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Athanasopoulos, N., & Bitsoris, G. (2011). Stability Analysis and Control of Bilinear Discrete-Time Systems: A Dual Approach. In *Proceedings of the 18th IFAC World Congress* Elsevier B.V.. https://doi.org/10.3182/20110828-6-IT-1002.03485

Published in:
Proceedings of the 18th IFAC World Congress

Document Version:
Peer reviewed version

Stability Analysis and Control of Bilinear Discrete-Time Systems: A Dual Approach

Nikolaos Athanasopoulos* **and George Bitsoris***

* *Control Systems Laboratory, University of Patras, 26500 Patras, Greece (e-mail: {nathanas,bitsoris}@ece.upatras.gr)*

Abstract: In this article, the stabilization problem of discrete-time bilinear systems by state-feedback control is investigated. First, necessary and sufficient conditions guaranteeing positive invariance or/and attractivity of polyhedral sets of general form with respect to quadratic systems are provided. These results are then used to establish systematic methods of determining linear and nonlinear state feedback control laws making a prespecified polyhedral set a domain of attraction with respect to the resulting closed-loop system, even in the presence of input constraints.

Keywords: bilinear discrete-time systems, constrained control, dual comparison principle.

1. INTRODUCTION

Bilinear systems are systems with a special type of nonlinearity in the dynamics, consisting of second order polynomial products between the input and state variables. Stability analysis as well as control design for this class of nonlinear systems still remains a topic of interest for practical reasons, as many processes in engineering and biology can be naturally modelled by bilinear systems (Mohler et al. (1980)). Also, bilinear approximations are better than the linear ones, especially when well established identification algorithms can be used (Favoreel et al. (1999)).

For the case of continuous-time systems, in recent works, (Amato et al (2009), Tarbouriech et al (2009)), quadratic Lyapunov functions were used, while in Athanasopoulos et al (2010) the authors choose polyhedral Lyapunov functions. For discrete-time systems, most approaches are based on model-based predictive control theory and feedback linearization, for instance Cannon et al. (2003), Fontes et al. (2008), Ekman (2005). Another possible approach, which leads to the computation of stabilizing linear state feedback control laws and polyhedral approximations of the domain of attraction can be found in Bitsoris et al. (2008), Athanasopoulos et al (2010), both for the unconstrained and constrained case. The control strategy is computed by applying established algebraic sufficient conditions for existence of polyhedral Lyapunov functions.

In this article, a novel approach, closely related to comparison systems, which leads to necessary and sufficient conditions of existence of stabilizing control laws for the closed-loop bilinear system, is adopted. In particular, the dual comparison principle, as stated in the companion paper Bitsoris et al. (2011), is exploited in order to establish conditions guaranteeing existence of stabilizing feedback control laws that can be either linear or nonlinear. The benefits of the approach are obvious since these conditions are necessary and sufficient and lead to the establishment of systematic methods leading to the computation of stabilizing control laws even in presence of input constraints.

The article is organized as follows: In section 2, necessary notations as well as the problem statement are given. In section 3, algebraic necessary and sufficient conditions guaranteeing positive invariance of polyhedral sets and the asymptotic stability of systems with second order polynomial nonlinearities are established. Then, in section 4, design techniques for the unconstrained and constrained stabilization problem are developed, using linear and nonlinear state-feedback control laws. Finally, in section 5, a numerical example illustrating the effectiveness of the proposed method is given, while in section 6 conclusions are drawn.

2. PROBLEM STATEMENT

Throughout the paper, capital letters denote real matrices and lower case letters denote column vectors or scalars. $\mathbb{R}^n$ denotes the real n-space and $\mathbb{R}^{n\times m}$ denotes the set of real $n \times m$ matrices. Given a real $n \times m$ matrix $A = (a_{ij})$, $A^+ = (a_{ij}^+)$ and $A^- = (a_{ij}^-)$ are $n \times m$ matrices with entries defined by the relations $a_{ij}^+ = \max\{a_{ij}, 0\}$ and $a_{ij}^- = -\min\{a_{ij}, 0\}$. Thus, $A = A^+ - A^-$. Inequality $A \leq B$ $(A < B)$ with $A, B \in \mathbb{R}^{n\times m}$ is equivalent to $a_{ij} \leq b_{ij}(a_{ij} < b_{ij})$. Similar notation holds for vectors. Given a function $v(x), v : \mathbb{R}^n \to \mathbb{R}^p$ and a set $X \subseteq \mathbb{R}^n$, then $v(X) = \{y \in \mathbb{R}^p : (\exists x \in \mathbb{R}^n : v(x) = y)\}$. Finally, T denotes the time set $T = \{0, 1, 2, ...\}$.

