq \mathcal{S}_x$ be an arbitrary proper $\mathcal{C}$–polytopic set and let $\varepsilon \in \mathbb{R}_{\geq 0}$. Then, $\mathcal{S}$ is an $\varepsilon-$contractive set if and only if there exists a Metzler matrix $P \in \mathbb{R}^{q\times q}$ such that
$$AV = VP \tag{3a}$$
$$\mathbf{1}_q^T P \leq -\varepsilon \mathbf{1}_q^T. \tag{3b}$$

The next result will provide the mechanism for the iterative enlargement of an $\varepsilon-$contractive set.

Lemma 1: Consider an ε–contractive proper $\mathcal{C}$–polytopic set $\mathcal{S}$ with respect to system (1) and constraints (2), and a vector $v \in \mathcal{S}_x$. Then, the set $\mathcal{S}^v = \mathrm{convh}(\{\mathcal{V}(\mathcal{S}), v\})$ is $\varepsilon-$contractive if and only if there exist a real scalar $\bar{p}$ and a nonnegative vector $p \in \mathbb{R}^q$ such that
$$Av = Vp + \bar{p}v \tag{4a}$$
$$\mathbf{1}_q^T p + \bar{p} \leq -\varepsilon. \tag{4b}$$

[1] The case where $i = \infty$ is excluded.

Proof: Since $\mathcal{S}$ is ε-contractive, there exists a Metzler matrix $P \in \mathbb{R}^{n\times q}$ such that relations (3a) and (3b) hold. Let $P^\star \in \mathbb{R}^{(q+1)\times(q+1)}$,

$$P^\star := \begin{bmatrix} P & p \\ \mathbf{0}_{1\times q} & \bar{p} \end{bmatrix}$$

and $V^\star := [V\ v]$. Taking into account relations (4a) and (4b), it follows that conditions (3) of Theorem 1 are also satisfied for $\mathcal{S}^v$, with $P = P^\star$, $V = V^\star$. Moreover, $S^v \subseteq \mathcal{S}_x$ since $v \in \mathcal{S}_x$ and $\mathcal{S} \subseteq \mathcal{S}_x$. Thus, $\mathcal{S}^v$ is also an $\varepsilon-$contractive set. Conversely, if S^v is ε-contractive, there exists a matrix $\hat{P} \in \mathbb{R}^{(q+1)\times(q+1)}$ satisfying conditions (3) with $V = V^\star$. Then, relations (4) are satisfied with $p = [\hat{P}]_{(q+1)\bullet}$ and $\bar{p} = [\hat{P}]_{(q+1)(q+1)}$. ■

The next result concerns the enlargement procedure that will be performed at each iteration of the algorithm:

Lemma 2: Let $\mathcal{S}_i \subseteq \mathcal{S}_x$, for all $i \in \mathbb{N}_{[1:M]}$, $M \in \mathbb{N}_{\geq 1}$, be an ε_i–contractive proper $\mathcal{C}$–polytopic set for system (1), where $\varepsilon_i \in \mathbb{R}_+$, for all $i \in \mathbb{N}_{[1:M]}$. Then, the set

$$\mathcal{S} := \mathrm{convh}(\{\mathcal{V}(\mathcal{S}_i)\}_{i\in\mathbb{N}_{[1:M]}})$$

is an ε–contractive proper $\mathcal{C}$–polytopic set for system (1), with $\varepsilon := \min_i \{\varepsilon_i\}$.

Proof: By definition, $\mathcal{S} = \mathrm{convh}(\{\mathcal{V}(\mathcal{S}_i)\}_{i\in\mathbb{N}_{[1:M]}}) = \mathrm{convh}(\{v^i_j\}_{i\in\mathbb{N}_{[1:M]},j\in\mathbb{N}_{[1:q_i]}})$. Thus, $\mathcal{S}$ is a polytopic set.

Moreover, $\mathcal{S}$ contains the origin in its interior since $\mathcal{S}_i \subseteq \mathcal{S}$, for all $i \in \mathbb{N}_{[1:M]}$, and each $\mathcal{S}_i$ is a proper $\mathcal{C}$–set. Also, $\mathcal{S} \subseteq \mathcal{S}_x$ since relation $\{v^i_j\}_{i\in\mathbb{N}_{[1:M]},j\in\mathbb{N}_{[1:q_i]}} \subseteq \mathcal{S}_x$ implies $\mathrm{convh}(\{v^i_j\}_{i\in\mathbb{N}_{[1:M]},j\in\mathbb{N}_{[1:q_i]}}) \subseteq \mathcal{S}_x$. Next, let the matrices $V^i \in \mathbb{R}^{n\times q_i}$, $i \in \mathbb{N}_{[1:M]}$ be defined by $[V^i]_{\bullet j} := v^i_j$, $i \in \mathbb{N}_{[1:M]}$, $j \in \mathbb{N}_{[1:q_i]}$ and let $V = [V^1\ V^2 ... V^M]$.

