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A CLASS OF ALMOST $C_0(K)$-C*-ALGEBRAS

J. INOUE, Y.-F. LIN AND J. LUDWIG

Abstract. We consider in this paper the family of exponential Lie groups $G_{n,\mu}$, whose Lie algebra is an extension of the Heisenberg Lie algebra by the reals and whose quotient group by the centre of the Heisenberg group is an $ax + b$-like group. The C*-algebras of the groups $G_{n,\mu}$ give new examples of almost $C_0(K)$-C*-algebras.

1. Introduction and notations

Let $\mathcal{A}$ be a C*-algebra and $\hat{\mathcal{A}}$ be its unitary spectrum. The C*-algebra $l^\infty(\hat{\mathcal{A}})$ of all bounded operator fields defined over $\hat{\mathcal{A}}$ is given by

$$l^\infty(\hat{\mathcal{A}}) := \{ A = (A(\pi) \in B(\mathcal{H}_\pi))_{\pi \in \hat{\mathcal{A}}}; \|A\|_{\infty} := \sup_{\pi} \|A(\pi)\|_{\text{op}} < \infty \},$$

where $\mathcal{H}_\pi$ is the Hilbert space on which $\pi$ acts. Let $\mathcal{F}$ be the Fourier transform of $\mathcal{A}$, i.e.,

$$\mathcal{F}(a) := \hat{a} := (\pi(a))_{\pi \in \hat{\mathcal{A}}} \quad \text{for} \quad a \in \mathcal{A}.$$ 

It is an injective, hence isometric, homomorphism from $\mathcal{A}$ into $l^\infty(\hat{\mathcal{A}})$. Hence one can analyze the C*-algebra $\mathcal{A}$ by recognizing the elements of $\mathcal{F}(\mathcal{A})$ inside the (big) C*-algebra $l^\infty(\hat{\mathcal{A}})$.

We know that the unitary spectrum $C^*(G)$ of the C*-algebra $C^*(G)$ of a locally compact group $G$ can be identified with the unitary dual $\hat{G}$ of $G$. If $G$ is an exponential Lie group, i.e., if the exponential mapping $\exp : \mathfrak{g} \rightarrow G$ from the Lie algebra $\mathfrak{g}$ to its Lie group $G$ is a diffeomorphism, then the Kirillov-Bernat-Vergne-Pukanszky-Ludwig-Leptin theory shows that there is a canonical homeomorphism $K : \mathfrak{g}^*/G \rightarrow \hat{G}$ from the space of coadjoint orbits of $G$ in the linear dual space $\mathfrak{g}^*$ onto the unitary dual space $\hat{G}$ of $G$ (see [Lep-Lud] for details and references). In this case, one can therefore identify the unitary spectrum $C^*(G)$ of the C*-algebra of an exponential Lie group with the space $\mathfrak{g}^*/G$ of coadjoint orbits of the group $G$.

The C*-algebra of an $ax + b$-like group was characterised in [Lin-Lud] and the C*-algebras of the Heisenberg group and of the threadlike groups were described in [Lu-Tu] as algebras of operator fields defined on the dual spaces of the groups. The method of describing group C*-algebras as algebras of operator fields defined on the dual spaces was first used in [Fell] and [Lee].

In this paper, we consider the exponential solvable Lie group $G_{n,\mu}$, whose Lie algebra is an extension of the Heisenberg Lie algebra $\mathfrak{h}_n$ by the reals, which means that $\mathbb{R}$ acts on $\mathfrak{h}_n$ by a diagonal matrix with real eigenvalues. The quotient group of $G_{n,\mu}$ by the centre of the Heisenberg group is then an $ax + b$-like group, whose C*-algebra has been determined in [Lin-Lud]. Since the orbit structure of exponential groups is well understood (see for instance [Ar-Lu-Sc]), we can write down the spectrum of the group $G_{n,\mu}$ explicitly and determine its topology.

In [ILL] the example of the group $N_{6,28}$ motivated the introduction of a special class of C*-algebras which we called almost $C_0(K)$-C*-algebra, where $K$ is the algebra of all compact operators on some Hilbert space. In Section 2, we recall the definition and the properties of almost $C_0(K)$-C*-algebras. In Section 3 we introduce the family of the $G_{n,\mu}$ groups and describe the space of coadjoint orbits $\mathfrak{g}^*_{n,\mu}/G_{n,\mu}$. We show that the spectrum $\hat{G}_{n,\mu}$ of $G_{n,\mu}$ is a disjoint union of the sets $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$, where $\Gamma_0$ is the set of the characters of $G_{n,\mu}$, $\Gamma_1$ and $\Gamma_2$ are the sets of the representations corresponding to the two-dimensional coadjoint orbits of $G_{n,\mu}$, and $\Gamma_3$ is the

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union of the two generic irreducible representations \( \pi_+, \pi_- \) which correspond to the two open orbits. Note that each of the sets \( \Gamma_i \) needs a special treatment. The sets \( \Gamma_1 \) and \( \Gamma_2 \) have been treated in the paper [Lin-Lud]. In Subsection 4.2, we discover the almost \( C_0(K) \) conditions for \( \Gamma_3 \). This is the most intricate part of the paper and the treatment is inspired by the study of the boundary condition for a class of 4-dimensional orbits in [ILL, Subsection 6.3]. At the end (Subsection 4.4), we describe the actual \( C^* \)-algebra of \( G_{n, \mu} \) as an algebra of operator fields and we see that this \( C^* \)-algebra has the structure of an almost \( C_0(K) \)-\( C^* \)-algebra.

2. Almost \( C_0(K) \)-\( C^* \)-algebras

The following definitions were given in [ILL]; for completeness, we recall them here.

**Definition 2.1.** Let \( A \) be a \( C^* \)-algebra and \( \hat{\mathcal{A}} \) be the spectrum of \( A \).

1. Suppose there exists a finite increasing family \( S_0 \subset S_1 \subset \ldots \subset S_d = \hat{\mathcal{A}} \) of subsets of \( \hat{\mathcal{A}} \) such that for \( i = 1, \ldots, d \), the subsets \( \Gamma_0 = S_0 \) and \( \Gamma_i := S_i \setminus S_{i-1} \) are Hausdorff in their relative topologies. Furthermore we assume that for every \( i \in \{0, \ldots, d\} \) there exists a Hilbert space \( H_i \) and a concrete realization \( (\pi_\gamma, H_i) \) of \( \gamma \) on the Hilbert space \( H_i \) for every \( \gamma \in \Gamma_i \). Note that the set \( S_0 \) is the collection \( \mathcal{X} \) of all characters of \( A \).
2. For a subset \( S \subset \hat{\mathcal{A}} \), denote by \( CB(S) \) the *-algebra of all uniformly bounded operator fields \( (\psi(\gamma) \in \mathcal{B}(H_i))_{\gamma \in S \cap \Gamma_i, i = 1, \ldots, d} \), which are operator norm continuous on the subsets \( \Gamma_i \cap S \) for every \( i \in \{1, \ldots, d\} \) for which \( \Gamma_i \cap S \neq \emptyset \). We provide the *-algebra \( CB(S) \) with the infinity-norm:

\[
\|\psi\|_S := \sup_{\gamma \in S} \|\psi(\gamma)\|_{op}.
\]

**Definition 2.2.** Let \( \mathcal{H} \) be a Hilbert space and \( K := \mathcal{K}(\mathcal{H}) \) be the algebra of all compact operators defined on \( \mathcal{H} \). A \( C^* \)-algebra \( \hat{\mathcal{A}} \) is said to be almost \( C_0(K) \) if for every \( a \in A \):

1. The mappings \( \gamma \mapsto \mathcal{F}(a)(\gamma) \) are norm continuous on the different sets \( \Gamma_i \), where \( \mathcal{F} : A \to \ell^\infty(\hat{\mathcal{A}}) \) is the Fourier transform given by

\[
\mathcal{F}(a)(\gamma) = \hat{\pi}_\gamma(a) \quad \text{for} \quad \gamma \in \hat{\mathcal{A}} \quad \text{and} \quad a \in A.
\]

2. For each \( i = 1, \ldots, d \), we have a sequence \( (\sigma_{i,k} : CB(S_{i-1}) \to CB(S_i))_k \) of linear mappings which are uniformly bounded in \( k \) (and independent of \( a \)) such that

\[
\lim_{k \to \infty} \text{dis} \left( (\sigma_{i,k}(\mathcal{F}(a)|_{S_{i-1}}) - \mathcal{F}(a)|_{\Gamma_i}), C_0(\Gamma_i, \mathcal{K}(H_i)) \right) = 0,
\]

and

\[
\lim_{k \to \infty} \text{dis} \left( (\sigma_{i,k}(\mathcal{F}(a)^*|_{S_{i-1}}) - (\mathcal{F}(a)^*)|_{\Gamma_i}), C_0(\Gamma_i, \mathcal{K}(H_i)) \right) = 0,
\]

where \( C_0(\Gamma_i, \mathcal{K}(H_i)) \) is the space of all continuous mappings \( \varphi : \Gamma_i \to \mathcal{K}(H_i) \) vanishing at infinity.

