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McKee, A., Todorov, I. G., & Turowska, L. (2018). Herz-Schur multipliers of dynamical systems. *Advances in Mathematics*, *331*, 387-438. https://doi.org/10.1016/j.aim.2018.04.002

Published in:
Advances in Mathematics

Document Version:
Peer reviewed version

HERZ-SCHUR MULTIPLIERS OF DYNAMICAL SYSTEMS

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Abstract. We extend the notion of a Herz-Schur multiplier to the setting of non-commutative dynamical systems: given a C*-algebra A, a locally compact group G, and an action α of G on A, we define transformations on the reduced crossed product of A by α which, in the case $A = \mathbb{C}$, reduce to the classical Herz-Schur multipliers. We introduce Schur A-multipliers, establish a characterisation that generalises the classical descriptions of Schur multipliers, and present a transference theorem in the new setting, identifying isometrically the Herz-Schur multipliers of the dynamical system (A, G, α) with the invariant part of the Schur A-multipliers. We discuss special classes of Herz-Schur multipliers, in particular, those associated to a locally compact abelian group G and its canonical action on the C*-algebra $C^*(\Gamma)$ of the dual group Γ.

Contents

1. Introduction

The notion of a Schur multiplier has its origins in the work of I. Schur in the early 20$^{\text{th}}$ century, and is based on entry-wise (or Hadamard) product of matrices. More specifically, a bounded function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is called a Schur multiplier if $(\varphi(i, j)a_{i,j})$ is the matrix of a bounded linear operator on ℓ^2 whenever $(a_{i,j})$ is such. A concrete description of Schur multipliers, which found numerous applications thereafter, was given by A. Grothendieck in his *Résumé* [12] (see also [32]). A measurable version of Schur multipliers was

2010 *Mathematics Subject Classification.* Primary 46L07, 47L05; Secondary 47L65, 47L10.

developed by M. S. Birman and M. Z. Solomyak (see [3] and the references therein) and V. V. Peller [30]. More concretely, given standard measure spaces (X,μ) and (Y,ν) and a function $\varphi : X \times Y \to \mathbb{C}$, one defines a linear transformation S_φ on the space of all Hilbert-Schmidt operators from $H_1 = L^2(X,\mu)$ to $H_2 = L^2(Y,\nu)$ by multiplying their integral kernels by φ; if S_φ is bounded in the operator norm (in which case φ is called a *measurable Schur multiplier*), it is extended to the space $\mathcal{K}(H_1,H_2)$ of all compact operators from H_1 into H_2 by continuity. The map S_φ is defined on the space $\mathcal{B}(H_1,H_2)$ of all bounded linear operators from H_1 into H_2 by taking the second dual of the constructed map on $\mathcal{K}(H_1,H_2)$. A characterisation of measurable Schur multipliers, extending Grothendieck's result, was obtained in [13] and [30] (see also [17] and [40]). Namely, a function $\varphi \in L^\infty(X \times Y)$ was shown to be a Schur multiplier if and only if φ coincides almost everywhere with a function of the form $\sum_{k=1}^\infty a_k(x)b_k(y)$, where $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are families of essentially bounded measurable functions such that $\operatorname{esssup}_{x\in X}\sum_{k=1}^\infty |a_k(x)|^2 < \infty$ and $\operatorname{esssup}_{y\in Y}\sum_{k=1}^\infty |b_k(y)|^2 < \infty$.

Among the large number of applications of Schur multipliers is the description of the space $M^{\mathrm{cb}}A(G)$ of completely bounded multipliers (also known as Herz-Schur multipliers) of the Fourier algebra $A(G)$ of a locally compact group G, introduced by J. de Cannière and U. Haagerup in [7]. Namely, as shown by M. Bożejko and G. Fendler [5] , $M^{\mathrm{cb}}A(G)$ can be isometrically identified with the space of all Schur multipliers on $G \times G$ of Toeplitz type. An alternative proof of this result was given by P. Jolissaint [15].

Herz-Schur multipliers have been highly instrumental in operator algebra theory, providing the route to defining and studying a number of approximation properties of group C*-algebras and group von Neumann algebras (we refer the reader to [20], [6] and [18]). Here one uses the fact that every Herz-Schur multiplier on a locally compact group G gives rise to a (completely bounded) map on the von Neumann algebra $\mathrm{VN}(G)$ of G, leaving invariant the reduced C*-algebra $C_r^*(G)$ of G.

In view of the large number of applications of Herz-Schur multipliers in operator algebra theory, it is natural to seek generalisations going beyond the context of group algebras. The main goal of this paper is to extend the notion of Herz-Schur multipliers to the setting of non-commutative dynamical systems. Given a C*-algebra A, a locally compact group G, and an action α of G on A, we define transformations on the (reduced) crossed product $A \rtimes_{r,\alpha} G$ of A by G, which, in the case $A = \mathbb{C}$, reduce to the classical Herz-Schur multipliers and, in the case of a discrete group G, to multipliers defined recently by E. Bedos and R. Conti in [2]. More generally, we introduce Schur A-multipliers which, in the case $A = \mathbb{C}$, reduce to the classical measurable Schur multipliers. In Section 2, we establish a characterisation of Schur A-multipliers that generalises the classical description of Schur multipliers (Theorem 2.6). We exhibit a large class of Schur A-multipliers defined in terms of Hilbert A-bimodules, and show that it exhausts all Schur

A-multipliers in the case A is finite-dimensional (Theorem 2.7). In Section 3, we prove a transference theorem in the new setting, identifying isometrically the Herz-Schur multipliers of the dynamical system (A, G, α) with the invariant part of the Schur A-multipliers (see Theorems 3.8 and 3.18). We introduce bounded multipliers of (A, G, α) and relate them to Herz-Schur (A, G, α)-multipliers, extending a corresponding result from [7]. In Section 4, we provide a description of a more general and closely related class of multipliers, namely, Herz-Schur multipliers associated with weak* closed crossed products, as the commutant of the scalar valued Herz-Schur multipliers associated with elements of $M^{\mathrm{cb}}A(G)$ (Theorem 4.3). While in the case $A = \mathbb{C}$ this description is straightforward, here we need to use structure theory of crossed products and some recent results from [1].

The rest of the paper is devoted to special classes of Herz-Schur multipliers. Namely, in Section 5, we consider multipliers naturally associated with the Haagerup tensor product of two copies of A, and multipliers defined on groupoids. In the former case, we relate our notion to examples of Herz-Schur multipliers exhibited in [2, Theorem 4.5] in the case of a discrete group. In the latter case, we show that completely bounded multipliers of the Fourier algebra of a groupoid, defined in [35], form a subclass of the class of Herz-Schur multipliers introduced in the present work.

The results in Section 6 were our original motivation for the present paper. Here, we consider the case of a locally compact abelian group G and its canonical action α on the C*-algebra $C^*(\Gamma)$ of the dual group Γ. We focus on a special class $\mathfrak{F}(G)$ of Herz-Schur multipliers, which we call convolution multipliers, and its natural subclass $\mathfrak{F}_\theta(G)$ of weak* extendible convolution multipliers. We show that the Fourier-Stieltjes algebras $B(G)$ and $B(\Gamma)$ can both be viewed as subspaces of $\mathfrak{F}_\theta(G)$, while $\mathfrak{F}_\theta(G)$ is a subspace of their Fubini product. When the crossed product of $C^*(\Gamma) \rtimes_\alpha G$ is canonically identified with the space $\mathcal{K}(L^2(G))$ of all compact operators on $L^2(G)$, the elements u of $B(G)$ give rise to the measurable Schur multipliers corresponding to u *via* the aforementioned Bożejko-Fendler classical transference theorem, while the elements of $B(\Gamma)$ correspond to a well-known class of completely bounded maps, arising from a representation of the measure algebra $M(G)$ of G on $\mathcal{K}(L^2(G))$, studied in a variety of contexts in both operator algebra theory and quantum information theory, and by a number of authors including F. Ghahramani [11], M. Neufang and V. Runde [26], M. Neufang, Zh.-J. Ruan and N. Spronk [27] and E. Størmer [41]. The main result of the section are Theorem 6.7 and 6.10, where we identify the set $\mathfrak{F}_\theta(G)$, and an associated subset, of convolution multipliers of G as subsets of the joint commutant of the two families described above.

The paper uses various notions from Operator Space Theory; we refer the reader to [4], [9], [28] or [33] for the basics. For background and notation on crossed products, which will be needed in Sections 3, 5 and 6, we refer the reader to [45].

We finish this section with setting some notation. If $\mathcal{E}$ and $\mathcal{F}$ are vector spaces, we let $\mathcal{E} \odot \mathcal{F}$ be their algebraic tensor product. For a Banach space $\mathcal{X}$, we let $\mathcal{B}(\mathcal{X})$ (resp. $\mathcal{K}(\mathcal{X})$) be the algebra of all bounded linear (resp. compact) operators on $\mathcal{X}$, and denote by $I_{\mathcal{X}}$ the identity operator on $\mathcal{X}$. If H and K are Hilbert spaces, we denote by $H \otimes K$ their Hilbertian tensor product; for operators $S \in \mathcal{B}(H)$ and $T \in \mathcal{B}(K)$, we let $S \otimes T$ be the bounded operator on $H \otimes K$ given by $(S \otimes T)(\xi \otimes \eta) = S\xi \otimes T\eta$. The (norm closed) spacial tensor product of two (norm closed) operator spaces $\mathcal{U} \subseteq \mathcal{B}(H)$ and $\mathcal{V} \subseteq \mathcal{B}(K)$ will be denoted by $\mathcal{U} \otimes \mathcal{V}$. If $\mathcal{U}$ and $\mathcal{V}$ are weak* closed, their weak* spacial tensor product will be denoted by $\mathcal{U} \bar{\otimes} \mathcal{V}$.

2. Schur A-multipliers

Let (X, μ) be a standard measure space; this means that μ is a Radon measure with respect to some complete metrisable separable locally compact topology (called an *admissible* topology) on X. For $p = 1, 2$ and a Banach space $\mathcal{E}$, we write $L^p(X, \mathcal{E})$ for the corresponding Lebesgue space of all (equivalence classes of) weakly measurable p-summable $\mathcal{E}$-valued functions on X (see *e.g.* [45, Appendix B]). If H and K are separable Hilbert spaces and $\mathcal{E} \subseteq \mathcal{B}(H, K)$ is a weak* closed subspace, let $L^\infty(X, \mathcal{E})$ be the space of all (equivalence classes of) bounded $\mathcal{E}$-valued functions T on X such that, for every $\xi \in H$ and every $\eta \in K$, the functions $x \to T(x)\xi$ and $x \to T(x)^*\eta$ are weakly measurable. Note that $L^\infty(X, \mathcal{E})$ contains all bounded weakly measurable functions from X into $\mathcal{E}$. Analogously to [42, Chapter IV, Section 7], we often identify an element g of $L^\infty(X, \mathcal{E})$ with the operator D_g from $L^2(X, H)$ into $L^2(X, K)$ given by $(D_g\xi)(x) = g(x)(\xi(x))$, $x \in X$.

We write $\|\cdot\|_p$ for the norm on $L^p(X, \mathcal{E})$, $p = 1, 2, \infty$. In the case $\mathcal{E}$ coincides with the complex field $\mathbb{C}$, we simply write $L^p(X)$. If $f \in L^p(X)$ and $a \in \mathcal{E}$, we let $f \otimes a \in L^p(X, \mathcal{E})$ be the function given by $(f \otimes a)(x) = f(x)a$, $x \in X$.

We fix throughout the section a separable Hilbert space H. For $a \in L^\infty(X)$, let $M_a \in \mathcal{B}(L^2(X))$ be the operator given by $M_a\xi = a\xi$; set

$$\mathcal{D}_X = \{M_a : a \in L^\infty(X)\}.$$

Note that the identification $L^2(X) \otimes H \equiv L^2(X, H)$ yields a unitary equivalence between $L^\infty(X, \mathcal{B}(H))$ and $\mathcal{D}_X \bar{\otimes} \mathcal{B}(H)$ [42, Theorem 7.10].

Let (Y, ν) be a(nother) standard measure space. We equip the direct products $X \times Y$ and $Y \times X$ with the corresponding product measures. It is easy to see that, if $k \in L^2(Y \times X, \mathcal{B}(H))$ and $\xi \in L^2(X, H)$ then, for almost all $y \in Y$, the function $x \to k(y, x)\xi(x)$ is weakly measurable; moreover,

$$\begin{aligned} \int_X \|k(y,x)\xi(x)\| d\mu(x) &\leq \int_X \|k(y,x)\| \|\xi(x)\| d\mu(x) \\ &\leq \|\xi\|_2 \left(\int_X \|k(y,x)\|^2 d\mu(x) \right)^{1/2}. \end{aligned} \tag{1}$$

It follows that the formula

$$(T_k\xi)(y) = \int_X k(y,x)\xi(x)d\mu(x), \quad y \in Y, \tag{2}$$

defines a (weakly measurable) function $T_k\xi : Y \to H$.

Lemma 2.1. *Let $k \in L^2(Y \times X, \mathcal{B}(H))$. Equation (2) defines a bounded operator $T_k : L^2(X,H) \to L^2(Y,H)$ with $\|T_k\| \le \|k\|_2$. Moreover, $T_k = 0$ if and only if $k = 0$ almost everywhere.*

Proof. Let $\xi \in L^2(X,H)$. Then, by (1),

$$\begin{aligned} \|T_k\xi\|^2 &= \int_Y \|T_k\xi(y)\|^2 d\nu(y) \\ &\le \|\xi\|_2^2 \int_Y \int_X \|k(y,x)\|^2 d\mu(x)d\nu(y) = \|k\|_2^2\|\xi\|_2^2. \end{aligned}$$

Thus, T_k is bounded and its norm does not exceed $\|k\|_2$.

It is clear that if $k = 0$ almost everywhere then $T_k = 0$. Conversely, suppose that $T_k = 0$. Choose a countable dense subset $\{e_i\}_{i\in\mathbb{N}}$ of H. If $\xi \in L^2(X)$ and $\eta \in L^2(Y)$ then

$$\int_{X\times Y} \langle k(y,x)e_i, e_j\rangle \xi(x)\overline{\eta(y)}dxdy = \langle T_k(\xi \otimes e_i), \eta \otimes e_j\rangle = 0,$$

and it follows that $\langle k(y,x)e_i, e_j\rangle = 0$ almost everywhere, for all i, j. Since $k(y,x)$ is a bounded operator, $k(y,x) = 0$ for almost all (x,y). $\square$

If $\mathcal{M} \subseteq \mathcal{B}(H)$ is a C*-subalgebra, let

$$\mathcal{S}_2(Y \times X, \mathcal{M}) = \{T_k : k \in L^2(Y \times X, \mathcal{M})\}.$$

Note that, if $w \in L^2(Y \times X)$ and $a \in \mathcal{M}$ then

$$T_{w\otimes a} = T_w \otimes a. \tag{3}$$

Letting

$$\mathcal{K} \stackrel{def}{=} \mathcal{K}(L^2(X), L^2(Y))$$

be the space of all compact operators from $L^2(X)$ into $L^2(Y)$, we have that $\mathcal{S}_2 \odot \mathcal{M}$ is norm dense in $\mathcal{K} \otimes \mathcal{M}$ (here $\mathcal{S}_2$ denotes the space of all Hilbert-Schmidt operators from $L^2(X)$ into $L^2(Y)$). We conclude that $\mathcal{S}_2(Y\times X, \mathcal{M})$ is norm dense in $\mathcal{K}\otimes\mathcal{M}$ and equip it with the operator space structure arising from its inclusion into $\mathcal{K} \otimes \mathcal{M}$.

We fix throughout the section a non-degenerate separable C-algebra $A \subseteq \mathcal{B}(H)$.* If B is a(nother) C*-algebra, we denote by $CB(A,B)$ the space of all completely bounded linear maps from A into B and write $CB(A) = CB(A,A)$. A function $\varphi : X \times Y \to CB(A, \mathcal{B}(H))$ will be called *pointwise measurable* if, for every $a \in A$, the function $(x,y) \to \varphi(x,y)(a)$ from $X \times Y$ into $\mathcal{B}(H)$ is weakly measurable. Let $\varphi : X \times Y \to CB(A, \mathcal{B}(H))$ be a bounded pointwise measurable function. For $k \in L^2(Y \times X, A)$, let $\varphi \cdot k : Y \times X \to \mathcal{B}(H)$ be the function given by

$$(\varphi \cdot k)(y,x) = \varphi(x,y)(k(y,x)), \quad (y,x) \in Y \times X.$$

It is easy to show that $\varphi \cdot k$ is weakly measurable; since φ is bounded, $\varphi \cdot k \in L^2(Y \times X, \mathcal{B}(H))$ and

$$\|\varphi \cdot k\|_2 \le \|\varphi\|_\infty \|k\|_2.$$

Let

$$S_\varphi : \mathcal{S}_2(Y \times X, A) \to \mathcal{S}_2(Y \times X, \mathcal{B}(H))$$

be the linear map given by

$$S_\varphi(T_k) = T_{\varphi \cdot k}, \quad k \in L^2(Y \times X, A).$$

By Lemma 2.1, S_φ is well-defined.

Definition 2.2. *A bounded pointwise measurable map*

$$\varphi : X \times Y \to CB(A, \mathcal{B}(H))$$

will be called a Schur A-multiplier *if the map S_φ is completely bounded.*

It follows from the discussion after Lemma 2.1 that a bounded pointwise measurable function $\varphi : X \times Y \to CB(A, \mathcal{B}(H))$ is a Schur A-multiplier if and only if the map S_φ possesses a completely bounded extension to a map from $\mathcal{K} \otimes A$ into $\mathcal{K} \otimes \mathcal{B}(H)$ (which will still be denoted by S_φ).

We let $\mathfrak{S}(X, Y; A)$ be the space of all Schur A-multipliers and endow it with the norm

$$\|\varphi\|_{\mathfrak{S}} = \|S_\varphi\|_{\mathrm{cb}}; \tag{4}$$

it follows from Lemma 2.1 that if $S_\varphi = 0$ then $\varphi = 0$ almost everywhere, and hence (4) indeed defines a norm on $\mathfrak{S}(X, Y; A)$.

Note that Schur $\mathbb{C}$-multipliers coincide with the classical (measurable) Schur multipliers [30], [17].

A special role in our considerations will be played by Schur A-multipliers φ for which $\varphi(x,y) \in CB(A)$ for all $(x,y) \in X \times Y$, that is, ones for which the range of $\varphi(x,y)$ is in A. In this case, S_φ is a map on $\mathcal{S}_2(Y \times X, A)$. We let $\mathfrak{S}_0(X, Y; A)$ be the space of all such Schur A-multipliers. The next proposition shows that $\mathfrak{S}_0(X, Y; A)$ does not depend on the faithful *-representation of A.

Proposition 2.3. *Let $\theta : A \to \mathcal{B}(K)$ be a faithful *-representation of A on a separable Hilbert space K. A bounded pointwise measurable map $\varphi : X \times Y \to CB(A)$ is a Schur A-multiplier if and only if the (bounded pointwise measurable) map $\varphi_\theta : X \times Y \to CB(\theta(A))$, given by $\varphi_\theta(x,y)(\theta(a)) = \theta(\varphi(x,y)(a))$, $a \in A$, is a Schur $\theta(A)$-multiplier. Moreover, $\|\varphi\|_{\mathfrak{S}} = \|\varphi_\theta\|_{\mathfrak{S}}$.*

Proof. Let $B = \theta(A)$. Note that the map $\mathrm{id} \otimes \theta : \mathcal{K} \odot A \to \mathcal{K} \odot B$ given by $(\mathrm{id} \otimes \theta)(T \otimes a) = T \otimes \theta(a)$ extends to a complete isometry from $\mathcal{K} \otimes A$

onto $\mathcal{K} \otimes B$ [4]. Let $k \in L^2(Y \times X, A)$; then the map $k_\theta = \theta \circ k$ belongs to $L^2(Y \times X, B)$. We claim that

$$T_{k_\theta} = (\mathrm{id} \otimes \theta)(T_k). \tag{5}$$

To see (5), note first that, by (3), it holds when $k = k' \otimes a$ for some $k' \in L^2(Y \times X)$ and $a \in A$; hence, by linearity, it holds if $k \in L^2(Y \times X) \odot A$. By [42, Proposition 7.4], there exists a sequence $(k_i)_{i \in \mathbb{N}} \subseteq L^2(Y \times X) \odot A$ such that $\|k_i - k\|_2 \to_{i\to\infty} 0$. It follows that $\|\theta \circ k_i - \theta \circ k\|_2 \to_{i\to\infty} 0$. By Lemma 2.1, $T_{k_i} \to T_k$ and $T_{\theta\circ k_i} \to T_{\theta \circ k}$ in the operator norm. Thus, $(\mathrm{id} \otimes \theta)(T_{k_i}) \to (\mathrm{id} \otimes \theta)(T_k)$ in the operator norm, and we conclude that $T_{k_\theta} = (\mathrm{id} \otimes \theta)(T_k)$.

By (5),

$$(\mathrm{id} \otimes \theta)(S_\varphi(T_k)) = S_{\varphi_\theta}(T_{k_\theta});$$

in other words,

$$S_{\varphi_\theta} = (\mathrm{id} \otimes \theta) \circ S_\varphi \circ (\mathrm{id} \otimes \theta^{-1}).$$

It follows that S_φ is completely bounded if and only if S_{φ_θ} is so and that, in this case, $\|\varphi\|_{\mathfrak{S}} = \|\varphi_\theta\|_{\mathfrak{S}}$. □

Proposition 2.3 allows us to view the elements of $\mathfrak{S}_0(X, Y; A)$ independently of the particular faithful representation of A on a separable Hilbert space; we will thus in the sequel refer to A-valued Schur A-multipliers without the need to specify a particular representation.

