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A. Sanand Amita Dilip
UC Louvain, ICTEAM, Belgium.
Email: sanand.athalye@uclouvain.be

Nikolaos Athanasopoulos
Queen's University Belfast, NI, UK.
Email: n.athanasopoulos@qub.ac.uk

Raphaël M. Jungers
UC Louvain, ICTEAM, Belgium.
Email: raphael.jungers@uclouvain.be

***Abstract*—We consider control systems where the input signal is transferred over a network and therefore, it is subject to *packet losses*. In this setting, the closed-loop behavior can be described as a constrained switching system. We investigate whether there exists a switching signal that prevents reachability of some target state, or alternatively, how much additional input energy is required to reach a target state in comparison to the dropout-free case. Mathematically, we formulate a reachability problem defined on a hybrid automaton and tackle an optimization problem, whose feasibility variants, the controllability and reachability properties, have been recently shown to be decidable. To do so, we provide *automata-theoretic* algorithms to study the properties of an appropriate generalization of the Controllability Gramian matrix. Additionally, we provide *polynomial time heuristics* for computations for a specific family of automata and show numerical evidence that they work well in practice. Last, we extend our observations to the analogous observability energy problem.**

Keywords*—Reachability, Controllability Gramian, Cyber-physical systems, Networks, Constrained switching, Packet dropouts.

I. Introduction

We consider discrete time linear systems, focusing on a common non-ideality in modern Cyber-Physical systems. In particular, we consider the control input to be transferred over a network that suffers from packet losses [4]-[6], [14]-[23] see Figure 1 for an illustration. Our goal is to quantify the degradation of performance by building on these models. The systems are described by the difference equations

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. Inhere, we consider time instants when there is a loss of communication between the plant and the controller. Following [4], [5], [6], we express the system as a switching system of the form $x(t+1) = Ax(t) + B\sigma(t)u(t)$, where

$$x(t+1) \quad = \quad \begin{cases} Ax(t) + Bu(t), & \text{if } \sigma(t) = 1, \\ Ax(t), & \text{if } \sigma(t) = 0 \end{cases} \tag{2}$$

and $\sigma(\cdot) : \mathbb{N} \to \{0, 1\}$ is a binary switching signal subject to constraints defined by a directed graph.

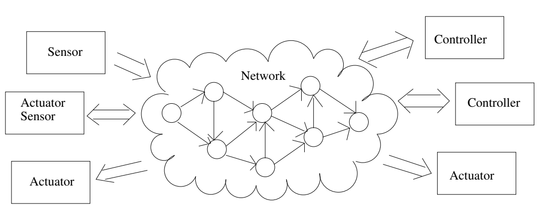

Fig. 1: Information exchange between sensors, actuators and controllers via a non ideal communication network with packet losses.

TABLE I: Assumptions on matrix A in (1)

Diagonalizable	Stable (infinite horizon)	Not anti-stable	Invertible
[8], [12], [25], [37]	[7], [9], [25]	[8], this work	this work

In [4]-[6], it is shown that there exists an algorithm deciding controllability and observability of (2) in finite time. In this work, we plan to characterize quantitatively the energy required for a state transfer starting from the origin. Alternatively, we aim to quantify the degradation of the performance for networked control systems, in terms of the required input energy in the presence of packet losses. Apart from the mathematical challenge and the joining of efforts in extending classical linear systems theory to networked control systems [4]-[9], [14], [17], [20], the main motivation for this work is to *obtain a measure of performance of the closed-loop system that depends on the characteristics of the communication network*. Controllability metrics are particularly important in *co-design problems* [7]-[9], [25], [26], [29], where a trade-off between the quality of the network (connected directly to the cost) and the energy required for regulation or tracking of the closed-loop system has to be taken into account.

The Controllability Gramian (CG) [1], [11] provides a quantitative measure of the energy required for a state transfer. In particular, the minimum eigenvalue of the CG is inversely proportional to the energy required for a state transfer in the direction most difficult to control. Additional controllability metrics are the determinant, the trace and the trace of the inverse of the CG [7], [9], [25], [29]. Note that for LTI systems, the determinant of the CG gives the volume of the ellipsoid that can be reached from the origin with unit energy input. On the other hand, the trace of the inverse of the CG gives the average energy for reachability when the final state lies on the unit sphere.

Although extending these metrics to systems with packet dropouts is conceptually straightforward, a significant computational challenge has to be addressed, namely, *all admissible switching signals have to be taken into account*.

TABLE II: Finite time reachability

CG metric	LTI	Large scale LTI	LTI with dropouts
Minimum eigenvalue ($\lambda(t)$)	[1], [29]	[8], [12] , [26], [37]	this work
Trace of the inverse ($\delta(t)$)	[29]	[9]	this work
Determinant ($\gamma(t)$)	[29]	–	this work

TABLE III: Infinite time reachability

CG metric	LTI	Large scale LTI	LTI with dropouts
Minimum eigenvalue (λ)	[1]	[8], [25], [26]	this work
Trace of the inverse (δ)	–	[7], [9], [25]	this work
Determinant (γ)	–	[7], [25]	this work

This observation immediately transforms the problem from a simple algebraic one where the properties of the CG are computed once, to a combinatorial one. This type of challenge has been met elsewhere in the literature, namely in sensor/actuator placement in large scale networks [7]-[9], [25], [37]. In that setting, the problem consists of optimally placing actuators/leaders in order to minimize the energy. The combinatorial aspect comes from the need to choose a subset out of all possible inputs. To tackle the problem, in [8] the authors exploit the network structure and the diagonalizability of the system matrix whereas in [7] sub-modularity properties of some of the controllability metrics were identified and used. The problem of minimizing the trace of the inverse of the CG for complex networks is studied in [9], where it is shown it can be solved efficiently by approximating algorithms. Upper bounds on the minimum eigenvalue of the CG for LTI systems are presented in [12], where it is shown that systems with clustered eigenvalues of the matrix A require more energy to control. Table 1 shows the assumptions made in the aforementioned works. [1] [2]. Compared to the above works, in our setting instead of choosing a subset of columns of the input matrix B, we have to choose directly subsets of columns of the controllability matrix. Thus, although our formulation looks similar to [7]-[9], [12], [25], [37], the techniques used there are not applicable.

Our contributions and related works on LTI systems and networks are summarized in Tables II and III.

Contributions: We provide algorithms to calculate energy metrics for reachability problems for (2). Specifically, by using the iterated dynamics of the system [30], we construct modified T-product lifted graphs that allow to restrict the search for switching sequences that maximize the reachability energy to strict subsets of *minimal* signals according to a well-defined partial order. Furthermore, we provide a heuristic for systems suffering from at most k successive dropouts, reducing in this case the computational complexity to a polynomial function of time. Our methods are applicable to the corresponding observability problem as well.

[1]The matrix A is intrinsically stable and symmetric in [25] because it is the negative of the Laplacian of an undirected graph. In [9], stability of A is assumed only for the infinite horizon problem.

[2]For infinite horizon problems with the determinant or the trace of the inverse of the CG as a metric, we assume that A is stable in accordance with [7], [25].

We focus our attention on the reachability problem i.e. we assume that $x(0) = 0$.