**Definition 2.3.** Let \( D^*(A) \) be the set of all operator fields \( \varphi \) defined over \( \hat{\mathcal{A}} \) such that

1. The field \( \varphi \) is uniformly bounded, i.e., we have that \( \|\varphi\| := \sup_{\gamma \in \hat{\mathcal{A}}} \|\varphi(\gamma)\|_{op} < \infty \).
2. \( \varphi|_{\Gamma_i} \in CB(\Gamma_i) \) for every \( i = 0, 1, \ldots, d \).
3. For every sequence \( (\gamma_k)_{k \in \mathbb{N}} \) going to infinity in \( \hat{\mathcal{A}} \), we have that \( \lim_{k \to \infty} \|\varphi(\gamma_k)\|_{op} = 0 \).
4. For each \( i = 1, 2, \ldots, d \),

\[
\lim_{k \to \infty} \text{dis} \left( (\sigma_{i,k}(\varphi|_{S_{i-1}}) - \varphi|_{\Gamma_i}), C_0(\Gamma_i, \mathcal{K}(H_i)) \right) = 0
\]

and

\[
\lim_{k \to \infty} \text{dis} \left( (\sigma_{i,k}(\varphi^*|_{S_{i-1}}) - (\varphi|_{\Gamma_i})^*), C_0(\Gamma_i, \mathcal{K}(H_i)) \right) = 0.
\]
We see immediately that if $A$ is almost $C_0(K)$, then for every $a \in A$, the operator field $F(a)$ is contained in the set $D^*(A)$. In fact it turns out that $D^*(A)$ is a $C^*$-subalgebra of $l_\infty(A)$ and that $A$ is isomorphic to $D^*(A)$.

**Theorem 2.4. (ILL, Theorem 2.6)** Let $A$ be a separable $C^*$-algebra which is almost $C_0(K)$. Then the subset $D^*(A)$ of the $C^*$-algebra $l_\infty(A)$ is a $C^*$-subalgebra which is isomorphic to $A$ under the Fourier transform.

3. The groups $G_{n,\mu}$

Let $n \in \mathbb{N}^*$, $V_n = \mathbb{R}^{2n}$ and denote by $\omega_n$ the canonical non-degenerate skew-symmetric bilinear form on $V_n$. Let

$$\mathfrak{h}_n := V_n \oplus \mathbb{R}.$$ 

Choose a symplectic basis $B := \{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ of $V_n$. Let

$$\mathfrak{g}_{n,\mu} := \mathbb{R} \times \mathfrak{h}_n$$

and $A = (1, 0_{V_n}, 0), Z = (0, 0_{V_n}, 1) \in \mathfrak{g}_{n,\mu}$. Then $\{A, X_1, \cdots, X_n, Y_1, \cdots, Y_n, Z\}$ is a basis of $\mathfrak{g}_{n,\mu}$. For

$$\mu := \{\lambda_1, \lambda'_1, \cdots, \lambda_n, \lambda'_n\} \subset \mathbb{R}$$

with $\lambda_i + \lambda'_i = 2$ for all $i = 1, \cdots, n$, we define the brackets

$$[A, X_i] = \lambda_i X_i, [A, Y_i] = \lambda'_i Y_i, [A, Z] = 2Z$$

for all $i = 1, \cdots, n$, and

$$[X_i, Y_j] = \delta_{i,j} Z \quad \text{for} \quad i, j = 1, \cdots, n.$$ 

Eventually by exchanging $X_j$ and $Y_j$ and replacing $X_j$ by $-X_j$ we can assume that $\lambda'_j \geq 0$ for all $j$. We then obtain a structure of an exponential solvable Lie algebra on $\mathfrak{g}_{n,\mu}$, and its subalgebra $\mathfrak{h}_n$ is the Heisenberg Lie algebra.

Define the diagonal operator $l_\mu : V_n \rightarrow V_n$ by

$$l_\mu (v) := \sum_{i=1}^n \lambda_i v_i X_i + \lambda'_i v'_i Y_i \quad \text{for} \quad v = \sum_{i=1}^n v_i X_i + \sum_{i=1}^n v'_i Y_i \in V_n.$$ 

For $v = \sum_{i=1}^n v_i X_i + v'_i Y_i \in V_n$ and $a \in \mathbb{R}$, we write

$$a \cdot v := \sum_{i=1}^n e^{a\lambda_i} v_i X_i + e^{a\lambda'_i} v'_i Y_i.$$ 

The corresponding simply connected Lie group $G_{n,\mu}$, which is exponential solvable, can be identified with the space $\mathbb{R} \times V_n \times \mathbb{R}$ equipped with the multiplication

$$(3.0.1) \quad (a, v, c) \cdot (a', v', c') := (a + a', -(a') \cdot v + v', e^{-2a'} c + c' + 2\omega_n((-a') \cdot v, v')).$$ 

The inner automorphism $\text{Ad} (a, u)$ on $\mathfrak{h}_n$ is given by

$$\text{Ad} (a, u)(0, v, z) = (a, u, 0)(0, v, z)(-a, -(a \cdot u), 0) = (a, 0, 0)(0, u, 0)(0, v, z)(0, -a, 0)(-a, 0, 0) = (a, 0, 0)(0, v, z + \omega_n(u, v)(-a, 0, 0) = (0, a \cdot v, e^{2a} z + e^{2a} \omega_n(u, v)) \quad \text{for} \quad (v, z) \in \mathfrak{h}_n.$$ 

The centre $Z$ of the normal subgroup $H_n := \{0\} \times V_n \times \mathbb{R}$ of $G_{n,\mu}$ is the subset $Z = \exp(\mathbb{R}Z) = \{0\} \times \{0_{V_n}\} \times \mathbb{R}$. Denote by $G_{n,\mu \setminus Z}$ the quotient group $G_{n,\mu} / Z$ which can be identified with $\mathbb{R} \times V_n$ equipped with the multiplication

$$(s, v) \cdot (t, w) := (s + t, (-t) \cdot v + w).$$
We write $V_n = V_0 \oplus V_+ \oplus V_-$ where

$V_+ := \text{span}\{X_j, Y_k; \lambda_j > 0, \lambda_k > 0\},$

$V_- := \text{span}\{X_j; \lambda_j < 0\},$

$V_0 := \text{span}\{X_j, Y_k; \lambda_j = 0, \lambda_k = 0\},$

and $V_1 := V_+ \oplus V_-$. Let

$\mu_+ := \mu \cap \mathbb{R}^+_*,$

$\mu_- := \mu \cap \mathbb{R}^-_*,$

$\mu_0 := \mu \cap \{0\},$

then we can write

$V_+ = \sum_{\lambda \in \mu_+} V_{+\lambda}$ and $V_- = \sum_{\lambda \in \mu_-} V_{-\lambda},$

where $V_{+\lambda}$ and $V_{-\lambda}$ are the respective eigenspaces of the operator $I_\mu$.