Lemma 2.4. *Let $\varphi \in \mathfrak{S}(X, Y; A)$. If $C \in \mathcal{D}_X$, $D \in \mathcal{D}_Y$ and $T \in \mathcal{K} \otimes A$ then*

$$S_\varphi((D \otimes I_H)T(C \otimes I_H)) = (D \otimes I_H)S_\varphi(T)(C \otimes I_H). \tag{6}$$

Proof. By continuity and linearity, it suffices to establish (6) in the case $T = T_k \otimes a$, where $k \in L^2(Y \times X)$ and $a \in A$. Assuming that $C = M_c$ and $D = M_d$, where $c \in L^\infty(X)$ and $d \in L^\infty(Y)$, we have that both $S_\varphi((D \otimes I_H)T(C \otimes I_H))$ and $(D \otimes I_H)S_\varphi(T)(C \otimes I_H)$ coincide with $T_{k'}$, where k' is the (weakly measurable) function from $Y \times X$ into $\mathcal{B}(H)$ given by

$$k'(y, x) = c(x)d(y)k(x, y)\varphi(x, y)(a).$$

□

Lemma 2.5. *Let $\mathcal{E}$ and $\mathcal{L}$ be separable Hilbert spaces and $\theta : \mathcal{K}(\mathcal{E}) \otimes A \to \mathcal{B}(\mathcal{L})$ be a non-degenerate *-representation. Then there exist a separable Hilbert space K, a unitary operator $U : \mathcal{L} \to \mathcal{E} \otimes K$ and a non-degenerate *-representation $\rho : A \to \mathcal{B}(K)$ such that*

$$U\theta(b \otimes a)U^* = b \otimes \rho(a), \qquad b \in \mathcal{K}(\mathcal{E}), a \in A.$$

Proof. Let $M(\mathcal{K}(\mathcal{E}) \otimes A)$ be the multiplier algebra of $\mathcal{K}(\mathcal{E}) \otimes A$. There exists a unital *-homomorphism $\hat{\theta} : M(\mathcal{K}(\mathcal{E}) \otimes A) \to \mathcal{B}(\mathcal{L})$ extending θ (see *e.g.* [29, Proposition 3.12.10]). The map $x \to \hat{\theta}(x \otimes I_H)$ is clearly a non-degenerate

*-representation of $\mathcal{K}(\mathcal{E})$ on $\mathcal{L}$. Thus, there exists a separable Hilbert space K and a unitary operator $U : \mathcal{L} \to \mathcal{E} \otimes K$ such that

$$U\hat{\theta}(b \otimes I_H)U^* = b \otimes I_K, \quad b \in \mathcal{K}(\mathcal{E}).$$

Let $\tilde{\theta} : M(\mathcal{K}(\mathcal{E}) \otimes A) \to \mathcal{B}(\mathcal{E} \otimes K)$ be given by

$$\tilde{\theta}(T) = U\hat{\theta}(T)U^*, \quad T \in M(\mathcal{K}(\mathcal{E}) \otimes A).$$

For $a \in A$ and $b \in \mathcal{K}(\mathcal{E})$, the operators $\tilde{\theta}(b \otimes I_H)$ and $\tilde{\theta}(I_{\mathcal{E}} \otimes a)$ commute. It follows that $\tilde{\theta}(I_{\mathcal{E}} \otimes a) = I_{\mathcal{E}} \otimes \rho(a)$, for some operator $\rho(a) \in \mathcal{B}(K)$. Since $\tilde{\theta}$ is a unital *-homomorphism, the map $\rho : A \to \mathcal{B}(K)$ is easily seen to be a non-degenerate *-homomorphism. Moreover, if $b \in \mathcal{K}(\mathcal{E})$ and $a \in A$ then

$$\begin{aligned} U\theta(b \otimes a)U^* &= U\hat{\theta}(b \otimes I_H)\hat{\theta}(I_{\mathcal{E}} \otimes a)U^* = U\hat{\theta}(b \otimes I_H)U^*U\hat{\theta}(I_{\mathcal{E}} \otimes a)U^* \\ &= \tilde{\theta}(b \otimes I_H)\tilde{\theta}(I_{\mathcal{E}} \otimes a) = (b \otimes I_K)(I_{\mathcal{E}} \otimes \rho(a)) = b \otimes \rho(a). \end{aligned}$$

□

Theorem 2.6. *Let $\varphi : X \times Y \to CB(A, \mathcal{B}(H))$ be a bounded pointwise measurable function. The following are equivalent:*

(i) φ is a Schur A-multiplier;

*(ii) there exist a separable Hilbert space K, a non-degenerate *-representation $\rho : A \to \mathcal{B}(K)$, $V \in L^\infty(X, \mathcal{B}(H, K))$ and $W \in L^\infty(Y, \mathcal{B}(H, K))$ such that, for all $a \in A$,*

$$\varphi(x, y)(a) = W^*(y)\rho(a)V(x),$$

for almost all $(x, y) \in X \times Y$.

Proof. (i)⇒(ii) Suppose that $\varphi \in \mathfrak{S}(X, Y; A)$. Let $\mathcal{E} = L^2(Y) \oplus L^2(X)$ and $\Phi : \mathcal{K}(\mathcal{E}) \otimes A \to \mathcal{K}(\mathcal{E}) \otimes \mathcal{B}(H)$ be given by

$$\Phi\left(\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \otimes a\right) = \begin{pmatrix} 0 & S_\varphi(x_{1,2} \otimes a) \\ 0 & 0 \end{pmatrix}. \tag{7}$$

It is clear that Φ is a completely bounded map with $\|\Phi\|_{\rm cb} = \|S_\varphi\|_{\rm cb}$. By the Haagerup-Paulsen-Wittstock Theorem, there exist a Hilbert space $\mathcal{L}$, a non-degenerate *-homomorphism $\theta : \mathcal{K}(\mathcal{E}) \otimes A \to \mathcal{B}(\mathcal{L})$ and operators $V_0, W_0 \in \mathcal{B}(\mathcal{E} \otimes H, \mathcal{L})$ such that

$$\Phi(T) = W_0^*\theta(T)V_0, \quad T \in \mathcal{K}(\mathcal{E}) \otimes A.$$

As $\mathcal{K}(\mathcal{E}) \otimes A$ is separable, we may assume that $\mathcal{L}$ is separable. By Lemma 2.5, there exist a separable Hilbert space K, a unitary operator $U : \mathcal{L} \to \mathcal{E} \otimes K$ and a *-representation $\rho : A \to \mathcal{B}(K)$ such that

$$U\theta(b \otimes a)U^* = b \otimes \rho(a), \quad b \in \mathcal{K}(\mathcal{E}), a \in A.$$

Let $\hat{W} = UW_0$ and $\hat{V} = UV_0$. Then

$$\Phi(b \otimes a) = \hat{W}^*(b \otimes \rho(a))\hat{V}, \quad b \in \mathcal{K}(\mathcal{E}), a \in A.$$

Writing $\hat{V}$ and $\hat{W}$ in two by two matrix form and recalling (7), we conclude that there exist bounded operators $\tilde{V} : L^2(X,H) \to L^2(X,K)$ and $\tilde{W} : L^2(Y,H) \to L^2(Y,K)$ such that

$$S_\varphi(b \otimes a) = \tilde{W}^*(b \otimes \rho(a))\tilde{V}, \quad b \in \mathcal{K}, a \in A. \tag{8}$$

Let

$$\mathcal{S} \stackrel{\text{def}}{=} \overline{[\{T\tilde{V}L^2(X,H) : T \in \mathcal{K}(L^2(X)) \otimes \rho(A)\}]}.$$

Clearly, $\mathcal{S}$ is invariant under $\mathcal{K}(L^2(X)) \otimes \rho(A)$. Thus, the projection onto $\mathcal{S}$ has the form $I_{L^2(X)} \otimes E$, for some projection $E \in \rho(A)'$. Moreover,

$$\tilde{V} = (I_{L^2(X)} \otimes E)\tilde{V}. \tag{9}$$

Setting $\tilde{\rho} = \mathrm{id} \otimes \rho$ (so that $\tilde{\rho}$ is a map from $\mathcal{K} \otimes A$ into $\mathcal{K} \otimes \mathcal{B}(K)$), by (8) and (9), we now have

$$S_\varphi(T) = \tilde{W}^*\tilde{\rho}(T)(I_{L^2(X)} \otimes E)\tilde{V}, \quad T \in \mathcal{K} \otimes A. \tag{10}$$

Note, further, that if $c \in L^\infty(X)$ and $d \in L^\infty(Y)$ then

$$\tilde{\rho}((M_d^* \otimes I_H)T(M_c \otimes I_H)) = (M_d^* \otimes I_K)\tilde{\rho}(T)(M_c \otimes I_K). \tag{11}$$

Let $W = (I_{L^2(Y)} \otimes E)\tilde{W}$. Since

$$\tilde{\rho}(T)(I_{L^2(X)} \otimes E) = (I_{L^2(Y)} \otimes E)\tilde{\rho}(T),$$

we conclude from (10) that

$$S_\varphi(T) = \tilde{W}^*(I_{L^2(Y)} \otimes E)\tilde{\rho}(T)\tilde{V} = W^*\tilde{\rho}(T)\tilde{V}, \tag{12}$$

for every $T \in \mathcal{K} \otimes A$.

Identities (11) and (12) and Lemma 2.4 imply that

$$W^*(M_d^* \otimes I_K)\tilde{\rho}(T)\tilde{V} = (M_d^* \otimes I_H)W^*\tilde{\rho}(T)\tilde{V}, \quad T \in \mathcal{K} \otimes A. \tag{13}$$

Thus,

$$\left\langle \tilde{\rho}(T)\tilde{V}\xi, (M_d \otimes I_K)W\eta \right\rangle = \left\langle \tilde{\rho}(T)\tilde{V}\xi, W(M_d \otimes I_H)\eta \right\rangle,$$

for all $\xi \in L^2(X,H)$ and all $\eta \in L^2(Y,H)$. We conclude that

$$(I_{L^2(Y)} \otimes E)(M_d \otimes I_K)W = (I_{L^2(Y)} \otimes E)W(M_d \otimes I_H)$$

and hence $(M_d \otimes I_K)W = W(M_d \otimes I_H)$ for all $d \in L^\infty(Y)$. It follows easily that $W \in L^\infty(Y, \mathcal{B}(H,K))$ (see [42, Theorem 7.10]). Let now

$$\mathcal{T} \stackrel{\text{def}}{=} \overline{[\{TWL^2(Y,H) : T \in \mathcal{K}(L^2(Y)) \otimes \rho(A)\}]}.$$

The projection onto $\mathcal{T}$ has the form $I_{L^2(Y)} \otimes F$ for some projection $F \in \rho(A)'$. Letting $V = (I_{L^2(X)} \otimes F)\tilde{V}$, and using similar arguments to the ones above, one shows that $(M_c \otimes I_K)V = V(M_c \otimes I_H)$ for all $c \in L^\infty(X)$ and hence that $V \in L^\infty(X, \mathcal{B}(H,K))$. Note that $W = (I_{L^2(Y)} \otimes F)W$ and hence, by (12),

$$S_\varphi(T) = W^*(I_{L^2(Y)} \otimes F)\tilde{\rho}(T)\tilde{V} = W^*\tilde{\rho}(T)(I_{L^2(Y)} \otimes F)\tilde{V} = W^*\tilde{\rho}(T)V, \tag{14}$$

for every $T \in \mathcal{K} \otimes A$.

Let $k \in L^2(Y \times X)$ and $a \in A$. For $\xi \in L^2(X,H)$ and $\eta \in L^2(Y,H)$ we have

$$\langle S_\varphi(T_k \otimes a)\xi, \eta\rangle = \int_Y \int_X k(y,x)\langle \varphi(x,y)(a)\xi(x), \eta(y)\rangle d\mu(x)d\nu(y). \tag{15}$$

On the other hand, by (14),

$$\begin{aligned}
\langle S_\varphi(T_k \otimes a)\xi, \eta\rangle &= \langle W^*(T_k \otimes \rho(a))V\xi, \eta\rangle \\
&= \langle (T_k \otimes \rho(a))V\xi, W\eta\rangle \\
&= \int_Y \int_X k(y,x)\langle \rho(a)(V(x)\xi(x)), W(y)\eta(y)\rangle d\mu(x)d\nu(y) \\
&= \int_Y \int_X k(y,x)\langle W(y)^*\rho(a)V(x)\xi(x), \eta(y)\rangle d\mu(x)d\nu(y).
\end{aligned}$$

Comparing the last identity with (15) and taking into account that these identities hold for all $k \in L^2(Y \times X)$, we conclude that

$$\langle \varphi(x,y)(a)\xi(x), \eta(y)\rangle = \langle W(y)^*\rho(a)V(x)\xi(x), \eta(y)\rangle \quad \text{almost everywhere,} \tag{16}$$

for all $\xi \in L^2(X,H)$ and all $\eta \in L^2(Y,H)$. If the measures μ and ν are finite, take $\xi = \chi_X \otimes \xi_0$ and $\eta = \chi_Y \otimes \eta_0$, where $\xi_0, \eta_0 \in H$. The separability of H and (16) imply that

$$\varphi(x,y)(a) = W(y)^*\rho(a)V(x), \quad \text{for almost all } (x,y) \in X \times Y.$$

If the measures μ and ν are not finite, the proof is completed by choosing increasing sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$, each of whose terms has finite measure, and letting $\xi = \chi_{X_n} \otimes \xi_0$ and $\eta = \chi_{Y_n} \otimes \eta_0$, with $\xi_0, \eta_0 \in H$.

(ii)$\Rightarrow$(i) The assumption shows that the mapping $S_\varphi : \mathcal{S}_2(Y \times X, A) \to \mathcal{S}_2(Y \times X, \mathcal{B}(H))$ satisfies

$$S_\varphi(T_h \otimes a) = W^*(T_h \otimes \rho(a))V, \quad h \in L^2(Y \times X), a \in A.$$

By linearity,

$$S_\varphi(T_k) = W^*T_{\rho\circ k}V, \tag{17}$$

whenever $k \in L^2(Y \times X) \odot A$.

Let $k \in L^2(Y \times X, A)$ be arbitrary. By [42, Proposition 7.4], there exists a sequence $(k_i)_{i\in\mathbb{N}} \subseteq L^2(Y \times X) \odot A$ with $\|k_i - k\|_2 \to_{i\to\infty} 0$. Using (5), (17), Lemma 2.1 and the fact that φ is bounded, we obtain

$$\begin{aligned}
S_\varphi(T_k) &= \lim_{i\to\infty} S_\varphi(T_{k_i}) = \lim_{i\to\infty} W^*T_{\rho\circ k_i}V \\
&= W^*(\lim_{i\to\infty} T_{\rho\circ k_i})V = W^*(\lim_{i\to\infty} \tilde{\rho}(T_{k_i})V = W^*\tilde{\rho}(T_k)V.
\end{aligned}$$

Thus, S_φ has a completely bounded extension to $\mathcal{K} \otimes \mathcal{A}$ (namely, the map $T \to W^*\tilde{\rho}(T))V$) and hence φ is a Schur A-multiplier. $\square$

Remarks (i) The proof of Theorem 2.6 shows that if $\varphi \in \mathfrak{S}(X, Y; A)$ then the operator valued functions V and W can be chosen so that

$$\|\varphi\|_{\mathfrak{S}} \leq \operatorname*{esssup}_{x \in X} \|W(x)\| \operatorname*{esssup}_{y \in Y} \|V(y)\|.$$

(ii) In the case $A = \mathbb{C}$, Theorem 2.6 reduces to the well-known characterisation of measurable Schur multipliers due to U. Haagerup [13] and V. V. Peller [30] (see also [17]). Indeed, in this case, ρ is equal to the identity representation of $\mathbb{C}$ and hence φ has a representation of the form

$$\varphi(x, y) = \langle w(y), v(x) \rangle, \tag{18}$$

where $v : X \to K$ and $w : Y \to K$ are weakly measurable essentially bounded functions, for some separable Hilbert space K.

If, in Theorem 2.6, the operator valued functions V and W can be chosen to be weakly measurable, then we will say that the Schur A-multiplier φ has a *weakly measurable representation.* In the next theorem we exhibit a class of A-valued Schur A-multipliers possessing a weakly measurable representation which exhausts all such multipliers in the case A is finite dimensional. Recall that a Hilbert A-bimodule is a right Hilbert A-module $\mathcal{N}$, equipped with a left A-module action given by $a \cdot \xi \stackrel{def}{=} \theta(a)(\xi)$, $a \in A$, $\xi \in \mathcal{N}$, for some *-representation θ of A into the C*-algebra of all adjointable operators on $\mathcal{N}$. As is customary in the literature on Hilbert modules, we assume linearity on the second variable of the A-valued inner product, denoted here by $\langle \cdot | \cdot \rangle_A$.

Theorem 2.7. *Let $A \subseteq \mathcal{B}(H)$ be a separable C*-algebra and $\varphi : X \times Y \to CB(A)$ be a bounded pointwise measurable function. Consider the conditions:*

(i) there exist a countably generated Hilbert A-bimodule $\mathcal{N}$ and bounded weakly measurable functions $v : X \to \mathcal{N}$ and $w : Y \to \mathcal{N}$ such that

$$\varphi(x, y)(a) = \langle w(y) | a \cdot v(x) \rangle_A, \quad \text{for almost all } (x, y) \in X \times Y \tag{19}$$

for every $a \in A$.

(ii) φ is a Schur A-multiplier possessing a weakly measurable representation.

Then (i) $\Rightarrow$ (ii). If A is finite-dimensional then (i) $\Leftrightarrow$ (ii).

Proof. (i)$\Rightarrow$(ii) It follows for instance from [8, Example 2.8] that there exist a separable Hilbert space K, an isometry $\tau : \mathcal{N} \to \mathcal{B}(K)$ and a faithful *-representation $\pi : A \to \mathcal{B}(K)$ such that $\tau(a \cdot z) = \pi(a)\tau(z)$, $\tau(z \cdot b) = \tau(z)\pi(b)$ and $\pi(\langle z_1, z_2 \rangle_A) = \tau(z_1)^* \tau(z_2)$, $z, z_1, z_2 \in \mathcal{N}$, $a, b \in A$. For all $a \in A$, we have that

$$\pi(\varphi(x, y)(a)) = \pi(\langle w(x) | a \cdot v(y) \rangle_A) = \tau(w(x))^* \pi(a) \tau(v(y)) \text{ a.e.}$$

Moreover, the maps $\tau \circ v$ and $\tau \circ w$ are weakly measurable. By Theorem 2.6, the map $\varphi_\pi : X \times Y \to CB(\pi(A))$, given by $\varphi_\pi(x, y)(\pi(a)) = \pi(\varphi(x, y)(a))$, is a Schur $\pi(A)$-multiplier. By Proposition 2.3, φ is a Schur A-multiplier.

Assume now that A is finite dimensional. By Proposition 2.3, we may identify A with the C*-algebra $\oplus_{k=1}^m M_{n_k} \subseteq \mathcal{B}(H)$, where M_n denotes, as customary, the n by n matrix algebra and $H = \oplus_{k=1}^m \mathbb{C}^{n_k}$, $n_k \in \mathbb{N}$, $k = 1, \dots, m$.

Suppose that φ is a Schur A-multiplier, K is a separable Hilbert space, $V : X \to \mathcal{B}(H, K)$, $W : Y \to \mathcal{B}(H, K)$ weakly measurable functions, and $\rho : A \to \mathcal{B}(K)$ a non-degenerate *-representation, such that, for every $a \in A$, we have $\varphi(x, y)(a) = W(y)^*\rho(a)V(x)$ for almost all $(x, y) \in X \times Y$. The space $\mathcal{B}(H, K)$ is an operator A-bimodule with respect to the actions $a \cdot T \stackrel{\text{def}}{=} \rho(a)T$ and $T \cdot a \stackrel{\text{def}}{=} Ta$, $a \in A$, $T \in \mathcal{B}(H, K)$. Let P_k be the projection in $\mathcal{B}(H)$ onto the summand $\mathbb{C}^{n_k}$ and $\Psi(T) = \sum_{k=1}^m P_k T P_k$, $T \in \mathcal{B}(H)$. Clearly, Ψ is a completely positive projection from $\mathcal{B}(H)$ onto A. We equip $\mathcal{B}(H, K)$ with the A-valued inner product given by $\langle S, T\rangle_A = \Psi(S^*T)$. As the projections P_k, $k = 1, \dots, m$, are mutually orthogonal and $\sum_{k=1}^m P_k = I$, we have $\langle S, S\rangle_A = 0$ if and only if $S = 0$. Moreover,

$$\langle S, T \cdot a\rangle_A = \Psi(S^*Ta) = \Psi(S^*T)a = \langle S, T\rangle_A a,$$

and hence $\mathcal{N} \stackrel{\text{def}}{=} \mathcal{B}(H, K)$ is a right Hilbert A-module. In addition,

$$\langle a \cdot S, T\rangle_A = \Psi(S^*\rho(a)^*T) = \langle S, a^* \cdot T\rangle_A,$$

showing that the map $\theta_a : S \to a \cdot S$ is adjointable and that the map $a \to \theta_a$ is a *-representation; thus, $\mathcal{N}$ is a Hilbert A-bimodule and $\varphi(x, y)(a) = \langle W(x), a \cdot V(x)\rangle_A$ for almost all $(x, y) \in X \times Y$. As H is finite dimensional and K is separable, $\mathcal{N}$ is countably generated. $\square$

Let $\mathcal{B} = \mathcal{B}(L^2(X), L^2(Y))$ for brevity.

Proposition 2.8. *If $\varphi \in \mathfrak{S}(X, Y; A)$ then the map S_φ has a unique extension to a completely bounded weak* continuous map from $\mathcal{B}\bar{\otimes}A^{**}$ into $\mathcal{B}\bar{\otimes}\mathcal{B}(H)$.*

Proof. Let $P : \mathcal{B}(H)^{**} \to \mathcal{B}(H)$ be the canonical projection (that is, the adjoint of the inclusion map of the trace class on H into $\mathcal{B}(H)^*$). Then the map $\mathrm{id} \otimes P : \mathcal{B}\bar{\otimes}\mathcal{B}(H)^{**} \to \mathcal{B}\bar{\otimes}\mathcal{B}(H)$ is weak* continuous and completely contractive (see *e.g.* [4] and [9, Proposition 7.1.6]).

Let $\tilde{\mathcal{K}} = \mathcal{K}(L^2(Y) \oplus L^2(X))$ and $\tilde{\mathcal{B}} = \mathcal{B}(L^2(Y) \oplus L^2(X))$. Write P_X (resp. P_Y) for the projection from $L^2(Y) \oplus L^2(X)$ onto $L^2(X)$ (resp. $L^2(Y)$). Let $\Phi : \tilde{\mathcal{K}} \otimes A \to \tilde{\mathcal{K}} \otimes \mathcal{B}(H)$ be given by

$$\Phi\left(\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \otimes a\right) = \begin{pmatrix} 0 & S_\varphi(x_{1,2} \otimes a) \\ 0 & 0 \end{pmatrix}. \tag{20}$$

By [14, Example 1], given a C^*-algebra B, there is a canonical normal *-isomorphism

$$(\tilde{\mathcal{K}} \otimes B)^{**} \cong \tilde{\mathcal{B}}\bar{\otimes}B^{**}. \tag{21}$$

Hence we may view the second dual Φ^{**} as a completely bounded map from $\tilde{\mathcal{B}}\bar{\otimes}A^{**}$ to $\tilde{\mathcal{B}}\bar{\otimes}\mathcal{B}(H)^{**}$, extending Φ. As $\tilde{\mathcal{K}}\otimes A$ is weak* dense in $\tilde{\mathcal{B}}\bar{\otimes}A^{**}$ we have that for any $T\in\tilde{\mathcal{B}}\bar{\otimes}A^{**}$ there exists $\Psi(T)\in\mathcal{B}\bar{\otimes}\mathcal{B}(H)^{**}$ such that

$$\Phi^{**}(T)=\begin{pmatrix}0 & \Psi(T)\\ 0 & 0\end{pmatrix}.$$

In particular,

$$\Phi^{**}((P_Y\otimes\mathrm{id})T(P_X\otimes\mathrm{id}))=\begin{pmatrix}0 & \Psi((P_Y\otimes\mathrm{id})T(P_X\otimes\mathrm{id}))\\ 0 & 0\end{pmatrix},$$

and the mapping $\tilde{\Psi}=\Psi|_{\mathcal{B}\bar{\otimes}A^{**}}:\mathcal{B}\bar{\otimes}A^{**}\to\mathcal{B}\bar{\otimes}\mathcal{B}(H)^{**}$ is completely bounded and weak* continuous. Hence the composition

$$(\mathrm{id}\otimes P)\circ\tilde{\Psi}:\mathcal{B}\bar{\otimes}A^{**}\to\mathcal{B}\bar{\otimes}\mathcal{B}(H)$$

is a completely bounded weak* continuous map, extending S_φ. The fact that this extension is unique follows by weak* density. □

We will use the same symbol, S_φ, to denote the map obtained in Proposition 2.8. We note that if S_φ satisfies equation (14), that is, if $S_\varphi(S)=W^*(\mathrm{id}\otimes\rho)(S)V$ for all $S\in\mathcal{K}\otimes A$, then $S_\varphi(T\otimes a)=W^*(T\otimes\rho(a))V$ for all $T\in\mathcal{B}$ and all $a\in A^{**}$, where ρ has been canonically extended to A^{**}.