In order to prove the contractivity property, we show that for the matrix $V \in \mathbb{R}^{n\times q}$ there exist a matrix $P \in \mathbb{R}^{q\times q}$ and a scalar $\varepsilon \in \mathbb{R}_{\geq 0}$ such that the relations (3a) are satisfied. Indeed, since each $\mathcal{S}_i$ is ε_i-contractive, there exist Metzler matrices $P^i \in \mathbb{R}^{q_i\times q_i}$, $i \in \mathbb{N}_{[1:M]}$, such that $AV^i = V^iP^i$ and $\mathbf{1}^T_{q_i}P^i \leq -\varepsilon_i\mathbf{1}^T_{q_i}$. Consequently, it follows that

$$\begin{aligned} AV =& A[V^1\ V^2 ... V^M] \\ =& [AV^1\ \ AV^2\ ...\ AV^M] \\ =& [V^1P^1\ V^2P^2\ ...\ V^MP^M] = VP, \end{aligned}$$

where P is the block diagonal matrix

$$P = \begin{bmatrix} P^1 & \dots & \mathbf{0}_{q_1\times q_M} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{q_M\times q_1} & \dots & P^M \end{bmatrix}.$$

By construction, P is a Metzler matrix. Moreover,

$$\begin{aligned} \mathbf{1}^T_q P =& [\mathbf{1}^T_{q_1}P^1\ \mathbf{1}^T_{q_2}P^2\ ...\ \mathbf{1}^T_{q_M}P^M] \\ \leq& [-\varepsilon_1\mathbf{1}^T_{q_1}\ \ -\varepsilon_2\mathbf{1}^T_{q_2} \dots -\varepsilon_M\mathbf{1}^T_{q_M}] \\ \leq& -\min_i\{\varepsilon_i\}\,\mathbf{1}^T_q. \end{aligned}$$

Thus, according to Theorem 1, $\mathcal{S}$ is ε–contractive with contraction factor $\varepsilon = \min_i\{\varepsilon_i\}$. ■

The next straightforward Fact will be used to obtain the main result.

Fact 1: Consider a proper $\mathcal{C}$–polytopic set $\mathcal{S}$ and a vector $x \notin \mathrm{interior}(\mathcal{S})$. Let $\alpha_i := [G]_{i\bullet}x$, for all $i \in \mathbb{N}_{[1:p]}$. Then, there exists at least one index $i^\star \in \mathbb{N}_{[1:p]}$, such that $\alpha_{i^\star} \geq 1$. Moreover, if $x \notin \mathrm{closure}(\mathcal{S})$, there exists at least an index $i^\star \in \mathbb{N}_{[1:p]}$ such that $\alpha_{i^\star} > 1$.

IV. Main algorithm

In this section we utilize the results of Section III in order to provide an algorithm which returns an admissible sequence of sets for system (1). First, we formulate the following problem:

Problem 2: Given system (1), a proper $\mathcal{C}$–polytopic constraint set $\mathcal{S}_x$, a scalar $\varepsilon \in \mathbb{R}_{\geq 0}$, a proper $\mathcal{C}$–polytopic ε–contractive set $\mathcal{S}$, and a fixed $j \in \mathbb{N}_{[1:p]}$, solve the optimization problem

$$\max_{v^j,p_j,\bar{p}_j} \{[G]_{j\bullet}v^j\}$$

subject to

$$Av^j = Vp_j + \bar{p}_jv^j \tag{5a}$$

$$\mathbf{1}^T_q p_j + \bar{p}_j \leq -\varepsilon \tag{5b}$$

$$G_xv^j \leq \mathbf{1}_{p_x} \tag{5c}$$

$$p_j \geq \mathbf{0}_q. \tag{5d}$$

Lemma 3: There exists a vector $\xi \in \mathcal{S}_x \setminus \mathcal{S}$ such that $\mathcal{S}^\xi$ is an ε–contractive set if and only if there exists an index $k \in \mathbb{N}_{[1:p]}$ such that $[G]_{k\bullet}v^{k\star} > 1$, where $v^{j\star}$, for all $j \in \mathbb{N}_{[1:p]}$, denotes the optimal solution of Problem 2.