Notation: For an input vector sequence $u(0), u(1), ..., u(T)$, $t \geq T$, $u(i) \in \mathbb{R}^m$, we denote the aggregated vector $\bar{u} := \begin{bmatrix} u(0)^\top & \cdots & u(T)^\top \end{bmatrix}^\top$. The $2-$norm of $\bar{u}$ is $\|\bar{u}\|_{l_2,T} = \sqrt{\bar{u}^\top \bar{u}}$. When $T \to \infty$, we write $\|\bar{u}\|_{l_2}$. We denote the n-dimensional unit ball and the unit sphere by $\mathbb{B}_1^n$ and $\mathbb{S}_1^n$ respectively. For a matrix $A \in \mathbb{R}^{n\times n}$, by $A \geq 0$, we mean it is positive semi-definite. The subsequence of a switching sequence $\sigma(0)\sigma(1)...$ from t_1 to t_2, with $0 \leq t_1 \leq t_2$, is denoted by $\sigma|_{t_1}^{t_2}$.

II. Preliminaries

We first express the energy required for a state transfer for LTI systems in terms of the CG. In subsection II-B we utilize the constrained switching modeling framework, while we extend the notion of reachability energy to systems with packet losses in subsection II-C.

A. Controllability Gramian properties for LTI systems

For the system (1), the state at time $t+1$ is connected to the initial condition by

$$x(t+1) = A^{t+1}x_0 + C_t(A,B)\bar{u},$$

with $x_0 := x(0) \in \mathbb{R}^n$, $\bar{u} := \begin{bmatrix} u(0)^\top & \cdots & u(t)^\top \end{bmatrix}^\top$, and

$$C_t(A,B) := \begin{bmatrix} A^tB & A^{t-1}B & \cdots & B \end{bmatrix}$$

is the *controllability matrix* at time $t+1$.

Definition 1 ([1], [10], [11]): The system (1) is reachable, if for the initial state $x_0 = 0$ and any final state $x_f \in \mathbb{R}^n$, there is an input sequence $u(0), ..., u(T)$ such that $x(T) = x_f$, for some $T \in \mathbb{N}$. If furthermore the property holds for all $x_0 \in \mathbb{R}^n$, then the system (1) is controllable.

Definition 2: Given $T \geq 1$, the least energy input $\bar{u}^* \in \mathbb{R}^T$ for a state transfer from $x(0) = x_0 \in \mathbb{R}^n$ to $x(T) = x_f \in \mathbb{R}^n$ with respect to the dynamics (1) is

$$\bar{u}^* = \text{arginf}_{\bar{u}}\{\|.\|_{l_2,T} : \bar{u} \to \|\bar{u}\|_{l_2,T}\}.$$

Definition 3: Given two compact sets $S_0, S_f \subseteq \mathbb{R}^n$ and an integer T, the minimum energy $E(S_0, S_f, T)$ required for a state transfer from S_0 to S_f in $T+1$ time instants is

$$\begin{aligned} E(S_0,S_f,T) \quad &= \sup_{x_0\in S_0, x_f \in S_f} \inf_{\bar{u}} \{\|\bar{u}\|^2_{l_2,T} \; : \text{(1) holds}, \\ &\qquad x(0) = x_0, x(T+1) = x_f\}. \end{aligned}$$

Definition 4 ([1]): The *Controllability Gramian at time t* with respect to (1) is $W_t(A,B) := \sum_{i=0}^t A^iBB^\top(A^\top)^i$. If A is stable, then the *CG at infinity*, or, simply the CG, is $W(A,B) := \lim_{t\to\infty}\sum_{i=0}^t A^iBB^\top(A^\top)^i$.

Remark 1: The Controllability gramian in Definition 4 is well defined for stable systems. However, we underline that characteristics of the CG such as its minimum eigenvalue, are well defined under weaker assumptions.

Remark 2: An alternative definition of the CG exists in the frequency-domain, that is valid also for unstable open-loop systems, see [1, Chapter 6.1] and [38].

When the system (1) is controllable, the least energy input for the state transfer from $x_0 \in \mathbb{R}^n$ to $x_f \in \mathbb{R}^n$ after $t+1$ instants is given directly from

$$\bar{u}^* = C_t(A,B)^\top W_t(A,B)^{-1}(x_f - A^{t+1}x_0) \tag{3}$$

and the least input energy required for a state transfer from $x(0) = x_0$ to $x(t+1) = x_f$ is given by [3] $E(x_0, x_f, t+1) := \|\bar{u}^*\|^2_{l_2,t}$. Correspondingly, from (3) we have

$$E(x_0, x_f, t{+}1) = (x_f - A^{t+1}x_0)^\top W_t(A,B)^{-1}(x_f - A^{t+1}x_0).$$

B. Constrained switching model

Calculating the energy required for a state transfer for the system (2) becomes non-trivial only if the switching signal satisfies some pattern, e.g., only a fixed number of consecutive packet dropouts is allowed, or, at least k packets are successfully received out of every m time instances. These constraints can be formally captured by an automaton [2]-[6], [30], [33], [34], [36].

Analogously to the LTI case, the state at time $t+1$ for system (2) is given by

$$x(t+1) = A^t x(0) + C_{\sigma(t)}(A,B)\bar{u}$$

where

$$C_{\sigma(t)}(A,B) := \begin{bmatrix} \sigma(0)A^tB & \cdots & \sigma(t-1)AB & \sigma(t)B \end{bmatrix}$$

is the controllability matrix associated with the switching sequence $\sigma(0)\sigma(1)...\sigma(t)$ at time t.

Definition 5: An automaton $\mathcal{A}$ is a directed labeled graph $G(V,E)$ with N_V nodes in V and N_E edges in E. Each edge $(v,w,\sigma) \in E$ is labeled by $\sigma \in \{0,1\}$. A sequence $\sigma(0)\sigma(1),\dots$ is *admissible if there is a path in* $G(V,E)$ *carrying the sequence as the succession of labels on its edges*. We denote by $\mathcal{L}(\mathcal{A})$ the set of all admissible switching sequences.

Example 1: Consider a control system where there can be at most 3 consecutive dropouts. This can be captured by an automaton containing 4 nodes $V = \{s_1, s_2, s_3, s_4\}$. Nodes s_i $(1 \le i \le 3)$ represent instants where a packet drop occurred at the last i successive instances and the one before arrived safely whereas s_4 represents the situation where the latest packet is successfully received. The automaton $\mathcal{A}$ is shown in Figure 2. An admissible switching signal of length t can be obtained from the edge labels of a walk of length t in the graph.

Definition 6 ([6]): The system (2) is reachable, if for any $\sigma \in \mathcal{L}(\mathcal{A})$, any final state $x_f \in \mathbb{R}^n$ and for $x_0 = 0$ there exists an input signal $u(\cdot) : \mathbb{N} \to \mathbb{R}^m$ such that $x(T) = x_f$ for some $T \in \mathbb{N}$. If, furthermore, the property holds for all $x_0 \in \mathbb{R}^n$, we say that (2) is controllable.

In [6] it is shown that (2) is controllable if and only if there exists $T \in \mathbb{N}$ for any switching signals σ such that $C_{\sigma(T)}(A,B)$ has rank equal to n, where n is the dimension of the state space.

[3]We abuse the notation $E(\{x_0\},\{x_f\},t+1)$ writing $E(x_0,x_f,t+1)$.