While Proposition 2.8 implies that, if $\varphi\in\mathfrak{S}_0(X,Y;A)$, then the map S_φ on $\mathcal{K}\otimes A$ has a weak* continuous extension to $\mathcal{B}\otimes A^{**}$, an analogous extension is not guaranteed to exist in representations of A different from the universal one. This motivates the following definition.

Definition 2.9. *Let A be a separable C^*-algebra and θ be a faithful *-representation of A on a separable Hilbert space. An element $\varphi\in\mathfrak{S}_0(X,Y;A)$ will be called a* Schur θ-multiplier *if the map $S_{\varphi_\theta}:\mathcal{K}\otimes\theta(A)\to\mathcal{K}\otimes\theta(A)$ can be extended to a weak* continuous map on $\mathcal{B}\bar{\otimes}\theta(A)''$.*

The notion of a Schur θ-multiplier will be used in the subsequent sections.

3. Herz-Schur multipliers and transference

In this section, we introduce and study Herz-Schur multipliers of crossed products. We assume throughout that G is a locally compact group. Left Haar measure on G will be denoted by m_G or m and integration with respect to m_G along the variable s will be denoted by ds. Let $\lambda^G:G\to\mathcal{B}(L^2(G))$ be the left regular representation of G; thus, $\lambda^G_t\xi(s)=\xi(t^{-1}s)$, $\xi\in L^2(G)$, $s,t\in G$. We write $C_r^*(G)$ (resp. $\mathrm{VN}(G)$) for the reduced group C*-algebra (resp. the von Neumann algebra) of G, that is, for the closure in the norm topology (resp. in the weak* topology) of $\lambda^G(L^1(G))$. As customary, we let $A(G)$ (resp. $B(G)$, $B_\lambda(G)$) be the Fourier (resp. the Fourier-Stieltjes, the reduced Fourier-Stieltjes) algebra of G. We note the canonical identifications $A(G)^*=\mathrm{VN}(G)$, $C^*(G)^*=B(G)$ and $C_r^*(G)^*=B_\lambda(G)$ [10].

Let A be a separable C*-algebra. In this section, unless otherwise stated, H will denote the Hilbert space of the universal representation of A; we consider A as a C*-subalgebra of $\mathcal{B}(H)$. Let $\alpha : G \to \mathrm{Aut}(A)$ be a continuous (with respect to point-norm topology) group homomorphism; thus, (A, G, α) is a C*-dynamical system. The space $L^1(G, A)$ is a *-algebra with respect to the product $\times$ given by $(f \times g)(t) = \int_G f(s)\alpha_s(g(s^{-1}t))ds$ and the involution $*$ given by $f^*(s) = \Delta(s)^{-1}\alpha_s(f(s^{-1})^*)$.

Let $\pi : A \to \mathcal{B}(L^2(G, H))$ be the *-representation defined by $(\pi(a)\xi)(t) = \alpha_{t^{-1}}(a)(\xi(t))$, $t \in G$, and $\lambda : G \to \mathcal{B}(L^2(G, H))$ be the (continuous) unitary representation given by $(\lambda_t\xi)(s) = \xi(t^{-1}s)$, $s, t \in G$. Note that

$$\pi(\alpha_t(a)) = \lambda_t\pi(a)\lambda_t^*, \quad t \in G;$$

thus, the pair (π, λ) is a covariant representation of (A, G, α) and hence gives rise to a *-representation $\pi \rtimes \lambda : L^1(G, A) \to \mathcal{B}(L^2(G, H))$ given by

$$(\pi \rtimes \lambda)(f) = \int_G \pi(f(s))\lambda_s ds, \quad f \in L^1(G, A).$$

The *reduced crossed product* $A \rtimes_{\alpha,r} G$ of A by α is, by definition, the closure of $(\pi \rtimes \lambda)(L^1(G, A))$ in the operator norm of $\mathcal{B}(L^2(G, H))$ [29, 7.7.4]. We let $A \rtimes_{\alpha,r}^{w^*} G$ be the weak* closure of $A \rtimes_{\alpha,r} G$.

A bounded function $F : G \to \mathcal{B}(A)$ will be called *pointwise measurable* if, for every $a \in A$, the map $s \to F(s)(a)$ is a weakly measurable function from G into A. Suppose that (ρ, τ) is a covariant representation of the dynamical system (A, G, α) on the Hilbert space K. We say that F is (ρ, τ)-*fiber continuous*, if the map

$$G \to \mathcal{B}(K), \qquad s \to \rho(F(s)(a))\tau_s,$$

is weak* continuous for every $a \in A$. We will say that F is *fiber continuous* if F is (π, λ)-fiber continuous. Note that if F is bounded and point norm continuous then it is pointwise measurable and fiber continuous.

We further say that F is *almost* (ρ, τ)-*fiber continuous* if, for every $\omega \in \mathcal{B}(K)_*$ and every $a \in A$, the function

$$s \to \langle \rho(F(s)(a))\tau_s, \omega\rangle$$

coincides, up to a null set, with a continuous function. Almost (π, λ)-fiber continuous functions will be referred to simply as *almost fiber continuous*.

For each $f \in L^1(G, A)$, let $F \cdot f \in L^1(G, A)$ be the function given by $(F \cdot f)(s) = F(s)(f(s))$, $s \in G$. It is easy to see that if F is pointwise measurable then $F \cdot f$ is weakly measurable and hence $F \cdot f \in L^1(G, A)$ for every $f \in L^1(G, A)$; in fact, $\|F \cdot f\|_1 \leq \|F\|_\infty \|f\|_1$, where $\|F\|_\infty = \sup_{s \in G} \|F(s)\|$.

Definition 3.1. *A pointwise measurable function $F : G \to CB(A)$ will be called a* Herz-Schur (A, G, α)-multiplier *if the map*

$$S_F : (\pi \rtimes \lambda)(L^1(G, A)) \to (\pi \rtimes \lambda)(L^1(G, A))$$

given by

$$S_F((\pi \rtimes \lambda)(f)) = (\pi \rtimes \lambda)(F \cdot f)$$

is completely bounded.

We denote by $\mathfrak{S}(A,G,\alpha)$ the set of all Herz-Schur (A,G,α)-multipliers. If $F \in \mathfrak{S}(A,G,\alpha)$ then the map S_F extends to a completely bounded map on $A \rtimes_{r,\alpha} G$. This (unique) extension will be denoted again by S_F. We let $\|F\|_{\mathrm{m}} = \|S_F\|_{\mathrm{cb}}$.

Remark 3.2. **(i)** If $F_1, F_2 \in \mathfrak{S}(A,G,\alpha)$, letting $F_1 + F_2 : G \to CB(A)$ and $F_1F_2 : G \to CB(A)$ be given by $(F_1 + F_2)(s) = F_1(s) + F_2(s)$ and $(F_1F_2)(s) = F_1(s) \circ F_2(s)$, we see that $S_{F_1+F_2} = S_{F_1} + S_{F_2}$ and $S_{F_1F_2} = S_{F_1}S_{F_2}$. Thus, $\mathfrak{S}(A,G,\alpha)$ is an algebra with respect to the operations just defined.

(ii) Recall that a bounded continuous function $u : G \to \mathbb{C}$ is called a *completely bounded* (or *Herz-Schur*) *multiplier* [7] of the Fourier algebra $A(G)$ of G if $uv \in A(G)$ for every $v \in A(G)$, and the map $m_u : v \to uv$ on $A(G)$ is completely bounded. The space of all Herz-Schur multipliers of $A(G)$ will be denoted as usual by $M^{\mathrm{cb}}A(G)$. If $u \in M^{\mathrm{cb}}A(G)$ then the dual S_u of m_u is a completely bounded (and weak* continuous) linear map on the von Neumann algebra $\mathrm{VN}(G)$ of G, such that $S_u(\lambda_t^G) = u(t)\lambda_t^G$, $t \in G$. Moreover, S_u leaves the reduced C*-algebra $C_r^*(G)$ of G invariant, and

$$S_u\left(\int_G f(s)\lambda_s ds\right) = \int_G u(s)f(s)\lambda_s ds, \quad f \in L^1(G).$$

The reduced crossed product of $\mathbb{C}$ by the (unique) action α of a locally compact group G on $\mathbb{C}$ coincides with $C_r^*(G)$. Identifying $\mathcal{B}(\mathbb{C})$ with $\mathbb{C}$ in the natural way, we have that a bounded continuous function $u : G \to \mathbb{C}$ is a Herz-Schur $(\mathbb{C},G,\alpha)$-multiplier if and only if u is a Herz-Schur multiplier.

(iii) Suppose that $\theta : A \to \mathcal{B}(K)$ is a faithful *-representation of A on a Hilbert space K. Let $\pi^\theta : A \to \mathcal{B}(L^2(G,K))$ be given by $(\pi^\theta(a)\xi)(t) = \theta(\alpha_{t^{-1}}(a))(\xi(t))$, $t \in G$, while $\lambda^\theta : G \to \mathcal{B}(L^2(G,K))$ be given by $(\lambda_t^\theta\xi)(s) = \xi(t^{-1}s)$, $s,t \in G$. Then the pair $(\pi^\theta, \lambda^\theta)$ is a covariant representation of (A,G,α). Since A is assumed to be universally represented, up to a *-isomorphism, K is a closed subspace of H that reduces A, $\pi^\theta(a)$ is the restriction of $\pi(a)$ to $L^2(G,K)$, while λ_s^θ is the restriction of λ_s to $L^2(G,K)$. In the sequel, we let $A \rtimes_{\alpha,\theta} G = (\pi^\theta \rtimes \lambda^\theta)(A \rtimes_\alpha G)$ and $A \rtimes_{\alpha,\theta}^{w^*} G = \overline{A \rtimes_{\alpha,\theta} G}^{w^*}$.

By [29, Theorem 7.7.5], the closure of $(\pi^\theta \rtimes \lambda^\theta)(L^1(G,A))$ is *-isomorphic to $A \rtimes_{r,\alpha} G$ and a pointwise measurable function $F : G \to CB(A)$ is a Herz-Schur (A,G,α)-multiplier if and only if the map

$$S_F^\theta : (\pi^\theta \rtimes \lambda^\theta)(f) \mapsto (\pi^\theta \rtimes \lambda^\theta)(F \cdot f) \tag{22}$$

is completely bounded. Thus, Herz-Schur (A,G,α)-multipliers can be defined starting with any faithful representation of A instead of its universal representation.

In the case $A = \mathbb{C}$, the maps on $C_r^*(G)$ associated with Herz-Schur multipliers automatically have a weak* continuous extension to (completely bounded) maps on the weak* closure $\mathrm{VN}(G)$ of $C_r^*(G)$. Such extension is not ensured to exist in the general case – this motivates the following definition.

Definition 3.3. *Let A be a separable C*-algebra, K be a Hilbert space and $\theta : A \to \mathcal{B}(K)$ be a faithful *-representation. A function $F : G \to CB(A)$ will be called a* θ-multiplier *if the map*

$$\Phi_F^\theta : \pi^\theta(a)\lambda_t^\theta \mapsto \pi^\theta(F(t)(a))\lambda_t^\theta,\ t \in G,\ a \in A,$$

has an extension to a bounded weak continuous map on $A \rtimes_{\alpha,\theta}^{w^*} G$.*

A θ-multiplier F will be called a Herz-Schur θ-multiplier *if the extension of Φ_F^θ to $A \rtimes_{\alpha,\theta}^{w^*} G$ is completely bounded.*

We note that, in Definition 3.3, we do not require the pointwise measurability of the function F. The weak* continuous extension of the map Φ_F^θ therein will still be denoted by the same symbol.

Remark 3.4. Let A be a separable C*-algebra, K be a Hilbert space and $\theta : A \to \mathcal{B}(K)$ be a faithful *-representation. Suppose that $F : G \to \mathcal{B}(A)$ is a bounded map and $\Phi : A \rtimes_{\alpha,\theta}^{w^*} G \to A \rtimes_{\alpha,\theta}^{w^*} G$ is a bounded weak* continuous map such that, for almost all $t \in G$,

$$\Phi(\pi^\theta(a)\lambda_t^\theta) = \pi^\theta(F(t)(a))\lambda_t^\theta, \quad a \in A.$$

Then, for any $\omega \in \mathcal{B}(L^2(G,K))_*$ and $f \in L^1(G,A)$, the function $s \mapsto \langle \pi^\theta(F(s)(f(s)))\lambda_s^\theta, \omega\rangle$ is measurable, and

$$\Phi((\pi^\theta \rtimes \lambda^\theta)(f)) = (\pi^\theta \rtimes \lambda^\theta)(F \cdot f), \quad f \in L^1(G,A).$$

Proof. Let $\omega \in \mathcal{B}(L^2(G,K))_*$ and $f \in L^1(G,A)$. Since the function $s \mapsto \langle \pi^\theta(f(s))\lambda_s^\theta, \Phi_*(\omega)\rangle$ is measurable so is $s \mapsto \langle \pi^\theta(F(s)(f(s)))\lambda_s^\theta, \omega\rangle$. We have

$$\begin{aligned}
\left\langle \Phi\left(\int \pi^\theta(f(s))\lambda_s^\theta ds\right), \omega\right\rangle &= \int \langle \pi^\theta(f(s))\lambda_s^\theta, \Phi_*(\omega)\rangle ds \\
&= \int \langle \Phi(\pi^\theta(f(s))\lambda_s^\theta), \omega\rangle ds \\
&= \int \langle \pi^\theta(F(s)(f(s)))\lambda_s^\theta, \omega\rangle ds \\
&= \langle (\pi^\theta \rtimes \lambda^\theta)(F \cdot f), \omega\rangle.
\end{aligned}$$

The claim follows. ☐

Lemma 3.5. *Let θ be a faithful *-representation of A on a Hilbert space K. Let $F : G \to CB(A)$ be a pointwise measurable map for which there exists $C > 0$ such that*

$$\|(\pi^\theta \rtimes \lambda^\theta)(F \cdot f)\| \le C\|(\pi^\theta \rtimes \lambda^\theta)(f)\|, \quad f \in L^1(G,A). \tag{23}$$

For $\omega \in \mathcal{B}(L^2(G,K))_*$ and $a \in A$, let $g_{\omega,a}(s) = \langle \pi^\theta(F(s)(a))\lambda_s^\theta, \omega\rangle$, $s \in G$. Then $g_{\omega,a}$ coincides with an element of $B_\lambda(G)$ up to a null set.

Proof. Fix $\omega \in \mathcal{B}(L^2(G,K))_*$ and $a \in A$, write $g = g_{\omega,a}$. Let $f \in L^1(G)$ and $\tilde{f}(s) = f(s)a$, $s \in G$; clearly, $\tilde{f} \in L^1(G,A)$ and, by (23), we have

$$\begin{aligned}\left|\int_G f(s)g(s)ds\right| &= \left|\int_G f(s)\langle\pi^\theta(F(s)(a))\lambda_s^\theta,\omega\rangle ds\right| \\ &= \left|\left\langle\int_G (\pi^\theta(F(s)(\tilde{f}(s)))\lambda_s^\theta ds,\omega\right\rangle\right| \\ &\leq C\|\omega\|\|(\pi^\theta \rtimes \lambda^\theta)(\tilde{f})\| \\ &\leq C\|\omega\|\left\|\pi^\theta(a)\int_G f(s)\lambda_s^\theta ds\right\| \\ &\leq C\|\omega\|\|\pi^\theta(a)\|\|\lambda^\theta(f)\| \\ &= C\|\omega\|\|\pi^\theta(a)\|\|\lambda^G(f)\|.\end{aligned}$$

Hence, the map $\lambda_G(f) \to \int_G f(s)g(s)ds$ can be extended to a bounded linear functional on $C_r^*(G)$ and therefore there exists $z \in B_\lambda(G)$ such that $\int_G f(s)g(s)ds = \int_G f(s)z(s)ds$ for any $f \in L^1(G)$. It follows that $z = g$ almost everywhere. □

Remark 3.6. Fix $a \in A$. For $\omega \in \mathcal{B}(L^2(G,K))_*$, let $g_\omega(s) = \langle\pi^\theta(F(s)(a))\lambda_s^\theta, \omega\rangle$ and let $b_\omega \in B_\lambda(G)$ be such that $b_\omega = g_\omega$ almost everywhere. If G is second countable and K is separable, then under the assumption of the previous lemma there exists a null subset $N \subseteq G$ such that $g_\omega(t) = b_\omega(t)$ for any $t \in G \setminus N$ and $\omega \in \mathcal{B}(L^2(G,K))_*$. Indeed, in this case we have that $\mathcal{B}(L^2(G,K)_*$ is separable. Let $\{\omega_n : n \in \mathbb{N}\}$ be a dense subset of $\mathcal{B}(L^2(G,K)_*$ and $a \in A$. Let $N_n \subseteq G$ be a null set such that $g_{\omega_n}(s) = b_{\omega_n}(s)$ for all $s \in G \setminus N_n$, and $N = \cup_{n\in\mathbb{N}} N_n$. Clearly, N is a null set. If $\omega \in \mathcal{B}(L^2(G,K)_*$, let $\{\omega_{n(k)}\}_k$ be a subsequence converging to ω in norm. Then $\{b_{\omega_{n(k)}}\}_k$ is a Cauchy sequence of bounded continuous functions: letting $C = \|\omega\| \sup_{s\in G} \|F(s)\|$, given $\varepsilon > 0$ there exists $L \in \mathbb{N}$ such that for any $l, k > L$, we have

$$|b_{\omega_{n(k)}}(t) - b_{\omega_{n(l)}}(t)| = |g_{\omega_{n(k)}}(t) - g_{\omega_{n(l)}}(t)| \leq C\|\omega_{n(k)} - \omega_{n(l)}\| \leq C\varepsilon,$$

whenever $t \in G \setminus N$. As $b_{\omega_{n(k)}}$ is continuous, $|b_{\omega_{n(k)}}(t) - b_{\omega_{n(l)}}(t)| < C\varepsilon$ for every $t \in G$. Thus, the sequence $\{b_{\omega_{n(k)}}\}_k$ converges to a continuous function, say b. On the other hand, $b_{\omega_{n(k)}}(t) \to g_\omega(t)$ whenever $t \in G \setminus N$. Therefore $g_\omega(t) = b(t)$ for $t \in G \setminus N$. As b_ω and b are continuous, and $b_\omega = g_\omega$ almost everywhere, we have $b = b_\omega$, giving the statement.

For the rest of the section we will assume that G is a second countable locally compact group. In this case, the measure space (G,m) is standard.

If $t \in G$, let us call a *Dirac family at* t a net $(f_U)_U \subseteq L^1(G)$ consisting of non-negative functions, indexed by the directed set of all open neighbourhoods of t with compact closure, with supp $f_U \subseteq U$ and $\|f_U\|_1 = 1$.

Lemma 3.7. *Let (A,G,α) be a C^*-dynamical system, F be a Herz-Schur (A,G,α)-multiplier, and θ be a faithful *-representation of A on a separable*

Hilbert space K. Then there exists a null set $N \subseteq G$ such that if $t \in G \setminus N$ and $(f_U)_U$ is a Dirac family at t then

$$S_F^\theta((\pi^\theta \rtimes \lambda^\theta)(f_U \otimes a)) \to_U \pi^\theta(F(t)(a))\lambda_t^\theta$$

in the weak topology, for every $a \in A$.*

Proof. Let $a \in A$. By Lemma 3.5 and Remark 3.6 there exists a null set $N \subseteq G$ such that for any ξ, $\eta \in L^2(G, K)$ there exists a continuous function $b_{\xi,\eta} : G \to \mathbb{C}$ such that

$$\langle \pi^\theta(F(s)(a))\lambda_s^\theta \xi, \eta\rangle = b_{\xi,\eta}(s) \quad \text{for all } s \in G \setminus N.$$

Fix ξ, $\eta \in L^2(G, K)$. For $t \in G \setminus N$, set

$$C_U = \sup_{s\in U} |b_{\xi,\eta}(s) - b_{\xi,\eta}(t)|.$$

Since $b_{\xi,\eta}$ is continuous, $C_U \to_U 0$. We have

$$\begin{aligned}
& \left|\langle S_F^\theta((\pi^\theta \rtimes \lambda^\theta)(f_U \otimes a))\xi, \eta\rangle - \langle \pi^\theta(F(t)(a))\lambda_t^\theta \xi, \eta\rangle\right| \\
= & \left|\int \langle \pi^\theta(F(s)(a)) f_U(s)\lambda_s^\theta \xi, \eta\rangle ds - \int \langle \pi^\theta(F(t)(a)) f_U(s)\lambda_t^\theta \xi, \eta\rangle ds\right| \\
\leq & \int f_U(s) |\langle \pi^\theta(F(s)(a))\lambda_s^\theta \xi, \eta\rangle - \langle \pi^\theta(F(t)(a))\lambda_t^\theta \xi, \eta\rangle| ds \\
= & \int f_U(s)|b_{\xi,\eta}(s) - b_{\xi,\eta}(t)| ds \leq C_U \int f_U(s) ds = C_U \longrightarrow_U 0.
\end{aligned}$$

The statement follows from the fact that the weak operator topology and the weak* topology coincide on bounded sets. □

If $\varphi : G \times G \to CB(A)$ is a bounded pointwise measurable function, let $\mathcal{T}(\varphi) : G \times G \to CB(A)$ be the function given by

$$\mathcal{T}(\varphi)(s, t)(a) = \alpha_t(\varphi(s, t)(\alpha_{t^{-1}}(a))), \quad a \in A.$$

It is easy to see that, for each $a \in A$, the function $(s, t) \to \mathcal{T}(\varphi)(s, t)(a)$ from $G \times G$ into A is bounded and weakly measurable. The inverse $\mathcal{T}^{-1}$ of $\mathcal{T}$ is given by $\mathcal{T}^{-1}(\varphi)(s, t)(a) = \alpha_{t^{-1}}(\varphi(s, t)(\alpha_t(a)))$, $a \in A$. For a map $F : G \to CB(A)$, let $N(F) : G \times G \to CB(A)$ be the function given by

$$N(F)(s, t) = F(ts^{-1}),$$

and

$$\mathcal{N}(F) = \mathcal{T}^{-1}(N(F));$$

thus, $\mathcal{N}(F) : G \times G \to CB(A)$ is the function given by

$$\mathcal{N}(F)(s, t)(a) = \alpha_{t^{-1}}(F(ts^{-1})(\alpha_t(a))), \quad a \in A, \ s, t \in G.$$

Note that if F is pointwise measurable then so is $\mathcal{N}(F)$.