Proof: Suppose the vector $v^{k\star} \in \mathbb{R}^n$ is the solution of Problem 2 satisfying $[G]_{k\bullet}v^{k\star} > 1$, for an index $k \in \mathbb{N}_{[1:p]}$. Since (5c) is satisfied with $v = v^{k\star}$, it follows that $v^{k\star} \in \mathcal{S}_x$. Moreover, according to Fact 1, since $[G]_{k\bullet}v^{k\star} > 1$, $k \in \mathbb{N}_{[1:p]}$, it follows that $v^{k\star} \notin \mathcal{S}$. Thus, $v^{k\star} \in \mathcal{S}_x \setminus \mathcal{S}$. Since relations (5a), (5b) and (5d) are satisfied, the set $\mathcal{S}^{v^{k\star}}$ is an ε-contractive set according to Lemma 1. Conversely, consider a vector $\xi \in \mathcal{S}_x \setminus \mathcal{S}$ such that $\mathcal{S}^\xi$ is ε–contractive. According to Fact 1, there exists an index $k \in \mathbb{N}_{[1:p]}$ such that $[G]_{k\bullet}\xi > 1$. Next, consider the set of constraints (5) and let $j = k$. According to Lemma 1, since $\mathcal{S}^\xi$ is an ε–contractive set, relations (5a),(5b) are satisfied with $v^i = \xi$. Also, since $\xi \in \mathcal{S}_x$, relation (5c) is satisfied. Hence, the vector ξ is a feasible solution of the constraints (5). Thus, the optimization problem (3) has an optimal solution $v^{k\star}$ such that $[G]_{k\bullet}v^{k\star} \geq [G]_{k\bullet}\xi > 1$. ■

The significance of Lemma 3 lies in the fact that existence of a nontrivial solution of Problem 2 is necessary and sufficient for the expansion of an ε–contractive set by adding a vertex to its convex hull.

The iterative method producing the sequence of sets is presented in an algorithmic structure in *Algorithm 1*.

Remark 2: Problem 2 is feasible for all $j \in \mathbb{N}_{[1:p_i]}$ and for all $i \in \mathbb{N}_{\geq 1}$. In fact, for each $j \in \mathbb{N}_{[1:p_i]}$, the set $\{x \in \mathcal{S}_i : [G_i]_{j\bullet}x = 1\}$ belongs to the feasible solution set. Also, it is worth noticing that the optimization criterion

Algorithm 1 **Input**: System (1), state constraint set $\mathcal{S}_x$, scalar $\varepsilon \in \mathbb{R}_{\geq 0}$, proper $\mathcal{C}$–polytopic ε–contractive set $\mathcal{S}_0$. **Output:** sequence of sets $\{\mathcal{S}_i\}_{i\in\mathbb{N}_{\geq 1}}$.

$i \leftarrow 0$
$f \leftarrow 1$
while $f = 1$ **do**
  for all $j \in \mathbb{N}_{[1:p_i]}$ **do**
    $v^{j\star} \leftarrow$ solution of Problem 2
  end for
  $\mathcal{S}_{i+1} \leftarrow \mathrm{convh}(\{\mathcal{V}(\mathcal{S}_i), \{v^{j\star}\}_{j\in\mathbb{N}_{[1:p_i]}}\})$
  if $\mathcal{S}_{i+1} = \mathcal{S}_i$ **then**
    f $\leftarrow 0$
  else
    $i \leftarrow i + 1$
  end if
end while

as well as the optimization constraints (5b),(5c) and (5d) are linear. Nevertheless, the equality constraint (5a) involves the product of the scalar $\bar{p}_j < 0$ and the vector v^j. However, Problem 2 can be reduced to a sequence of linear programs by fixing the variable $\bar{p}^j$: Initially, a feasible solution can be found by setting $\bar{p}^j$ to be sufficiently small. The cost function is increased iteratively using a standard line search algorithm, by changing the value of $\bar{p}^j$ and solving the corresponding linear program.