We can also identify $g^*_n, \mu$ with $\mathbb{R}A^* \oplus V^*_0 \oplus \mathbb{R}Z^* \cong \mathbb{R} \times V_0 \times \mathbb{R}$, and then

$\langle \text{Ad}^*(a, u)(a^*, v^*, \lambda^*), (0, v, z) \rangle = \langle (a^*, v^*, \lambda^*), \text{Ad}((a, u)^{-1})(0, v, z) \rangle$

$= \langle (a^*, v^*, \lambda^*), (0, (-a) \cdot v, e^{-2\theta}z + e^{-2\theta}\omega_n(-a \cdot u, v)) \rangle$

$= (0, v^*, (-a) \cdot v) + \lambda^* e^{-2\theta}z + \lambda^* e^{-2\theta}\omega_n(-a \cdot u, v).$

Hence

$\text{Ad}^*(a, u)(a^*, v^*, \lambda^*)|_{\mathcal{B}_n} = (a^*, (-a) \cdot v^* - \lambda^* e^{-2\theta}(a \cdot u) \times \omega_n, \lambda^* e^{-2\theta}).$

Here we denote by $u \times \omega_n$ the linear functional on $V_0$ as

$u \times \omega_n(v) := \omega_n(u, v)$ for all $v \in V_0$.

The coadjoint orbit $\Omega_\ell$ of an element $\ell = (a^*, v^*, \lambda^*) \in g^*_n, \mu$ is given by

$\Omega_\ell = \{ (a^* + v^*([A, u]) + 2z\lambda^*, (-a) \cdot v^* - \lambda^* e^{-2\theta}(a \cdot u) \times \omega_n, \lambda^* e^{-2\theta}) : a, z \in \mathbb{R}, u \in V_0 \}.$

Hence if $\lambda^* \neq 0$ then the corresponding coadjoint orbit is the subset

$\Omega_{\lambda^*} = \mathbb{R} \times V^*_0 \times \mathbb{R}^+_\lambda^*,$

where $V^*_0$ is the linear dual space of $V_0$. Therefore we have two open coadjoint orbits

$\Omega_{\epsilon} := \text{Ad}^*(G_{n, \epsilon})\ell = \mathbb{R} \times V^*_0 \times \mathbb{R}^+_{\epsilon}$ for $\epsilon \in \{+, -\}$,

where $\ell_{\epsilon} = \epsilon Z^*$. The other orbits are contained in $Z^+$ with the form

$\Omega_{\epsilon^*} = \mathbb{R}A^* + \mathbb{R} \cdot v^*$ for $v^* \in V^*_0 \setminus V^*_0$,

or the one point orbits

$\{ a^* A^* + v^* \} \quad \text{for} \quad a^* \in \mathbb{R}, v^* \in V^*_0.$

We can decompose the linear dual space $V^*_0$ of $V_0$ into

$V^*_+ := \{ f \in V^*_n : f(V_- \cup V_0) = \{0\}\}$,

$V^*_- := \{ f \in V^*_n : f(V_+ \cup V_0) = \{0\}\}$,

$V^*_0 := \{ f \in V^*_n : f(V_+ \cup V_-) = \{0\}\}$.

The following definition was given in [Lin-Lud2].

**Definition 3.1.** Denote by $\| \cdot \|$ the norm on $V^*_0$ coming from the scalar product defined by the basis $\{ X_1, \ldots, X_n, Y_1, \ldots, Y_n \}$. For $f_+ = \sum_{\lambda \in \mu_+} f_\lambda \in V^*_+$ and $f_- = \sum_{\lambda \in \mu_-} f_\lambda \in V^*_-$, let

$|f_+|_\mu = |f_+| := \max_{\lambda \in \mu_+} \| f_\lambda \|^{1/\lambda}$ and $|f_-|_\mu = |f_-| := \max_{\lambda \in \mu_-} \| f_\lambda \|^{1/\lambda}.$

Then for $t \in \mathbb{R}$, we have the relation

$|t \cdot f_+| = e^{t}|f_+|$ and $|t \cdot f_-| = e^{-t}|f_-|$ for $f_+ \in V^*_+, f_- \in V^*_-. $

On $V^*_0$ we shall use the norm coming from the scalar product. This gives us a global gauge on $V^*_0$:

$\langle |f_0, f_+, f_-| \rangle := \max\{ \| f_0 \|, \| f_+ \|, \| f_- \| \}.$
We denote by $V^*_{gen}$ the open subset of $V^*_0$ consisting of all the $f = (f_0, f_+, f_-) \in V^*_0 \times V^*_+ \times V^*_-$ for which $f_+ \neq 0$ and $f_- \neq 0$. The subset $V^*_{sin}$ consists of all the $f = (f_0, f_+, f_-)$ for which either $f_+ \neq 0, f_- = 0$ or $f_+ = 0, f_- \neq 0$. We see that for every $f = (f_0, f_+, f_-) \in V^*_{gen}$ there exists exactly one element $f' = (f'_0, f'_+, f'_-) \in V^*_0$ in its $G_{n, \mu}$-orbit such that $|f'_+| = |f'_-|$. In the same way, for $f = (f_0, f_+, 0)$ (resp. $f = (f_0, 0, f_-)$) $\in V^*_{sin}$, there exists exactly one element $f' = (f'_0, f'_+, 0)$ (resp. $f' = (f_0, 0, f'_-)$) in its $G_{n, \mu}$-orbit for which $|f'_+| = 1$ (resp. $|f'_-| = 1$).

For $f_+ \in V^*_+ \setminus \{0\}$, let us denote by $r(f_+)$ the unique real number for which the vector $r(f_+) \cdot f_+$ in $V^*_+$ has gauge 1. This means that
\[ r(f_+) := -\ln(|f_+|). \]
Similarly, for $f_- \in V^*_- \setminus \{0\}$ we define the number $q(f_-)$ by
\[ q(f_-) := \ln(|f_-|) \]
such that $|q(f_-) \cdot f_-| = 1$. Let
\[
D = \{(f_0, f_+, f_-) : |f_+| = |f_-| \neq 0\},
\]
\[
S_+ = \{(f_0, f_+, 0) : |f_+| = 1\}, S_- = \{(f_0, 0, f_-) : |f_-| = 1\}, \text{ and}
\]
\[
S = S_+ \cup S_-.
\]
The orbit space $\mathfrak{g}_{n, \mu}/G_{n, \mu}$ can then be written as the disjoint union $\Gamma$ of the sets
\[
\begin{align*}
\Gamma_0 &= \mathbb{R} \times V^*_0, \text{ corresponding to the unitary characters of } G_{n, \mu}, \\
\Gamma_1 &= S \simeq V^*_{sin}/G_{n, \mu}, \\
\Gamma_2 &= D \simeq V^*_{gen}/G_{n, \mu}, \\
\Gamma_3 &= \{+, -\} \simeq \{\Omega_+, \Omega_-\}/G_{n, \mu},
\end{align*}
\]
in the case where $V^*_{gen} \neq \emptyset$, i.e., $\mu_+ \neq \emptyset$ and $\mu_- \neq \emptyset$. In case $V^*_{gen} = \emptyset$, we have $\Gamma$ as the union of
\[
\begin{align*}
\Gamma_0 &= \mathbb{R} \times V^*_0, \text{ corresponding to the unitary characters of } G_{n, \mu}, \\
\Gamma_1 &= S \simeq V^*_+/G_{n, \mu}, \\
\Gamma_2 &= \{+, -\} \simeq \{\Omega_+, \Omega_-\}/G_{n, \mu}.
\end{align*}
\]
In order to simplify notations, we shall treat only the first case in the following, i.e., we shall assume that $V^*_{gen}$ is nonempty. The other case is similar and easier.