The next theorem is a dynamical system version of the well-known description of Herz-Schur multipliers in terms of Schur multipliers [5]. Recall that, given a map $\varphi : X \times Y \to CB(A)$ and a faithful *-representation θ of

A, we let $\varphi_\theta : X \times Y \to CB(\theta(A))$ be the map given by $\varphi_\theta(x, y)(\theta(a)) = \theta(\varphi(x, y)(a))$, $a \in A$. Note that, if φ is pointwise measurable then so is φ_θ.

Theorem 3.8. *Let (A, G, α) be a C^*-dynamical system and $F : G \to CB(A)$ be a pointwise measurable map. The following are equivalent:*
(i) F is a Herz-Schur (A, G, α)-multiplier;
(ii) $\mathcal{N}(F)$ is a Schur A-multiplier.
Moreover, if (i) holds then $\|F\|_{\rm m} = \|\mathcal{N}(F)\|_{\mathfrak{S}}$.

Proof. (i)⇒(ii) Suppose that F is a Herz-Schur (A, G, α)-multiplier and let θ be a faithful *-representation of A on a separable Hilbert space H'. By the Haagerup-Paulsen-Wittstock Theorem, there exist a separable Hilbert space K, a *-representation $\rho : A \rtimes_{\alpha,\theta} G \to \mathcal{B}(K)$ and operators $V, W : L^2(G, H') \to K$ such that

$$S_F^\theta(T) = W^* \rho(T) V, \quad T \in A \rtimes_{\alpha,\theta} G, \tag{24}$$

and $\|S_F^\theta\|_{\rm cb} = \|V\|\|W\|$. Let $A \rtimes_\alpha G$ be the full crossed product of A by α, $q : A \rtimes_\alpha G \to A \rtimes_{\alpha,\theta} G$ be the quotient map and $\tilde{\rho} = \rho \circ q$; thus, $\tilde{\rho} : A \rtimes_\alpha G \to \mathcal{B}(K)$ is a *-representation. Let $\rho_A : A \to \mathcal{B}(K)$ be a *-representation and $\rho_G : G \to \mathcal{B}(K)$ be a strongly continuous unitary representation such that $\tilde{\rho} = \rho_A \rtimes \rho_G$. For $f \in L^1(G, A)$, we have

$$\tilde{\rho}(f) = \int \rho_A(f(s))\rho_G(s)ds.$$

Setting $T = (\pi^\theta \rtimes \lambda^\theta)(f)$ in equation (24), we have

$$\int \pi^\theta(F(s)(f(s)))\lambda_s^\theta ds = W^* \left(\int \rho_A(f(s))\rho_G(s)ds \right) V. \tag{25}$$

Standard arguments show that, if $a \in A$ and $(f_U)_U$ is a Dirac family at the point $t \in G$, then

$$\int \rho_A((f_U \otimes a)(s))\rho_G(s)ds \longrightarrow_U \rho_A(a)\rho_G(t) \tag{26}$$

in the weak operator topology. Taking $f = f_U \otimes a$ in (25) and using Lemma 3.7, we obtain a null set N such that

$$\pi^\theta(F(t)(a))\lambda_t^\theta = W^* \rho_A(a)\rho_G(t)V, \text{ for all } t \in G \setminus N. \tag{27}$$

For s and t in G, let $\alpha(s), \beta(t) \in \mathcal{B}(L^2(G, H'), K)$ be given by

$$\alpha(s) = \rho_G(s^{-1})V\lambda_s^\theta, \quad \beta(t) = \rho_G(t^{-1})W\lambda_t^\theta;$$

then for every $\xi \in L^2(G, H')$, the functions $s \to \alpha(s)\xi$ and $s \to \beta(s)\xi$ are weakly continuous and hence α and β belong to $L^\infty(G, \mathcal{B}(L^2(G, H'), K))$.

Using (27), for all $(s,t) \in G \times G$ with $ts^{-1} \in G \setminus N$, we obtain

$$\begin{aligned}\beta(t)^*\rho_A(a)\alpha(s) &= \lambda^\theta_{t^{-1}} W^* \rho_G(t)\rho_A(a)\rho_G(s^{-1})V\lambda^\theta_s \\ &= \lambda^\theta_{t^{-1}} W^* \rho_A(\alpha_t(a))\rho_G(ts^{-1})V\lambda^\theta_s \\ &= \lambda^\theta_{t^{-1}} \pi^\theta(F(ts^{-1})(\alpha_t(a)))\lambda^\theta_{ts^{-1}}\lambda^\theta_s \\ &= \pi^\theta(\alpha_{t^{-1}}(F(ts^{-1})(\alpha_t(a))) = \pi^\theta(\mathcal{N}(F)(s,t)(a)).\end{aligned}$$

As $\{(s,t) : ts^{-1} \in N\}$ is a null set for the product measure $m \times m$, by Theorem 2.6, $\mathcal{N}(F)$ is a Schur A-multiplier. Moreover,

$$\operatorname*{esssup}_{s\in G} \|\alpha(s)\| = \|V\| \quad \text{and} \quad \operatorname*{esssup}_{t\in G} \|\beta(t)\| = \|W\|$$

and hence

$$\|\mathcal{N}(F)\|_{\mathfrak{S}} \le \|V\|\|W\| = \|F\|_{\mathrm{m}}. \tag{28}$$

(ii)⇒(i) Let $\theta : A \to \mathcal{B}(H')$ be a faithful *-representation, where H' is a separable Hilbert space. Suppose that $\varphi \stackrel{def}{=} \mathcal{N}(F)_\theta$ is a Schur A-multiplier. Fix $f \in C_c(G,A)$. A straightforward calculation shows that, if $\xi \in L^2(G,H')$ then, for almost all $t \in G$ we have

$$(\pi^\theta \rtimes \lambda^\theta)(f)\xi(t) = \int \Delta(r)^{-1}\theta(\alpha_{t^{-1}}(f(tr^{-1})))(\xi(r))dr.$$

Fix a compact set $K \subseteq G$. Then the function

$$h_K : (t,r) \to \chi_{K\times K}(t,r)\Delta(r)^{-1}\theta(\alpha_{t^{-1}}(f(tr^{-1}))) \tag{29}$$

belongs to $L^2(G \times G, \theta(A))$. Note that

$$T_{h_K} = (M_{\chi_K} \otimes I_{H'})(\pi^\theta \rtimes \lambda^\theta)(f)(M_{\chi_K} \otimes I_{H'}). \tag{30}$$

We have

$$\begin{aligned}\varphi \cdot h_K(t,r) &= \chi_{K\times K}(t,r)\Delta(r)^{-1}\theta(\alpha_{t^{-1}}(F(tr^{-1})(\alpha_t(\alpha_{t^{-1}}(f(tr^{-1})))) \\ &= \chi_{K\times K}(t,r)\Delta(r)^{-1}\theta(\alpha_{t^{-1}}(F(tr^{-1})(f(tr^{-1})))).\end{aligned}$$

Let $\xi, \eta \in L^2(G,H')$ have compact support and $(K_n)_{n=1}^\infty$ be an increasing sequence of compact sets such that $G = \cup_{n=1}^\infty K_n$ (such a sequence exists since G is second countable and hence σ-compact). Then

$$\langle S_\varphi(T_{h_{K_n}})\xi, \eta\rangle = \int_{K_n\times K_n} \Delta(r)^{-1}\langle\theta(\alpha_{t^{-1}}(F(tr^{-1})(f(tr^{-1}))))\xi(r), \eta(t)\rangle drdt. \tag{31}$$

On the other hand,

$$\begin{aligned}&|\Delta(r)^{-1}\langle\theta(\alpha_{t^{-1}}(F(tr^{-1})(f(tr^{-1}))))\xi(r), \eta(t)\rangle| \\ \le\ &\Delta(r)^{-1}\|F\|_\infty\|f(tr^{-1})\|\|\xi(r)\|\|\eta(t)\|,\end{aligned}$$

while

$$\int_{G\times G} \Delta(r)^{-1}\|f(tr^{-1})\|\|\xi(r)\|\|\eta(t)\|drdt = \langle f' * \xi', \eta'\rangle,$$

where $f', \xi', \eta' : G \to \mathbb{R}$ are the functions given by $f'(s) = \|f(s)\|$, $\xi'(s) = \|\xi(s)\|$ and $\eta'(s) = \|\eta(s)\|$, $s \in G$. Thus, an application of the Lebesgue Dominated Convergence Theorem shows that the right hand side of (31) converges to

$$\int \Delta(r)^{-1}\langle \theta(\alpha_{t^{-1}}(F(tr^{-1})(f(tr^{-1}))))\xi(r), \eta(t)\rangle drdt = \langle (\pi^\theta \rtimes \lambda^\theta)(F\cdot f))\xi, \eta\rangle;$$

thus,

$$\lim_{n\to\infty} \langle S_\varphi(T_{h_{K_n}})\xi, \eta\rangle = \langle (\pi^\theta \rtimes \lambda^\theta)(F\cdot f)\xi, \eta\rangle. \tag{32}$$

By (30), $T_{h_{K_n}} \to_{n\to\infty} (\pi^\theta \rtimes \lambda^\theta)(f)$ in the weak* topology. It follows that

$$\begin{aligned} |\langle (\pi^\theta \rtimes \lambda^\theta)(F\cdot f))\xi, \eta\rangle| &\leq \limsup_{n\in\mathbb{N}} |\langle S_\varphi(T_{h_{K_n}})\xi, \eta\rangle| \\ &\leq \|\varphi\|_{\mathfrak{S}}\|\xi\|\|\eta\| \limsup_{n\in\mathbb{N}} \|T_{h_{K_n}}\| \\ &\leq \|\varphi\|_{\mathfrak{S}}\|\xi\|\|\eta\|\|(\pi^\theta \rtimes \lambda^\theta)(f)\|. \end{aligned}$$

Thus,

$$\|(\pi^\theta \rtimes \lambda^\theta)(F\cdot f))\| \leq \|\varphi\|_{\mathfrak{S}}\|(\pi^\theta \rtimes \lambda^\theta)(f)\|,$$

and hence the map S_F^θ is bounded. Similar arguments show that, in fact, S_F^θ is completely bounded and $\|S_F^\theta\|_{\mathrm{cb}} \leq \|\varphi\|_{\mathfrak{S}}$. By Remark 3.2 (iii), F is a Herz-Schur multiplier and

$$\|F\|_{\mathrm{m}} \leq \|\mathcal{N}(F)\|_{\mathfrak{S}}.$$

The last inequality and (28) show that $\|F\|_{\mathrm{m}} = \|\mathcal{N}(F)\|_{\mathfrak{S}}$ and the proof is complete. □

Remark 3.9. In the case $A = \mathbb{C}$, Theorem 3.8 reduces to the classical transference theorem for Herz-Schur multipliers [5]: a continuous function $u : G \to \mathbb{C}$ is a Herz-Schur multiplier of $A(G)$ if and only if $N(u)$ is a Schur multiplier on $G \times G$.

Corollary 3.10. *Let θ be a faithful *-representation of A on a separable Hilbert space and $F : G \to CB(A)$ be a pointwise measurable function. The following are equivalent:*

(i) F is a Herz-Schur multiplier such that S_F^θ can be extended to a weak continuous map on $A \rtimes_{\alpha,\theta}^{w^*} G$;*

(ii) there exists a weak continuous completely bounded linear map Φ on $A \rtimes_{\alpha,\theta}^{w^*} G$ such that for almost all $t \in G$*

$$\Phi(\pi^\theta(a)\lambda_t^\theta) = \pi^\theta(F(t)(a))\lambda_t^\theta, \quad a \in A. \tag{33}$$

In particular, (i) holds true if $\mathcal{N}(F)$ is a Schur θ-multiplier.

Proof. (i)⇒(ii) Suppose that $F : G \to CB(A)$ is a Herz-Schur multiplier and let Φ be the (unique) weak* continuous extension of S_F^θ to $A \rtimes_{\alpha,\theta}^{w^*} G$.

By Lemma 3.7, there exists a null set $N \subseteq G$ such that, if $(f_U)_U$ is a Dirac family at $t \in G \setminus N$ then, for every $a \in A$,

$$\Phi((\pi^\theta \rtimes \lambda^\theta)(f_U \otimes a)) \to_U \pi^\theta(F(t)(a))\lambda_t^\theta,$$

while, as can be easily checked,

$$(\pi^\theta \rtimes \lambda^\theta)(f_U \otimes a) \to_U \pi^\theta(a)\lambda_t^\theta$$

(both limits are in the weak operator topology). It follows that (33) holds for all $t \in G \setminus N$.

(ii)⇒(i) By Remark 3.4, if $f \in L^1(G, A)$, then

$$\Phi\left(\int_G \pi^\theta(f(t))\lambda_t^\theta dt\right) = \int_G \pi^\theta(F(t)(f(t)))\lambda_t^\theta dt,$$

i.e. Φ is a weak* continuous extension of S_F^θ. Since Φ is completely bounded, Remark 3.2 (iii) implies that $F \in \mathfrak{S}(G, A, \alpha)$.

Suppose that the map $S_{\mathcal{N}(F)_\theta}$ has a weak* continuous extension to a map on $\mathcal{B}(L^2(G))\bar{\otimes}\theta(A)''$. By the proof of Theorem 3.8, if $G = \cup_{n=1}^\infty K_n$, where K_n is an increasing sequence of compact subsets of G, $f \in C_c(G, A)$ and T_{h_K} is the operator with the kernel h_K given by (29), then $T_{h_{K_n}} \to (\pi^\theta \rtimes \lambda^\theta)(f)$ in the weak* topology. As $S_{\mathcal{N}(F)_\theta}$ has a weak* continuous extension, we have

$$\langle S_{\mathcal{N}(F)_\theta}(T_{h_{K_n}})\xi, \eta\rangle \to \langle S_{\mathcal{N}(F)_\theta}((\pi^\theta \rtimes \lambda^\theta)(f))\xi, \eta\rangle, \quad \xi, \eta \in L^2(G, H').$$

On the other hand, by (32),

$$\langle S_{\mathcal{N}(F)_\theta}(T_{h_{K_n}})\xi, \eta\rangle \to \langle S_F^\theta(\pi^\theta \rtimes \lambda^\theta(f))\xi, \eta\rangle.$$

Thus, S_F^θ is the restriction of $S_{\mathcal{N}(F)_\theta}$ to $A \rtimes_{\alpha,\theta} G$, and hence S_F^θ possesses a weak* continuous extension to $A \rtimes_{\alpha,\theta}^{w^*} G$. $\square$

Remark 3.11. We remark that if in Corollary 3.10 we assume also that $F : G \to CB(A)$ is $(\pi^\theta, \lambda^\theta)$-fiber continuous then condition (ii) is equivalent to F being a Herz-Schur θ-multiplier. Therefore, in this case, F is a Herz-Schur θ-multiplier if and only if F is a Herz-Schur multiplier such that S_F^θ possesses a weak*-continuous extension to $A \rtimes_{\alpha,\theta}^{w^*} G$.

Corollary 3.12. *Let θ be a faithful *-representation of A on a separable Hilbert space and $F : G \to CB(A)$ be a Herz-Schur θ-multiplier. Then* $\sup_{t\in G} \|F(t)\|_{\mathrm{cb}} \le \|F\|_{\mathrm{m}}$.

Proof. Immediate from Corollary 3.10 and the fact that λ_s^θ and π^θ are isometries. $\square$

An *equivariant representation* of the dynamical system (A, G, α) on a Hilbert A-module $\mathcal{N}$ is a pair (ρ, τ), where $\rho : A \to \mathcal{L}(\mathcal{N})$ is a *-representation of A on $\mathcal{N}$ and τ is a homomorphism from G into the group $\mathcal{I}(\mathcal{N})$ of all $\mathbb{C}$-linear, invertible, bounded maps on $\mathcal{N}$, which satisfy:

(1) $\rho(\alpha_s(a)) = \tau(s)\rho(a)\tau(s)^{-1}$ for all $s \in G$ and $a \in A$;

(2) $\alpha_s(\langle \xi, \eta\rangle_A) = \langle \tau(s)\xi, \tau(s)\eta\rangle_A$, for all $s \in G$ and ξ, $\eta \in \mathcal{N}$;
(3) $\tau(s)(\xi \cdot a) = (\tau(s)\xi) \cdot \alpha_s(a)$, for all $s \in G$, $\xi \in \mathcal{N}$ and $a \in A$;
(4) the map $s \mapsto \tau(s)\xi$ is continuous for every $\xi \in \mathcal{N}$.

This definition was given in [2] for discrete twisted dynamical systems. An example of such an equivariant representation can be obtained as follows. Let

$$H_A^G = \{\xi : G \to A : \xi(\cdot)^*\xi(\cdot) \in L^1(G, A)\}.$$

We have that H_A^G is a Hilbert A-module with respect to the right action given by $(\xi \cdot a)(s) := \xi(s)a$ and the inner product given by $\langle \xi, \eta\rangle_A = \int_G \xi(s)^*\eta(s)ds$. The regular equivariant representation of (A, G, α) on H_A^G is the pair (ρ, τ) given by

$$\rho(a)\xi(h) = a\xi(h), \quad (\tau(t)\xi)(s) = \alpha_t(\xi(t^{-1}s)).$$

It is easy to check that (ρ, τ) satisfies the conditions (1)-(4).

The following corollary was proved in [2, Theorem 4.8] for discrete dynamical systems using different arguments.

Corollary 3.13. *Let (ρ, τ) be an equivariant representation of (A, G, α) on a countably generated Hilbert A-module $\mathcal{N}$, and let ξ, $\eta \in \mathcal{N}$. Define*

$$F(t)(a) = \langle \xi, \rho(a)\tau(t)\eta\rangle_A, \quad t \in G, a \in A.$$

Then F is a Herz-Schur (A, G, α)-multiplier.

Proof. By Theorem 3.8, it suffices to show that $\mathcal{N}(F)$ is a Schur A-multiplier. We have

$$\begin{aligned}\mathcal{N}(F)(s,t)(a) &= \alpha_{t^{-1}}(F(ts^{-1})(\alpha_t(a))) = \alpha_{t^{-1}}(\langle \xi, \rho(\alpha_t(a))\tau(ts^{-1})\eta\rangle_A) \\ &= \langle \tau(t^{-1})\xi, \tau(t)^{-1}\rho(\alpha_t(a))\tau(t)\tau(s^{-1})\eta\rangle_A \\ &= \langle \tau(t^{-1})\xi, \rho(a)\tau(s^{-1})\eta\rangle_A\end{aligned}$$

As

$$\|\tau(t)\xi\|^2 = \|\langle \tau(t)\xi, \tau(t)\xi\rangle_A\| = \|\alpha_t(\langle \xi, \xi\rangle_A)\| = \|\langle \xi, \xi\rangle_A\| = \|\xi\|^2$$

for all $t \in G$, the statement follows from Theorem 2.7. $\square$

We next identify the Schur multipliers of the form $\mathcal{N}(F)$ as the "invariant" part of $\mathfrak{S}_0(G, G; A)$. Let ρ be the right regular representation of G on $L^2(G)$, *i.e.* $(\rho_r\xi)(s) = \Delta(r)^{1/2}\xi(sr)$, $\xi \in L^2(G)$, $s, r \in G$. Let $\tilde{\alpha}_r = (\mathrm{Ad}\rho_r) \otimes \alpha_r$, where, as usual, $\mathrm{Ad}U$ is the map given by $\mathrm{Ad}U(T) = UTU^*$; we have that $\tilde{\alpha}_r$ is a *-automorphism of $\mathcal{K}(L^2(G)) \otimes A$ [16, Theorem 11.2.9].

Definition 3.14. *A Schur A-multiplier $\varphi : G \times G \to CB(A)$ will be called* invariant *if S_φ commutes with $\tilde{\alpha}_r$ for every $r \in G$.*

We denote by $\mathfrak{S}_{\mathrm{inv}}(G, G; A)$ the set of all invariant Schur A-multipliers. If w is a (possibly vector-valued) function defined on $G \times G$, for $r \in G$, we let w_r be the function given by $w_r(s, t) = w(sr, tr)$.

Lemma 3.15. *If $k \in L^2(G \times G, A)$ then $\tilde{\alpha}_r(T_k) = T_{\tilde{k}}$, where $\tilde{k} \in L^2(G \times G, A)$ is given by $\tilde{k}(t, s) = \Delta(r)\alpha_r(k_r(t, s))$, $s, t \in G$.*

Proof. First note that if $k \in L^2(G \times G, A)$ then $\tilde{k} \in L^2(G \times G, A)$ and

$$\|\tilde{k}\|_2 = \|k\|_2; \tag{34}$$

indeed,

$$\|\tilde{k}\|_2 = \int \Delta^2(r)\|\alpha_r(k(tr, sr))\|^2 dt ds = \int \|k(t', s')\|^2 dt' ds'.$$

Let

$$\Theta, \Theta' : L^2(G \times G, A) \to \mathcal{B}(L^2(G, H))$$

be the maps defined by

$$\Theta(k) = T_{\tilde{k}}, \quad \Theta'(k) = \tilde{\alpha}_r(T_k).$$

By Lemma 2.1 and (34), Θ and Θ' are continuous.

Suppose that $k \in L^2(G \times G, A)$ is given by $k = h \otimes a$, where $h \in L^2(G \times G)$ and $a \in A$. As $\rho_r^*\xi(s) = \Delta(r)^{-1/2}\xi(sr^{-1})$, $\xi \in L^2(G)$, $s \in G$, we have

$$\begin{aligned}(\rho_r T_h \rho_r^* \xi)(s) &= \Delta(r)^{1/2}(T_h \rho_r^*)\xi(sr) = \Delta(r)^{1/2} \int h(sr, x)(\rho_r^*\xi)(x) dx \\ &= \int h(sr, x)\xi(xr^{-1}) dx = \int h(sr, yr)\xi(y)\Delta(r) dy,\end{aligned}$$

that is, $\rho_r T_h \rho_r^* = T_{\tilde{h}_r}$, where $\tilde{h}_r(t, s) = \Delta(r)h(tr, sr)$, $s, t \in G$. Thus,

$$\tilde{\alpha}_r(T_k) = \tilde{\alpha}_r(T_h \otimes a) = T_{\tilde{h}_r} \otimes \alpha_r(a),$$

and so $\Theta'(k) = \Theta(k)$. As h and a vary, the functions k span a dense subspace of $L^2(G \times G, A)$, and hence $\Theta = \Theta'$ by continuity. $\square$

In the proof of Theorem 3.18 below, we will need the following improvement of [43, Lemma 3.9].