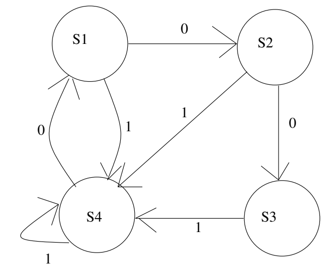

Fig. 2: Automaton $\mathcal{A}$ modeling a communication network where no more than 3 successive dropouts are allowed.

C. Controllability Gramian for systems with packet dropouts

Similarly to the LTI case, for a given switching signal $\sigma(\cdot)$, the minimum energy required for a state transfer from a set $S_0 \subset \mathbb{R}^n$ to a set $S_f \subset \mathbb{R}^n$ in $T+1$ time instances is

$$\begin{aligned} E_\sigma(S_0, S_f, T) \;\; = \;\; & \sup_{x_0 \in S_0, x_f \in S_f} \inf_{\bar{u}} \{\|u\|^2_{l_2,T} \; : \text{(2) holds}, \\ & x(0) = x_0, x(T+1) = x_f\}. \end{aligned} \tag{4}$$

Definition 7: Consider the system (2), two compact sets $S_0, S_f \subset \mathbb{R}^n$ and $T \ge 1$. The worst case energy required for a state transfer from S_0 to S_f over all admissible switching sequences of length T is

$$E(S_0, S_f, T) = \sup_{\sigma \in \mathcal{L}(\mathcal{A})} \{E_\sigma(S_0, S_f, T)\}.$$

We can obtain the least input energy $E_\sigma(x_0, x_f, T)$ required for a state transfer from the Controllability Gramian, the initial and the final state. We give an expression for the CG of a system for a fixed signal σ and time t below.

Definition 8: Given system (2) and a signal σ of length $t+1$, we define the CG to be

$$W_{\sigma(t)} \;\; := \;\; \sum_{i=0}^{t} \sigma(t-i) A^i B B^\top (A^\top)^i. \tag{5}$$

Furthermore, if A is stable, we define $W_\sigma := \lim_{t\to\infty} W_{\sigma(t)}$.

Using similar arguments as with the LTI case [11, Chapter 3], for a fixed signal σ, $x(0) = x_0$ and $x(t) = x_f$, we have

$$E_\sigma(x_0, x_f, t) := (x_f - A^t x_0)^\top W^{-1}_{\sigma(t-1)} (x_f - A^t x_0).$$

We next define the minimum eigenvalue $\lambda_{\min}(t)$ of the CG $W_{\sigma(t)}$, $t \ge 0$ and the switching signal/signals for which it is attained. Taking into account that the CG is a symmetric matrix, we define $\lambda_{\min}(W_{\sigma(t)})$ via the Rayleigh quotient

$$\lambda_{\min}(W_{\sigma(t)}) := \min_{\|x\|=1} x^\top W_{\sigma(t)} x. \tag{6}$$

The relationship between the least input energy (Definition 7) and the spectral properties of the CG is presented below without a proof as it is a direct extension of the LTI case [1], [10], [11].

Lemma 1: Consider system (2). For any integer $t \ge 1$, let

$$\lambda(t) := \min_{\{\sigma(t) \in \mathcal{L}(\mathcal{A}), |\sigma| = t\}} \lambda_{\min}(W_{\sigma(t)}). \tag{7}$$

Then,

$$E(0, \mathbb{S}_1^n, t+1) = \lambda(t)^{-1}, \quad \forall t \geq t^\star,$$

where $t^\star \geq 1$ is the time instant for which the system (2) becomes controllable.

For every admissible signal σ, there exists t_σ such that $\forall t \geq t_\sigma$, $C_\sigma(t)$ is full rank. It is clear that $t^\star = \max_{\sigma \in \mathcal{A}}\{t_\sigma\}$.

Throughout the paper, we assume for simplicity that the system matrix A is nonsingular.

Assumption 1: A is nonsingular.

Assumption 1 excludes special cases for which additional refinements to the presented theory are required. For example, consider the controllable system (2) with $A = 0$, $B = I$ and $\mathcal{A}$ as defined in Example 1. For any time t, for the admissible signal σ for which at time instant t, $\sigma(t) = 0$, it follows that $\lambda_{min}(W_{\sigma(t)}) = 0$. As we will see in the sequel, this would imply that the energy required for a state transfer to be arbitrarily large, which is clearly not the case. In the following Proposition we show that the minimum eigenvalue of the CG is well-defined.

Lemma 2: Let $A \in \mathbb{R}^{n\times n}$. If $A = A_1 + A_2$ with $A_1, A_2 \geq 0$, then $\lambda_{min}(A) \geq \lambda_{min}(A_1) + \lambda_{min}(A_2)$.

Proof: Let v be a vector such that $\|v\| = 1$ and $v^\top A v = \lambda_{min}(A)$. Clearly, $v^\top A_1 v \geq \lambda_{min}(A_1)$ and $v^\top A_2 v \geq \lambda_{min}(A_2)$, thus, $\lambda_{min}(A) \geq \lambda_{min}(A_1) + \lambda_{min}(A_2)$. ■

Proposition 1: Consider the system (2) and suppose that the LTI system (1) is controllable. For any integer $t \geq 1$, consider the corresponding sequence $\{\lambda(t)\}$, $t \geq 1$. Then the following hold.

(i) The sequence $\{\lambda(t)\}$ is monotonically non-decreasing.

(ii) The sequence is convergent, i.e., there exists $\lambda \in \mathbb{R}$ such that

$$\lambda := \lim_{t\to\infty} \lambda(t) \tag{8}$$

if and only if at least one eigenvalue of the matrix A lies strictly inside the unit circle.

The proof of Proposition 1 is in the Appendix. Consequently, throughout the paper, we make the following assumption.

Assumption 2: The matrix A is not anti-stable, i.e., A has an eigenvalue strictly inside the unit circle.

It is worth noting that Assumption 2 is not restrictive in our context since it rules out only the non-interesting cases for which the least input energy becomes arbitrarily small for a large time horizon t.

We can also utilize the metric which corresponds to the average energy, given by the trace of the inverse of the CG, and define a relevant quantity associated with the determinant of the CG.

Definition 9: Given a system (2), we define the set $S_\sigma(t)$ to contain all switching sequences that correspond to the minimum eigenvalue of the CG at time t as

$$S_\sigma(t) := \text{ argmin}_{\sigma(t)}(\lambda_{min}(W_{\sigma(t)})).$$

Analogously, we define the set $\bar{S}_\sigma(t)$ to contain all switching signals that correspond to the *maximum average energy*

$$\bar{S}_\sigma(t) := \text{ argmax}_\sigma(\text{tr}(W_{\sigma(t)}^{-1})),$$

where tr denotes the trace of a matrix. Let $\bar{\sigma} \in \bar{S}_\sigma(t)$. We define the maximum trace of the inverse of the CG

$$\delta(t) \quad := \text{tr}(W_{\bar{\sigma}(t)}^{-1}). \tag{9}$$

Similarly for the determinant of the CG,

$$\begin{aligned} \hat{S}_\sigma(t) &:= \text{argmin}_{\sigma(t)}(\det(W_{\sigma(t)})), \\ \gamma(t) &:= \det(W_{\hat{\sigma}(t)}), \end{aligned} \tag{10}$$

where $\hat{\sigma} \in \hat{S}_\sigma(t)$. Under the additional assumption of stability of the matrix A, we define $\delta := \lim_{t\to\infty} \delta(t)$ and $\gamma := \lim_{t\to\infty} \gamma(t)$.