We consider bilinear discrete-time systems described by difference equations of the form

$$x(t+1) = Ax(t) + Bu(t) + \begin{bmatrix} x^T(t)C_1 \\ x^T(t)C_2 \\ \vdots \\ x^T(t)C_n \end{bmatrix} u(t) \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $t \in T$ is the time variable and $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{n\times m}$, $i = 1, 2, \dots, n$.

Also, a bounded polytopic subset of the state space defined by inequalities

$$Gx \leq d \tag{2}$$

with $G \in \mathbb{R}^{q\times n}$, $d \in \mathbb{R}^q$, $d > 0$, is given. This set can also be described as the convex hull of its vertices w_i $i = 1, 2, ..., p$, i.e.

$$conv(w_1, w_2, ..., w_p) \tag{3}$$

where w_i $i = 1, 2, ..., p$ denote the vertices of the polytopic set.

The unconstrained stabilization problem to be investigated is formulated as follows: Given system (1) and a bounded polytopic subset of the state space defined by (3), determine a state-feedback control law making this set a domain of attraction of the resulting closed-loop system.

In the constrained stabilization problem, control constraints $u \in S(L, \rho)$ of the form

$$S(L, \rho) = \{u \in \mathbb{R}^m : Lu(t) \leq \rho\} \tag{4}$$

with $\rho \in \mathbb{R}^q, \rho > 0$, $L \in \mathbb{R}^{c\times m}$ are also imposed. Thus, the problem is the determination of a state-feedback control law such that all initial states belonging to the set defined by (3) transferred asymptotically to the origin while the control constraints (4) are satisfied.

3. POSITIVE INVARIANCE AND STABILITY

Given a dynamical system, a subset of its state space is said to be *positively invariant* if all trajectories starting from this set remain in it for all future instances. Thus, if the state constraints define an admissible subset of the state space then a solution to the control problem under state constraints is a stabilizing linear control law making this admissible set positively invariant with respect to the resulting closed-loop system. Since in practical control problems the state constraints are usually expressed by linear inequalities, the admissible set is a polyhedron. When linear state-feedback control laws of the form

$$u(t) = Kx(t) \tag{5}$$

are chosen, $K \in \mathbb{R}^{m\times n}$, the resulting closed-loop system is described by equation

$$x(t+1) = (A + BK)x(t) + \begin{bmatrix} x^T(t)C_1Kx(t) \\ x^T(t)C_2Kx(t) \\ \vdots \\ x^T(t)C_nKx(t) \end{bmatrix}. \tag{6}$$

This equation describes a nonlinear system with second order polynomial nonlinearity. Therefore, it is very important to establish conditions guaranteeing positive invariance of polyhedral sets of the form (3) with respect to nonlinear systems with second order polynomial nonlinearities of the general form

$$x(t+1) = Ax(t) + \begin{bmatrix} x^T(t)M_1x(t) \\ x^T(t)M_2x(t) \\ \vdots \\ x^T(t)M_nx(t) \end{bmatrix}, \tag{7}$$

$A \in \mathbb{R}^{n\times n}$, $M_i \in \mathbb{R}^{n\times n}$, $i = 1, .., n$. Using the notation adopted in the companion paper Bitsoris et al. (2011) set (3) can be written as

$$Q(W, g, 1) \triangleq \{x \in \mathbb{R}^n : (\exists y \in \mathbb{R}^p_+ : g(y) \leq 1,\ x = Wy)\}$$

where $W = [\, w_1\ w_2\ \cdots\ w_p \,]$ and $g(y) = e^T y$. The following Lemma provides necessary and sufficient conditions for a set of the form $Q(W, g, r)$ to be positively invariant with respect to a nonlinear discrete-time system $x(t+1) = f(x(t))$ and is very important for the development of the results of this paper.