Lemma 3.16. *Let $\mathcal{X}$ be a separable Banach space and $w : G \times G \to \mathcal{B}(\mathcal{X})$ be a bounded function, such that, for every $a \in \mathcal{X}$, the function $(s, t) \to w(s, t)(a)$ is weakly measurable, and $w_r = w$ almost everywhere, for every $r \in G$. Then there exists a bounded function $u : G \to \mathcal{B}(\mathcal{X})$ such that, for every $a \in \mathcal{X}$, the function $s \to u(s)(a)$ is measurable and, up to a null set, $w = N(u)$.*

Proof. The map $\phi : G \times G \to G \times G$, given by $\phi(y, x) = (y, xy)$, is continuous (and hence measurable) and bijective, and Fubini's Theorem shows that it preserves null sets in both directions. By assumption, for all $r \in G$, we have that $w_r(s, x) = w(s, x)$ for almost all $(s, x) \in G \times G$. Thus, $w_r(\phi(s, x)) = w(\phi(s, x))$ for almost all $(s, x) \in G \times G$, that is, $w(sr, xsr) = w(s, xs)$ for almost all $(x, s) \in G \times G$. We claim that $w(sr, xsr) = w(s, xs)$ for almost all $(x, s, r) \in G \times G \times G$. In fact, let $\mathcal{S} \subseteq \mathcal{X}$ be a countable dense subset.

For every $a \in \mathcal{S}$, we have

$$\begin{aligned}&\int_{G\times G\times G} \|w(sr, xsr)(a) - w(s, xs)(a)\| dxdsdr \\ = &\int_G \left(\int_{G\times G} \|w(sr, xsr)(a) - w(s, xs)(a)\| dxds \right) dr = 0.\end{aligned}$$

Thus, there exists a null set $N_a \subseteq G \times G \times G$ such that $w(sr, xsr)(a) = w(s, xs)(a)$ for all $(x, s, r) \notin N_a$. Let $N = \cup_{a\in\mathcal{S}} N_a$. Then $w(sr, xsr)(a) = w(s, xs)(a)$ for all $(x, s, r) \notin N$ and all $a \in \mathcal{S}$. Since $w(sr, xsr)$ and $w(s, xs)$ are bounded operators on $\mathcal{X}$, we conclude that $w(sr, xsr) = w(s, xs)$ for all $(x, s, r) \notin N$. Thus, there exists $s_0 \in G$ such that

$$w(s_0 r, xs_0 r) = w(s_0, xs_0), \text{ for almost all } (x, r) \in G \times G. \tag{35}$$

For each $x \in G$, let $u(x) = w(s_0, xs_0)$. Clearly, $u : G \to \mathcal{B}(\mathcal{X})$ is a bounded function such that, for every $a \in \mathcal{X}$, the function $x \to u(x)(a)$ is weakly measurable. Now (35) implies that $w(y, xy) = u(x)$ for almost all $(x, y) \in G \times G$. Letting $\tilde{u} : G \times G \to \mathcal{X}$ be the map given by $\tilde{u}(s, t) = u(t)$, we thus have that $w(y, xy) = \tilde{u}(y, x)$ for almost all $(x, y) \in G \times G$. It follows that $w(\phi^{-1}(y, xy)) = \tilde{u}(\phi^{-1}(y, x))$ for almost all $(x, y) \in G \times G$, that is, $w(y, x) = u(xy^{-1})$ for almost all $(x, y) \in G \times G$. The proof is complete. □

Lemma 3.17. *Let $\varphi \in \mathfrak{S}_0(G, G; A)$. The following are equivalent:*
(i) φ is an invariant Schur A-multiplier;
(ii) $\mathcal{T}(\varphi)_r = \mathcal{T}(\varphi)$ almost everywhere, for every $r \in G$.

Proof. Assume that A is faithfully represented on a separable Hilbert space.

(i)⇒(ii) Let $r \in G$, $a \in A$ and $k \in L^2(G \times G)$. By Lemma 3.15, $(S_\varphi \circ \tilde{\alpha}_r)(T_k \otimes a) = T_{k_1}$, where $k_1 : G \times G \to A$ is the function given by

$$k_1(t, s) = \Delta(r) k(tr, sr) \varphi(s, t)(\alpha_r(a)), \quad s, t \in G,$$

and $(\tilde{\alpha}_r \circ S_\varphi)(T_k \otimes a) = T_{k_2}$, where $k_2 : G \times G \to A$ is the function given by

$$k_2(t, s) = \Delta(r) k(tr, sr) \alpha_r(\varphi(sr, tr)(a)), \quad s, t \in G.$$

By Lemma 2.1, $k_1 = k_2$ almost everywhere, and hence

$$\varphi(sr, tr)(a) = \alpha_{r^{-1}}(\varphi(s, t)(\alpha_r(a))),$$

for almost all $(s, t) \in G \times G$. Thus, for every $a \in A$,

$$\begin{aligned}\mathcal{T}(\varphi)(sr, tr)(a) &= \alpha_{tr}(\varphi(sr, tr)(\alpha_{r^{-1}t^{-1}}(a))) \\ &= \alpha_{tr}(\alpha_{r^{-1}}(\varphi(s, t)(\alpha_r(\alpha_{r^{-1}t^{-1}}(a))))) \\ &= \alpha_t(\varphi(s, t)(\alpha_{t^{-1}}(a))) = \mathcal{T}(\varphi)(s, t)(a)\end{aligned}$$

for almost all $(s, t) \in G \times G$. Since A is separable, we conclude that $\mathcal{T}(\varphi)(sr, tr) = \mathcal{T}(\varphi)(s, t)$ for almost all $(s, t) \in G \times G$.

(ii)⇒(i) follows by reversing the steps in the previous paragraph and using the density in $\mathcal{K}(L^2(G)) \otimes A$ of the linear span of the operators of the form $T_k \otimes a$, with $k \in L^2(G \times G)$ and $a \in A$. □

Theorem 3.18. *The map $\mathcal{N}$ is a linear isometry from $\mathfrak{S}(A,G,\alpha)$ onto $\mathfrak{S}_{\rm inv}(G,G;A)$.*

Proof. By Theorem 3.8, the map $\mathcal{N}$ is a linear isometry from $\mathfrak{S}(A,G,\alpha)$ into $\mathfrak{S}_0(G,G;A)$. By the definition of $\mathcal{N}$, we have that $\mathcal{T}(\mathcal{N}(F))_r = \mathcal{T}(\mathcal{N}(F))$ almost everywhere for every $r \in G$ and every $F \in \mathfrak{S}(A,G,\alpha)$. By Lemma 3.17, the image of $\mathcal{N}$ is in $\mathfrak{S}_{\rm inv}(G,G;A)$.

It remains to show that $\mathcal{N}$ is surjective. To this end, let θ be a faithful *-representation of A on a separable Hilbert space K; we identify A with its image $\theta(A)$ under θ and let $\varphi \in \mathfrak{S}_{\rm inv}(G,G;A)$. By Lemmas 3.17 and 3.16, there exists a bounded function $F : G \to \mathcal{B}(A)$ such that $N(F) = \mathcal{T}(\varphi)$ almost everywhere and such that, for every $a \in A$, the function $s \to F(s)(a)$, is weakly measurable. It follows that $\mathcal{N}(F) = \varphi$ almost everywhere. Since $\varphi(x,y)$ is completely bounded for all (x,y), we have that $F(s) \in CB(A)$ for all $s \in G$. As φ is a Schur A-multiplier, it follows from the proof of Theorem 3.8 that the map $(\pi^\theta \rtimes \lambda^\theta)(f) \to (\pi^\theta \rtimes \lambda^\theta)(F\cdot f)$ on $(\pi^\theta \rtimes \lambda^\theta)(L^1(G,A))$ is completely bounded. By Remark 3.2 (iii), $F \in \mathfrak{S}(A,G,\alpha)$. $\square$

We now consider bounded, as opposed to completely bounded, multipliers, and use them to characterise Herz-Schur θ-multipliers in the spirit of [7] in Proposition 3.19 below. Let Γ be a a locally compact group. For a function $F : G \to CB(A)$, let $F^\Gamma : \Gamma \times G \to CB(A)$ be given by $F^\Gamma(x,s) = F(s)$, $x \in \Gamma$, $s \in G$. If α is an action of G on A, we let $\alpha^\Gamma : \Gamma \times G \to {\rm Aut}(A)$ be given by $\alpha^\Gamma_{(x,s)}(a) = \alpha_s(a)$, $a \in A$, $x \in \Gamma$, $s \in G$. We have the following characterisation of Herz-Schur θ-multipliers, similar to [7, Theorem 1.6].

Proposition 3.19. *Let $F : G \to CB(A)$ and $\theta : A \to \mathcal{B}(K)$ be a faithful *-representation. The following are equivalent:*
(i) F is a Herz-Schur θ-multiplier;
(ii) for each locally compact group Γ, the function F^Γ is a θ-multiplier;
(iii) for $\Gamma = SU(2)$, the function F^Γ is a θ-multiplier.

Proof. Let $\pi^{\Gamma,\theta} : A \to \mathcal{B}(L^2(\Gamma\times G,K))$ be the *-representation given by $\pi^{\Gamma,\theta}(a)\xi(x,t) = \alpha^\Gamma_{(x^{-1},t^{-1})}(a)(\xi(x,t))$. Identifying the Hilbert space $L^2(\Gamma \times G,K)$ with $L^2(\Gamma)\otimes L^2(G,K)$ in the natural way, we see that

$$\pi^{\Gamma,\theta}(a) = I_{L^2(\Gamma)} \otimes \pi^\theta(a), \quad a \in A.$$

On the other hand,

$$\lambda^\theta_{(x,t)} = \lambda^\Gamma_x \otimes \lambda^\theta_t, \quad x \in \Gamma, t \in G.$$

Suppose that $f \in L^1(\Gamma\times G,A)$ has the form $f(x,s) = g(x)h(s)$, where $f \in L^1(\Gamma)$ and $h \in L^1(G,A)$. We have

$$\begin{aligned}\int_{\Gamma\times G}\pi^{\Gamma,\theta}(f(x,s))\lambda^\theta_{(x,s)}dxds &= \int_{\Gamma\times G}(g(x)I_{L^2(\Gamma)}\otimes\pi^\theta(f(s)))(\lambda^\Gamma_x\otimes\lambda^\theta_s)dxds \\ &= \left(\int_\Gamma g(x)\lambda^\Gamma_x dx\right)\otimes\left(\int_G \pi^\theta(f(s))\lambda^\theta_s ds\right). \end{aligned} \tag{36}$$

Thus,

$$A \rtimes^{w^*}_{\alpha^\Gamma,\theta} (\Gamma \times G) = \mathrm{VN}(\Gamma)\bar{\otimes}(A \rtimes^{w^*}_{\alpha,\theta} G). \tag{37}$$

(i)⇒(ii) Suppose that $F : G \to CB(A)$ is a Herz-Schur θ-multiplier. Since the map Φ^θ_F on $A \rtimes^{w^*}_{\alpha,\theta} G$, given by $\Phi^\theta_F(\pi^\theta(a)\lambda^\theta_t) = \pi^\theta(F(t)(a))\lambda^\theta_t$, is completely bounded and weak* continuous, $\mathrm{id} \otimes \Phi^\theta_F$ is a (completely) bounded map on $\mathrm{VN}(\Gamma)\bar{\otimes}(A \rtimes^{w^*}_{\alpha,\theta} G)$ (see *e.g.* [7, Lemma 1.5]). Moreover,

$$\pi^{\Gamma,\theta}(F^\Gamma(a))\lambda^\theta_{(x,s)} = (\mathrm{id} \otimes \Phi^\theta_F)(\pi^{\Gamma,\theta}(a)\lambda^\theta_{(x,s)}). \tag{38}$$

It follows that the map $\Phi^\theta_{F^\Gamma} : \pi^{\Gamma,\theta}(a)\lambda^\theta_{(x,s)} \to \pi^{\Gamma,\theta}(F^\Gamma(a))\lambda^\theta_{(x,s)}$ extends to a bounded weak* continuous map on $A \rtimes^{w^*}_{\alpha^\Gamma,\theta} (\Gamma \times G)$; in other words, F^Γ is a θ-multiplier.

(ii)⇒(iii) is trivial.

(iii)⇒(i) We have that $\mathrm{VN}(SU(2)) \equiv \oplus_{n\in\mathbb{N}} M_n$, where M_n is the n by n matrix algebra. Hence

$$\mathrm{VN}(SU(2))\bar{\otimes}(A \rtimes^{w^*}_{\alpha,\theta} G) \equiv \oplus_{n=1}^\infty \left(M_n \otimes (A \rtimes^{w^*}_{\alpha,\theta} G) \right).$$

As $\Phi^\theta_{F^\Gamma}$ is a bounded weak* continuous map on $\mathrm{VN}(SU(2))\bar{\otimes}(A \rtimes^{w^*}_{\alpha,\theta} G)$, equations (37) and (38) now imply that $\| \mathrm{id}_{M_n} \otimes \Phi^\theta_F\| \leq \|\Phi^\theta_{F^\Gamma}\|$ for all n and hence Φ^θ_F is completely bounded. □

4. Multipliers of the weak* crossed product

In this section, we consider the weak* extendable Herz-Schur multipliers introduced in Definition 3.3, and characterise them as the commutator of the "scalar valued" multipliers described in Proposition 4.1 below. We fix, throughout the section, a C^*-dynamical system (A, G, α). As before, A is assumed to be separable, while G is assumed to be second countable. If $\theta : A \to \mathcal{B}(K)$ is a faithful *-representation, where K is a separable Hilbert space, we let $\alpha^\theta : G \to \mathrm{Aut}(\theta(A))$ be given by $\alpha^\theta_t(\theta(a)) = \theta(\alpha_t(a))$, $t \in G$, $a \in A$. We call α a *θ-action*, if α^θ_t can be extended to a weak* continuous automorphism (which we will denote in the same fashion) of $\theta(A)''$, such that the map $s \to \alpha^\theta_s(x)$ from G into $\theta(A)''$ is weak*-continuous for each $x \in \theta(A)''$.

Proposition 4.1. *Let $u : G \to \mathbb{C}$ be a bounded continuous function, and let $F_u : G \to CB(A)$ be given by $F_u(t)(a) = u(t)a$, $a \in A, t \in G$. The following are equivalent:*

(i) F_u is a Herz-Schur (A, G, α)-multiplier;

(ii) $u \in M^{\mathrm{cb}}A(G)$.

Moreover, if (i) holds then F_u is a Herz-Schur θ-multiplier for every faithful representation θ of A on a separable Hilbert space.

Proof. Set $F = F_u$. We have

$$\mathcal{N}(F)(s,t)(a) = u(ts^{-1})a, \quad a \in A, \ s, t \in G. \tag{39}$$

We assume, without loss of generality, that A is a non-degenerate C*-subalgebra of $\mathcal{B}(H)$, for a separable Hilbert space H.

(i)⇒(ii) By Theorems 2.6 and 3.8, there exist a separable Hilbert space K, a non-degenerate *-representation $\rho : A \to \mathcal{B}(K)$ and elements V, W of $L^\infty(G, \mathcal{B}(H, K))$ such that

$$u(ts^{-1})a = W(t)^*\rho(a)V(s), \quad \text{for almost all } s, t \in G \text{ and all } a \in A.$$

Let $(a_i)_{i=1}^\infty$ be a bounded approximate identity for A. Then, for a unit vector $\xi \in H$ and every $i \in \mathbb{N}$, we have

$$\langle u(ts^{-1})a_i\xi, \xi\rangle = \langle \rho(a_i)V(s)\xi, W(t)\xi\rangle, \quad \text{for almost all } s, t \in G.$$

Since $A \subseteq \mathcal{B}(H)$ is non-degenerate and ρ is a non-degenerate representation, passing to a limit along i, we obtain

$$u(ts^{-1}) = \langle V(s)\xi, W(t)\xi\rangle, \quad \text{for almost all } s, t \in G.$$

By [5], $u \in M^{\text{cb}}A(G)$.

(ii)⇒(i) As G is second countable, by [5], there exist bounded, weakly measurable functions $\xi, \eta : G \to \ell^2$, such that

$$u(ts^{-1}) = \langle \xi(s), \eta(t)\rangle, \quad \text{for almost all } s, t \in G.$$

Let $\rho : A \to \mathcal{B}(H^\infty)$ be the countable ampliation of the identity representation of A. Write $\xi(s) = (\xi_i(s))_{i\in\mathbb{N}}$ and $\eta(t) = (\eta_i(t))_{i\in\mathbb{N}}$, $s, t \in G$. Let $V(s) : H \to H^\infty$ (resp. $W(t) : H \to H^\infty$) be given by $V(s) = (\xi_i(s)I_H)_{i\in\mathbb{N}}$ (resp. $W(t) = (\eta_i(t)I_H)_{i\in\mathbb{N}}$). Then $V, W \in L^\infty(G, \mathcal{B}(H, H^\infty))$ and

$$W(t)^*\rho(a)V(s) = \sum_{i=1}^\infty \xi_i(s)\overline{\eta_i(t)}a = u(ts^{-1})a,$$

for almost all $s, t \in G$ and all $a \in A$. It follows by (39) and Theorems 2.6 and 3.8 that F_u is a Herz-Schur (A, G, α)-multiplier.

Now suppose that $u \in M^{\text{cb}}A(G)$ and denote by Ψ_u the weak* continuous completely bounded map on $\mathcal{B}(L^2(G))$ corresponding to the function u *via* classical transference [5] (see Remark 3.9). Let $\theta : A \to \mathcal{B}(K)$ be a faithful *-representation of A, for some separable Hilbert space K. Note that $\mathcal{N}(F)$ is a Schur θ-multiplier; indeed, we have that $\mathcal{N}(F)_\theta(s,t)(\theta(a)) = u(ts^{-1})\theta(a)$, $a \in A$, and hence $S_{\mathcal{N}(F)_\theta} = \Psi_u|_{\mathcal{K}(L^2(G))} \otimes \text{id}_{\theta(A)}$. It follows that $S_{\mathcal{N}(F)_\theta}$ is the restriction to $\mathcal{K}(L^2(G)) \otimes \theta(A)$ of the weak* continuous map $\Psi_u \otimes \text{id}_{\theta(A)''}$. By Corollary 3.10 and Remark 3.11, F_u is a Herz-Schur θ-multiplier. □

In what follows we denote by S_u^θ the weak* continuous map on $A \rtimes_{\alpha,\theta}^{w^*} G$ arising from the previous proposition.

It is well-known that an essentially bounded function on G that is invariant under right translations agrees almost everywhere with a constant function. The next lemma is a dynamical system version of this fact. For a *-representation $\theta : A \to \mathcal{B}(K)$ such that α is a θ-action, let $\bar{\pi}^\theta$ be the *-representation of $\theta(A)''$ on $L^2(G, K)$ given by $(\bar{\pi}^\theta(a)\xi)(s) = \alpha_s^{-1}(a)(\xi(s))$,

$a \in \theta(A)''$, $\xi \in L^2(G, K)$. Recall that if $\rho : G \to \mathcal{B}(L^2(G))$ is the right regular representation of G we write $\tilde{\alpha}_r$ for the map $\mathrm{Ad}\rho_r \otimes \alpha_r$ on $\mathcal{K}(L^2(G)) \otimes \theta(A)$; we have that the map $\tilde{\alpha}_r$ can be extended to a weak* continuous map $\mathcal{B}(L^2(G)\bar{\otimes}\theta(A)''$, denoted in the same fashion.

Lemma 4.2. *Let (A, G, α) be a C^*-dynamical system and $\theta : A \to \mathcal{B}(K)$ be a faithful *-representation, where K is a separable Hilbert space, such that α is a θ-action. Then*

$$(\mathcal{D}_G\bar{\otimes}\theta(A)'') \cap (A \rtimes^{w^*}_{\alpha,\theta} G) = \bar{\pi}^\theta(\theta(A)''). \tag{40}$$

Proof. Suppose that D belongs to both $L^\infty(G, \theta(A)'')$ and $A \rtimes^{w^*}_{\alpha,\theta} G$. By [25, Theorem 1.2, Chapter II], there exists a null set $M \subseteq G$ such that $\tilde{\alpha}_r(D) = D$ whenever $r \in M^c$. However, $\tilde{\alpha}_r(D) = D_r$, where $D_r \in L^\infty(G, \theta(A)'')$ is given by $D_r(s) = \alpha_r(D(sr))$, $s \in G$. Let $\tilde{D} \in L^\infty(G, \theta(A)'')$ be defined by $\tilde{D}(s) = \alpha_s(D(s))$, $s \in G$. For every $r \in M^c$,

$$\tilde{D}(sr) = \alpha_{sr}(D(sr)) = \alpha_s(\alpha_r(D(sr)) = \alpha_s(D(s)) = \tilde{D}(s), \text{ for almost all } s.$$

As in the proof of Lemma 3.16, there exists $s_0 \in G$ such that $\tilde{D}(s_0r) = \tilde{D}(s_0)$ for almost all $r \in G$. Thus, there exists $a \in \theta(A)''$ such that $\tilde{D}(t) = a$ for almost all $t \in G$, and hence $D(t) = \alpha_{t^{-1}}(a)$ for almost all $t \in G$; in other words, $D = \bar{\pi}^\theta(a)$. We thus showed that the intersection on the left hand side of (40) is contained in $\bar{\pi}^\theta(\theta(A)'')$. The converse inclusion is trivial. □

Let K be a Hilbert space. If $\omega \in \mathcal{B}(H)_*$, we let L_ω be the (unique) weak* continuous linear map from $\mathcal{B}(K\otimes H)$ into $\mathcal{B}(K)$ such that $L_\omega(b\otimes a) = \omega(a)b$, $a \in \mathcal{B}(H)$, $b \in \mathcal{B}(K)$. Recall that a weak* closed subspace $\mathcal{U} \subseteq \mathcal{B}(H)$ is said to have property S_σ [19] if

$$\mathcal{V}\bar{\otimes}\mathcal{U} = \{T \in \mathcal{B}(K)\bar{\otimes}\mathcal{U} : L_\omega(T) \in \mathcal{V} \text{ for all } \omega \in \mathcal{B}(H)_*\},$$

for every weak* closed subspace $\mathcal{V} \subseteq \mathcal{B}(K)$.

For the proof of the next theorem, we recall that an operator $T \in \mathcal{B}(L^2(G))$ is said to be supported on a measurable subset $E \subseteq G\times G$ if $M_{\chi_\beta}TM_{\chi_\alpha} = 0$ whenever $\alpha, \beta \subseteq G$ are measurable sets with $(\alpha \times \beta) \cap E = \emptyset$. It is easy to see that the space of operators supported on the set $\{(s, ts) : s \in G\}$ coincides with $\mathcal{D}_G\lambda^G_t$.