Theorem 2: Consider system (1), a proper $\mathcal{C}$–polytopic constraint set $\mathcal{S}_x$, a scalar $\varepsilon \in \mathbb{R}_+$ and a proper $\mathcal{C}$–polytopic ε–contractive set $\mathcal{S}_0$. Then, Algorithm 1 produces an admissible sequence of sets $\{S_i\}_{i\in\mathbb{N}_{\geq 1}}$ for system (1).

Proof: By definition of the convex hull, the following holds, for all $i \in \mathbb{N}$:

$$\begin{aligned}\mathcal{S}_{i+1} &= \mathrm{convh}(\{\mathcal{V}(\mathcal{S}_i), \{v^{j\star}\}_{j\in\mathbb{N}_{[1:p_i]}}\})\\ &= \mathrm{convh}(\{\mathcal{V}(\mathcal{S}_i), v^{j\star}\}_{j\in\mathbb{N}_{[1:p_i]}})\\ &= \mathrm{convh}(\{\mathcal{V}(\mathcal{S}_i^{v^{j\star}})\}_{j\in\mathbb{N}_{1:p_i}}). \end{aligned}\tag{6}$$

By definition of $\mathcal{S}_i^{j\star}$, for all $j \in \mathbb{N}_{[1:p_i]}$, it follows that $\mathcal{S}_i \subset \mathcal{S}_{i+1}$, for all $i \in \mathbb{N}$. Thus, property *1)* holds. Moreover, due to (5c), it follows that $G_x v_i^{j\star} \leq 1$, for all $i \in \mathbb{N}$, $j \in \mathbb{N}_{[1:p_i]}$. Since $\mathcal{S}_0 \subseteq \mathcal{S}_x$, we obtain $\mathcal{S}_i \subseteq \mathcal{S}_x$, for all $i \in \mathbb{N}$. Thus, property *2)* holds. Since $\mathcal{S}_0$ is a proper $\mathcal{C}$–polytopic set, property *1)* further yields that $0 \in \mathrm{interior}(\mathcal{S}_i)$, for all $i \in \mathbb{N}$. Furthermore, by (6), $\mathcal{S}_i$ is a polytopic set, hence a proper $\mathcal{C}$-polytopic set for all $i \in \mathbb{N}$. Thus, property *3)* holds. Lastly, note that $\mathcal{S}_0$ is an ε-contractive set and suppose $\mathcal{S}_i$ is ε-contractive also. Observing that the sets $\mathcal{S}_i^{j\star}$, $j \in \mathbb{N}_{[1:p_i]}$ are proper $\mathcal{C}$–polytopic ε-contractive sets and combining (6) and Lemma 2, it follows that $\mathcal{S}_{i+1}$ is also an ε–contractive set. Hence, it follows by induction that property *4)* holds. Thus, the set sequence $\{S_i\}_{i\in\mathbb{N}_{\geq 1}}$ is an admissible sequence of sets for system (1). ■

The algorithm terminates if there exists an $N \in \mathbb{N}_{\geq 1}$ such that $\mathcal{S}_{N+1} = \mathcal{S}_N$. Nevertheless, artificial termination criteria can be introduced in order to ensure finite termination. Employment of different criteria can be made according to the problem considered. We examine the following two frequently encountered cases:

a) *Approximation of the domain of attraction.* It is well known that the domain of attraction coincides with the maximal positively invariant set for linear continuous-time systems. Thus, to obtain an approximation of the domain of attraction, one can apply Algorithm 1 starting from a given proper $\mathcal{C}$–polytopic positively invariant set and setting $\varepsilon = 0$. In this case, the finite termination is not guaranteed. Thus, a maximum number of iterations N_{max} as an additional termination criterion can be introduced. Alternatively, Algorithm 1 can be terminated when a distance function between two consecutive sets of the sequence is less than a prespecified value $d \in \mathbb{R}_+$. We propose the Hausdorff distance, denoted as $\mathrm{distH}(\mathcal{S}_i, \mathcal{S}_{i+1})$, $i \in \mathbb{N}_{\geq 1}$. For each pair $(\mathcal{S}_i, \mathcal{S}_{i+1})$ of the sequence, it is easy to show that

$$\begin{aligned}\mathrm{distH}(\mathcal{S}_i, \mathcal{S}_{i+1}) &= \max_{j\in\mathbb{N}_{[1:q_i]}} \min_{k\in\mathbb{N}_{[1:q_{i+1}]}} \|[V_i]_{\bullet j} - [V_{i+1}]_{\bullet k}\|_\infty\\ &= \max_{j\in\mathbb{N}_{[1:q_i]}} \min_{k\in\mathbb{N}_{[1:p_i]}} \left\|[V_i]_{\bullet j} - v^{k\star}\right\|_\infty,\end{aligned}$$

for all $i \in \mathbb{N}$, where $v^{k\star}$, $k \in \mathbb{N}_{[1:p_i]}$, are the corresponding solutions of Problem 2. Thus, checking the termination criterion $\mathrm{distH}(\mathcal{S}_i, \mathcal{S}_{i+1}) \leq d$ is equivalent to solving a single linear program.