III. Lower and upper bounds on the Controllability Gramian metrics

We provide constructive upper and lower bounds for the metrics of the CG. Additionally to Assumptions 1 and 2, for the determinant and the trace of the inverse of CG we assume strict stability of the matrix A. From Proposition 1, we have that $\lambda(t)$ is a lower bound on λ. In what follows, we provide also an upper bound on λ by concatenating σ with an infinite sequence of ones.

Theorem 1: Consider system (2) and assume it is controllable. Let $V \in \mathbb{R}^{n\times n}$ be such that

$$V^{-1}AV = \bar{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \bar{B} = V^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where A_1 is strictly stable and A_2 has eigenvalues on or outside the unit circle. Let $W_{\sigma^*(t)}$ be the CG associated with the signal $\sigma^*(t)$ such that $\lambda(t) = \lambda_{min}(W_{\sigma^*(t)})$ and set $\bar{W}_{\sigma^*(t)} := V^{-1}W_{\sigma^*(t)}V^{-\top}$, $\hat{W}_{\sigma^*(t)} := \begin{bmatrix} I & 0 \end{bmatrix} \bar{W}_{\sigma^*(t)} \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $K = \|V\|_2^2$. Then,

$$\lambda \leq K\lambda_{min}(\hat{W}_{\sigma^*(t)} - W_t(A_1, B_1) + W(A_1, B_1)).$$

Proof: From (7), (8) we have

$$\begin{aligned}
\lambda &\leq \lambda_{min}(V(\bar{W}_{\sigma^*(t)} + \sum_{i=t+1}^{\infty} \bar{A}^i \bar{B}\bar{B}^\top (\bar{A}^\top)^i)V^\top) \\
&= \min_{\|x\|=1} x^\top V(\bar{W}_{\sigma^*(t)} + \sum_{i=t+1}^{\infty} \bar{A}^i \bar{B}\bar{B}^\top (\bar{A}^\top)^i)V^\top x \\
&\leq \|V\|_2^2 \min_{\|x\|=1} x^\top (\bar{W}_{\sigma^*(t)} + \sum_{i=t+1}^{\infty} \bar{A}^i \bar{B}\bar{B}^\top (\bar{A}^\top)^i) x \\
&\leq K \min_{\|y\|=1} \begin{bmatrix} y & 0 \end{bmatrix} (\bar{W}_{\sigma^*(t)} + \\
&\quad \sum_{i=t+1}^{\infty} \bar{A}^i \bar{B}\bar{B}^\top (\bar{A}^\top)^i) \begin{bmatrix} y \\ 0 \end{bmatrix} \\
&= K\lambda_{\min}(\hat{W}_{\sigma^*(t)} + \sum_{i=t+1}^{\infty} (A_1)^i B_1 B_1^\top ({A_1}^\top)^i) \\
&= K\lambda_{\min}(\hat{W}_{\sigma^*(t)} - W_t(A_1, B_1) + W(A_1, B_1)).
\end{aligned}$$

■

Remark 3: It is worth noting that in the proof of Theorem 1(ii) we can choose any admissible signal to concatenate with σ^* in order to to compute, possibly tighter, upper bounds[4]. For clarity, we present the simple choice of concatenating σ^* with the signal of all ones.

Remark 4: When[5] $\rho(A) < 1$, the upper bound becomes

$$\begin{aligned}\bar{\lambda}(t) &= \lambda_{min}(W_{\sigma^*(t)} + \sum_{i=t+1}^{\infty} A^i BB^\top (A^\top)^i). \\ &= \lambda_{min}(W_{\sigma^*(t)} - W_t(A,B) + W(A,B)).\end{aligned}$$

When A is stable, $W(A,B)$ can be computed directly by solving the Lyapunov equation [1]. Moreover, in this case it additionally holds that $\lim_{t\to\infty} \bar{\lambda}(t) = \lambda$.

Under the additional assumption $\rho(A) < 1$, we can provide upper and lower bounds for the other CG metrics as well. The following result is a straightforward extension of Theorem 1.

Corollary 1.1: Suppose (2) is controllable and furthermore $\rho(A) < 1$. Suppose $\sigma_1 \in \bar{S}_{\sigma(t)}$ and $\sigma_2 \in \hat{S}_{\sigma(t)}$. Let

$$\begin{aligned}\underline{\delta}(t) &:= \operatorname{tr}(W_{\sigma_1(t)} - W_t(A,B) + W(A,B))^{-1}, \\ \bar{\gamma}(t) &:= \det(W_{\sigma_2(t)} - W_t(A,B) + W(A,B)).\end{aligned}$$

Then, it holds that **(i)** $\underline{\delta}(t) \le \delta \le \delta(t)$ and **(ii)** $\gamma(t) \le \gamma \le \bar{\gamma}(t)$. Moreover, $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \underline{\delta}(t) = \delta$ and $\lim_{t\to\infty} \gamma(t) = \lim_{t\to\infty} \underline{\gamma}(t) = \gamma$.

IV. Computing the reachability energy

As one can observe from (7), (9), (10), the computation of the CG metrics $\lambda(t)$, $\delta(t)$ and $\gamma(t)$ necessitates the enumeration of all admissible signals of length t. To reduce this computational cost, we show that only a strict subset of these signals should be analyzed. Formally, we introduce a natural notion of partial ordering on our signals.

Definition 10: Given two switching signals σ_1 and σ_2, we write $\sigma_1 \preceq \sigma_2$ if for all $i \in \mathbb{Z}$ for which $\sigma_1(i) = 1$ it follows that $\sigma_2(i) = 1$.

Theorem 2: Consider a switching system given by (2). Suppose time t is given and $\sigma_1 \preceq \sigma_2$. Then,

(i) $\lambda_{min}(W_{\sigma_1(t)}) \le \lambda_{min}(W_{\sigma_2(t)})$.

(ii) $\operatorname{tr}(W^{-1}_{\sigma_1(t)}) \ge \operatorname{tr}(W^{-1}_{\sigma_2(t)})$.

(iii) $\det(W_{\sigma_1(t)}) \le \det(W_{\sigma_2(t)})$.

Proof: **(i)** We can write $W_{\sigma_2(t)} = W_{\sigma_1(t)} + P$ where $P \ge 0$. Let v_1 ($\|v_1\| = 1$) be the eigenvector $W_{\sigma_1(t)}$ such that $v_1^\top W_{\sigma_1(t)} v_1 = \lambda_{min}(W_{\sigma_1(t)})$. Let v_2 ($\|v_2\| = 1$) be the eigenvector $W_{\sigma_2(t)}$ such that $v_2^\top W_{\sigma_2(t)} v_2 = \lambda_{min}(W_{\sigma_2(t)})$. We have

$$\begin{aligned}&\lambda_{min}(W_{\sigma_2(t)}) = v_2^\top W_{\sigma_2(t)} v_2 = v_2^\top (W_{\sigma_1(t)} + P) v_2 \\ &\quad \ge v_2^\top (W_{\sigma_1(t)}) v_2 \ge v_1^\top W_{\sigma_1(t)} v_1 \ge \lambda_{min}(W_{\sigma_1(t)}).\end{aligned}$$

(ii) Since $W_{\sigma_2(t)} \ge W_{\sigma_1(t)}$, it follows that $W^{-1}_{\sigma_2(t)} \le W^{-1}_{\sigma_1(t)}$ [27, Proposition 7.23] and $\operatorname{tr}(W^{-1}_{\sigma_1(t)} - W^{-1}_{\sigma_2(t)}) \ge 0$. **(iii)** Since $W_{\sigma_2(t)} = W_{\sigma_1(t)} + P$ and $P \ge 0$, by Minkowski's determinant theorem [28, p. 115] we have $\det(W_{\sigma_2(t)}) = \det(W_{\sigma_1(t)} + P) \ge \det(W_{\sigma_1(t)})$. ■

Definition 11: We say that a signal σ is minimal, if there does not exist any other admissible signal $\bar{\sigma}$ ($\bar{\sigma} \neq \sigma$) such that $\bar{\sigma} \preceq \sigma$. We denote by $M_{\sigma(t)}$ the set of all minimal signals of length $t+1$.