Lemma 1:(Bitsoris et al. (2011)). The set

$$Q(W, g, r) \triangleq \{x \in \mathbb{R}^n : (\exists y \in \mathbb{R}^p : g(y) \leq r,\ x = Wy)\}$$

with $W \in \mathbb{R}^{n\times p}$ and $g(y)$, $g : \mathbb{R}^p \to \mathbb{R}$ is a positively invariant set of system

$$x(t+1) = f(x(t)) \tag{8}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$, if and only if there exists a function $h(y), h : \mathbb{R}^p_+ \to \mathbb{R}^p_+$ such that

$$f(Wy) = Wh(y)$$

and set

$$R(g, r) \triangleq \{y \in \mathbb{R}^p : g(y) \leq r\} \tag{9}$$

is a positively invariant set of system

$$y(t+1) = h(y(t)).$$

We shall use this result to establish conditions guaranteeing that a polyhedral set defined by linear inequalities (2) or equivalently as the convex hull of its vertices is positively invariant with respect to the nonlinear system (7):

Theorem 2: The polytope with vertices w_i $i = 1, 2, ..., p$ is positively invariant with respect to system (7) if and only if there exist matrices $D_j \in \mathbb{R}^{p\times p}$ $j = 1, 2, \dots, p$ and a nonnegative matrix $P \in \mathbb{R}^{p\times p}$ such that

$$AW = WP \tag{10}$$

$$W^T M_i W = \sum_{j=1}^{p} w_{ij} D_j \qquad i = 1, 2, \dots, n \tag{11}$$

and function $h : \mathbb{R}^p \to \mathbb{R}^p$,

$$h(y) = Py + \begin{bmatrix} y^T D_1 y \\ y^T D_2 y \\ \vdots \\ y^T D_p y \end{bmatrix} \tag{12}$$

is nonnegative and satisfies inequality

$$0 \leq e^T Py + y^T Dy \leq 1 \qquad \forall y \in \mathbb{R}^p_+,\quad e^T y \leq 1 \tag{13}$$

where

$$D = \sum_{j=1}^{n} D_j \ . \tag{14}$$

Proof: According to Lemma 1, the polytope with vertices w_i $i = 1, 2, ..., p$ is positively invariant with respect to system (7) if and only if there exists a function $h(y)$, $h : \mathbb{R}^p_+ \longrightarrow \mathbb{R}^p_+$, that is a nonnegative function $h(y)$, such that

$$AWy + \begin{bmatrix} y^T W^T M_1 W y \\ y^T W^T M_1 W y \\ \vdots \\ y^T W^T M_1 W y \end{bmatrix} = Wh(y) \tag{15}$$

and set $P(g,1) = \left\{y \in \mathbb{R}^p_+ : e^T y \le 1\right\}$ is positively invariant w.r.t. system $y(t+1) = h(y(t))$

$$e^T h(y) \le 1 \qquad \forall y \in \mathbb{R}^p_+,\ e^T y \le 1 \tag{16}$$

It is clear that a function $h(y)$ satisfying (15) is of the form (12). Indeed,

$$AWy + \begin{bmatrix} y^T W^T M_1 W y \\ y^T W^T M_1 W y \\ \vdots \\ y^T W^T M C_1 W y \end{bmatrix} = WPy + W \begin{bmatrix} y^T D_1 y \\ y^T D_2 y \\ \vdots \\ y^T D_p y \end{bmatrix}$$

which is equivalent to relations (10) and (11). Furthermore, set $P(g,1)$ is positively invariant w.r.t. system $y(t+1) = h(y(t))$ if and only if $e^T h(y) \le 1\ \ \forall y \in \mathbb{R}^p_+,\ e^T y \le 1$, that is if and only if

$$e^T Py + e^T \begin{bmatrix} y^T D_1 y \\ y^T D_2 y \\ \vdots \\ y^T D_p y \end{bmatrix} \le 1 \qquad \forall y \in \mathbb{R}^p_+, \quad e^T y \le 1,$$

which is equivalent to

$$e^T Py + y^T Dy \le 1 \qquad \forall y \in \mathbb{R}^p_+,\ e^T y \le 1$$

with matrix D given by (14) ■.