The topology of the orbit space $\mathfrak{g}_{V_0}/G_{V_0}$ of the quotient group $G_{n, \mu}/\mathbb{Z}$ has been described in [Lin-Lud]. We recall that a sequence $y = (y_k)_k$ is called properly converging if $y$ has limit points and if every cluster point of the sequence is a limit point, i.e., the set of limit points of any subsequence is always the same, indeed, it equals to the set of all limit points of the sequence $y$.

**Theorem 3.2.** ([Lin-Lud, Theorem 2.3])

1. A properly converging sequence $(\Omega_{f_k})_k$ with $f_k = (f_{k,0}, f_{k,+}, f_{k,-}) \in D$ has either a unique limit point $\Omega_f$ for some $f \in D$ and then $f = \lim_k f_k$, or $\lim_k (f_{k,+}, f_{k,-}) = 0$ and then the limit set $L$ of the sequence is given by
\[
L = \{\Omega_{(f_0, f_+, 0)}, \Omega_{(f_0, 0, f_-)}, \mathbb{R}\},
\]
where $f_0 = \lim_k f_{k,0}, f_+ = \lim_k r(f_{k,+}) \cdot f_{k,+} \in S_+$ and $f_- = \lim_k q(f_{k,-}) \cdot f_{k,-} \in S_-.$

2. A properly converging sequence $(\Omega_{f_k})_k$ with $f_k = (f_{k,0}, f_{k,+}, f_{k,-}) \in S$ has the limit set
\[
L = \{\Omega_f, \mathbb{R}\},
\]
where $f = \lim_k f_k \in S$.

**Corollary 3.3.** The orbit $\Omega_f$ for $f \in D$ is closed in $\mathfrak{g}_{n, \mu}$. The closure of the orbit $\Omega_f$ for $f \in S$ is the set $\{\Omega_f, \mathbb{R}\}$.

From the description (3.0.2) of the open orbits $\Omega_{\varepsilon}, \varepsilon = \pm$, we have the boundary of $\Omega_{\varepsilon}$ as the following.

**Corollary 3.4.** For $\varepsilon \in \{+, -, \}$, the boundary of the open orbit $\Omega_{\varepsilon}$ is the subset $\mathbb{R} \times V^*_0 \times \{0\} = Z^1 \simeq \mathfrak{g}_{V_0}$.
On the other hand, for every coadjoint orbit we can write down a corresponding irreducible representation as an induced representation by using Kirillov’s orbit theory.

(1) Let \( P_n = \exp(\sum_{j=1}^{n} \mathbb{R}Y_j + \mathbb{R}Z) \). This is a closed connected normal abelian subgroup of \( G_{n,\mu} \). Let also \( \mathfrak{p}_n := \sum_{j=1}^{n} \mathbb{R}X_j \) and \( \mathfrak{n}_n := \sum_{j=1}^{n} \mathbb{R}Y_j \subset \mathfrak{a}_{n,\mu} \) (an abelian subalgebra of \( \mathfrak{g}_{n,\mu} \)), then \( \mathcal{X}_n := \exp(\mathfrak{p}_n) \) and \( \mathcal{Y}_n = \exp(\mathfrak{n}_n) \) are closed connected abelian subgroups of \( G_{n,\mu} \). We have

\[
G_{n,\mu} = \exp(\mathbb{R}A) \cdot \mathcal{X}_n \cdot P_n = \mathcal{S}_n \cdot P_n,
\]

where \( \mathcal{S}_n := \exp(\mathbb{R}A) \cdot \mathcal{X}_n \) is a subgroup of \( G_{n,\mu} \). The irreducible representations \( \pi_{\varepsilon, \zeta} \), corresponding to the orbits \( \Omega_{\varepsilon, \zeta} \) are of the form

\[
\pi_{\varepsilon} := \text{ind}_{\mathcal{P}_n}^{G_{n,\mu}} \chi \xi \mathfrak{z} \mathfrak{z}. \]

The Hilbert space of \( \pi_{\varepsilon} \) is the \( \mathbb{D}^2 \)-space \( \mathbb{D}^2(G_{n,\mu}/\mathcal{P}_n, \chi \xi) \simeq \mathbb{D}^2(\mathcal{S}_n) \), where \( \chi \xi(y, z) := e^{-12\pi \varepsilon z} \) for \( (y, z) \in \mathcal{P}_n \). The elements of this space are the measurable functions \( \xi : G_{n,\mu} \rightarrow \mathbb{C} \) satisfying the relations

\[
\xi(gp) = \chi \xi(p^{-1})\xi(g) \text{ for } g \in G_{n,\mu}, p \in P_n, \text{ and }
\int_{G_{n,\mu}/\mathcal{P}_n} |\xi(g)|^2 d\mathcal{g} < \infty,
\]

where \( d\mathcal{g} \) is the left invariant measure on \( G_{n,\mu}/\mathcal{P}_n \). For \( F \in L^1(G_{n,\mu}) \) and \( \xi \in L^2(G_{n,\mu}/\mathcal{P}_n) \), we have

\[
\pi_{\varepsilon}(F)\xi(s') = \int_{\mathcal{S}_n \cdot \mathcal{P}_n} F(sp)\xi(p^{-1}s^{-1}s')d\mathcal{p}
\]

where

\[
F := \int_{\mathbb{S}_n \cdot \mathcal{P}_n} \int_{\mathcal{S}_n \cdot \mathcal{P}_n} F(sp)\xi(p^{-1}s^{-1}s')d\mathcal{p}
\]

Here \( \hat{F}^{\varepsilon}_{\mathfrak{p}_n} \) is the partial Fourier transform of \( F \) in the direction \( \mathfrak{p}_n \) given by

\[
\hat{F}^{\varepsilon}_{\mathfrak{p}_n}(s; \ell) := \int_{\mathcal{P}_n} F(sp) e^{-2\pi i (\ell, \log(p))} \; dp \quad \text{for } s \in \mathbb{S}_n, \ell \in \mathfrak{p}_n^*.
\]

Hence the operator \( \pi_{\varepsilon}(F) \) is given by the kernel function

\[
F_{\varepsilon}((a', x'), (a, x)) = \hat{F}^{\varepsilon}_{\mathfrak{p}_n}(a' - a, a \cdot (x' - x) ; (-\varepsilon e^{-2\alpha} (a \cdot x) \times \omega_n, \varepsilon e^{-2\alpha}) | \ell | \alpha),
\]

where \( |\ell| := \sum_{j=1}^{n} \lambda_j \). In fact the linear functional \( \varepsilon e^{-2\alpha} (a \cdot x) \times \omega_n \) is given by

\[
\varepsilon e^{-2\alpha} (a \cdot x) \times \omega_n = \varepsilon \left( \sum_{j=1}^{n} e^{(\lambda_j - 2\alpha)x_j} Y_j^* \right) \text{ for } a \in \mathbb{R}, x \in \mathcal{X}_n.
\]

Therefore,

\[
F_{\varepsilon}((a', x'), (a, x)) = \hat{F}^{\varepsilon}_{\mathfrak{p}_n}(a' - a, a \cdot (x' - x) ; (-\varepsilon \left( \sum_{j=1}^{n} e^{(\lambda_j - 2\alpha)x_j} Y_j^* \right), \varepsilon e^{-2\alpha}) | \ell | \alpha).
\]
For $v^* \in V_0^*$, we have the irreducible representation $\pi_{v^*}$ on $L^2(\mathbb{R})$ defined by

$$\pi_{v^*} := \text{ind}^{G_{n,\mu}}_{H_n} \chi_{v^*},$$

where $H_n := \exp(h_n)$. The kernel function $F_{v^*}$ of the operator $\pi_{v^*}(F)$, $F \in L^1(G_{n,\mu})$, is given by

$$F_{v^*}(a, b) = \hat{F}_{n,\mu}(a - b, b \cdot v^*, 0) \quad \text{for} \quad a, b \in \mathbb{R}.$$  

(3.0.4) Finally, for $(a^*, v_0^*) \in \mathbb{R} \times V_0^*$ we have the unitary characters

$$\chi_{(a^*, v_0^*)}(a, r, v, c) := e^{-2\pi i \langle a^* a + v_0^*(v) \rangle} \quad \text{for} \quad a, c \in \mathbb{R}, r, v \in V_0, v \in V_1.$$