Example 2: Suppose $t = 5$ and at most two consecutive packet drops are allowed by the automaton. Let $\sigma_1 = 001001$, $\sigma_2 = 001100$ and $\sigma_3 = 101001$. Then, we can verify that $\sigma_1, \sigma_2 \in M_{\sigma(t)}$. On the other hand, σ_3 is not minimal, since $\sigma_1 \preceq \sigma_3$.

By Theorem 2(i) and Definition 11, we see immediately that

$$\lambda(t) = \min_{\sigma \in M_{\sigma(t)}} \{\lambda_{min}(W_{\sigma(t)})\} \tag{11}$$

Remark 5: Suppose E is a fixed amount of energy available for control in t time instants. We can compute $\lambda(t)$ using only the minimal signals of length $t+1$. If $\lambda(t)^{-1} > E$, then the reachability problem is infeasible. Since $S_{\sigma(t)}$ is the set of switching signals for which $\lambda(t)$ is attained, in this case it provides a subset of switching signals for which certain state transfers from the origin become infeasible.

Although a significant reduction of the switching sequences that have to be checked is achieved using the minimal signals, the challenge is transferred to retrieving them. In the following subsection, we enumerate all elements of the set $M_{\sigma(t)}$ by utilizing the so-called T-product lifts of a graph. In subsection IV.B, we provide a heuristic of computing fast a subset of the minimal signals, based on a notion of sparsity.

A. Exact algorithm for enumeration of the minimal signals

Given an automaton $\mathcal{A}$, we aim to construct an automaton $\mathcal{A}_{min}(t)$ for which each minimal switching sequence of a fixed length $t+1$ appears in at least one label. We achieve this by iteratively calculating $T-$product lifts [30] of the associated graph $G(V,E)$ and by removing dominant signals with respect to the partial order (Definition 10). We provide Algorithm 1 to construct a directed graph $G_t(V_t, E_t)$ associated with $\mathcal{A}_{min}(t)$ from $G(V,E)$. We refer to the graph $G_i(V_i, E_i)$ along the iterative procedure of Algorithm 1 as the reduced $i-$lifted graph. The following result states that all the minimal signals of length t are captured by the labels of $\mathcal{A}_{\min}(t)$.

Theorem 3: Consider the automaton $\mathcal{A}$ and let $\mathcal{A}_{\min}(t)$, $t \ge 2$, be the automaton with a graph $G_t(V_t, E_t)$ computed with the procedure followed in Algorithm 1. Then, there exists at least one label of $\mathcal{A}_{\min}(t)$ for each minimal admissible signal of length t.

Proof: Let σ be a minimal admissible signal of length t. From Definition 5, there exists a path in $G(V,E)$ carrying σ. Suppose $e = (a, b, \sigma) \notin E_t$, for some nodes $a, b \in V_t$. This implies that during the iterative procedure of Algorithm 1, at least one of the following holds:

- there exists an instant $t_1 < t$, and $e' = (a, c, \sigma|_0^{t_1}), e_1 = (a, c, \sigma_1) \in E_{t_1}$ such that $\sigma_1 \preceq \sigma|_0^{t_1}$,
- there exists an edge $e_1 = (c, d, \sigma_1) \in E_t$ such that $\sigma_1 \preceq \sigma$.

[4] We thank Professor W.P.M.H. Heemels for the suggestion.

[5] $\rho(\cdot)$ stands for the spectral radius of a matrix.

Algorithm 1 Constructing minimal signals of length t

Input : $G(V,E), t$
Output : Minimal signals $M_{\sigma(t)}$

$G_1(V_1,E_1) \leftarrow G(V,E)$
for $i \leftarrow 2$ to t **do**
  $V_i \leftarrow V$
  $E_i \leftarrow \emptyset$
  for every $e_1 = (v,w,\sigma_1) \in E_{i-1}$ **do**
    if $e_2 = (w,z,\sigma_2) \in E$ **then**
      $e \leftarrow (v,z,\sigma_1\sigma_2)$
      $E_i \leftarrow E_i \cup e$
    end if
    if $e_1 = (v,w,\sigma_1), e_2 = (v,w,\sigma_2) \in E_i$ & $\sigma_1 \preceq \sigma_2$ **then**
      $E_i \leftarrow E_i \setminus e_2$
    end if
  end for
end for
$M_{\sigma(t)} \leftarrow \{\sigma^\star : (\exists a, b, \in V_t : (a,b,\sigma^\star) \in E_t)\}$

Both of these cases contradict the minimality of σ. Thus, there exists an edge $e = (a,b,\sigma) \in E_t$. ∎

Example 3: Consider automaton $\mathcal{A}$ where no more than one successive dropout is allowed (Figure 3). Node a represents a state where a packet dropout has just happened. In Figure 4, the figure on the left represents the $2-$lift of

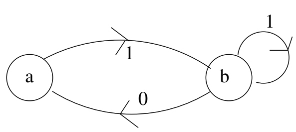

Fig. 3: $G(V,E)$: No more than one consecutive dropout is allowed

$G(V,E)$ without removing the dominant edges and the figure on the right represents the reduced $2-$lift $G_2(V_2,E_2)$. For example, among the two edges $(b,b,10)$ and $(b,b,11)$ that correspond to a self-loop in node b, we remove $(b,b,11)$. Similarly, Figure 5 represents the $3-$lift and the reduced $3-$lift

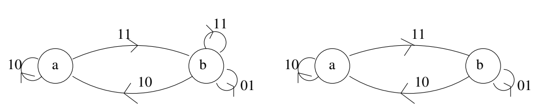

Fig. 4: $2-$lift and the reduced $2-$lift of $G(V,E)$

$G_3(V_3,E_3)$. Suppose we want minimal signals of length 3,

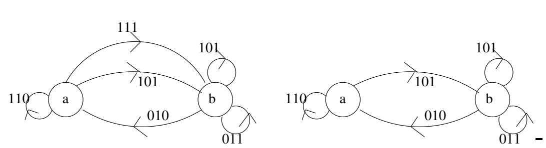

Fig. 5: $3-$lift and the reduced $3-$lift of $G(V,E)$

then we stop iterating at $G_3(V_3,E_3)$. We compare all edges of E_3 independent of the starting and ending node, and remove the dominant ones. From Figure 5, since $010 \prec 110$ and $010 \prec 011$, we remove the dominant ones and the minimal signals are $010, 101$.

