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Complexity and capacity bounds for quantum channels

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Abstract—We generalise some well-known graph parameters to operator systems by considering their underlying quantum channels. In particular, we introduce the quantum complexity as the dimension of the smallest co-domain Hilbert space a quantum channel requires to realise a given operator system as its non-commutative confusability graph. We describe quantum complexity as a generalised minimum semidefinite rank and, in the case of a graph operator system, as a quantum intersection number. The quantum complexity and a closely related quantum version of orthogonal rank turn out to be upper bounds for the Shannon zero-error capacity of a quantum channel, and we construct examples for which these bounds beat the best previously known general upper bound for the capacity of quantum channels, given by the quantum Lovász theta number.

I. INTRODUCTION

In 1956, Shannon [Sha56] initiated zero-error information theory, introducing the concept of the *confusability graph* $G_{\mathcal{N}}$ of a finite input-output information channel $\mathcal{N} : X \rightarrow Y$. The vertex set of $G_{\mathcal{N}}$ is the input alphabet X , and two vertices form an edge in $G_{\mathcal{N}}$ if they can result in the same symbol from the output alphabet Y after transmission via $\mathcal{N}$. Shannon showed that the zero-error behaviour of $\mathcal{N}$, and various measures of its capacity, depend only on the graph $G_{\mathcal{N}}$. In particular, if two channels have the same confusability graphs, then they have the same one-shot zero-error capacity and the same Shannon capacity. It is not hard to see that every graph with vertex set X is the confusability graph of some—and in fact many—information channels. Thus, a natural measure of the *complexity* of a graph G on X is the minimal cardinality of Y over all realisations of G as the confusability graph of a channel $\mathcal{N} : X \rightarrow Y$.

Similarly, Duan, Severini and Winter [DSW13] showed that every quantum channel $\Phi : M_n \rightarrow M_m$, where M_m denotes the set of complex $m \times m$ matrices, has an associated *non-commutative confusability graph* $\mathcal{S}_{\Phi}$, which they defined as a certain operator subsystem of M_n . In the case Φ is a classical channel, $\mathcal{S}_{\Phi}$ coincides with the *graph operator system* of G_{Φ} (see [DSW13, equation (3)]). As in Shannon's case, they proved that many natural measures of the quantum capacity of such a channel depend only on the operator system $\mathcal{S}$. It is again the case that every operator subsystem of M_n arises as the non-commutative confusability graph of potentially many quantum channels.

Thus, we are lead to define the *quantum complexity* $\gamma(\mathcal{S})$ of an operator subsystem $\mathcal{S}$ of M_n as the least positive integer k for which there exists a quantum channel $\Phi : M_n \rightarrow M_k$ such that $\mathcal{S} = \mathcal{S}_{\Phi}$.

The goal of this paper is to study this and other, closely related, measures of complexity and to derive their relationships with various measures of capacity for classical and quantum channels. We will show, in particular, that the measures of complexity we introduce give upper bounds on the zero-error capacity of a quantum channel.

One of the most useful general bounds on the Shannon capacity of a classical channel comes from ϑ , the Lovász theta function [Lo79]. While, for classical channels, the complexity based bound is outperformed by the Lovász number (see [Lo79, Theorem 11]), we will show that there exist quantum channels for which the quantum complexity bound on capacity we suggest is better than the bounds arising from the non-commutative analogue $\tilde{\vartheta}$ of the Lovász number introduced in [DSW13]. In fact, we will show that there exist quantum channels Φ_k for $k \in \mathbb{N}$ for which the ratio of the quantum Lovász theta number $\tilde{\vartheta}(\Phi_k)$ to the quantum complexity $\gamma(\Phi_k)$ introduced herein is arbitrarily large, while the upper bound $\gamma(\Phi_k)$ for the quantum Shannon zero-error capacity $\Theta(\Phi_k)$ is accurate to within a factor of two (see Corollary V.3).

We will see that the classical complexity of a graph G is a familiar parameter which coincides with its *intersection number* (provided G lacks isolated vertices). For operator systems, the measure of quantum complexity we propose has not been previously studied. We will characterise it in several different ways. Since every graph G gives rise to a canonical operator system $\mathcal{S}_G$, it can in addition be endowed with a *quantum complexity*, which can be strictly smaller than its classical counterpart, and can be equivalently characterised as a *quantum intersection number* of G .