Asserting whether a given polytope $Q(W,g,1)$ is positively invariant depends on our ability to check whether hypotheses of Theorem 1 and in particular condition $e^T Py + y^T Dy \le 1$ is verified for all $y \in P(g,1)$ where $g(y) = e^T y$. Taking into account that matrix P is nonnegative and that function $e^T Py + y^T Dy$ is nonnegative in $y \in P(g,1)$ we establish the following result:

Theorem 3: The polytope with vertices $w_i\ i = 1,2,...,p$ is positively invariant with respect to system (7) if and only if there exist matrices $D_j \in \mathbb{R}^{p\times p}\ j = 1,2,\dots,p$ and a nonnegative matrix $P \in \mathbb{R}^{p\times p}$ such that

$$AW = WP \tag{17}$$

$$W^T M_i W = \sum_{j=1}^{p} w_{ij} D_j \qquad i = 1,2,\dots,n \tag{18}$$

and

$$e^T Py^* \le 2 \tag{19}$$

if there exists a $y^* \ge 0$ satisfying relations

$$P^T e + (D + D^T)y^* = 0 \tag{20}$$

$$e^T y^* \le 1 \tag{21}$$

or

$$e^T Py + y^T Dy \le 1 \qquad \forall y:\ e^T y = 1,\ y \ge 0. \tag{22}$$

if there does not exist a $y^* \ge 0$ satisfying relations (20),(21).

Proof: By virtue of Theorem 1, it is sufficient to prove that condition (13) is equivalent either to the existence of a vector $y^* \ge 0$, $e^T y^* \le 1$ satisfying relations (20) and (21), or relation (22). Relation (20) is the condition for $e^T Py^* + y^{*T} Dy^*$ to be an extreme value of function $e^T Py + y^T Dy$. Then

$$\begin{aligned} e^T Py^* + y^{*T} Dy^* &= e^T Py^* + 0.5 y^{*T}(D + D^T)y^* = \\ &= e^T Py^* - 0.5 e^T Py^* = \\ &= 0.5 e^T Py^* \end{aligned}$$

If $y^* \ge 0$, then this extreme value is a maximum because matrix P is nonnegative. Thus, if $e^T y^* \le 1$, then condition

$$e^T Py + y^T Dy \le 1 \qquad \forall y \in \mathbb{R}^p_+,\ e^T y \le 1 \tag{23}$$

is satisfied if and only if $e^T Py^* \le 2$. If $e^T y^* > 1$, then condition (23) is satisfied if and only if $e^T Py + y^T Dy \le 1 \quad \forall y:\ e^T y = 1,\ y \ge 0$. Finally, if there does not exist $y^* \ge 0$ satisfying (20), then $e^T Py + y^T Dy$ is increasing in $\mathbb{R}^p_+$ and again condition (23) is satisfied if and only if $e^T Py + y^T Dy \le 1$, $\forall y:\ e^T y = 1,\ y \ge 0$.

The above theorem is useful in order to check if set $Q(W,g,r)$ is positively invariant w.r.t. system (7), as it is explained below: First, relation (17),(18),(20), (21) are solved in order to find suitable matrices P, D_j, $j = 1,...,p$ and a vector y^*. If a solution can be found and in addition relation (19) holds, set $Q(W,g,r)$ is positively invariant w.r.t. system (7). If relation (19) is not satisfied, then set $Q(W,g,r)$ is not positively invariant. If relations (17),(18),(20), (21) are not compatible, the following optimization problem is solved

$$\max_{P,D_1,\dots,D_p,y} \{e^T Py + y^T Dy\} \tag{24}$$

subject to relations (17), (18) and the additional constraints $e^T y = 1$, $y \ge 0$. If the optimal value of the objective function is less than one, set $Q(W,g,r)$ is positively invariant w.r.t. (7).

It is worth noticing that the previous results provide necessary and sufficient conditions guaranteeing positive invariance of $Q(W,g,r)$ with respect to (7). However, it is possible to establish simpler sufficient conditions guaranteeing invariance. This is done in the following Corollary of Theorem 2.