B. Approximation of minimal signals using sparsity

In this subsection we propose a method which approximates $\lambda(t)$ by evaluating $\lambda_{min}(W_{\sigma(t)})$ over a set of switching signals whose cardinality is bounded by a polynomial function of t. The methods applies to a particular class of automata where at most k consecutive dropouts are allowed, see, e.g., the automata in Figures 2 and 3. We denote by S_m the set of all $m-$minimal signals for an automaton with m number of nodes, ordered by the lexicographic order[6]. We present Algorithm 2 that generates a set Sp_t of sparse signals of length $t \geq m$. The main idea behind the heuristic is to evaluate $\lambda_{min}(W_{\sigma(t)})$ for only those admissible switching signals which have as many zeros as possible. We call signals constructed using Algorithm 2 as sparse minimal signals.

Algorithm 2 Constructing sparse minimal signals of length t

Input : $G(V,E), t$
Output : Sparse minimal signals Sp_t

$m \leftarrow |V|$
$n \leftarrow \lceil \frac{t}{m} \rceil$
$S_m \leftarrow$ minimal signals of length m constructed by Algorithm 1 (ordered by the lex. order)
$k \leftarrow |S_m|$
$Sp_m \leftarrow S_m$
for $i \leftarrow 2$ to n **do**
  $Sp_{i.m} \leftarrow \emptyset$
  for every $\sigma_j \in S_m$ **do**
    $T_{\sigma_j} \leftarrow \{\sigma \in Sp_{(i-1).m} | \sigma = \sigma_j \hat{\sigma}\}$
    $\bar{\sigma} \leftarrow$ minimum signal in T_{σ_j} (lex. order)
    $Sp^{\bar{\sigma}}_{(i-1).m} \leftarrow \{\sigma \in Sp_{(i-1).m} \,|\bar{\sigma} \preceq_{lex} \sigma\}$
    $S_c \leftarrow \{\sigma_j\sigma, \forall \sigma \in Sp^{\bar{\sigma}}_{(i-1).m}\}$
    $Sp_{i.m} \leftarrow Sp_{i.m} \cup S_c$
  end for
end for
$Sp_t \leftarrow Sp_{m.n}(: \,,\; 1:t)$

Lemma 3: Given $k \geq 1$, let $\mathcal{A}$ be the automaton representing the case where not more than k successive dropouts are allowed. Let m be the number of nodes of the automaton, with $m = k+1$. Let $n = \lceil \frac{t}{m} \rceil$ and $t \geq m$. Then $|Sp_t| \leq \binom{m-1+n}{n}$.

Proof: We proceed by induction. For $n=1$, there are $m = k+1$ possibilities and $|Sp_m| = \binom{m}{1}$. Assuming for $n-1$ it holds $|Sp_{(n-1).m}| = \binom{m-1+n-1}{n-1}$, we have $|Sp_{n.m}| = \binom{m-1+n-1}{n-1} + \binom{m-1+n-2}{n-1} + \ldots + 1 = \binom{m-1+n}{n}$. Since $|Sp_t| \leq |Sp_{n.m}|$ the result follows. ∎

It follows from the above lemma that the computational complexity of Algorithm 2 is of the order of $\mathcal{O}(m^n)$.

Lemma 4: For any automaton which allows at most k consecutive dropouts ($k \in \mathbb{N}$), all signals constructed by Algorithm 2 are admissible.

[6] if σ_1 and σ_2 are $m-$minimal signals, then $\sigma_1 \preceq_{lex} \sigma_2$ if and only if there exists $i \in \mathbb{N} \cup \{0\}$ such that $\sigma_1(i) = 0$ and $\sigma_2(i) = 1$ and $\sigma_1(j) = \sigma_2(j)$, for all $j < i, j \in \mathbb{N} \cup \{0\}$.

Proof: Let $m = k+1$ and $n = \lceil \frac{t}{m} \rceil$. We proceed by induction. For $n = 1$, all the signals in Sp_m are admissible by construction. Suppose the result is true for $Sp_{(n-1).m}$. Suppose $i = n$ and $\sigma \in Sp_{n.m}$. Let $\sigma = \sigma_j \hat{\sigma}$ where $\hat{\sigma} \in Sp_{(n-1).m}$. From Algorithm 2, by the lexicographic order used to construct $Sp_{n.m}$, σ is admissible. Hence all signals of $Sp_{n.m}$ are admissible. Therefore from the construction, all the signals in Sp_t are admissible. ■

Naturally, there are minimal signals which are not sparse minimal, as illustrated below.

Example 4: Let $\mathcal{A}$ be an automaton where at most two successive dropouts are allowed. This automaton is defined by a graph consisting of three nodes, hence $m = 3$. Let $\sigma|_0^8 = 100110010$. σ is a minimal signal but it is not sparse minimal since it can not be generated from $3-$minimal signals. For illustration purposes, we present below the sparse minimal signals of length 6,

$$001001, 001010, 001100, 010010, 010100, 100100.$$

Definition 12: We define an upper bound on $\lambda(t)$ as follows

$$\bar{\lambda}(t) := \min_{\sigma \in Sp_{\sigma(t)}} \{\lambda_{min}(W_{\sigma(t)})\} \tag{12}$$

Since $Sp_{\sigma(t)} \subset M_{\sigma(t)}$, it is clear that $\lambda(t) \leq \bar{\lambda}(t)$.

In the following proposition, we identify a simple class of automata where the heuristic is exact and provides all minimal signals.

Proposition 2: Let $\mathcal{A}$ be an automaton whose graph consists of m nodes. Assume that the automaton only allows periodic signals of period m. Then $M_{\sigma(t)} = Sp_{\sigma(t)}$. Moreover, $\lambda(t) = \bar{\lambda}(t)$.

Proof: Note that for any automaton, $Sp_{\sigma(t)} \subseteq M_{\sigma(t)}$. Note that since all the signals are periodic of period m, they are defined uniquely by the first m values. Thus a signal is minimal if and only if the first block of length m is $m-$minimal. Thus any minimal signal is also sparse minimal. Hence, $Sp_{\sigma(t)} \subseteq M_{\sigma(t)}$. The second statement follows from Equations (12) and (11). ■

V. Examples

Example 5: We consider the automaton in Figure 3, capturing the case for which at most one consecutive dropout is allowed. For this example, and for a range of time intervals, we compute the admissible switching sequences, the minimal switching sequence (Algorithm 1) and the sparse minimal switching sequences (Algorithm 2). Figure 6 shows a plot of the number of all signals (blue curve), minimal signals (red curve) and sparse minimal signals (green curve) as a function of time.

The cardinality of the generated switching sequences for the case where at most two consecutive dropouts are allowed is presented in Figure 7 in logarithmic scale.

Moreover, by considering the (A, B) pair with

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 4 \\ 0 & -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ 4 & 5 \\ -1 & 2 \end{bmatrix},$$

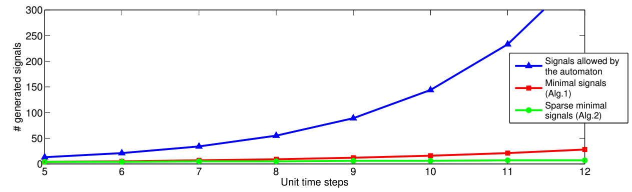

Fig. 6: The number of signals vs time for the case of at most one consecutive dropouts.