**Definition 3.5.** We denote by $l^\infty(\Gamma)$ the C*-algebra

$$l^\infty(\Gamma) = \{(\phi, \gamma) \in B(H_\gamma)_\gamma \in \Gamma; \|\phi\| := \sup_{\gamma \in \Gamma} \|\phi(\gamma)\|_{op} < \infty\}.$$ 

The Fourier transform $F_{n,\mu} : C^*(G_{n,\mu}) \to l^\infty(\Gamma)$ for $C^*(G_{n,\mu})$ is given by

- $F_{n,\mu}(a)(\varepsilon) := \pi_\varepsilon(a)$ for $\varepsilon \in \{+,-\}$,
- $F_{n,\mu}(a)(f) := \pi_f(a)$ for $f \in D \cup S$,
- $F_{n,\mu}(a^*, v_0^*) := \chi_{(a^*, v_0^*)}(a)$ for $(a^*, v_0^*) \in \mathbb{R} \times V_0^*$,

$$\langle F(s, v_0, v_1, z)e^{-2\pi i a^* s}e^{-2\pi i v_0^*(v) ds}dv_0dv_1dz \quad \text{for} \quad F \in L^1(G_{n,\mu}) \rangle.$$ 

4. **The C*-conditions**

4.1. **The continuity and infinity conditions.**

**Theorem 4.1.** For every $a \in C^*(G_{n,\mu})$, the mapping

$$S \cup D \ni B(L^2(\mathbb{R})): f \mapsto \hat{a}(f),$$

is norm continuous. We also have that

$$\lim_{\|f\| \to \infty} \|\pi_f(a)\|_{op} = 0$$

**Proof.** See [Lin-Lud, Proposition 4.2].

4.2. **The condition for the open orbits $\Omega_x$.** To understand the case of open orbits, we have to take into account the boundary points of such an orbit. It is well known that for $a \in C^*(G)$ the operator $\pi_\varepsilon(a)$ is compact if and only if $\pi(a) = 0$ for every $\pi$ in the boundary of the representation $\pi_\varepsilon$, i.e., if $\pi(a) = 0$ for every $\gamma \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$. In this subsection we shall give a description of the algebra of operators $\pi_\varepsilon(C^*(G_{n,\mu}))$.

**Definition 4.2.** For $k \in \mathbb{Z}$ and $r \in \mathbb{R}$, let $I_{r,k}$ be the half-open interval:

$$I_{r,k} := [kr, kr + r[ \subset \mathbb{R}.$$ 

(1) Let $S_{\delta,1} := \{(a, x) \in \mathbb{R} \times \mathcal{X}_\alpha; e^{-a} > \delta^3\}$.

(2) Let $\delta \mapsto r_\delta \in \mathbb{R}_+$ be such that $\lim_{\delta \to 0} r_\delta = +\infty$ and $\lim_{\delta \to 0} e^{m r_\delta} \delta^{1/2} = 0$, where $1 \leq m := \max_j (2 - \lambda_j)$.

(3) For constants $D = (D_1, \ldots, D_n) \in (\mathbb{R}_+^*)^n$ and $\underline{c} = (k_0, k_1, \ldots, k_n) \in \mathbb{Z}^{n+1}$ let

$$S_{\delta,D,\underline{c},2} := \{(a, x_1, \ldots, x_n) \in \mathbb{R} \times \mathcal{X}_\alpha; e^{-a} \leq \delta^3, a \in I_{r_\delta,k_0}, x_j \in I_{D_je^{-r_j(2-\lambda_j)}k_0,k_j}, j = 1, \ldots, n\}.$$

**Proposition 4.3.** For every compact subset $K \subseteq \mathbb{R} \times \mathcal{X}_\alpha$ and $\delta > 0$ small enough, we have that

$$KS_{\delta,D,\underline{c},2} \subseteq \bigcup_{\underline{c} \in \delta,\underline{c},2} S_{\delta,D,\underline{c},2} =: R_{\delta,D,\underline{c},2},$$

where $D_{\delta,\underline{c}} = (D_1 e^{-r_j(2-\lambda_j)(k_0)}, \ldots, D_n e^{-r_j(2-\lambda_j)(k_0)}) \in (\mathbb{R}_+^*)^n$. 


Proof. Indeed, there is an $M > 0$ such that $K \subset [-M, M]^{n+1} \subset \mathbb{R}^{n+1}$. Let $r_\delta > M$. For $(s, u) \in K$ and $(a, x) \in S_{\delta, D, \underline{k}}^2$, it follows that
\[
\zeta := (s, u) \cdot (a, x) = (s + a, (-a) \cdot u + x),
\]
and $(k_0 + j_0)r_\delta \leq s + a < (k_0 + j_0 + 1)r_\delta$ for some $k_0 \in \mathbb{Z}$ and $j_0 \in \{-1, 0, 1\}$. Furthermore
\[
|e^{-a\lambda_j}u_j| = |u_j|e^{-2a_\lambda_j(s-\lambda_j)a} \leq Me^{-2a_\lambda_j(s-\lambda_j)a_{k_0+1}} \leq D_\delta e^{-2\delta e|s-\lambda_j|(k_0+j_0)}(k_0+j_0),
\]
since for $\delta$ small enough
\[
Me^{-2a_\lambda_j(s-\lambda_j)a} \leq M\delta e\sup|a\lambda_j| < D_\delta^2 \text{ for every } j.
\]
Hence this operator has a kernel function given by
\[
x_j + e^{-a\lambda_j}u_j \leq (k_0 + j_0 + 1)D_\delta e^{-2\delta e|s-\lambda_j|(k_0+j_0)} + e^{-a\lambda_j}u_j,
\]
and also
\[
x_j + e^{-a\lambda_j}u_j \geq (k_0 + j_0)D_\delta e^{-2\delta e|s-\lambda_j|(k_0+j_0)} - e^{-a\lambda_j}u_j.
\]
Therefore $\zeta$ is contained in the set $R_{\delta, D, \underline{k}}^2$. \hfill $\square$

Remark 4.4.

(1) The family of sets \{\(S_{\delta, 1}, S_{\delta, D, \underline{k}}^2; \delta > 0, \underline{k} \in \mathbb{Z}^{n+1}\)\} forms a partition of $\mathbb{R}^{n+1}$.

(2) Denote by $M_{\delta, 1}$ the multiplication operator in \(L^2(\mathbb{R}^{n+1}) \simeq L^2(G_{n, \mu}/P_n, \chi_\varepsilon)\) with the characteristic function of the set $S_{\delta, 1}$. Similarly let $M_{\delta, D, \underline{k}}^2$ be the multiplication operator on \(L^2(G_{n, \mu}/P_n, \chi_\varepsilon)\) with the characteristic function of the set $S_{\delta, D, \underline{k}}^2$. These multiplication operators are pairwise disjoint orthogonal projections and the sum of them is the identity operator.

Let $N_{\delta, D, \underline{k}}^2$ be the multiplication operator with the characteristic function of the set $R_{\delta, D, \underline{k}}^2$ for $\delta > 0$ and $\underline{k} \in \mathbb{Z}^{n+1}$. We have the following property of the operator $N_{\delta, D, \underline{k}}^2$.