b) *Stability analysis of an assigned set of initial conditions*. This case can be considered as the autonomous continuous-time version of the problem studied in [17]: Given a continuous-time linear system (1), a proper $\mathcal{C}$–polytopic constraint set $\mathcal{S}_x$ and a proper $\mathcal{C}$–polytopic initial condition set $\mathcal{S}_0$, the problem consists in determining if $\mathcal{S}_{x_0}$ belongs to the admissible domain of attraction. For the case of linear continuous-time systems and taking into account Assumption 1, it follows that $\mathcal{S}_{x_0}$ belongs to the admissible domain of attraction if and only if there exists a positively invariant proper $\mathcal{C}$–set $\mathcal{S}$ that satisfies the set relation $\mathcal{S}_{x_0} \subseteq \mathcal{S} \subseteq \mathcal{S}_x$. Thus, one can employ Algorithm 1 with the additional input $\mathcal{S}_0$ and stop when $\mathcal{S}_0 \subseteq \mathcal{S}_i$. For proper $\mathcal{C}$–polytopic sets, checking this termination criterion is equivalent to solving a linear program by directly applying the extended Farkas' Lemma [18].

Remark 3: The developed results and algorithm can be extended to polytopic differential inclusions of the form

$$\dot{x}(t) \in \Phi(x(t)),\tag{7}$$

where $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$,

$$\Phi(x) := \{Ax : A \in \mathrm{convh}(\{A_i\})_{i\in\mathbb{N}_{[1:M]}}\},$$

$A_i \in \mathbb{R}^{n\times n}$, for all $i \in \mathbb{N}_{[1:M]}$, $M \in \mathbb{N}_{\geq 1}$. Notice that an admissible sequence of sets for system (7) is also admissible for the corresponding continuous-time switched system with arbitrary switching. Equivalently to Assumption 1, system (7) is considered to be asymptotically stable.

It has been shown (see e.g. [1]) that a proper $\mathcal{C}$–polytopic set is ε–contractive for system (7) if and only if the conditions of Theorem 1 are satisfied for each vertex A_i, $i \in \mathbb{N}_{[1:M]}$, of the matrix polytope $\mathrm{convh}(\{A_i\})_{i\in\mathbb{N}_{[1:M]}}$. Taking this into account and modifying Lemma 1 and Lemma 2 *mutatis mutandis*, it is straightforward to obtain the correspondent of Lemma 3. We state the following problem:

Problem 3: Given system (7), a proper $\mathcal{C}$–polytopic constraint set $\mathcal{S}_x$, a scalar $\varepsilon \in \mathbb{R}_{\geq 0}$, a proper $\mathcal{C}$–polytopic ε–contractive set $\mathcal{S}$, and a fixed $j \in \mathbb{N}_{[1:p]}$, compute the solution of the optimization problem

$$\max_{v^j, p_j^l, \bar{p}_j^l, l \in \mathbb{N}_{[1:M]}} \{[G]_{j\bullet} v^j\}$$

subject to

$$A_l v^j = V p_j^l + \bar{p}_j^l v^j, \quad \forall l \in \mathbb{N}_{[1:M]}$$

$$\mathbf{1}_q^T p_j^l + \bar{p}_j^l \leq -\varepsilon, \quad \forall l \in \mathbb{N}_{[1:M]}$$

$$G_x v^j \leq \mathbf{1}_{p_x}$$

$$p_j^l \geq \mathbf{0}_q, \quad \forall l \in \mathbb{N}_{[1:M]}.$$

Lemma 4: There exists a vector $\xi \in \mathcal{S}_x \setminus \mathcal{S}$ such that $\mathcal{S}^\xi$ is an ε–contractive set for system (7) if and only if there exists an index $k \in \mathbb{N}_{[1:p]}$ such that $[G]_{k\bullet} v^{k\star} > 1$, where $v^{j\star}$, for all $j \in \mathbb{N}_{[1:p]}$, denotes the optimal solution of Problem 3.
For brevity, the proof of Lemma is 4 omitted. An immediate consequence of this result is that Algorithm 1 can be used for system (7) with a single modification concerning the computation of the new vectors $\{v^{j\star}\}_{i\in\mathbb{N}_{[1:p_i]}}$, which in this case is given by the optimal solution of Problem 3 instead of Problem 2.