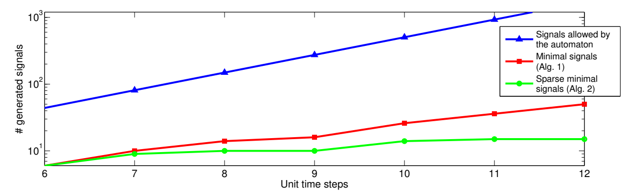

Fig. 7: Logarithmic plot of the number of signals vs time for the case of at most two consecutive dropouts.

we compute the CG metric $\lambda(t)$ (7) by calculating the minimum eigenvalues of the CG for the generated switching sequences for all three cases, for both systems having at most one or two consecutive dropouts. Figures 8 and 9 show the respective computational times in logarithmic scale.

Example 6: In this numerical experiment, we compute the relative error between the minimum eigenvalue $\lambda(t)$ and the approximating quantity $\bar{\lambda}(t)$ that is calculated utilizing the switching sequences generated by Algorithm 2. We denote the relative error percentage by ϵ. For details regarding the generation of random (A, B) pairs, we refer the reader to Appendix B. In Table IV, we list the relative error percentage between $\lambda(t)$ (computed by Algorithm 1) and $\bar{\lambda}(t)$ (computed by Algorithm 2) for random pairs of A and B of different sizes where the switching signals can have at most 1, 2 or 3 successive dropouts. We list the sample size and the percentage of the number of samples with relative percentage error ranging from zero to more than 90% relative error.

We observe very good performances for our heuristic in most cases (Table IV). Nevertheless, in some cases, the performances of the heuristic is degraded. The idea behind the heuristic was to exploit the sparsity in the minimal signals. However, in some cases sparsity is not directly related with minimum energy and this explains a poorer performance in these cases. In the future, we plan to investigate the reasons for this behavior.

Example 7: The following example gives an instance where a difference between $\lambda(t)$ and $\bar{\lambda}(t)$ exists. Let

$$A = \begin{bmatrix} -1.6747 & -0.4747 & -0.1930 & -0.0423 & -0.3439 \\ -0.4747 & -0.0990 & 1.5974 & -0.1492 & -0.2069 \\ -0.1930 & 1.5974 & -0.3087 & -0.1067 & 0.0434 \\ -0.0423 & -0.1492 & -0.1067 & -1.8911 & 0.5842 \\ -0.3439 & -0.2069 & 0.0434 & 0.5842 & 1.3735 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4497 & 0.2410 \\ 0.8504 & -0.5313 \\ 0.6386 & -0.3696 \\ -0.0176 & 2.6961 \\ -0.1881 & -1.1107 \end{bmatrix}.$$

TABLE IV: Relative % error ϵ between $\lambda(t)$ (using minimal signals, Alg. 1) and $\bar{\lambda}(t)$ (using sparse minimal signals, Alg. 2)

(# dropouts, dim, # inputs, # samples)	$\epsilon = 0$	$0\% \leq \epsilon \leq 20\%$	$21\% \leq \epsilon \leq 60\%$	$61\% \leq \epsilon \leq 90\%$	$\epsilon > 90\%$
(1,5,3,2903)	95.63	99.38	0.45	0.14	0.035
(1,10,4,2822)	97.45	99.58	0.35	0.071	0
(2,5,2,2906)	94.83	96.56	1.14	0.24	2.065
(2,10,5,2260)	97.39	99.47	0.27	0	0.27
(3,5,3,2320)	100	100	0	0	0
(3,8,3,2878)	99.90	99.93	0.03	0.03	0
(1,500,250,200)	88	91.5	1.5	1.5	5.5
(1,1000,500,20)	80	95	0	0	5

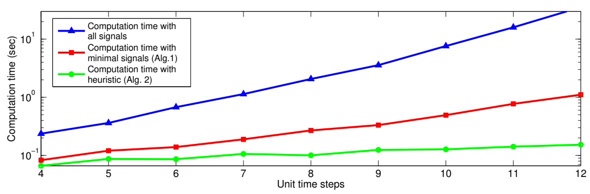

Fig. 8: Computation time in seconds vs number of time steps for the case of at most one consecutive dropouts.

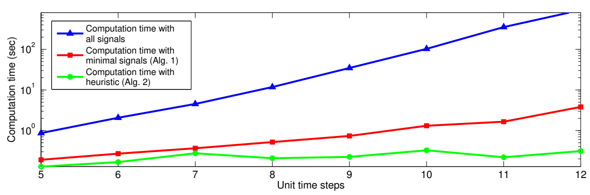

Fig. 9: Computation time in seconds vs number of time steps for the case of at most two consecutive dropouts.

We consider the case where nor more than two successive dropouts are allowed. We observe that for $t = 11$, the actual minimum eigenvalue of the CG is $\lambda(t) = 0.0088$, while $\bar{\lambda}(t) = 0.4332$. The corresponding minimizing switching signals are $\sigma^* = 001010101010$ and $\bar{\sigma} = 001001010100$ respectively, while the time required to compute the minimum eigenvalue was $6.6906s$ and $0.7605s$. Note that even though there are more non zero entries in the signal σ^* than $\bar{\sigma}$, $\lambda_{min}(W_{\sigma^*(11)}) < \lambda_{min}(W_{\bar{\sigma}(11)})$.

VI. Observability metrics and computations

We consider non-idealities in the transmission of information obtained by sensors over a communication network for discrete LTI systems [5], [6], [18], [22]. We consider the discrete linear system subject to packet dropouts

$$\begin{aligned} x(t+1) &= Ax(t) \\ y(t) &= \begin{cases} Cx(t), & \text{if } \sigma(t) = 1, \\ \emptyset, & \text{if } \sigma(t) = 0. \end{cases} \end{aligned} \tag{13}$$

where the switching signal σ is governed by an automaton. The Observability Gramian (OG) associated with discrete LTI systems is defined as $W_t^o := \sum_{i=0}^{t} (A^\top)^i C^\top C A^i$ [1]. If W_t^o is singular, then the states in the null space of W_t^o are unobservable. Nearly singular OG results in poor results for the state estimation algorithms [31]. Suppose $y = \begin{bmatrix} y(0)^\top & y(1)^\top & \cdots & y(t)^\top \end{bmatrix}^\top$. It follows that $y^\top y = x(0)^\top W_t^o x(0)$. A state lying in the direction associated with the eigenvector corresponding to the minimum eigenvalue of the OG corresponds to the least output energy among all states hence it is the least observable state [13]. The inverse of the minimum eigenvalue of the OG gives the maximum estimation uncertainty whereas the trace of the inverse of the OG gives the average estimation uncertainty [32].

The problem of optimal sensor placements, dual to the optimal actuator placement, is studied in the literature for large scale systems [29], [26], [31], [32]. We refer the reader to [35] for works on optimal sensor/actuator placement for descriptor systems. In [18], [22], a similar problem is studied, however with a probabilistic model, while our non-idealities are generated by an automaton. In the future, we plan to investigate the relations between both models.