Proposition 4.5. There exists a constant $C > 0$ such that for any bounded linear operator $L$ on the Hilbert space $L^2(G_{n, \mu}/P_n, \chi_\varepsilon)$, we have that
\[
\| \sum_{\underline{k} \in \mathbb{Z}^{n+1}} N_{\delta, D, \underline{k}}^2 \circ L \circ M_{\delta, D, \underline{k}}^2 \|_{\text{op}} \leq C \sup_{\underline{k}} \| N_{\delta, D, \underline{k}}^2 \circ L \circ M_{\delta, D, \underline{k}}^2 \|_{\text{op}}.
\]

Proof. See Proposition 6.2 and 6.18 in [ILL]. \hfill $\square$

Definition 4.6. For $\underline{k} \in \mathbb{Z}^{n+1}$ and $\delta > 0$, let
\[
\ell_{\delta, \underline{k}} := -\varepsilon \sum_{j=1}^n D_\delta e\sup|\lambda_j|k_j Y_j^* \in \mathfrak{h}_n^*.
\]
Let $\sigma_{\delta, \underline{k}} := \text{ind}_{\delta P_n}^G \chi_{\ell_{\delta, \underline{k}}}$. The Hilbert space of this representation is the space
\[
\mathcal{H}_{\delta, \underline{k}} = L^2(G_{n, \mu}/P_n, \chi_{\ell_{\delta, \underline{k}}})
\]
and for $F \in L^1(G_{n, \mu})$, $\xi \in \mathcal{H}_{\delta, \underline{k}}$ we have that
\[
\sigma_{\delta, \underline{k}}(F)(\xi(a', x')) = \int_{\mathbb{S}} \hat{F}^n(s^{-1}) \Delta_{\delta}(s^{-1}) ds.
\]
Hence this operator has a kernel function given by
\[
F_{\delta, \underline{k}}(a', x', (a, x)) = \hat{F}^n(a' - a \cdot (x' - x); ((-a) \cdot \ell_{\delta, \underline{k}}, 0)) e^{\lambda a}.
\]
Moreover, the representation $\sigma_{\delta, \underline{k}}$ is equivalent to the representation
\[
\sigma_{n, \ell_{\delta, \underline{k}}} := \int_{P_n \subset V_n^*} \pi_f e^{\lambda a} df,
\]
and an equivalence is given by

\[
U_{n,t_2} : L^2(\mathbb{R} \times \mathcal{X}) \equiv L^2(G_{n,\mu}/P_\alpha, \chi_{f(t_2)}) \to \int_{p_\alpha} \mathcal{F} L^2(G_{n,\mu}/H_n, \chi_f) df
\]

(4.2.1)

\[
U_{n,t_2}(\xi)(g) := \int_{H_n/P_\alpha} \chi_f + t_2(h_n) \xi(g h_n) d\mu_n \text{ for } g \in G, f \in p_\alpha.
\]

Let \( C_{S\cup D} \) be the C*-algebra of all uniformly bounded continuous mappings from \( S \cup D \) into \( B(L^2(\mathbb{R})) \). It follows from Theorem 4.1 that for every \( a \in C^*(G_{n,\mu}) \) we have that \( \hat{a}_{S\cup D} \) is contained in \( C_{S\cup D} \).

For each \( f = (f_0, f_+, f_-) \in V_{\alpha}^\ast \), we denote by \( f_1 \) the unique element in its coadjoint orbit \( \Omega_f \) contained in \( S \cup D \). Let \( U_{n,\xi}(f) : L^2(G_{n,\mu}/H_n, \chi_f) \to L^2(G_{n,\mu}/H_n, \chi_f) \) be the canonical intertwining operator of \( \pi_1 \) and \( \pi_2 \). Formula (4.2.1) allows us to define a representation of the algebra \( C_{S\cup D} \) on the space \( L^2(\mathbb{R}) \) by

\[
\tau_{n,\xi}(a) := U_{n,\xi}(a) \circ \phi((a + t_2) \xi) \circ U_{n,\xi}(f) df \circ U_{n,\xi}(a).
\]

We have that

(4.2.2)

\[
\sigma_{n,\xi}(a) = \tau_{n,\xi}(f)(a) \text{ for all } a \in C^*(G_{n,\mu}).
\]

**Definition 4.7.** For \( \delta > 0, k \in \mathbb{Z}^{n+1} \) and \( a \in C^*(G_{n,\mu}) \), let

\[
\sigma_{n,k}(a) := \tau_{n,k}(a) \circ M_{k,D,k} \circ \sigma_{n,k}(a).
\]

**Proposition 4.8.** Let \( a \in C^*(G_{n,\mu}) \) and \( \epsilon \in \{+,-\} \). Then

\[
\lim_{\delta \to 0} \text{dis}(\pi_{\epsilon}(a) - \sigma_{n,\delta}(a)), K(L^2(\mathbb{R} \times \mathcal{X}))) = 0.
\]

**Proof.** Let \( L_{1}^1 \) be the space of all \( F \in L^1(G_{n,\mu}) \) for which the partial Fourier transform \( \hat{F}^\mu((a,x),(v^* , s)) \) is a \( C^\infty \)-function with compact support on \( S_n \times p_\alpha^* \). Take \( F \in L^1 \) and choose \( C > 0 \) such that \( \hat{F}^\mu((a,x),(v^* , s)) = 0 \), whenever \( |a| + |x| > C \) or \( ||v^*|| + |s| > C \). By Proposition 4.3, for \( \delta > 0 \) small enough, we have that

\[
\pi_{\epsilon}(F) = M_{k,D,\delta} \circ \sigma_{n,\delta}(F) \circ M_{k,D,\delta} \circ \sigma_{n,\delta}(F)
\]

for every \( k \) and hence

\[
\pi_{\epsilon}(F) \circ (I - M_{k,1}) - \sigma_{n,\delta}(F) = \pi_{\epsilon}(F) \circ \left( \sum_{k} M_{k,D,\delta} \right) - \sigma_{n,\delta}(F)
\]

\[
= \sum_{k \in \mathbb{Z}^{n+1}} N_{k,D,\delta} \circ \left( \pi_{\epsilon}(F) - \sigma_{n,k}(F) \right) \circ M_{k,D,\delta}.
\]

and the kernel function \( F_{k,\xi} \) of the operator \( a_{\xi,\delta} := N_{k,D,\delta} \circ \left( \pi_{\epsilon}(F) - \sigma_{n,k}(F) \right) \circ M_{k,D,\delta} \) is therefore given by

\[
F_{k,\xi}(a',x',(a,x)) = \left( \hat{F}^\mu(a' - a, a \cdot (x' - x); (-\epsilon \sum_{j=1}^n e^{(\lambda_j - 2)a_j}X_jY^*), \epsilon e^{-2a}) \right)
\]

\[
- \hat{F}^\mu(a' - a, a \cdot (x' - x); (-\epsilon(a), \xi, 0))
\]

\[
e^{\lambda_j a_1 S_{k,n}D}((a,x)1R_{k,n}D)(a',x') \text{ for } a, a' \in \mathbb{R}, x, x' \in V_n.
\]

We see that

\[
e^{(\lambda_j - 2)a_j}X_j - e^{-\lambda_j a_j}D_j \delta^2 e^{(\lambda_j - 2)a_j}k_0 k_j = e^{-\lambda_j a_j}(x_j - D_j \delta^2 e^{(\lambda_j - 2)a_j}k_0 k_j).
\]
Therefore,  

\[ |e^{(\lambda_j-2)\alpha}x_j - e^{-\lambda_j\alpha}D_j\delta^2 e^{r_s(2-\lambda_j)k_0k_j}| \]

\[ \leq e^{-\lambda_j\alpha}D_j\delta^2 e^{r_s(2-\lambda_j)k_0} \]

\[ = D_j\delta^2(e^{r_s(2-\lambda_j)k_0}k_0) - a) \]

\[ \leq e^{r_s(2-\lambda_j)}D_j\delta^2 \]

\[ \leq e^{r_{sc}m}D_j\delta^2 \]

\[ \leq \delta. \]

Since \( F \in L_c^1 \), there exists a continuous function \( \varphi : S_n \to \mathbb{R}^+ \) with compact support such that

\[ |\hat{F}_{F_n}(s; t) - \hat{F}_{F_n}(s; t')| \leq \varphi(s)\|t - t'\| \quad \text{for } t, t' \in \mathfrak{p}_n, s \in S_n. \]

Hence for any \((a, x), (a', x') \in S_n\) and any \(\delta > 0\) small enough,

\[ |F(a, x; \delta)\|_{op} \leq \varphi(a', x; \delta)\|_{op} \]

for some constant \( C > 0 \) independent of \( \delta \) by (4.2.3). Therefore by Young’s inequality we have that

\[ \|a_{F, \delta}\|_{op} \leq C\delta \quad \text{for } k \in \mathbb{Z}^{n+1}, \]

and finally

\[ \|\pi_{(F)}(I - M_{\delta, 1}) - \sigma_{n, \delta}(F)\|_{op} \leq C'\delta \]

for a new constant \( C' \), by Proposition 4.5.