Theorem 4.3. *Let (A, G, α) be a C^*-dynamical system and θ be a faithful *-representation of A on a separable Hilbert space K such that α is a θ-action and $\theta(A)''$ possesses property S_σ. Let Φ be a completely bounded weak* continuous map on $A \rtimes^{w^*}_{\alpha,\theta} G$. The following are equivalent:*

(i) $\Phi S^\theta_u = S^\theta_u\Phi$ for all $u \in M^{\mathrm{cb}}A(G)$;

(ii) For each $t \in G$, there exists completely bounded map $F_t : \theta(A)'' \to \theta(A)''$ such that $\Phi(\bar{\pi}^\theta(a)\lambda^\theta_t) = \bar{\pi}^\theta(F_t(a))\lambda^\theta_t$, $a \in \theta(A)''$.

Proof. (i)⇒(ii) Given $u \in M^{\mathrm{cb}}A(G)$, let Ψ_u be the weak* continuous completely bounded map on $\mathcal{B}(L^2(G))$ corresponding to the Schur multiplier

$N(u)$ (see Remark 3.9). We claim that

$$(L_\omega \circ S_u^\theta)(T) = (\Psi_u \circ L_\omega)(T), \quad T \in A \rtimes_{\alpha,\theta}^{w^*} G, \ \omega \in \mathcal{B}(K)_*. \tag{41}$$

For $S \in \mathcal{B}(K)$ and $T \in \mathcal{B}(L^2(G))$, we have

$$L_\omega((S \otimes T)(I_K \otimes \lambda_t^G)) = \omega(S)T\lambda_t^G = L_\omega(S \otimes T)\lambda_t^G;$$

by weak* continuity, we obtain

$$L_\omega(R(I_K \otimes \lambda_t^G)) = L_\omega(R)\lambda_t^G, \quad R \in \mathcal{B}(L^2(G,K)). \tag{42}$$

On the other hand, if $t \in G$ then $\lambda_t^\theta = I_K \otimes \lambda_t^G$, and it follows that

(43)

$$L_\omega(S_u^\theta(\pi^\theta(a)\lambda_t^\theta)) = u(t)L_\omega(\pi^\theta(a)\lambda_t^\theta) = u(t)L_\omega(\pi^\theta(a))\lambda_t^G, \quad a \in A, t \in G.$$

Since $\pi^\theta(a) \in L^\infty(G, \theta(A)'')$, we have that $L_\omega(\pi^\theta(a)) \in \mathcal{D}_G$, for every $\omega \in \mathcal{B}(K)_*$. Since Ψ_u is a $\mathcal{D}_G$-bimodule map, using equation (42) we obtain

$$\begin{aligned} \Psi_u(L_\omega(\pi^\theta(a)\lambda_t^\theta)) &= \Psi_u(L_\omega(\pi^\theta(a))\lambda_t^G) = L_\omega(\pi^\theta(a))\Psi_u(\lambda_t^G) \\ &= u(t)L_\omega(\pi^\theta(a))\lambda_t^G. \end{aligned} \tag{44}$$

Equation (41) follows from the weak* continuity of L_ω and S_u^θ (see Proposition 4.1), after comparing (43) and (44).

Let $a \in \theta(A)''$, $t \in G$ and $T = \bar{\pi}^\theta(a)\lambda_t^\theta$. Set

$$J = \{u \in M^{\mathrm{cb}}A(G) : u(t) = 1\}.$$

If $u \in J$ then $S_u^\theta(T) = T$ and hence, by (41) and the fact that Φ commutes with S_u^θ, we have

$$\Psi_u(L_\omega(\Phi(T))) = L_\omega(\Phi(T)).$$

Thus, for every $u \in J$, the operator $L_\omega(\Phi(T))$ is u-harmonic in the sense of [26]. It follows from [1, Corollary 3.7] that $L_\omega(\Phi(T))$ is supported on the set $\{(x,y) \in G \times G : yx^{-1} \in Z\}$, where $Z = \{s \in G : u(s) = 1, \text{ for all } u \in J\}$. By the regularity of $A(G)$ and the fact that $A(G) \subseteq M^{\mathrm{cb}}A(G)$, we have that $Z = \{t\}$ and hence, by (42) and the paragraph before the statement of the theorem,

$$L_\omega(\Phi(T)\lambda_{t^{-1}}^\theta) = L_\omega(\Phi(T))\lambda_{t^{-1}}^G \in \mathcal{D}_G.$$

Since this holds for every $\omega \in \mathcal{B}(K)_*$ and $\theta(A)''$ is assumed to possess property S_σ, we conclude that $\Phi(T)\lambda_{t^{-1}}^\theta \in \mathcal{D}_G \bar{\otimes} \theta(A)''$.

On the other hand, $\Phi(T)\lambda_{t^{-1}}^\theta \in A \rtimes_{\alpha,\theta}^{w^*} G$. By Lemma 4.2, $\Phi(T)\lambda_{t^{-1}}^\theta = \bar{\pi}^\theta(a_t)$ for some $a_t \in \theta(A)''$, and hence $\Phi(T) = \bar{\pi}^\theta(a_t))\lambda_t^\theta$. Writing $F_t(a) = a_t$, we have $\Phi(T) = \bar{\pi}^\theta(F_t(a)))\lambda_t^\theta$. The map F_t is linear and completely bounded since Φ is so.

(ii)⇒(i) For $t \in G$ and $a \in \theta(A)''$, we have

$$\Phi(S_u^\theta(\bar{\pi}^\theta(a)\lambda_t^\theta)) = u(t)\Phi(\bar{\pi}^\theta(a)\lambda_t^\theta) = u(t)\bar{\pi}^\theta(F_t(a))\lambda_t^\theta = S_u^\theta(\Phi(\bar{\pi}^\theta(a)\lambda_t^\theta)).$$

The commutation relations now follow by linearity and weak* continuity. □

5. Two classes of multipliers

In this section, we describe two special classes of Herz-Schur multipliers and relate them to maps that have been studied previously.

5.1. **Multipliers from the Haagerup tensor product.**

Multipliers of the type studied in this subsection have been considered in the case of a discrete group in [2]. Let A be a separable non-degenerate C*-subalgebra of $\mathcal{B}(H)$, where H is a separable Hilbert space, and $C_\infty(A)$ be the column operator space over A; thus, the elements of $C_\infty(A)$ are the sequences $(a_i)_{i\in\mathbb{N}} \subseteq A$ such that the series $\sum_{i=1}^\infty a_i^* a_i$ converges in norm. Recall [4] that the Haagerup tensor product $A \otimes_{\mathrm{h}} A$ consists, by definition, of all sums $u = \sum_{i=1}^\infty b_i \otimes a_i$, where $(a_i)_{i\in\mathbb{N}}, (b_i^*)_{i\in\mathbb{N}} \in C_\infty(A)$. Let $\beta : X \to C_\infty(A)$ and $\gamma : Y \to C_\infty(A)$ be bounded weakly measurable functions. Write $\beta(x) = (\beta_i(x))_{i\in\mathbb{N}}$, $x \in X$, and $\gamma(y) = (\gamma_i(y))_{i\in\mathbb{N}}$, $y \in Y$. It is clear that, in particular, $\beta_i \in L^\infty(X, A)$ and $\gamma_i \in L^\infty(Y, A)$ for each $i \in \mathbb{N}$.

Let $\varphi_{\beta,\gamma} : X \times Y \to A \otimes_{\mathrm{h}} A$ be given by

$$\varphi_{\beta,\gamma}(x, y) = \sum_{i=1}^\infty \gamma_i(y)^* \otimes \beta_i(x), \quad (x, y) \in X \times Y. \tag{45}$$

Note that $A \otimes_{\mathrm{h}} A$ embeds canonically into $CB(A)$; for an element $u = \sum_{i=1}^\infty b_i \otimes a_i$ of $A \otimes_{\mathrm{h}} A$, the corresponding map $\Phi_u : A \to A$ is given by $\Phi_u(a) = \sum_{i=1}^\infty b_i a a_i$, $a \in A$. We thus view $\varphi_{\beta,\gamma}(x, y)$ as a completely bounded map on A. It is easy to see that the partial sums of (45) define weakly measurable functions, and since the convergence of the series is in norm, [45, Lemma B.17] shows that the function $\varphi_{\beta,\gamma}$ is weakly measurable. In particular, $\varphi_{\beta,\gamma}$ is pointwise measurable.

Proposition 5.1. *Let $\beta : X \to C_\infty(A)$ and $\gamma : Y \to C_\infty(A)$ be bounded weakly measurable functions. Then $\varphi_{\beta,\gamma}$ is a Schur* id*-multiplier. Moreover,*

$$S_{\varphi_{\beta,\gamma}}(T) = \sum_{i=1}^\infty \gamma_i^* T \beta_i, \quad T \in \mathcal{K} \otimes A, \tag{46}$$

where the series converges in norm.

Proof. First note that

$$\left\| \sum_{i=1}^\infty \beta_i^* \beta_i \right\| = \mathrm{essup}_{x\in X} \left\| \sum_{i=1}^\infty \beta_i^*(x) \beta_i(x) \right\| = \mathrm{essup}_{x\in X} \|\beta(x)\|^2,$$

and that a similar estimate holds for $\sum_{i=1}^\infty \gamma_i \gamma_i^*$. It follows that the series on the right hand side of (46) converges in norm. Let $k \in L^2(Y \times X, A)$,

$\xi \in L^2(X,H)$ and $\eta \in L^2(Y,H)$. Then

$$\begin{aligned}\left\langle \sum_{i=1}^{\infty} \gamma_i^* T_k \beta_i \xi, \eta \right\rangle &= \int_{X\times Y} \sum_{i=1}^{\infty} \langle k(y,x)\beta_i(x)\xi(x), \gamma_i(y)\eta(y)\rangle dxdy \\ &= \int_{X\times Y} \langle \varphi_{\beta,\gamma}(x,y)(k(y,x))\xi(x), \eta(y)\rangle dxdy \\ &= \langle T_{\varphi_{\beta,\gamma}\cdot k}\xi, \eta\rangle.\end{aligned}$$

It follows that $\varphi_{\beta,\gamma}$ is a Schur A-multiplier. Identity (46) now follows by boundedness. Since the map expressed by the right hand side of (46) is weak* extendible, we conclude that $\varphi_{\beta,\gamma}$ is in fact a Schur id-multiplier. $\square$

Proposition 5.2. *Let $\beta : G \to C_\infty(A)$ and $\gamma : G \to C_\infty(A)$ be bounded weakly measurable functions. The following are equivalent:*

(i) there exists $F \in \mathfrak{S}(A,G,\alpha)$ such that $S_F^{\rm id}$ coincides with the restriction of $S_{\varphi_{\beta,\gamma}}$ to $A \rtimes_{\alpha,\rm id} G$;

(ii) for every $a \in A$, the function $\varphi_a : G\times G \to A$ given by

$$\varphi_a(s,t) = \sum_{i=1}^{\infty} \alpha_t(\gamma_i(t))^* a \alpha_t(\beta_i(s)), \quad s,t \in G,$$

has the property that, for every $r \in G$, $\varphi_a(sr,tr) = \varphi_a(s,t)$ for almost all (s,t).

Moreover, if (i) holds then the map $S_F^{\rm id}$ has an extension to a bounded weak continuous map on $A \rtimes_{\alpha,\rm id}^{w^*} G$.*

Proof. (i)⇒(ii) By Proposition 5.1, the map $S_{\varphi_{\beta,\gamma}}$ has a weak* continuous extension to a completely bounded map on $\mathcal{B}(L^2(G))\bar{\otimes}A''$. Since $S_F^{\rm id}$ is the restriction of $S_{\varphi_{\beta,\gamma}}$, it possesses a weak* continuous extension to a completely bounded map on $A \rtimes_{\alpha,\rm id}^{w^*} G$.

Let $a \in A$ and $s \in G$. Note that, if $\beta_i^s \in L^\infty(G,A)$ is given by $\beta_i^s(t) = \beta_i(s^{-1}t)$, then $\lambda_s^{\rm id}\beta_i = \beta_i^s\lambda_s^{\rm id}$. By Corollary 3.10, for almost all $s \in G$, we have

$$\pi^{\rm id}(F(s)(a)) = \left(\sum_{i=1}^{\infty} \gamma_i^* \pi^{\rm id}(a)\lambda_s^{\rm id}\beta_i\right)(\lambda_s^{\rm id})^* = \sum_{i=1}^{\infty} \gamma_i^* \pi^{\rm id}(a)\beta_i^s, \ a \in A.$$

Therefore, if $\xi \in L^2(G,H)$ then, for almost all $s,t \in G$, we have

$$\begin{aligned}\alpha_{t^{-1}}(F(s)(a))(\xi(t)) &= \left(\sum_{i=1}^{\infty} \gamma_i^* \pi^{\rm id}(a)\beta_i^s\xi\right)(t) \\ &= \sum_{i=1}^{\infty} \gamma_i(t)^* \alpha_{t^{-1}}(a)\beta_i(s^{-1}t)(\xi(t)).\end{aligned}$$

A standard argument using the separability of H now shows that, for almost all $s,t\in G$, we have

$$\alpha_{t^{-1}}(F(s)(a)) = \sum_{i=1}^{\infty}\gamma_i(t)^*\alpha_{t^{-1}}(a)\beta_i(s^{-1}t),$$

and, since the series on the right hand side converges is norm,

$$\sum_{i=1}^{\infty}\alpha_t(\gamma_i^*(t))a\alpha_t(\beta_i(s^{-1}t)) = F(s)(a), \tag{47}$$

i.e. $\varphi_a(s^{-1}t,t) = F(s)(a)$ for almost all $s,t\in G$. As the map $(s,t)\mapsto(t,s^{-1}t)$ is continuous, bijective and preserves null sets in both directions, we obtain $\varphi_a(s,t) = F(ts^{-1})(a)$ for almost all $(s,t)\in G\times G$. Hence, for each $r\in G$, $\varphi_a(sr,tr) = \varphi_a(s,t)$ almost everywhere on $G\times G$.

(ii)⇒(i) As $\gamma(t),\beta(s)\in C_\infty(A)$ for all (s,t), we have that

$$\varphi_{\beta,\gamma}(s,t)(a) = \lim_{n\to\infty}\sum_{i=1}^{n}\gamma_i(t)^*a\beta_i(s)$$

in norm and, in particular, $\varphi_{\beta,\gamma}(s,t)(a)\in A$ for all $a\in A$. Hence

$$\alpha_t(\varphi_{\beta,\gamma}(s,t)(\alpha_{t^{-1}}(a)) = \sum_{i=1}^{\infty}\alpha_t(\gamma_i(t))^*a\alpha_t(\beta_i(s)) = \varphi_a(s,t).$$

Thus $\mathcal{T}(\varphi_{\beta,\gamma})(s,t)(a) = \varphi_a(s,t)$ for all $s,t\in G$. Fix $r\in G$ and let $\mathcal{S}\subseteq A$ be a countable dense subset. Then, for every $a\in\mathcal{S}$ we have

$$\begin{aligned}\mathcal{T}(\varphi_{\beta,\gamma})_r(s,t)(a) &= \mathcal{T}(\varphi_{\beta,\gamma})(sr,tr)(a) = \varphi_a(sr,tr)\\ &= \varphi_a(s,t) = \mathcal{T}(\varphi_{\beta,\gamma})(s,t)(a),\end{aligned}$$

for almost all $(s,t)\in G\times G$. It follows that there exists a set $E\subseteq G\times G$ whose complement is null, such that $\mathcal{T}(\varphi_{\beta,\gamma})_r(s,t)(a) = \mathcal{T}(\varphi_{\beta,\gamma})(s,t)(a)$ for all $(s,t)\in E$ and all $a\in\mathcal{S}$. Fix $(s,t)\in E$. By the boundedness of the maps $\mathcal{T}(\varphi_{\beta,\gamma})_r(s,t)$ and $\mathcal{T}(\varphi_{\beta,\gamma})(s,t)$, we have that

$$\mathcal{T}(\varphi_{\beta,\gamma})_r(s,t)(a) = \mathcal{T}(\varphi_{\beta,\gamma})(s,t)(a),$$

for all $a\in A$. Thus, $\mathcal{T}(\varphi_{\beta,\gamma})_r = \mathcal{T}(\varphi_{\beta,\gamma})$ almost everywhere, for all $r\in G$. By Lemma 3.17, Theorem 3.18 and Proposition 5.1, there exists $F\in\mathfrak{S}(A,G,\alpha)$ such that $\mathcal{N}(F) = \varphi_{\beta,\gamma}$ almost everywhere. □

5.2. **Groupoid multipliers.** In this subsection, we relate Herz-Schur multipliers to the multipliers of the Fourier algebra of a groupoid. We refer the reader to [23] and [34] for more details on the background, which we now recall.

Let G be a locally compact group acting on a locally compact Hausdorff space X; thus, we are given a map $X\times G\to X$, $(x,s)\to xs$, jointly continuous and such that $x(st) = (xs)t$ for all $x\in X$ and all $s,t\in G$.

The set $\mathcal{G} = X\times G$ is a groupoid, where the set $\mathcal{G}^2$ of composable pairs is given by $\mathcal{G}^2 = \{[(x_1,t_1),(x_2,t_2)] : x_2 = x_1t_1\}$, and if $[(x_1,t_1),(x_2,t_2)]\in\mathcal{G}^2$, the product $(x_1, t_1) \cdot (x_2, t_2)$ is defined to be $(x_1, t_1 t_2)$, while the inverse $(x, t)^{-1}$ of (x, t) is defined to be (xt, t^{-1}). The domain and range maps are given by

$$d((x, t)) := (x, t)^{-1} \cdot (x, t) = (xt, e), \quad r((x, t)) := (x, t) \cdot (x, t)^{-1} = (x, e).$$

The unit space $\mathcal{G}_0$ of the groupoid, which is by defnition equal to the common image of the maps d and r, can therefore be canonically identified with X.

Let λ be the left Haar measure on G. The groupoid $\mathcal{G}$ can be equipped with the Haar system $\{\lambda^x : x \in X\}$, where $\lambda^x = \delta_x \times \lambda$ and δ_x is the point mass at x. The space $C_c(\mathcal{G})$ of compactly supported continuous functions on $\mathcal{G}$ is a $*$-algebra with respect to the convolution product given by

$$(f * g)(x, t) = \int f(x, s) g(xs, s^{-1}t) ds,$$

and the involution given by $f^*(x, s) = \overline{f(xs, s^{-1})}$. We equip $C_c(\mathcal{G})$ with the norm

$$\|f\|_I = \max\left\{\sup_{x\in X} \int |f| d\lambda^x, \sup_{x\in X} \int |f^*| d\lambda^x\right\}.$$

The completion of $C_c(\mathcal{G})$ with respect to this norm is denoted by $L^I(\mathcal{G})$, and its enveloping C^*-algebra $C^*(\mathcal{G})$ is called the *groupoid C^*-algebra* of $\mathcal{G}$.

Let $A = C_0(X)$ and $\alpha_t(a)(x) = a(xt)$, $t \in G$, $x \in X$. Then $\alpha : t \mapsto \alpha_t$ is a continuous homomorphism from G to $\mathrm{Aut}(A)$. Identifying $C_c(X \times G) = C_c(\mathcal{G})$ with a subspace of $C_c(G, A)$, we see that the $*$-algebra structure on $C_c(G, A)$, associated with the action α (see the beginning of Section 3), coincides with the one on $C_c(\mathcal{G})$ except for the absence of the modular function in the definition of the involution. However, the C^*-algebras $C^*(\mathcal{G})$ and the full crossed product $A \rtimes_\alpha G$ are isomorphic *via* the map ϕ given by $\phi(f)(x, s) = \Delta^{-1/2}(s) f(x, s)$, $f \in C_c(X \times G)$. In fact, for $f, g \in C_c(X \times G)$, we have

$$\phi(f * g)(x, s) = \Delta^{-1/2}(s)(f * g)(x, s) = \Delta^{-1/2}(s) \int f(x, t) g(xt, t^{-1}s) dt,$$

while

$$\phi(f) \times \phi(g)(x, s) = \Delta^{-1/2}(s) \int f(x, t) g(xt, t^{-1}s) dt;$$

hence, $\phi(f * g) = \phi(f) \times \phi(g)$. In addition,

$$\phi(f^*)(x, s) = \Delta^{-1/2}(s) \overline{f(xs, s^{-1})},$$

while

$$\begin{aligned} \phi(f)^*(x, s) &= \Delta^{-1}(s) \overline{\phi(f)(xs, s^{-1})} = \Delta^{-1}(s) \Delta^{-1/2}(s^{-1}) \overline{f(xs, s^{-1})} \\ &= \Delta^{-1/2}(s) \overline{f(xs, s^{-1})}, \end{aligned}$$

giving $\phi(f)^* = \phi(f^*)$. By [23, p. 9], the map ϕ extends to a *-isomorphism from $C^*(\mathcal{G})$ onto $A \rtimes_\alpha G$.

Let μ be a measure on X and $\nu = \mu \times \lambda$; thus, for a measurable subset E of $X \times G$, we have $\nu(E) = \int \lambda^x(E^x) d\mu(x)$ (for $x \in X$, we have set $E^x = E \cap (\{x\} \times G)$). For a measurable subset E, set $\nu^{-1}(E) = \int \lambda^x((E^{-1})^x) d\mu(x)$.

Let $\mathrm{Ind}(\mu)$ be the *-representation of $C_c(\mathcal{G})$ on $L^2(\mathcal{G}, \nu^{-1})$ given by

$$(\mathrm{Ind}(\mu)(f)\xi)(x,t) = \int f(x,s)\xi(xs, s^{-1}t)ds, \quad f \in C_c(\mathcal{G}).$$

One can check that $\|\mathrm{Ind}(\mu)(f)\| \leq \|f\|_I$ [23]; hence $\mathrm{Ind}(\mu)$ can be extended to $C^*(\mathcal{G})$.

If $\operatorname{supp}\mu = X$ then the map $f \mapsto M_f$, where M_f is the operator of multiplication by f on $L^2(X, \mu)$, is a faithful *-representation θ of $C_0(X)$. The corresponding regular representation $\pi^\theta \rtimes \lambda^\theta$ of the crossed product $A \rtimes_\alpha G$ on $L^2(G, L^2(X, \mu)) = L^2(X \times G, \mu \times \lambda)$ is given by

$$(\pi^\theta \rtimes \lambda^\theta)(f)\xi(x,t) = \int \alpha_{t^{-1}}(f(x,s))\xi(x, s^{-1}t)ds = \int f(xt^{-1}, s)\xi(x, s^{-1}t)ds,$$

for $f \in C_c(X \times G)$ and $\xi \in L^2(X \times G, \mu \times \lambda)$. Let $J\xi(x,t) = \xi(xt, t^{-1})$; then J is a unitary operator from $L^2(\mathcal{G}, \nu)$ to $L^2(\mathcal{G}, \nu^{-1})$ with $J^{-1}\eta(x,t) = \eta(xt, t^{-1})$. Let also $U\xi(x,t) = \Delta^{-1/2}(t)\xi(x, t^{-1})$; thus, U is a unitary operator on $L^2(\mathcal{G}, \nu)$. We have

$$(U^{-1}J^{-1}\mathrm{Ind}(\mu)(f)JU\xi)(x,t) = \int f(xt^{-1}, s)\xi(x, s^{-1}t)\Delta^{-1/2}(s)ds;$$

we thus see that $(\pi^\theta \rtimes \lambda^\theta) \circ \phi$ is unitarily equivalent to $\mathrm{Ind}(\mu)$.