Remark 4: The developed results and algorithm can be also modified in a straightforward manner to deal with continuous-time non autonomous linear systems with additive disturbances.

V. Illustrative examples

Example 1. We consider the linear continuous-time system (1) with system matrix

$$A = \begin{bmatrix} -0.1 & 1.0 \\ -2.0 & -0.4 \end{bmatrix},$$

with eigenvalues $-0.25 \pm 1.4062i$. Also, we assume the proper $\mathcal{C}$–polytopic constraint set $\mathcal{S}_x$, where $G_x = [I_2 - I_2]^T$. The initial proper $\mathcal{C}$–polytopic positively invariant set $\mathcal{S}_0$ is computed applying the results from [10], resulting in a symmetric proper $\mathcal{C}$–polytopic set with 26 vertices. Setting $\varepsilon = 0$, we aim to approximate the admissible domain of attraction. As artificial termination criterion, the maximum number of iterations was chosen to be equal to 30. Applying Algorithm 1, an admissible sequence of sets is produced. The last element of the sequence $\mathcal{S}_{30}$ has 128 vertices and is shown in Figure 1 in yellow. In the same figure, the initial set $\mathcal{S}_0$ is shown in blue and the constraint set $\mathcal{S}_x$ is shown in grey.

In Figure 2, the trajectories of the system are shown for initial conditions starting from the vertices of set $\mathcal{S}_{30}$.

Example 2. We consider the switched linear system under arbitrary switching, also studied in [9], [10], with matrices

$$A_1 = \begin{bmatrix} 0.3 & 0.7 \\ -2.3 & -2.3 \end{bmatrix}, A_2 = \begin{bmatrix} -1.8 & 1.0 \\ -0.8 & 0.1 \end{bmatrix}.$$

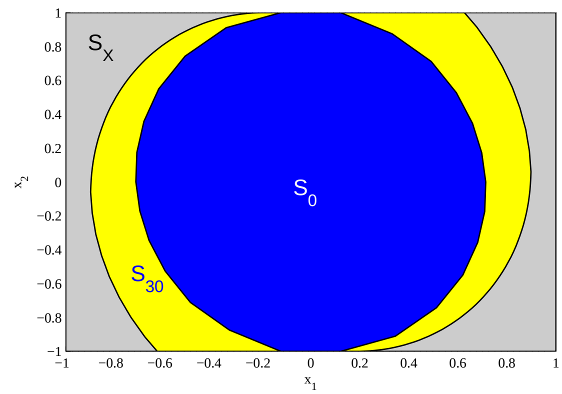

Fig. 1. State constraint set $\mathcal{S}_x$ (grey), starting positively invariant set $\mathcal{S}_0$ (blue), and the last element of the admissible sequence of set $\mathcal{S}_{30}$ (yellow).

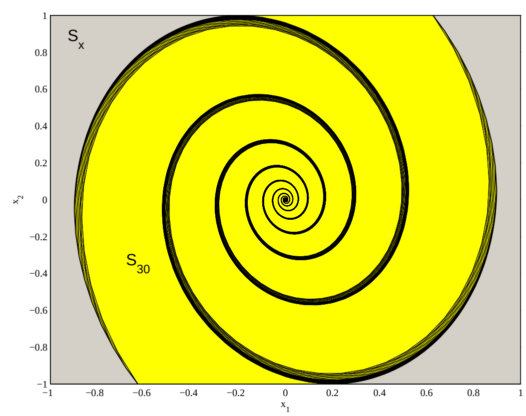

Fig. 2. Trajectories of the system (Example 1) starting from the vertices of set $\mathcal{S}_{30}$.