Corollary 4: The polytope with vertices $w_i\ i = 1,2,...,p$ is positively invariant with respect to system (7) if and only if there exist matrices $D_j \in \mathbb{R}^{p\times p}\ j = 1,2,\dots,p$ and a nonnegative matrix $P \in \mathbb{R}^{p\times p}$ such that

$$AW = WP \tag{25}$$

$$W^T M_i W = \sum_{j=1}^{n} w_{ij} D_j \qquad i = 1,2,\dots,p \tag{26}$$

and function $h : \mathbb{R}^p \to \mathbb{R}^p$,

$$h(y) = Py + \begin{bmatrix} y^T D_1 y \\ y^T D_2 y \\ \vdots \\ y^T D_p y \end{bmatrix} \tag{27}$$

is nonnegative and satisfies inequality

$$e^T Py + y^T D^+ y \le 1, \quad \forall e^T y = 1 \tag{28}$$

where

$$D = \sum_{j=1}^{n} D_j \tag{29}$$

Proof: Condition (12) of Theorem 2 can be equivalently expressed as

$$e^T Py + y^T (D^+ - D^-)y \le 1 \qquad \forall y \in \mathbb{R}^p_+,\ e^T y \le 1 \tag{30}$$

Since matrices P, D^+ and D^- are nonnegative, condition (30) is satisfied if

$$e^T Py + y^T D^+ y \le 1 \qquad \forall y \in \mathbb{R}^p_+,\ e^T y = 1$$

■

Using this result, checking whether a given polytope $Q(W, g, 1)$ is positively invariant w.r.t. system (6) can be carried out by solving a convex optimization problem, that is

$$\max_{P,D_1,\dots,D_p,y} \{e^T Py + y^T Dy\} \tag{31}$$

subject to relations (25),(26), (29) and $e^T y = 1$.

It is a well known fact that positive invariance is closely related to the stability of a dynamical system. The following lemma can be used for stability analysis purposes:

Lemma 5: (Bitsoris et al. (2011)). Suppose that the origin is an equilibrium point of system (7), $g(y)$ is a continuous function, $g : \mathbb{R}^p \to \mathbb{R}^s$, $g(0) = 0$, and there exists matrix $W, W \in \mathbb{R}^{n\times p}$, $rankW = n$. If there exists a function $h(y)$, $h : \mathbb{R}^p_+ \to \mathbb{R}^p_+$ such that function $g(h(y))$ is nondecreasing,

$$g(f(Wy)) = g(Wh(y)) \tag{32}$$

and $P(g, r)$ is a contractive set with respect to system

$$y(t + 1) = h(y(t))$$

then the equilibrium $x = 0$ of system (7) is asymptotically stable and set $Q(W, g, r)$ is a domain of attraction.

Following similar steps as in Theorem 3, we are in a position to establish necessary and sufficient conditions guaranteeing asymptotic stability for system (7), as shown in the next result:

Theorem 6: The polytope with vertices w_i $i = 1, 2, ..., p$ is a domain of attraction with respect to system (7) if and only if there exist matrices $D_j \in \mathbb{R}^{p\times p}$ $j = 1, 2, \dots, p$ a nonnegative matrix $P \in \mathbb{R}^{p\times p}$ and a positive scalar $\varepsilon > 0$ such that

$$AW = WP \tag{33}$$

$$W^T M_i W = \sum_{j=1}^{p} w_{ij} D_j \qquad i = 1, 2, \dots, n \tag{34}$$

and

$$e^T Py^* \leq 2 - \varepsilon \tag{35}$$

if there exists a $y^* \geq 0$ satisfying relations

$$P^T e + (D + D^T)y^* = 0 \tag{36}$$

$$e^T y^* \leq 1 \tag{37}$$

or

$$e^T Py + y^T Dy \leq 1 - \varepsilon \qquad \forall y : \ e^T y = 1, \ y \geq 0. \tag{38}$$

if there does not exist a y^* satisfying relations (36),(37).