We define the associated OG for (13) as

$$W^o_{\sigma(t)} := \sum_{i=0}^{t} \sigma(i)(A^\top)^i C^\top C A^i.$$

Analogously to (7) and Definition 9, we define $\lambda^o(t)$ and $S^o_{\sigma(t)}$ respectively for the OG. The exact counterpart of the controllability Theorem 2, holds for the OG. Using results of previous sections, one can compute and approximate $\lambda^o(t)$ using the minimal signals and the sparse minimal signals respectively. We can also compute quantities $\gamma^o(t)$ and $\delta^o(t)$. The observability and controllability problems are perfect duals of each other in LTI systems. However, it was observed in [6] that the duality between controllability and observability for (2) only holds when A is invertible, which is our standing Assumption 1.

VII. Conclusions

We characterized quantitatively the impact of packet dropouts in the communication link of closed-loop LTI systems by studying several metrics of an appropriately defined Controllability Gramian. Using fairly standard tools from Matrix and Systems theory, we provided upper and lower bounds on these metrics that approximate efficiently their asymptotic behavior. Moreover, to deal with the, unavoidable, combinatorial explosion of admissible switching sequences, we defined a partial order and showed that it is sufficient only to compute a subset of minimal signals to evaluate the CG metrics. These signals are constructed using the method of $T-$product lifts on graphs. We also gave a heuristic for a class of automata (which allows no more than a fixed number of consecutive dropouts) to reduce the computational complexity further and showed by numerical examples that it works well in general.

It is in our future research plans to exploit the theoretical and computational framework established to approach co-design problems in networked control systems, by identifying trade-offs between the quality of the communication networks and the degradation of performance in terms of the reachability and controllability energy. Moreover, we wish to explore the possible connection of our approach to the probabilistic framework for packet dropouts. Last, we would like to investigate the implications on our results when a more general notion of reachability is considered, namely when no or partial information of the occurrence of packet dropouts is available for control design.

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Appendix

A. Proof of Proposition 1

(i) Let $t_1 \leq t_2$ and $\sigma|_i^j$ denote subsequence of σ from i to j. Let $S_{\sigma(t_1)} := \text{argmin}_{\sigma(t_1)}(\lambda_{min}(W_{\sigma(t_1)}))$, $S_{\sigma(t_2)} := \text{argmin}_{\sigma(t_2)}(\lambda_{min}(W_{\sigma(t_2)}))$ and consider $\sigma_1^* \in S_{\sigma(t_1)}$ and $\sigma_2^* \in S_{\sigma(t_2)}$. Thus $\lambda(t_1) = \lambda_{min}(W_{\sigma_1^*(t_1)})$ and $\lambda(t_2) = \lambda_{min}(W_{\sigma_2^*(t_2)})$. We assume that $\lambda(t_2) < \lambda(t_1)$. Then, from Lemma 2, it follows

$$\begin{aligned}\lambda(t_2) &= \lambda_{min}(W_{\sigma_2^*(t_2)}) = \lambda_{min}(W_{\sigma_2^*|_0^{t_2-t_1-1}} + W_{\sigma_2^*|_{t_2-t_1}^{t_2}}) \\ &\geq \lambda_{min}(W_{\sigma_2^*|_{t_2-t_1}^{t_2}}), \end{aligned} \tag{14}$$

where $W_{\sigma_2^*|_{t_2-t_1}^{t_2}} = \sigma_2^*(t_2)BB^\top + \sigma_2^*(t_2-1)ABB^\top A^\top + \ldots + \sigma_2^*(t_2-t_1)A^{t_1}BB^\top(A^\top)^{t_1}$. From (5), we have $W_{\sigma_1^*(t_1)} = \sigma_1^*(t_1)BB^\top + \sigma_1^*(t_1-1)ABB^\top A^\top + \ldots + \sigma_1^*(0)A^{t_1}BB^\top(A^\top)^{t_1}$. Next, define $\sigma_3|_0^{t_1} = \sigma_2^*|_{t_2-t_1}^{t_2}$. From (14) it follows that $\lambda_{min}(W_{\sigma_3(t_1)}) = \lambda_{min}(W_{\sigma_2^*|_{t_2-t_1}^{t_2}}) \leq \lambda(t_2) < \lambda(t_1)$ which is a contradiction to the assumption that $\lambda(t_1) = \lambda_{min}(W_{\sigma_1^*(t_1)}) = \min_\sigma(\lambda_{min}(W_{\sigma(t_1)}))$. Thus, $\lambda(t_1) \leq \lambda(t_2)$.

(ii) By assumption, there is at least one eigenvalue $|\alpha| < 1$ of A. Let v be the associated unitary eigenvector. For any $t \geq 1$, by setting $\sigma^\star(t) = 11...1$ we have from (6), (7) $\lambda(t) \leq \lambda_{\min}(W_{\sigma^\star(t)}) \leq v^\top W_{\sigma^\star(t)} v = \sum_{i=1}^t v^\top A^i BB^\top (A^\top)^i v = \sum_{i=1}^t \alpha^{2i} v^\top BB^\top v \leq \sum_{i=1}^t \alpha^{2i}\|B^\top v\|$, consequently, taking the limit as $t \to \infty$ it follows that $\lim_{t\to\infty}\lambda(t) \leq \frac{\|B^\top v\|}{1-\alpha^2}$. Thus, by taking into account (i), $\{\lambda(t)\}$ necessarily converges to a finite number λ. What remains is to show that the sequence $\{\lambda(t)\}$ is not convergent when A has all eigenvalues lying on or outside the unit circle. We assume that λ is bounded in this case. Then, there is a unitary vector $v \in \mathbb{R}^n$ and a monotonically increasing sequence $\{n_i\}$ of integers such that the sequence $\{v^\top A^{n_i} BB^\top (A^\top)^{n_i} v\}$ converges to 0 as $n_i \to \infty$. However, by assumption, $\lim_{n_i\to\infty} v^\top A^{n_i} \neq 0$. Thus, it necessarily holds that $\lim_{n_i\to\infty} v^\top A^{n_i} \in \ker(B)$. Let $\bar{v}^\top := \lim_{n_i\to\infty} v^\top A^{n_i}$. Then, $\bar{v}^\top$ is left $A-$invariant and also $\bar{v}^\top B = 0$, contradicting controllability of (1). Therefore, the sequence $\{\lambda(t)\}$ does not converge when all eigenvalues of A are outside the unit circle. ■

B. Construction of the (A,B) pairs, Example 6

We considered random (A, B) pairs such that A is non singular and (A, B) is controllable. Along each row of Table IV, the total number of (A, B) pairs were generated using the following three cases: (i) First, we used `randi` from Matlab to generate diagonal matrices with entries in the interval $[-25, 25]$ and scaled them so the eigenvalues of A were in $[-2.5, 2.5]$. We then used random orthogonal matrices (`orth(randn(.))` from Matlab) to generate A matrices from these diagonal matrices. Entries of B were chosen to be random (using randn from Matlab). (ii) Second, we used `randn` from Matlab to generate (A, B) pairs. (iii) Third, we again used `randn` from Matlab to generate (A, B) pairs but scaled the entries of A by a factor of 10 so that A has unstable eigenvalues. For the first three instances and the sixth instance, 1000 samples were generated for each of the three cases mentioned above from which (A, B) pairs with singular A matrix were removed. The total sample size was formed by adding the samples from these three cases. For the fourth and the fifth instance, we took 800 samples each. For the seventh row in the table, we took 80 samples from each type and for the last row, we took 10 samples each to compute $\lambda(t)$ and $\bar{\lambda}(t)$. All computations were performed in Matlab 2012a in a laptop computer, with 4G memory and a AMD Quad A4-5000 APU processor.