Moreover, we consider a non symmetric constraint set $\mathcal{S}_x$ described by

$$V_x = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 2 & -1 & 1 \end{bmatrix}.$$

The initial positively invariant set $\mathcal{S}_0$ was taken from [10] and is shown in Figure 3 in blue. Applying Algorithm 1 and setting as termination criterion the Hausdorff distance $\mathrm{distH}(\mathcal{S}_i, \mathcal{S}_{i+1})$ to be less than $d = 0.001$, the algorithm terminates after 25 iterations. In Figure 3, the set $\mathcal{S}_{25}$ is shown in yellow, the initial proper $\mathcal{C}$–polytopic positively invariant set $\mathcal{S}_0$ is shown in blue, and the constraint set $\mathcal{S}_x$ is shown in grey. Finally, in Figure 4 and Figure 5, the trajectories of initial conditions starting from the vertices of the set $\mathcal{S}_{25}$ are shown for systems $\dot{x}(t) = A_1 x(t)$ and $\dot{x}(t) = A_2 x(t)$ respectively.

VI. Conclusions

A method for iteratively enlarging a given invariant (contractive) proper $\mathcal{C}$–polytopic set for *continuous-time* linear systems was presented. It was proven that the proposed

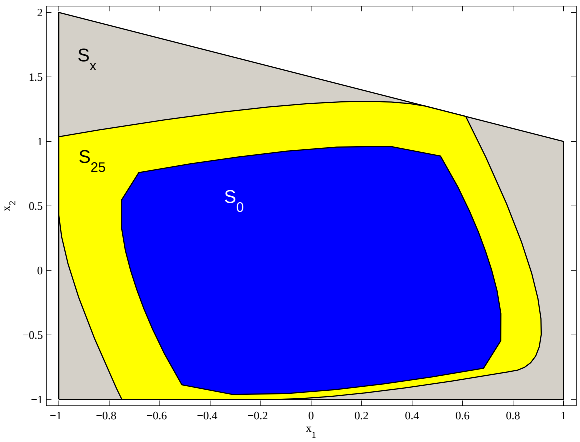

Fig. 3. State constraint set $\mathcal{S}_x$ (grey), initial condition set $\mathcal{S}_{x_0}$ (blue), and the positively invariant set $\mathcal{S}_{25}$ (yellow).

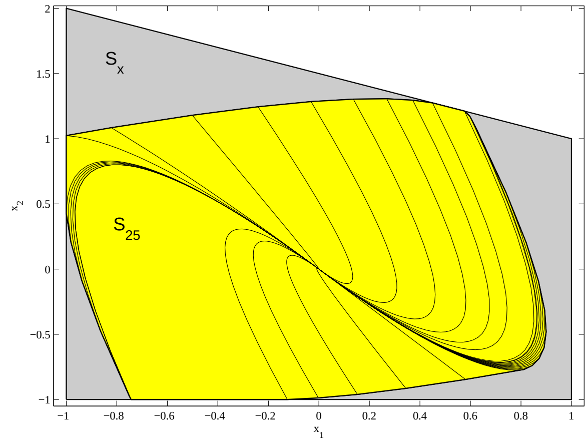

Fig. 5. State constraint set $\mathcal{S}_x$ (grey), the positively invariant set $\mathcal{S}_{25}$ (yellow), and trajectories for the A_2 dynamics starting from the vertices of $\mathcal{S}_{25}$.

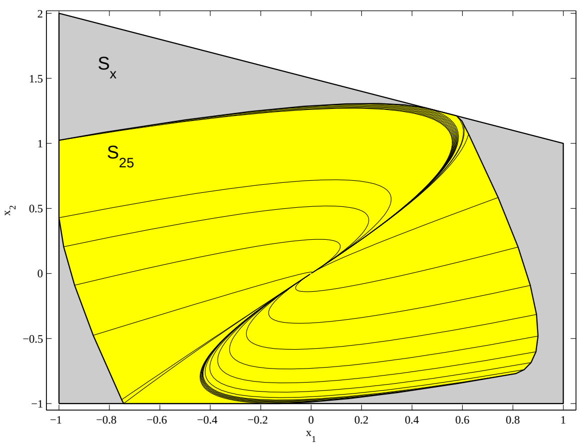

Fig. 4. State constraint set $\mathcal{S}_x$ (grey), the positively invariant set $\mathcal{S}_{25}$ (yellow), and trajectories for the A_1 dynamics starting from the vertices of $\mathcal{S}_{25}$.

algorithmic procedure generates a monotonic sequence of invariant (contractive) proper $\mathcal{C}$–polytopic sets. A formal characterization of the limit of the *admissible sequence of sets* makes the object of future research work.

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