On the other hand, the operator \( \pi_{(F)}(I - M_{\delta, 1}) \) is compact since

\[ \|\pi_{(F)}(I - M_{\delta, 1})\|_{H^{-}\delta} \]

\[ = \int_{\mathbb{R}^n} \left( |F(a' - a, a \cdot (x' - x), (x_j, x_j) = e^{-\lambda_j\alpha}D_j\delta^2 e^{r_s(2-\lambda_j)k_0k_j} | \right) \int_{\mathbb{R}^n} \left( |F(a' - a, a \cdot (x' - x), (x_j, x_j) = e^{-\lambda_j\alpha}D_j\delta^2 e^{r_s(2-\lambda_j)k_0k_j} | \right) \]

\[ \leq \int_{\mathbb{R}^n} \left( |F(a' - a, a \cdot (x' - x), (x_j, x_j) = e^{-\lambda_j\alpha}D_j\delta^2 e^{r_s(2-\lambda_j)k_0k_j} | \right) \int_{\mathbb{R}^n} \left( |F(a' - a, a \cdot (x' - x), (x_j, x_j) = e^{-\lambda_j\alpha}D_j\delta^2 e^{r_s(2-\lambda_j)k_0k_j} | \right) \]

\[ < \infty. \]

Therefore,

\[ \text{dis}(\pi_{\pi_{(F)}}(F) - \sigma_{n, \delta}(F), K(L^2(\mathbb{R} \times \mathcal{X}))) \]

\[ \leq \|\pi_{(F)}(I - M_{\delta, 1}) - \sigma_{n, \delta}(F)\|_{op} \]

\[ \to 0 \quad \text{as } \delta \to 0. \]

The Proposition follows, since \( L_c^1 \) is dense in \( C^*(G_{n, \mu}) \).
4.3. The two-dimensional orbits $\Omega_{\ast}$ and the characters. The \(C^\ast\)-algebras of the groups $G_{V_k} = G_{n,\mu}/\mathbb{Z}$ have been determined as algebras of operator fields in \cite{Lin-Lud}. We adapt this result to our present setting of almost $C_0(K)$-$C^\ast$-algebras.

**Definition 4.9.** For $a \in C^\ast(G_{n,\mu})$, let $\Phi(a)$ be the element of $C^\ast(\mathbb{R} \times V_0)$ defined by $\widehat{\Phi(a)}(\theta) := (\chi_a, a)$ for all $\theta \in \mathbb{R} \times V_0$. The mapping $\Phi : C^\ast(G_{n,\mu}) \to C^\ast(\mathbb{R} \times V_0)$ is a surjective homomorphism. Let the kernel of $\Phi$ be denoted by $I_K$, then $C^\ast(G_{n,\mu})/I_K \cong C^\ast(\mathbb{R} \times V_0)$. For $\eta \in C_c(G_{n,\mu})$, the element $\Phi(\eta) \in C^\ast(\mathbb{R} \times V_0)$ is the continuous function with compact support given by

$$\Phi(\eta)(t, v_0) = \int_{V_0 \times \mathbb{R}} \eta(t, v_0, v, s)dvds \quad \text{for} \quad t \in \mathbb{R}, v_0 \in V_0.$$

Choose $\zeta \in C_c(V_1 \times \mathbb{R})$ with $\zeta \geq 0$ and $\int_{V_1 \times \mathbb{R}} \zeta(v, s)dvds = 1$, define the mapping $\beta : C_c(\mathbb{R} \times V_0) \to C_c(G_{n,\mu}) \subset C^\ast(G_{n,\mu})$ by

$$\beta(\varphi)(a, v_0, s) = \varphi(a, v_0)\zeta(v, s) \quad \text{for} \quad \varphi \in C_c(\mathbb{R} \times V_0), s \in \mathbb{R} \text{ and } v \in V_1.$$

It has been shown in \cite{Lin-Lud} that $\beta$ can be extended to a linear mapping bounded by 1 from $C^\ast(\mathbb{R} \times V_0)$ into $C^\ast(G_{n,\mu})$, such that for every $\varphi \in C^\ast(\mathbb{R} \times V_0)$ we have $\Phi(\beta(\varphi)) = \varphi$.

**Definition 4.10.** Let $(\Omega_{\ast}, k) \in \mathcal{D}$ of all $k$ be a properly converging sequence in $\widehat{G_{n,\mu}}$, whose limit set contains the orbits $\Omega_{(f_0, 0)}$ and $\Omega_{(0, f_\ast)}$. Let $r_k, q_k \in \mathbb{R}$ be such that $|r_k \cdot f_k| = 1$ and $q_k \cdot f_{k-1} = 1$ for all $k \in \mathbb{N}$. Then $\lim_{k} r_k = -\infty$ and $\lim_{k} q_k = +\infty$. Choose two positive sequences $(\rho_k)_k, (\kappa_k)_k$ such that $\kappa_k > q_k, -r_k < \rho_k$ for all $k \in \mathbb{N}$, $\lim_{k} \kappa_k - q_k = \infty, \lim_{k} \kappa_k - r_k = \infty$ and $\lim_{k} \frac{\kappa_k - r_k}{q_k} = 0, \lim_{k} \frac{\rho_k - r_k}{q_k} = 0$. We say that the sequences $(\rho_k, \kappa_k)_k$ are adapted to the sequence $(f_k)_k$.

For $r \in \mathbb{R}$, let $U(r)$ be the unitary operator on $L^2(\mathbb{R})$ defined by

$$U(r)\xi(s) := \xi(s + r) \quad \text{for all} \quad \xi \in L^2(\mathbb{R}) \text{ and } s \in \mathbb{R}.$$

**Definition 4.11.** Let $A = (A(f) \in \mathcal{B}, f \in \Gamma)$ be a field of bounded operators. We say that $A$ satisfies the generic condition if for every properly converging sequence $(\pi_{f_k})_k \subset \widehat{G_{n,\mu}}$ with $f_k \in \mathcal{D}$ for every $k \in \mathbb{N}$, which admits limit points $\pi_{(f_0, 0, f_\ast)}, \pi_{(0, f_\ast, 0)}$ and for every pair of sequences $(\rho_k, \kappa_k)_k$ adapted to the sequence $(f_k)_k$ we have that

\begin{align}
(1) \quad &\lim_{k \to \infty} \|U(\rho_k) \circ A(f_k) \circ U(-\rho_k) \circ M_{(\rho_k, +\infty)} - A(f_0, f_+, 0) \circ M_{(\rho_k, +\infty)}\|_{\text{op}} = 0, \\
(2) \quad &\lim_{k \to \infty} \|U(q_k) \circ A(f_k) \circ U(-q_k) \circ M_{(-\infty, \kappa_k)} - A(f_0, 0, f_-) \circ M_{(-\infty, \kappa_k)}\|_{\text{op}} = 0.
\end{align}

The following proposition had been proved in \cite[Proposition 5.2]{Lin-Lud}.