Let I be the intersection of the kernels of $\mathrm{Ind}(\mu)$ as μ varies over the measures of X. The quotient $C^*(\mathcal{G})/I$ is called the reduced C^*-algebra of $\mathcal{G}$ and denoted by $C^*_{\mathrm{red}}(\mathcal{G})$. It follows from [23, Proposition 2.17] that $\mathrm{Ind}(\mu)$ is a faithful representation of $C^*_{\mathrm{red}}(\mathcal{G})$ if $\operatorname{supp}\mu = X$. Therefore $C^*_{\mathrm{red}}(\mathcal{G})$ is isomorphic to the reduced crossed product C^*-algebra of the C^*-dynamical system $(C_0(X), G, \alpha)$.

A measure μ on X is called quasi-invariant if the measures ν and ν^{-1} are equivalent. It is known that μ is quasi-invariant if and only if the measures μ and $\mu \cdot s$ are equivalent for any $s \in G$ (here $\mu \cdot s(E) = \mu(Es^{-1})$). If $\delta(\cdot, s)$ is the Radon-Nikodym derivative $d(\mu \cdot s^{-1})/d\mu$ and D is the Radon-Nikodym derivative $d\nu/d\nu^{-1}$ then $D(x,s) = \Delta(s)/\delta(x,s)$, $x \in X$, $s \in G$ (see [34, Chapter I, 3.21]). In what follows we will asume that X possesses a quasi-invariant measure μ such that $\operatorname{supp}\mu = X$.

The groupoid $\mathcal{G}$ equipped with such a measure μ is called a *measured groupoid* [34]. Next we would like to point out a connection between its multipliers, studied in [35], and Herz-Schur $(C_0(X), G, \alpha)$-multipliers.

The Hilbert space $L^2(\mathcal{G}, \nu)$ carries a representation Reg of $C_c(\mathcal{G})$ defined by

$$(\mathrm{Reg}(f)\xi)(x,s) = \int f(x,t)\xi(xt, t^{-1}s)D^{-1/2}(x,t)dt,$$

and unitarily equivalent to $\mathrm{Ind}(\mu)$ *via* the unitary operator V from $L^2(\mathcal{G},\nu)$ to $L^2(\mathcal{G},\nu^{-1})$ given by $V\xi = D^{1/2}\xi$. The von Neumann algebra $\mathrm{VN}(\mathcal{G})$ of $\mathcal{G}$ is defined to be the bicommutant $\mathrm{Reg}(C_c(\mathcal{G}))''$ [35, 2.1].

The Fourier algebra $A(\mathcal{G})$ of the measured groupoid $\mathcal{G}$ was defined in [35] and is, similarly to the case where $\mathcal{G}$ is a group, a Banach algebra of complex-valued continuous functions on $\mathcal{G}$. By [35, Propsition 3.1], the operator M_φ of multiplication by $\varphi \in L^\infty(\mathcal{G})$ is a bounded linear map on $A(\mathcal{G})$ if and only if the map $\mathrm{Reg}(f) \to \mathrm{Reg}(\varphi f)$, $f \in C_c(\mathcal{G})$, is bounded. The function φ is in this case called a multiplier of $A(\mathcal{G})$. If the map M_φ is moreover completely bounded then φ is called a *completely bounded multiplier* of $A(\mathcal{G})$.

For a bounded continuous function $\varphi : X \times G \to \mathbb{C}$ and $t \in G$, let $F_\varphi(t)$ be the linear map on $C_0(X)$ given by $F_\varphi(t)(a)(x) = \varphi(x,t)a(x)$, $a \in C_0(X)$, $x \in X$.

Proposition 5.3. *Let $\varphi : X \times G \to \mathbb{C}$ be a bounded continuous function. Then*

(i) the map F_φ is a θ-multiplier if and only if φ is a multiplier of $A(\mathcal{G})$;

(ii) the map F_φ is a Herz-Schur $(C_0(X), G, \alpha)$-multiplier if and only if φ is a completely bounded multiplier of $A(\mathcal{G})$.

Proof. Both statements follow from the previous paragraphs, Remark 3.2 (iii), the definition of (Herz-Schur) θ-multipliers and the fact that $\|\mathrm{Reg}(f)\| = \|(\pi^\theta \rtimes \lambda^\theta)(\phi(f))\|$, $f \in C_c(\mathcal{G})$. ☐

The following statement gives the result of [35, Proposition 3.8] in case G is a locally compact second countable group.

Corollary 5.4. *Let $\theta : G \to \mathbb{C}$ be a bounded continuous function and $\varphi : X \times G \to \mathbb{C}$ be the function given by $\varphi(x,t) = \theta(t)$. Then φ is a completely bounded multiplier of $A(\mathcal{G})$ if and only if $\theta \in M^{\mathrm{cb}}A(G)$.*

Proof. The statement follows from Proposition 5.3 and Proposition 4.1. ☐

The next corollary provides a new description of the completely bounded multipliers of $A(\mathcal{G})$. We write $H = L^2(X,\mu)$.

Corollary 5.5. *Let $\varphi : X \times G \to \mathbb{C}$ be a bounded continuous function. Assume that φ is a completely bounded multiplier of $A(\mathcal{G})$. Then there exist a separable Hilbert space K and functions $V, W \in L^\infty(G, \mathcal{B}(H,K))$ such that, for almost all $s,t \in G$, we have that $W^*(t)V(s) \in \mathcal{D}_X$ and $\varphi(xt^{-1}, ts^{-1}) = (W^*(t)V(s))(x)$, for almost all $x \in X$.*

Proof. By Proposition 5.3, $F = F_\varphi$ is a Herz-Schur $(C_0(X), G, \alpha)$-multiplier. By Theorem 3.8, $\mathcal{N}(F)$ is a Schur $C_0(X)$-multiplier. We have

$$\mathcal{N}(F)(s,t)(a)(x) = \varphi(xt^{-1}, ts^{-1})a(x). \tag{48}$$

Hence there exist a separable Hilbert space K, a non-degenerate $*$-representation $\rho : C_0(X) \to \mathcal{B}(K)$ and functions $V, W \in L^\infty(G, \mathcal{B}(H,K))$ such that

$$\varphi(xt^{-1}, ts^{-1})a(x)\xi(x) = W^*(t)\rho(a)V(s)\xi(x), \quad a \in C_0(X), \xi \in L^2(X,\mu).$$

Taking an approximate unit $(a_n)_{n\in\mathbb{N}}$ of $C_0(X)$ in (48) and letting $n\to\infty$, we obtain the statement. □

6. Convolution multipliers

Throughout this section, we will assume that G is an abelian locally compact group, and will write the group operations additively. Let A be a separable C*-algebra and (A,G,α) be a C*-dynamical system. For a measure $\mu\in M(G)$, let $\alpha_\mu : A\to A$ be the completely bounded map given by $\alpha_\mu(a)=\int_G \alpha_r(a)d\mu(r)$ (see [39],[41]).

Definition 6.1. *A family $\Lambda=(\mu_t)_{t\in G}$, where $\mu_t\in M(G)$, $t\in G$, will be called a* convolution (A,G,α)-multiplier *(or simply a* convolution multiplier*), if the map $F_\Lambda : G\to CB(A)$ given by $F_\Lambda(t)=\alpha_{\mu_t}$, $t\in G$, is a Herz-Schur (A,G,α)-multiplier.*

For a convolution multiplier $\Lambda=(\mu_t)_{t\in G}$, we let $\|\Lambda\|_{\mathrm{m}}=\|F_\Lambda\|_{\mathrm{m}}$. Since G is assumed to be abelian, we have that $\alpha_\mu\circ\alpha_r=\alpha_r\circ\alpha_\mu$ for every $r\in G$ and every $\mu\in M(G)$. It is well-known that in this case the map $\alpha_\mu : A\to A$ lifts to a completely bounded map on th the crossed product; the following proposition provides a concrete route to this fact.

Proposition 6.2. *Let $\mu\in M(G)$, $\mu_t=\mu$ for every $t\in G$, and $\Lambda=(\mu_t)_{t\in G}$. Then Λ is a convolution multiplier and $\|\Lambda\|_{\mathrm{m}}\le\|\mu\|$.*

Proof. Note that $\pi(\alpha_r(a))=\lambda_r\pi(a)\lambda_r^*$, $r\in G$. Set $F=F_\Lambda$. If $a\in A$ then

$$\pi(F(s)(a))=\pi(\alpha_\mu(a))=\int_G \pi(\alpha_r(a))d\mu(r)=\int_G \lambda_r\pi(a)\lambda_r^* d\mu(r).$$

A straightforward calculation now shows that if $f\in L^1(G,A)$ then

$$S_F((\pi\rtimes\lambda)(f))=\int_G \lambda_r\left(\int_G \pi(f(s))\lambda_s ds\right)\lambda_r^* d\mu(r).$$

The claims follow from the fact that the mapping $T\mapsto\int\lambda_r T\lambda_r^* d\mu(r)$ is a completely bounded map on $\mathcal{B}(L^2(G,H))$ with completely bounded norm dominated by $\|\mu\|$ (see [39]). □

In this section, we will be concerned with a special class of convolution multipliers, which we now define. Let Γ be the dual group of G. The C*-algebra $C^*(\Gamma)$ of Γ is canonically *-isomorphic to its reduced C*-algebra $C_r^*(\Gamma)$ (see *e.g.* [29, Theorem 7.3.9]). We let $\theta : C^*(\Gamma)\to\mathcal{B}(L^2(\Gamma))$ be the associated (faithful) *-representation. An element $s\in G$ will be viewed as a character (and, in particular, a unimodular function) on Γ. For $s\in G$, let $\alpha_s : \lambda^\Gamma(L^1(\Gamma))\to\lambda^\Gamma(L^1(\Gamma))$ be the map given by

$$\alpha_s(\lambda^\Gamma(f))=\lambda^\Gamma(sf),\quad f\in L^1(\Gamma).$$

Note that, if $f\in L^1(\Gamma)$, $s\in G$ and $x\in\Gamma$, then

$$\alpha_s(\lambda^\Gamma(f))=M_s\lambda^\Gamma(f)M_{-s}. \tag{49}$$

Thus, α_s extends canonically to an automorphism of $C_r^*(\Gamma)$. By abuse of notation, we consider α_s as an automorphism of $C^*(\Gamma)$; thus, $(C^*(\Gamma), G, \alpha)$ is a C*-dynamical system. By (49), α is a θ-action. Moreover, by [29, Theorem 7.7.7], $C^*(\Gamma) \rtimes_\alpha G$ is *-isomorphic to the C*-subalgebra $C^*(\Gamma) \rtimes_{\alpha,\theta} G$ of $\mathcal{B}(L^2(G\times\Gamma))$.

Given a bounded measurable function $\psi : G\times\Gamma \to \mathbb{C}$ and $t \in G$ (resp. $x \in \Gamma$), let the function $\psi_t : \Gamma \to \mathbb{C}$ (resp. $\psi^x : G \to \mathbb{C}$) given by $\psi_t(y) = \psi(t,y)$ (resp. $\psi^x(s) = \psi(s,x)$). We call ψ *admissible* if $\psi_t \in B(\Gamma)$ for every $t \in G$ and $\sup_t \|\psi_t\|_{B(\Gamma)} < \infty$. Assuming that ψ is admissible, let $F_\psi(t) : C_r^*(\Gamma) \to C_r^*(\Gamma)$ be the map given by

$$F_\psi(t)(\lambda^\Gamma(g)) = \lambda^\Gamma(\psi_t g), \quad g \in L^1(\Gamma).$$

By abuse of notation, we consider $F_\psi(t)$ as a map on $C^*(\Gamma)$. Set

$$\begin{aligned}\mathfrak{F}(G) = \{\psi : G\times\Gamma \to \mathbb{C}\ : \psi \text{ is admissible and}\\ F_\psi \text{ is a Herz-Schur } (C^*(\Gamma), G, \alpha)\text{-multiplier}\}\end{aligned}$$

and

$$\begin{aligned}\mathfrak{F}_\theta(G) = \{\psi : G\times\Gamma \to \mathbb{C}\ : \psi \text{ is admissible and}\\ F_\psi \text{ is a Herz-Schur } \theta\text{-multiplier}\}.\end{aligned}$$

Clearly, the space $\mathfrak{F}(G)$ is an algebra with respect to the operations of pointwise addition and multiplication, and $\mathfrak{F}_\theta(G)$ is a subalgebra of $\mathfrak{F}(G)$. For $\psi \in \mathfrak{F}(G)$, let $\|\psi\|_{\mathrm{m}} = \|F_\psi\|_{\mathrm{m}}$, and use S_ψ to denote the map S_{F_ψ}.

For $\mu \in M(G)$, set $\check{\mu}(x) = \int_G \langle x, s\rangle d\mu(s)$, $x \in \Gamma$.

Proposition 6.3. *Let $\psi : G\times\Gamma \to \mathbb{C}$ be an admissible function. The following are equivalent:*

(i) $\psi \in \mathfrak{F}(G)$;

(ii) for each $t \in G$, there exists $\mu_t \in M(G)$ such that $\psi(t,x) = \check{\mu}_t(x)$, $t \in G$, $x \in \Gamma$, and the family $(\mu_t)_{t\in G}$ is a convolution $(C^*(\Gamma), G, \alpha)$-multiplier.

Proof. Note that, if $\mu \in M(G)$ and $g \in L^1(\Gamma)$ then

$$\begin{aligned}(50)\ \alpha_\mu(\lambda^\Gamma(g)) &= \int \alpha_r(\lambda^\Gamma(g))d\mu(r) = \int_G \left(\int_G \langle s, r\rangle g(s)\lambda_s^\Gamma ds\right) d\mu(r)\\ &= \int_G \check{\mu}(s)g(s)\lambda_s^\Gamma ds = \lambda^\Gamma(\check{\mu}g).\end{aligned}$$

(i)⇒(ii) If ψ is admissible then, for every $t \in G$, $\psi_t \in B(\Gamma)$ and hence, by Bochner's theorem, there exists $\mu_t \in M(G)$ such that $\psi_t = \check{\mu}_t$ (see *e.g.* [37, Section I]). It follows from (50) that the family $(\mu_t)_{t\in G}$ is a convolution multiplier.

(ii)⇒ (i) By (50), $F_\psi(t) = \alpha_{\mu_t}$. The claim now follows from the definition of a convolution multiplier. ☐

For a family $\Lambda = (\mu_t)_{t\in G}$ of measures in $M(G)$, let $\psi_\Lambda : G\times\Gamma \to \mathbb{C}$ be the function given by $\psi_\Lambda(t,x) = \check{\mu}_t(x)$. Call Λ admissible if the function ψ_Λ

is admissible. By Proposition 6.3, an admissible family of measures Λ is a Herz-Schur $(C^*(\Gamma), G, \alpha)$-multiplier if and only if $\psi_\Lambda \in \mathfrak{F}(G)$. By abuse of terminology, we will hence call the elements of $\mathfrak{F}(G)$ convolution multipliers.

Corollary 6.4. *Let $g \in L^\infty(\Gamma)$ and let $\psi : G \times \Gamma \to \mathbb{C}$ be given by $\psi(s,x) = g(x)$, $s \in G$, $x \in \Gamma$. The following are equivalent:*
(i) $\psi \in \mathfrak{F}(G)$;
(ii) $g \in B(\Gamma)$.
Moreover, if (i) holds then $\psi \in \mathfrak{F}_\theta(G)$.

Proof. The equivalence of (i) and (ii) follows from Propositions 6.2 and 6.3. Suppose that $g \in B(\Gamma)$. Then the map on $C_r^*(\Gamma)$ corresponding to g *via* classical transference has a (completely bounded) weak* continuous extension $\Phi_g : \mathrm{VN}(\Gamma) \to \mathrm{VN}(\Gamma)$. Thus, the restriction of the map $\Phi_g \otimes \mathrm{id}$ to $C^*(\Gamma) \rtimes_{\alpha,\theta}^{w^*} G$ is a weak* continuous extension of $S_{F_\psi}^\theta$. By Remark 3.11, F_ψ is a Herz-Schur θ-multiplier. $\square$

It will be convenient, in the sequel, to denote by S_g the map S_ψ, where ψ and g are as in Corollary 6.4.

The rest of the paper will be devoted to properties of the spaces $\mathfrak{F}(G)$ and $\mathfrak{F}_\theta(G)$. In the next theorem, we identify an elementary tensor $u \otimes h$, where $u \in B(G)$ and $h \in B(\Gamma)$, with the function $(s,x) \to u(s)h(x)$, $s \in G$, $x \in \Gamma$. Let $\mathfrak{F}(B(G), B(\Gamma))$ be the complex vector space of all separately continuous functions $\psi : G \times \Gamma \to \mathbb{C}$ such that, for every $s \in G$ (resp. $x \in \Gamma$), the function $\psi_s : \Gamma \to \mathbb{C}$ (resp. $\psi^x : G \to \mathbb{C}$) belongs to $B(\Gamma)$ (resp. $B(G)$).

Theorem 6.5. *(i) The inclusions*

$$B(G) \odot B(\Gamma) \subseteq \mathfrak{F}_\theta(G) \subseteq \mathfrak{F}(B(G), B(\Gamma))$$

hold.
(ii) Suppose that $\psi \in \mathfrak{F}_\theta(G)$. Then $\|\psi^x\|_{B(G)} \leq \|\psi\|_{\mathrm{m}}$ for every $x \in \Gamma$ and $\|\psi_s\|_{B(\Gamma)} \leq \|\psi\|_{\mathrm{m}}$ for every $s \in G$.
(iii) Let $\psi : G \times \Gamma \to \mathbb{C}$ be an admissible function, such that the function $G \to B(\Gamma)$, sending s to ψ_s, is continuous. Suppose that $(\psi_k)_{k\in\mathbb{N}} \subseteq \mathfrak{F}(G)$, $\sup_{k\in\mathbb{N}} \|\psi_k\|_{\mathrm{m}} < \infty$ and $\psi_k \to \psi$ pointwise. Then $\psi \in \mathfrak{F}(G)$.

Proof. (i) The first inclusion follows from Proposition 4.1 and Corollary 6.4.
Let $\psi \in \mathfrak{F}_\theta(G)$ and fix $x \in \Gamma$. The map Ψ_{ψ_s} corresponding to ψ_s *via* classical transference satisfies the identities $\Psi_{\psi_s}(\lambda_x^\Gamma) = \psi_s(x)\lambda_x^\Gamma$, $x \in \Gamma$. Thus

$$\begin{aligned}\Phi_\psi^\theta(\bar{\pi}^\theta(\lambda_x^\Gamma)\lambda_s^\theta) &= \bar{\pi}^\theta(\psi_s(x)\lambda_x^\Gamma)\lambda_s^\theta \\ &= \psi_s(x)\bar{\pi}^\theta(\lambda_x^\Gamma)\lambda_s^\theta = \bar{\pi}^\theta(\lambda_x^\Gamma)(\psi^x(s)\lambda_s^G \otimes I).\end{aligned}$$

On the other hand,

$$\bar{\pi}^\theta(\lambda_x^\Gamma)\lambda_s^\theta = \bar{\pi}^\theta(\lambda_x^\Gamma)(\lambda_s^G \otimes I).$$

It follows that the map

$$\lambda_s^G \to \psi^x(s)\lambda_s^G$$

is bounded, and hence ψ^x is a Herz-Schur multiplier giving $\psi^x \in B(G)$. The fact that $\psi_s \in B(\Gamma)$ for every $s \in G$ is implicit in the definition of the space $\mathfrak{F}(G)$.

(ii) The inequalities $\|\psi^x\|_{B(\Gamma)} \leq \|\psi\|_{\mathrm{m}}$, $x \in \Gamma$, follow from the proof of (i). The inequalities $\|\psi_s\|_{B(\Gamma)} \leq \|\psi\|_{\mathrm{m}}$, $s \in G$, follow from Corollary 3.12 and the fact that $\|\psi_s\|_{B(\Gamma)} = \|F_\psi(s)\|_{\mathrm{cb}}$, $s \in G$.

(iii) Suppose that $\|\psi_k\|_{\mathrm{m}} \leq C$ for every $k \in \mathbb{N}$. Then it follows from (27) that $\|F_{\psi_k}(s)\| \leq \|F_{\psi_k}\|_{\mathrm{m}} = \|\psi_k\|_{\mathrm{m}}$ for almost all $s \in \Gamma$. As $\|F_{\psi_k}(s)\| = \|(\psi_k)_s\|_{B(\Gamma)}$, we obtain $\|\psi_k\|_\infty \leq \|\psi_k\|_{\mathrm{m}} \leq C$.

Let $\xi, \eta \in L^2(G \times \Gamma)$ and $f \in L^1(G, C_c(\Gamma))$. A direct verification shows that

$$\begin{aligned} &\langle S^\theta_{\psi_k}((\pi^\theta \rtimes \lambda^\theta)(f))\xi, \eta\rangle = \langle (\pi^\theta \rtimes \lambda^\theta)(\psi_k f)\xi, \eta\rangle \\ = &\int \overline{\langle t, x\rangle}\psi_k(t,y)f(t,y)\xi(s-t, x-y)\overline{\eta(s,x)}dsdtdxdy. \end{aligned}$$

On the other hand,

$$|\overline{\langle t, x\rangle}\psi_k(t,y)f(t,y)\xi(s-t, x-y)\overline{\eta(s,x)}| \leq C|f(t,y)||\xi(s-t,x-y)||\eta(s,x)|,$$

and the L^1-norm of the latter function is equal to $C\langle |f| * |\xi|, |\eta|\rangle$. By the Lebesgue Dominated Convergence Theorem,

$$\langle S^\theta_{\psi_k}((\pi^\theta \rtimes \lambda^\theta)(f))\xi, \eta\rangle \to \langle (\pi^\theta \rtimes \lambda^\theta)(\psi f)\xi, \eta\rangle. \tag{51}$$

Now let $f_{i,j} \in L^1(G, C_c(\Gamma))$, $i, j = 1, \ldots, m$. By (51), the operator matrix $(S^\theta_{\psi_k}((\pi^\theta \rtimes \lambda^\theta)(f)))_{i,j}$ converges weakly to $((\pi^\theta \rtimes \lambda^\theta)(\psi f))_{i,j}$. The fact that the map $s \to \psi_s$ is continuous implies that ψ is weakly measurable. Since $\|S^\theta_{\psi_k}\|_{\mathrm{cb}} \leq C$ for all k, we conclude that $\psi \in \mathfrak{F}(G)$ and that $\|\psi\|_{\mathrm{m}} \leq C$. ☐

In view of Theorem 6.5, it is natural to ask the following question.