Proof: It is easy to show that if relations (33)-(37) hold, the maximum value of $e^T h(y)$ is strictly less than one, thus set $P(g, r)$ is contractive with contraction factor $(1 - \varepsilon)$ and according to Lemma 5 set $Q(W, g, r)$ is a domain of attraction w.r.t. (7). If relations (33),(34),(36),(37) hold but (35) does not, then set $Q(W, g, r)$ is not a domain of attraction. Otherwise, if (38) holds for $\varepsilon > 0$, set $Q(W, g, r)$ is contractive w.r.t. (7).

4. CONTROL DESIGN TECHNIQUES

4.1 Linear state feedback stabilization

We first consider the unconstrained control problem. A linear control law $u = Kx$ is a solution to the unconstrained control problem for system (1) if set defined by (2) or (3) is a domain of attraction of the resulting closed-loop bilinear system (6). Thus, by applying the previous result stated in Theorem 6 to system (6), we establish the following:

Theorem 7: The linear control law $u(t) = Kx(t)$ is a solution to the unconstrained control problem if there exist matrices $D_j \in \mathbb{R}^{p\times p}$ $j = 1, 2, \dots, p$, a nonnegative matrix $P \in \mathbb{R}^{p\times p}$, a control gain $K \in \mathbb{R}^{m\times n}$ and a scalar ε such that

$$(A + BK)W = WP \tag{39}$$

$$W^T C_i KW = \sum_{j=1}^{p} w_{ij} D_j \qquad i = 1, 2, \dots, n \tag{40}$$

$$\varepsilon > 0 \tag{41}$$

and

$$e^T Py^* \leq 2 - \varepsilon \tag{42}$$

if there exists a $y^* \geq 0$ satisfying relations

$$P^T e + (D + D^T)y^* = 0 \tag{43}$$

$$e^T y^* \leq 1 \tag{44}$$

or

$$e^T Py + y^T Dy \leq 1 - \varepsilon \tag{45}$$

$$\forall y : \ e^T y = 1, \ y \geq 0 \tag{46}$$

if there does not exist a y^* satisfying relations (43),(44).

Thus, a possible approach to the determination of a linear state feedback control law which is a solution to the unconstrained optimization problem, is to consider the above relations as constraints of an optimization problem. In specific, we first solve the optimization problem with objective function

$$\max_{K,P,D_1,\dots,D_p,y,\varepsilon} \{e^T Py + y^T Dy\}$$

subject to constraints (39)-(44). If no solution can be found, another optimization problem is solved, with the same objective functions and constraints (39)-(41), (45),(46). It can be easily seen proven that quantity $(1-\varepsilon)$ is a measure of the performance of the closed-loop bilinear system in terms of convergence speed to the equilibrium point. Thus, ε can be a fixed quantity if we are interested in finding a linear control law with a prespecified rate of convergence.

4.2 Nonlinear state feedback stabilization

In this section, we consider nonlinear state feedback control laws of the form

$$u(x) = U\lambda(x), \quad \forall x \in conv\{w_1, w_2, .., w_p\}, \tag{47}$$

where $U \in \mathbb{R}^{m\times p}$, $W\lambda(x) = x$, $e^T \lambda(x) \leq 1$, $\lambda \geq 0$, $\lambda \in \mathbb{R}^p$. The nonlinear control law (47) is a solution to the unconstrained control problem for system (1) if set defined by (2) or (3) is a domain of attraction of the resulting closed-loop bilinear system (6). Taking into account Lemma 1, we establish the following

Theorem 8: The nonlinear control law (47) is a solution to the unconstrained control problem if there exist matrices $D_j \in \mathbb{R}^{p\times p}$ $j = 1, 2, \dots, p$, a nonnegative matrix $P \in \mathbb{R}^{p\times p}$, a matrix $U \in \mathbb{R}^{m\times p}$ and a scalar ε such that

$$AW + BU = WP \tag{48}$$

$$W^T C_i U = \sum_{j=1}^{p} w_{ij} D_j \qquad i = 1, 2, \dots, n \tag{49}$$

and function $h : \mathbb{R}^p \to \mathbb{R}^p$,

$$h(y) = Py + \begin{bmatrix} y^T D_1 y \\ y^T D_2 y \\ \vdots \\ y^T D_p y \end{bmatrix} \quad (50)$$