**Proposition 4.12.** For every $a \in C^\ast(G_{n,\mu})$, the operator field $F(a)$ satisfies the generic condition.

We must show that on $\mathcal{D}$, our $C^\ast$-algebra satisfies the almost $C_0(K)$ conditions given in Definition 2.2. For $a \in C^\ast(G_{n,\mu})$ and $f = (f_0, f_+, f_-) \in V^\ast_{\text{gen}}$, we define the operator

$$\sigma_f(a) := U(-r(f)) \circ \pi_{(f_0, f_+, 0)}(a) \circ U(r(f)) \circ M_{(-\infty, \kappa(f) + r(f))} + U(-q(f)) \circ \pi_{(f_0, 0, f_-)}(a) \circ U(q(f)) \circ M_{(q(f) - \rho(f), +\infty]},$$

where

$$r(f) = -\ln(|f_+|), \quad q(f) = \ln(|f_-|), \quad \rho(f) = q(f)\frac{1}{3} - r(f), \quad \kappa(f) = q(f) - r(f)^{1/3}.$$

We have the following proposition.

**Proposition 4.13.** For all $f \in \mathcal{D}$, the operator field

$$f \mapsto \sigma_f(a) := \pi_f(a) - \sigma_f(a) \quad (a \in C^\ast(G_{n,\mu}))$$

is contained in $C_0(\mathcal{D}, \mathcal{K}(L^2(\mathbb{R})))$. 
Proof. Let \( a \in C^*(G_{n, \mu}) \). We know that \( \pi_f(a) \) is a compact operator for any \( f \in V_{gen}^\ast \), that the mapping \( f \mapsto \pi_f(a) \) is norm continuous and that \( \lim_{f \to \infty} \pi_f(a) = 0 \) by Corollary 3.2 and Proposition 4.2 in [Lin-Lud]. If \( F \in L^1_c \), then the kernel function \( \tilde{F}_{f_0, f_+} \) of the operator \( \pi_{(f_0, f_+, 0)} \circ M_{[\rho(f), \infty]} \) is given by

\[
\tilde{F}_{f_0, f_+}(s, t) = \mathcal{F}_{\infty}^n(s - t, t \cdot f_+)1_{[\rho(f), \infty]}(t).
\]

The function \( F_{f_0, f_+} \) is of compact support and \( \rho \) is continuous. Hence the mapping \( f \mapsto \pi_{(f_0, f_+, 0)} \circ M_{[\rho(f), \infty]} \) is norm continuous on \( D \) and for every \( f \in D \), the operator \( \pi_{(f_0, f_+, 0)} \circ M_{[\rho(f), \infty]} \) is compact. Since

\[
\rho(f) = \ln(|f_-|) + \ln(|f_+|) = \ln(|f_-|) + \ln(|f_+|)
\]

go to infinity as \( ||f|| \) goes to infinity, it follows that \( \pi_{(f_0, f_+, 0)} \circ M_{[\rho(f), \infty]} = 0 \) if \( ||f|| \) is big enough. Similar properties hold for the mapping \( f \mapsto \pi_{(f_0, f_+, 0)} \circ M_{[\rho(f), \infty]} \) on \( \mathcal{D} \).

Since the boundary \( \partial \mathcal{D} \) of \( \mathcal{D} \) is the set \( \mathcal{S} \cup \mathbb{R} \), the generic condition tells us that

\[
\lim_{f \to \partial \mathcal{D}} ||\pi_\mathcal{D}(f)(a)|| = 0.
\]

Hence the mapping \( f \mapsto \pi_\mathcal{D}(f)(\mathcal{F}) \) is contained in \( C_0(\mathcal{D}, \mathcal{K}(L^2(\mathbb{R}))) \). The proposition follows from the density of \( L^1_c \) in \( C^*(G_{n, \mu}) \). □

4.4. The C*-algebras of the groups \( G_{n, \mu} \). Let \( \Gamma_i \subseteq g_{n, \mu}^\ast \) be given as in Section 3.5 and \( \Gamma = \cup \Gamma_i \).

Definition 4.14. (1) For \( f \in \mathcal{D} \) and \( \phi \in l^\infty(\Gamma) \), let

\[
\sigma_f(\phi) := U(-r(f)) \circ \phi(f_0, f_+, 0) \circ U(r(f)) \circ M_{[\rho(f), \infty]} \circ U(q(f)) \circ M_{[\rho(f), -\rho(f)]}.
\]

(2) Let \( \varphi = (\varphi(f) \in \mathcal{B}, f \in \Gamma) \) be a field of bounded operators such that the restriction of the field \( \varphi \) to the set of characters \( \Gamma_0 \) is contained in \( C_0(\Gamma_0) \). We get the element \( \varphi(0) \in C^*(\mathbb{R} \times V_0) \) determined as in Definition 4.9 by the condition \( \gamma(\varphi(0)) = \varphi(\gamma) \) for \( \gamma \in \Gamma_0 \). We can then define as in Definition 4.9 that

\[
\sigma_f(\varphi) := \beta(\varphi(0)) \in \mathcal{B}(L^2(\mathbb{R})) \text{ for } f \in \mathcal{S}.
\]

Definition 4.15. Let \( D^*(G_{n, \mu}) \) be the subset of \( l^\infty(\Gamma) \) defined as a set of all the operator fields \( \phi \) defined over \( \widetilde{G_{n, \mu}} \) such that the mappings \( \gamma \mapsto \phi(\gamma) \) are norm continuous and vanish at infinity on the sets \( \Gamma_0 \) and \( \Gamma_2 \) and such that \( \phi(f) \in \mathcal{K}(L^2(\mathbb{R})) \) for all \( f \in \mathcal{D} \). Moreover, each \( \phi \) must fulfills the following conditions:

(1) For \( \varepsilon \in \{+, -\} \),

\[
\lim_{\delta \to 0} \text{dis}(\phi(\varepsilon) - \sigma_n, \delta(\phi)), \mathcal{K}(L^2(\mathbb{R} \times X)) = 0,
\]

and

\[
\lim_{\delta \to 0} \text{dis}(\phi^*(\varepsilon) - \sigma_n, \delta(\phi^*)), \mathcal{K}(L^2(\mathbb{R} \times X)) = 0.
\]

(2) The mappings

\[
\mathcal{D} \ni f \mapsto (\phi(f) - \sigma_f(\phi)) \text{ and } \mathcal{D} \ni f \mapsto (\phi(f)^* - \sigma_f(\phi^*))
\]

are contained in \( C_0(\mathcal{D}, \mathcal{K}(L^2(\mathbb{R}))) \).

(3) The mappings

\[
\mathcal{S} \ni f \mapsto (\phi(f) - \sigma_f(\phi)) \text{ and } \mathcal{S} \ni f \mapsto (\phi(f)^* - \sigma_f(\phi^*))
\]

are contained in \( C_0(\mathcal{S}, \mathcal{K}(L^2(\mathbb{R}))) \).

Theorem 4.16. The C*-algebra of \( G_{n, \mu} \) is an almost \( C_0(\mathcal{K}) \)-C*-algebra. In particular, the Fourier transform maps \( C^*(G_{n, \mu}) \) onto the subalgebra \( D^*(G_{n, \mu}) \) of \( l^\infty(\Gamma) \).

Proof. Propositions 4.8 and 4.13 show that the Fourier transform maps \( C^*(G_{n, \mu}) \) into \( D^*(G_{n, \mu}) \). The conditions on \( D^*(G_{n, \mu}) \) imply that \( D^*(G_{n, \mu}) \) is a closed involutive subspace of \( l^\infty(\Gamma) \). It follows from [ILL] that \( D^*(G_{n, \mu}) \) is a C*-subalgebra of \( l^\infty(\Gamma) \) and that \( \mathcal{F}_{n, \mu}(C^*(G_{n, \mu})) = D^*(G_{n, \mu}) \). □
A CLASS OF ALMOST $C_0(K)$-$C^*$-ALGEBRAS

References


Junko Inoue, Education Center, Organization for Supporting University Education, Tottori University, Tottori 680-8550, Japan. E-mail: inoue@uec.tottori-u.ac.jp

Ying-Fen Lin, Pure Mathematics Research Centre, Queen’s University Belfast, BT7 1NN, U.K. E-mail: y.lin@qub.ac.uk

Jean Ludwig, Université de Lorraine, Institut Elie Cartan de Lorraine, UMR 7502, Metz, F-57045, France. E-mail: jean.ludwig@univ-lorraine.fr