Question 6.6. *Can $\mathfrak{F}_\theta(G)$ be characterised as a topological closure of $B(G) \odot B(\Gamma)$?*

Let $CB_{w^*}(C^*(\Gamma) \rtimes_{\alpha,\theta} G)$ be the space of all completely bounded maps on $C^*(\Gamma) \rtimes_{\alpha,\theta} G$ which admit a weak* continuous extension to $C^*(\Gamma) \rtimes^{w^*}_{\alpha,\theta} G$. Set

$$\mathcal{S} = \{S^\theta_\psi : \psi \in \mathfrak{F}_\theta(G)\}.$$

As usual, if $\mathcal{J}$ is a family of linear transformations acting on a vector space, we denote by $\mathcal{J}'$ its commutant. We recall that, for $g \in B(\Gamma)$, we let S^θ_g denote the map on $C^*_r(\Gamma)$ given by $S^\theta_g(\lambda^\Gamma(f)) = \lambda^\Gamma(gf)$, $f \in L^1(\Gamma)$.

Theorem 6.7. *We have*

$$\mathcal{S} = CB_{w^*}(C^*(\Gamma) \rtimes_{\alpha,\theta} G) \cap \{S^\theta_u, S^\theta_v : u \in B(G), v \in B(\Gamma)\}'. \tag{52}$$

In particular, $\mathcal{S}$ is a maximal abelian subalgebra of $CB_{w^*}(C^*(\Gamma) \rtimes_{\alpha,\theta} G)$.

Proof. Note that $\mathcal{S}$ is a commutative subalgebra of $CB_{w^*}(C^*(\Gamma) \rtimes_{\alpha,\theta} G)$ and, by Theorem 6.5 (i), contains the maps of the form S_u^θ and S_v^θ, where $u \in B(G)$ and $v \in B(\Gamma)$. It follows that it is contained in the commutant on the right hand side of (52).

Let $A = C^*(\Gamma)$. To establish the reverse inclusion, suppose that $\Phi \in CB_{w^*}(C^*(\Gamma) \rtimes_{\alpha,\theta} G)$ commutes with the operators of the form S_u^θ and S_v^θ, where $u \in B(G)$ and $v \in B(\Gamma)$. Since G is abelian, $\mathrm{VN}(G)$ possesses property S_σ (see *e.g.* [19, Theorem 1.9]). By Theorem 4.3, for each $t \in G$, there exists a completely bounded map $F_t : \mathrm{VN}(\Gamma) \to \mathrm{VN}(\Gamma)$ such that

$$\Phi(\bar{\pi}^\theta(a)\lambda_t^\theta) = \bar{\pi}^\theta(F_t(a))\lambda_t^\theta, \quad a \in \mathrm{VN}(\Gamma).$$

If $v \in B(\Gamma)$, $f \in L^1(\Gamma)$ and $a = \lambda^\Gamma(f)$ then

$$\bar{\pi}^\theta(F_t(\lambda^\Gamma(vf)))\lambda_t^\theta = \Phi(S_v^\theta(\bar{\pi}^\theta(a)\lambda_t^\theta)) = S_v^\theta(\Phi(\bar{\pi}^\theta(a)\lambda_t^\theta)) = \bar{\pi}^\theta(vF_t(\lambda^\Gamma(f))\lambda_t^\theta.$$

Thus, $F_t(\lambda^\Gamma(vf)) = vF_t(\lambda^\Gamma(f))$ for each $v \in B(\Gamma)$. Let $\tilde{F}_t$ be the restriction of F_t to $C_r^*(\Gamma)$. Then $\tilde{F}_t^*$ is a map from $\mathrm{VN}(\Gamma)^*$ to $B(\Gamma)$. In particular, $\tilde{F}_t^*(A(\Gamma)) \subseteq B(\Gamma)$ and $\tilde{F}_t^*(vu) = v\tilde{F}_t^*(u)$ for any $u \in A(\Gamma)$ and $v \in B(\Gamma)$. It follows that $\tilde{F}_t^*(u) = \psi_t u$ for some function $\psi_t : \Gamma \to \mathbb{C}$. As $\psi_t u \in B(\Gamma)$ for all $u \in A(\Gamma)$, by [37, Theorem 3.8.1], $\psi_t \in B(\Gamma)$. Hence, for $u \in A(\Gamma)$, we have

$$\langle F_t(\lambda^\Gamma(f)), u\rangle = \langle \lambda^\Gamma(f), \tilde{F}_t^*(u)\rangle = \langle \lambda^\Gamma(f), \psi_t u\rangle = \langle \lambda^\Gamma(\psi_t f), u\rangle$$

and $F_t(\lambda^\Gamma(f)) = \lambda^\Gamma(\psi_t f)$. The proof is complete. $\square$

Our next aim is to identify the joint commutant of two families of completely bounded maps on $\mathcal{K}(L^2(G))$, in terms of multipliers of Herz-Schur type. Recall that, for $a \in L^\infty(G)$, we denote by M_a the operator on $L^2(G)$ given by $M_a f = af$, $f \in L^2(G)$, and set

$$\mathcal{C} = \{M_a : a \in C_0(G)\}.$$

We let id be the identity representation of $\mathcal{C}$. For $t \in G$, let $\beta_t : C_0(G) \to C_0(G)$ be given by $\beta_t(h)(s) = h(s - t)$, $h \in C_0(G)$. By abuse of notation, we denote by β_t the corresponding map on the C*-algebra $\mathcal{C}$. Note that

$$\beta_t(T) = \lambda_t^G T \lambda_{-t}^G, \quad T \in \mathcal{C}, \tag{53}$$

and that $(\mathcal{C}, G, \beta)$ is a C*-dynamical system. Note also that

$$\beta_\mu(M_a) = M_{\mu * a}, \quad \mu \in M(G), a \in L^\infty(G). \tag{54}$$

Note that $\mathcal{C} \rtimes_{\beta,\mathrm{id}} G$ is a C*-subalgebra of $\mathcal{B}(L^2(G \times G))$.

Let $\mathcal{F} : L^2(G) \to L^2(\Gamma)$ be the Fourier transform, so that $\mathcal{F}\xi(x) = \int_G \langle x, s\rangle \xi(s) ds$, $\xi \in L^1(G) \cap L^2(G)$, $x \in \Gamma$. Then

$$\mathcal{F}^* M_t \mathcal{F} = \lambda_t^G \text{ and } \mathcal{F}^* \lambda^\Gamma(f) \mathcal{F} = M_{\hat{f}}, \quad t \in G, f \in L^1(\Gamma), \tag{55}$$

where $\hat{f} : G \to \mathbb{C}$ is the function given by $\hat{f}(t) = \int_\Gamma \overline{\langle t, x\rangle} f(x) dx$. In particular, $\mathcal{F}^* C_r^*(\Gamma) \mathcal{F} = \mathcal{C}$. Moreover, if $f \in L^1(\Gamma)$ then

$$\mathcal{F}^* \alpha_t(\lambda^\Gamma(f)) \mathcal{F} = \beta_t(\mathcal{F}^* \lambda^\Gamma(f) \mathcal{F}), \tag{56}$$

giving

$$\mathcal{F}^*\alpha_t(T)\mathcal{F} = \beta_t(\mathcal{F}^*T\mathcal{F}), \quad T \in C_r^*(\Gamma), t \in G.$$

Let $\tilde{\mathcal{F}} = I \otimes \mathcal{F}^*$; thus, $\tilde{\mathcal{F}}$ is a unitary operator from $L^2(G \times \Gamma)$ onto $L^2(G \times G)$. It is well-known that

$$\tilde{\mathcal{F}}(C^*(\Gamma) \rtimes_{\alpha,\theta} G)\tilde{\mathcal{F}}^* = \mathcal{C} \rtimes_{\beta,\mathrm{id}} G.$$

Let $\Lambda = (\mu_t)_{t \in G}$ be a family of measures in $M(G)$, such that the function ψ_Λ is in $\mathfrak{F}(G)$. Then ψ_Λ gives rise to a Herz-Schur $(\mathcal{C}, G, \beta)$-multiplier given by

$$\pi(M_{\hat{g}})\lambda_t = \pi(\mathcal{F}^*\lambda^\Gamma(g)\mathcal{F})\lambda_t \mapsto \pi(\mathcal{F}^*F_{\psi_t}(\lambda^\Gamma(g))\mathcal{F})\lambda_t = \pi(M_{\mu_t * \hat{g}})\lambda_t.$$

This observation was our motivation for the chosen terminology for convolution multipliers. Note that the convolution multipliers are of different nature than Herz-Schur $(\mathcal{C}, G, \beta)$-multipliers considered in Section 5.2.

The pair (id, λ^G) is a covariant representation of $(\mathcal{C}, G, \beta)$ (see (53)); in addition, $\mathrm{id} \rtimes \lambda^G$ is a faithful representation of $\mathcal{C} \rtimes_\beta G$ on $L^2(G)$ and its image coincides with the algebra $\mathcal{K}(L^2(G))$ of all compact operators on $L^2(G)$ (see [36] and [45]).

For $\psi \in \mathfrak{F}(G)$, let $\mathcal{E}_\psi : \mathcal{K}(L^2(G)) \to \mathcal{K}(L^2(G))$ be the (completely bounded) map given by

$$\mathcal{E}_\psi((\mathrm{id} \rtimes \lambda^G)(\tilde{\mathcal{F}}T\tilde{\mathcal{F}}^*)) = (\mathrm{id} \rtimes \lambda^G)(\tilde{\mathcal{F}}S_\psi^\theta(T)\tilde{\mathcal{F}}^*), \quad T \in \mathcal{C}^*(\Gamma) \rtimes_{\alpha,\theta} G.$$

We extend $\mathcal{E}_\psi$ to a weak* continuous map on $\mathcal{B}(L^2(G))$, denoted in the same fashion. For $r \in G$, let $\rho_r^G \in \mathcal{B}(L^2(G))$ be the corresponding right regular unitary on $L^2(G)$, that is, $\rho_r^G f(s) = f(s + r)$, $s, r \in G$, $f \in L^2(G)$. For a measure $\mu \in M(G)$, consider the map $\Theta(\mu) \in CB(\mathcal{B}(L^2(G))$, given by

$$\Theta(\mu)(T) = \int_G \rho_r^G T \rho_{-r}^G d\mu(r), \quad T \in \mathcal{B}(L^2(G)). \tag{57}$$

It is easy to see that

$$\Theta(\mu)(M_a) = M_{\mu \cdot a} = M_{\tilde{\mu} * a}, \quad a \in L^\infty(G), \tag{58}$$

where $(\mu \cdot a)(s) = \int_G a(s + r)d\mu(r)$ and $\tilde{\mu}$ is the measure on G given by $\tilde{\mu}(E) = \mu(-E)$. Note that $\Theta(\mu)$ is a $\mathrm{VN}(G)$-bimodule map and leaves $\mathcal{D}_G$ invariant.

Recall that, for every $u \in B(G)$, the function $N(u)$ given by $N(u)(s,t) = u(t-s)$, is a Schur multiplier [5] (see also Remark 3.9). Thus, the corresponding map $\Psi_u : \mathcal{B}(L^2(G)) \to \mathcal{B}(L^2(G))$ is a completely bounded $\mathcal{D}_G$-bimodule map that leaves $\mathrm{VN}(G)$ invariant.

Proposition 6.8. *Suppose that $\mu \in M(G)$ and $u \in B(G)$. Let ψ_μ and ψ_u be the elements of $\mathfrak{F}(G)$ given by $\psi_\mu(s,x) = \check{\mu}(x)$ and $\psi_u(s,x) = u(s)$, $s \in G$, $x \in \Gamma$. Then*

(i) $\mathcal{E}_{\psi_\mu} = \Theta(\tilde{\mu})$, and
(ii) $\mathcal{E}_{\psi_u} = \Psi_u$.

Proof. Let $f \in C_c(G, C_c(\Gamma))$ be given by $f(s) = f_0(s)g$, for a certain $g \in C_c(\Gamma)$ and a certain $f_0 \in C_c(G)$. Let $i : C_c(G, C_c(\Gamma)) \to C^*(\Gamma) \rtimes_{\alpha,\theta} G$ be the embedding map given by

$$i(h)\xi(t) = \int_G \alpha_{-t}(\lambda^\Gamma(h(s)))\lambda_s^\theta \xi(t)ds, \quad \xi \in L^2(G \times \Gamma), h \in C_c(G \times \Gamma).$$

We have

$$i(f)\xi(t) = \int_G \alpha_{-t}(\lambda^\Gamma(g)) f_0(s)\lambda_s^\theta \xi(t)ds$$

and, by (55) and (56),

$$\begin{aligned} \tilde{\mathcal{F}} i(f)\tilde{\mathcal{F}}^*\xi(t) &= \int_G \mathcal{F}^*\alpha_{-t}(\lambda^\Gamma(g))\mathcal{F} f_0(s)\lambda_s^\theta \xi(t)ds \\ &= \int_G \beta_{-t}(\mathcal{F}^*(\lambda^\Gamma(g))\mathcal{F}) f_0(s)\lambda_s^\theta \xi(t)ds \qquad (59) \\ &= \int_G \beta_{-t}(M_{\hat{g}}) f_0(s)\lambda_s^\theta \xi(t)ds. \end{aligned}$$

Let $T, T_\mu : G \to \mathcal{C}$ be the maps given by $T(s) = M_{f_0(s)\hat{g}}$ and $T_\mu(s) = M_{f_0(s)\mu * \hat{g}}$, $s \in G$; clearly, $T, T_\mu \in L^1(G, \mathcal{C})$. Let $j : L^1(G, \mathcal{C}) \to \mathcal{C} \rtimes_{\beta,\mathrm{id}} G$ be the canonical injection. By (59), $\tilde{\mathcal{F}} i(f)\tilde{\mathcal{F}}^* = j(T)$ and $\tilde{\mathcal{F}} i(\psi_\mu f)\tilde{\mathcal{F}}^* = j(T_\mu)$.

(i) Note that

$$S_{\psi_\mu}^\theta(i(f)) = i(\psi_\mu f) = i(f_0 \otimes (\check{\mu} g)).$$

We have

$$\begin{aligned} \mathcal{E}_{\psi_\mu}(M_{\hat{g}}\lambda^G(f_0)) &= \mathcal{E}_{\psi_\mu}((\mathrm{id} \rtimes \lambda^G)(j(T))) = \mathcal{E}_{\psi_\mu}((\mathrm{id} \rtimes \lambda^G)(\tilde{\mathcal{F}} i(f)\tilde{\mathcal{F}}^*)) \\ &= (\mathrm{id} \rtimes \lambda^G)(\tilde{\mathcal{F}} S_{\psi_\mu}^\theta(i(f))\tilde{\mathcal{F}}^*) = (\mathrm{id} \rtimes \lambda^G)(\tilde{\mathcal{F}} i(f_0 \otimes \check{\mu} g)\tilde{\mathcal{F}}^*) \\ &= (\mathrm{id} \rtimes \lambda^G)(j(T_\mu)) = M_{\mu * \hat{g}}\lambda^G(f_0). \end{aligned}$$

By (58) and the modularity of $\Theta(\mu)$ over $\mathrm{VN}(G)$ we now have

$$\mathcal{E}_{\psi_\mu}(M_{\hat{g}}\lambda^G(f_0)) = M_{\mu * \hat{g}}\lambda^G(f_0) = M_{\tilde{\mu} \cdot \hat{g}}\lambda^G(f_0) = \Theta(\tilde{\mu})(M_{\hat{g}}\lambda^G(f_0)).$$

Since the operators of the form $M_{\hat{g}}\lambda^G(f_0)$ span a dense subspace of $\mathcal{K}(L^2(G))$, it follows that $\mathcal{E}_{\psi_\mu} = \Theta(\tilde{\mu})$.

(ii) Similarly to (i), we have

$$\mathcal{E}_{\psi_u}(M_{\hat{g}}\lambda^G(f_0)) = (\mathrm{id} \rtimes \lambda^G)(\tilde{\mathcal{F}} i(uf_0 \otimes g)\tilde{\mathcal{F}}^*) = M_{\hat{g}}\lambda^G(uf_0).$$

Since, by [15], $\lambda^G(uf_0) = \Psi_u(\lambda^G(f_0))$, and Ψ_u is a $\mathcal{C}$-bimodule map, we obtain that $\mathcal{E}_{\psi_u}(M_{\hat{g}}\lambda^G(f_0)) = \Psi_u((M_{\hat{g}}\lambda^G(f_0))$. The statement now follows by the density of the linear span of the operators of the form $M_{\hat{g}}\lambda^G(f_0)$ in $\mathcal{K}(L^2(G))$. ☐

Definition 6.9. *Let (θ, τ) be a covariant representation of the dynamical system (A, G, α). We say that $F : G \to CB(A)$ is a Herz-Schur (θ, τ)-multiplier if the map*

$$\theta(a)\tau_s \to \theta(F(s)(a)\tau_s, \ s \in G, \ a \in A$$

can be extended to a weak-continuous completely bounded map on the weak* closed hull of* $(\theta \rtimes \tau)(A \rtimes_\alpha G)$.

Let $\mathfrak{F}_{\mathrm{id},\lambda^G}(G)$ be the set of admissible functions $\psi : G \times \Gamma \to \mathbb{C}$ such that the corresponding $F_\psi : G \to CB(\mathcal{C})$ is a Herz-Schur (id, λ^G)-multiplier for the dynamical system $(\mathcal{C}, G, \beta)$.

Theorem 6.10. *Let* $\mathcal{E} = \{\mathcal{E}_\psi : \psi \in \mathfrak{F}_{\mathrm{id},\lambda^G}(G)\}$. *Then*

$$\mathcal{E} = CB(\mathcal{K}(L^2(G))) \cap \{\Theta(\mu), \Psi_u : \mu \in M(G), u \in B(G)\}'.$$

In particular, $\mathcal{E}$ *is a maximal abelian subalgebra of* $CB(\mathcal{K}(L^2(G))$.

Proof. The fact that $\mathcal{E}$ is contained in the right hand side follows from the fact that $\mathcal{E}$ is a commutative subalgebra of $CB(\mathcal{K}(L^2(G)))$ and, by Proposition 6.8, contains the maps $\Theta(\mu)$ and Ψ_u, where $u \in B(G)$ and $\mu \in M(G)$.

To prove the reverse inclusion, we modify the arguments in the proof of Theorem 4.3. Let $\Phi \in CB(\mathcal{K})$ commute with Ψ_u and $\Theta(\mu)$, for all $\mu \in M(G)$ and all $u \in B(G)$. The map Φ has a unique extension to a weak* continuous completely bounded map on $\mathcal{B}(L^2(G))$, which will be denoted by the same symbol. Let $t \in G$, $a \in L^\infty(G)$ and $T = M_a\lambda_t^G$. Set

$$J = \{u \in B(G) : u(t) = 1\}.$$

Then for $u \in J$ we have

$$\Psi_u(M_a\lambda_t^G) = u(t)M_a\lambda_t^G = M_a\lambda_t^G.$$

As $\Phi\Psi_u = \Psi_u\Phi$, we obtain

$$\Psi_u(\Phi(T)) = \Phi(T).$$

By [1] and the remark before Theorem 4.3, we conclude that $\Phi(T)\lambda_{-t}^G \in \mathcal{D}_G$. Therefore, there exists $a_t \in L^\infty(G)$ such that $\Phi(M_a\lambda_t^G) = M_{a_t}\lambda_t^G$. Let $F_t(a) = a_t$, $t \in G$. Then F_t is a linear map on $\mathcal{D}_G$. Since Φ is completely bounded and weak* continuous, F_t is so, too.

Since Φ commutes with $\Theta(\mu)$, $\mu \in M(G)$, we have

$$\Phi(M_{\mu\cdot a}\lambda_t^G) = M_{\mu\cdot F_t(a)}\lambda_t^G,$$

giving $F_t(\mu \cdot a) = \mu \cdot F_t(a)$. The map F_t is the adjoint of a bounded linear map $\Psi_t : L^1(G) \to L^1(G)$ such that $\Psi_t(\mu * f) = \mu * \Psi_t(f)$, $\mu \in M(G)$, $f \in L^1(G)$. By [37, 3.8.4], there exists $\nu_t \in M(G)$ such that $F_t(a) = \nu_t * a$, $a \in L^\infty(G)$. Let $\psi(t, x) = \check{\nu}_t(x)$, $t \in G$, $x \in \Gamma$. Then $\psi \in \mathfrak{F}_{\mathrm{id},\lambda^G}(G)$ and Φ is the weak* extension of $\mathcal{E}_\psi$. $\square$

Remark 6.11. Assume G is arbitrary and let $\beta_t : \mathcal{C} \to \mathcal{C}$ be given by $\beta_t(M_a) = \lambda_t^G M_a \lambda_{t^{-1}}^G$. Then $(\mathcal{C}, G, \beta)$ is a C^*-dynamical system. For a measure μ we consider the map $\Theta(\mu) \in CB(L^2(G))$ given by (57). We have

$\Theta(\mu)(\mathcal{C}) \subseteq \mathcal{C}$. Moreover, for each $r \in G$ we have $\beta_r \circ \Theta(\mu) = \Theta(\mu) \circ \beta_r$:

$$\begin{aligned}\beta_r(\Theta(\mu)(M_a)) &= \lambda_r^G \left(\int_G \rho_s^G M_a \rho_{s^{-1}}^G d\mu(s) \right) \lambda_{r^{-1}}^G \\ &= \int_G \rho_s^G \lambda_r^G M_a \lambda_{r^{-1}}^G \rho_{s^{-1}} d\mu(s) = \Theta(\mu)(\beta_r(M_a)).\end{aligned}$$

Hence $\Theta(\mu)$ gives rise to a completely bounded map on $\mathcal{C} \rtimes_{\beta,r} G$, *i.e.* the function F, given by $F(t)(M_a) := \Theta(\mu)(M_a)$, is a Herz-Schur $(\mathcal{C}, G, \beta)$-multiplier. For $\Lambda = \{\mu_t\}_{t\in G}$ we let $F_\Lambda(t)(M_a) = \Theta(\mu_t)(M_a)$. The class of Herz-Schur multipliers F_Λ includes the convolution multipliers examined in the present section, whose study will be pursued elsewhere.

Acknowledgement. We would like to thank Sergey Neshveyev and Adam Skalski for many helpful conversations during the preparation of this paper. We are grateful to the referee for a number of suggestions that improved the exposition.

Note. After the completion of the paper, two preprints have appeared that have made essential use of the notions and results of the present work: in [21], Herz-Schur multipliers were used to characterise the weak amenability of crossed products, while in [22], they were utilised to provide characterisations of nuclearity and the Haagerup property.

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