is nonnegative and satisfies inequality

$$0 \le e^T Py + y^T Dy \le 1 - \varepsilon \qquad \forall y \in \mathbb{R}^p_+, \quad e^T y \le 1 \quad (51)$$

where

$$D = \sum_{j=1}^{n} D_j \quad (52)$$

and

$$\varepsilon > 0. \quad (53)$$

Proof: All $x \in conv\{w_1, w_2, .., w_p\}$ can be expressed as a linear combination of the set vertices $x = W\lambda$, $e^T\lambda \le 1$, $\lambda \ge 0$. Thus, $x(t+1) = f(x(t), u(x(t))$ can be written as

$$f(x(t), u(x(t)) = AW\lambda + BU\lambda + \begin{bmatrix} \lambda^T W^T C_1 U\lambda \\ \lambda^T W^T C_2 U\lambda \\ \vdots \\ \lambda^T W^T MC_n U\lambda \end{bmatrix} \quad (54)$$

Setting $y = \lambda$ and using relations (48),(49) it can be easily seen that

$$f(x(t), u(x(t)) = f(Wy) = Wh(y). \quad (55)$$

Furthermore, set $P(g, r)$ is a contractive set with respect to system $y(t+1) = h(y(t))$ since relations (51)-(53) hold. Thus, according to Lemma 5, the equilibrium $x = 0$ is asymptotically stable and set $Q(W, g, r)$ is a domain of attraction with respect to the closed-loop system.

Equivalent algebraic relations that lead to the computation of a stabilizing control law (47) can be formulated, similar than those in Theorem 7.

4.3 The constrained control problem

Let us now consider the case where control constraints of the form (4) are also imposed. It is known(Bitsoris et al. (1995)) that a state feedback control law $u(t)$ is a solution to the constrained control problem if and only if there exists a subset Ω of the state space which is both a positively invariant set and a domain of attraction of the resulting closed-loop system and satisfies the set relation

$$Q(W, g, 1) \subseteq \Omega \subseteq S_x(L, \rho), \quad (56)$$

where $S_x(L, \rho) = \{x \in \mathbb{R}^n : Lu(x) \le \rho\}$, $L \in \mathbb{R}^{c\times m}$, $\rho \in \mathbb{R}^c$, $\rho > 0$. Many different approaches for the determination of such a control law can be developed by combining this result with those concerning the positive invariance of polyhedral sets. An interesting special case is when $Q(W, g, 1) = \Omega$, that is when the stabilizing linear control law $u(t) = Kx(t)$ renders the desired domain of attraction positively invariant w.r.t. the closed-loop system. Then, set relation (56) becomes

$$Q(W, g, 1) \subseteq S_x(L, \rho). \quad (57)$$

Choosing a linear control law, set relation (57) can be verified through linear algebraic relations established using the extended Farkas' lemma. Thus, the input constraints are satisfied if and only if there exists a nonnegative matrix $E \in \mathbb{R}^{c\times q}$ such that

$$EG = LK \quad (58)$$

$$Ed \le \rho. \quad (59)$$

In the case of nonlinear state feedback control laws of the form (47), set relation (57) holds if the following inequalities are verified:

$$LU_i \le \rho, \quad i = 1, .., p. \quad (60)$$

This is true because by definition the control law (57) is a convex combination of the column vectors of matrix U.

■

5. NUMERICAL EXAMPLE

We consider a bilinear system (1) with the same data matrices as in Bitsoris et al. (2008):

$$A = \begin{bmatrix} 0.8 & 0.5 \\ 0.4 & 1.2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.45 \\ 0.45 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.3 \\ -0.3 \end{bmatrix}$$

To illustrate the ability to cope with polyhedral sets of general form, the desired domain of attraction is a non symmetric polyhedral set having as vertices the columns of matrix W,

$$W = \begin{bmatrix} -1.2 & -1.44 & 0.6 & 1.8 & 0 \\ 1.2 & -0.6 & 0.6 & -1.2 & -1.56 \end{bmatrix}.$$

Also, the control input must satisfy physical constraints

$$-u_m \le u \le u_M$$

with $u_M = u_m = 0.5$. We consider a nonlinear state feedback control law of the form (47). In order to find the input matrix U, the following optimization problem is solved:

$$\max_{U,\varepsilon,y^* D_j, j=1,..,5} \{\varepsilon\}$$

subject to

$$AW + BU = WP$$

$$W^T C_i U = \sum_{j=1}^{p} w_{ij} D_j \qquad i = 1, 2, \ldots, n$$

$$\varepsilon > 0$$

$$e^T Py^* \le 2 - \varepsilon$$

$$y^* \ge 0$$

$$P^T e + (D + D^T)y^* = 0$$

$$e^T y^* \le 1$$

$$-u_m \le U_i \le u_M, \quad i = 1, .., p$$

In this case, no feasible solution can be found. Thus, we solve the following optimization problem

$$\max_{U,\varepsilon,y D_j, j=1,..,5} \{e^T Py + y^T Dy\}$$

subject to

$$AW + BU = WP$$

$$W^T C_i U = \sum_{j=1}^{p} w_{ij} D_j \qquad i = 1, 2, \ldots, n$$

$$0 < \varepsilon < 1$$

$$0 \le e^T Py + y^T Dy \le 1 - \varepsilon$$

$$y \geq 0$$
$$e^T y = 1$$
$$-u_m \leq U_i \leq uM, \quad i = 1, .., p$$

This optimization problem is feasible, and the optimal values of the input matrix U are

$$U = [\, -0.20 \ 0.50 \ -0.50 \ 0.06 \ 0.50 \,] \,.$$

Since the control law $U\lambda(x) = x$, $\lambda(x) \geq 0$, $e^T\lambda \leq 1$ is not uniquely defined, we can choose among many different stabilizing strategies. In this example, we choose an online control law, solving at each time instant t the following linear programming problem:

$$\min_{\lambda} \left\{ Ax\,(t) + Bu\,(\lambda) + \begin{bmatrix} x(t)^T C_1 \\ x(t)^T C_2 \end{bmatrix} u\,(\lambda) \right\}$$

subject to

$$x(t) = W\lambda$$
$$u(\lambda) = U\lambda$$
$$\lambda \geq 0$$
$$e^T \lambda \leq 1$$

In Fig. 1 the state trajectories of the closed loop system starting from the vertices of the polyhedral set are shown.

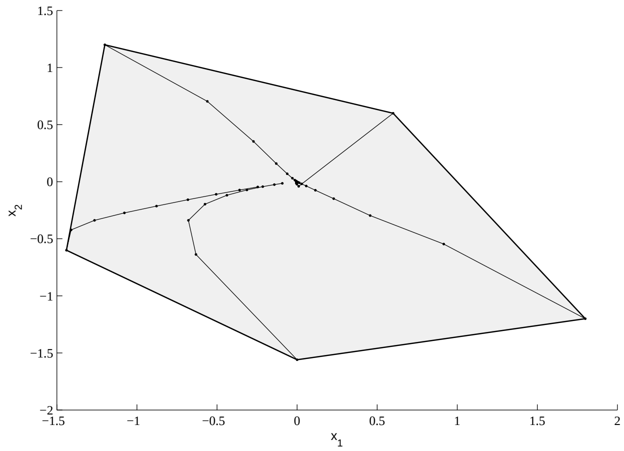

Fig. 1. Contractive polyhedral set having as vertices the columns of matrix W and state trajectories of the closed-loop system starting from the vertices of the set.

6. CONCLUSION

In this article a novel approach regarding the unconstrained and constrained stabilization of bilinear discrete-time system, based on the dual comparison principle, is presented. First, necessary and sufficient conditions guaranteeing the positive invariance of a polyhedral set of general form with respect to a quadratic system are presented. By slightly modifying these conditions, necessary and sufficient conditions were established guaranteeing asymptotic stability of the zero equilibrium point together with a polyhedral approximation of the domain of attraction. Finally, systematic design methods were presented which lead to the computation of stabilizing linear state feedback laws and a special family of nonlinear control laws, even in the presence of input constraints. The control laws can be computed so that the closed loop system has a guaranteed rate of convergence to the equilibrium point.

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