The paper is organised as follows. In Section II, we begin by recalling the graph theoretic parameters needed in the sequel and show that our measure of the classical complexity of a graph coincides with its intersection number. In Section III, we turn to the quantum complexity of a graph, and show that it coincides with its minimum semi-definite rank (modulo any isolated vertices). In Section IV, we achieve a parallel development for operator systems, considering simultaneously a closely related notion of subcomplexity that coincides with

the quantum chromatic number introduced in [Sta16]. We show that our operator system parameters are genuine extensions of the graph theoretic ones (Theorem IV.10) and explore similarities and differences between their behaviour on commutative and non-commutative graphs. In Section V, we establish the bounds on capacities in terms of complexities (Theorem V.1) and show by example that these bounds can improve dramatically on the Lovász ϑ bound. Finally, in Appendix A we establish the partial ordering among various bounds on the quantum Shannon zero-error capacity, from this paper and elsewhere.

In the sequel, we employ standard notation from linear algebra: we denote by $M_{k,n}$ the space of all k by n matrices with complex entries, and set $M_n = M_{n,n}$. We let $\|X\|$ be the operator norm of a matrix $X \in M_{k,n}$, so that $\|X\|^2$ is the largest eigenvalue of X^*X . We equip M_n with the inner product given by $\langle X, Y \rangle = \text{tr}(Y^*X)$, where $\text{tr}(Z)$ is the trace of a matrix $Z \in M_n$. We write I_n (or simply I) for the identity matrix in M_n . The positive cone of M_n (that is, the set of all positive semi-definite n by n matrices) will be denoted by M_n^+ ; if $S \subseteq M_n$, we let $S^+ = S \cap M_n^+$. We write $\mathbb{R}_+^k$ for the cone of all vectors in $\mathbb{R}^k$ with non-negative entries, and let $(e_i)_{i=1}^k$ be the standard basis of $\mathbb{C}^k$. If $v, w \in \mathbb{C}^n$, we denote by vw^* the rank one operator on $\mathbb{C}^n$ given by $(vw^*)(z) = \langle z, w \rangle v$, $z \in \mathbb{C}^n$. The cardinality of a set S will be denoted by $|S|$.

II. GRAPH PARAMETERS

In this section, we recall some graph theoretic parameters and point out their connection with Shannon's confusability graphs and channel capacities. We start by establishing notation and terminology. Unless otherwise stated, all graphs in this paper will be simple graphs: undirected graphs without loops and at most one edge between any pair of vertices. Let $n \in \mathbb{N}$ and let G be a graph with vertex set $[n] := \{1, \dots, n\}$. For $i, j \in [n]$ we write $i \simeq j$ or $i \simeq_G j$ to denote non-strict adjacency: either $i = j$, or G contains the edge ij . We denote by G^c the *complement* of the graph G ; by definition, G^c has vertex set $[n]$ and, for distinct $i, j \in [n]$, we have $i \simeq_{G^c} j$ if and only if $i \not\simeq_G j$. For graphs H, G with vertex set $[n]$, we write $H \subseteq G$, and say that H is a subgraph of G , if every edge of H is an edge of G . An *independent set* in G is a subset of its vertices between which there are no edges of G . Let $k \in \mathbb{N}$, and for an n -tuple $x = (x_1, \dots, x_n)$, where each $x_i \in \mathbb{C}^k$ is a non-zero vector, we define $G(x)$ to be the *non-orthogonality graph* of x , with vertex set $[n]$ and adjacency relation given by

$$i \simeq_{G(x)} j \iff \langle x_j, x_i \rangle \neq 0.$$

Let G be a graph with vertex set $[n]$. Consider the following graph parameters:

- (a) the *independence number* $\alpha(G)$ of G , given by

$$\alpha(G) = \max\{|S| : S \text{ is independent in } G\};$$

- (b) the *quantum complexity*

$$\gamma(G) = \min\{k \in \mathbb{N} : G(x) = G \text{ for some } x \in (\mathbb{C}^k \setminus \{0\})^n\}$$

and the *quantum subcomplexity*

$$\begin{aligned} \beta(G) &= \min\{k \in \mathbb{N} : G(x) \subseteq G \text{ for some } x \in (\mathbb{C}^k \setminus \{0\})^n\} \\ &= \min\{\gamma(H) : H \text{ is a subgraph of } G\}; \end{aligned}$$

- (c) the *intersection number* $\text{int}(G)$ of G , given by

$$\begin{aligned} \text{int}(G) &= \min\{k \in \mathbb{N} : G(x) = G \\ &\quad \text{for some } x \in (\mathbb{R}_+^k \setminus \{0\})^n\}. \end{aligned}$$

Remark II.1. The graph parameters defined above are well known, and will be generalised to non-commutative graphs in Section IV.

- (i) The independence number $\alpha(G)$ is standard in graph theory [GR01].

- (ii) Writing G^c for the complement of G , we have $\beta(G) = \xi(G^c)$ where ξ is the orthogonal rank (see, for example, [SS15]). The parameter $\gamma(G)$ is the minimum vector rank of G in the terminology of [JMN08], and is equal to $\text{msr}(G) + |\text{iso}(G)|$ where $\text{msr}(G)$ is the classical minimum semidefinite rank of G [FH13], [HPRS15] and $\text{iso}(G)$ is the set of isolated vertices of G .

- (iii) Let $\text{int}_{\text{st}}(G)$ be the set-theoretic intersection number of G ; thus, $\text{int}_{\text{st}}(G)$ is the smallest positive integer k for which there exist non-empty sets $R_i \subseteq [k]$, $i = 1, \dots, n$, such that $i \simeq_G j$ if and only if $R_i \cap R_j \neq \emptyset$. (Note that usually in the literature one relaxes the assumption that the sets R_i be non-empty [MM99]; however, it is more convenient for us to work with the definition above.) We claim that $\text{int}(G) = \text{int}_{\text{st}}(G)$. Indeed, let $k = \text{int}_{\text{st}}(G)$ and choose $\emptyset \neq R_i \subseteq [k]$ for $i \in [n]$ so that $i \simeq_G j \iff R_i \cap R_j \neq \emptyset$. Defining $x_i = \sum_{r \in R_i} e_r$ for $i \in [n]$, we have that $x_i \in \mathbb{R}_+^k$, $i \in [n]$, and $G(x) = G$; hence $\text{int}(G) \leq k$. Conversely, suppose that $k \in \mathbb{N}$ and $x = (x_1, \dots, x_n)$ is a tuple of non-zero vectors in $\mathbb{R}_+^k$ such that $G(x) = G$. Let $R_i = \{r \in [k] : \langle x_i, e_r \rangle \neq 0\}$. The non-negativity of the entries of x_i , $i = 1, \dots, n$, implies that $i \simeq_G j$ if and only if $R_i \cap R_j \neq \emptyset$; thus, $\text{int}_{\text{st}}(G) \leq \text{int}(G)$ and so $\text{int}_{\text{st}}(G) = \text{int}(G)$.

It is straightforward from the definitions that

$$\alpha(G) \leq \beta(G) \leq \gamma(G) \leq \text{int}(G) \quad (1)$$

for every graph G . We note that in Theorem IV.4 below, these inequalities will be generalised to arbitrary operator systems in M_n .

We now review some of Shannon's ideas [Sha56]. Suppose that we have a finite set X , which we view as an alphabet that we wish to send through a noisy channel $\mathcal{N}$ in order to obtain symbols from another alphabet, say Y . We let $p(y|x)$ denote the probability that, if we started with the symbol $x \in X$, then after this process, the symbol $y \in Y$ is received. We require that every $x \in X$ is transformed into some $y \in Y$, that is, $\sum_{y \in Y} p(y|x) = 1$, for all $x \in X$. The column-stochastic matrix $\mathcal{N} = (p(y|x))$, indexed by $Y \times X$, is often referred to as the *noise operator* of the channel. We will write $\mathcal{N} : X \rightarrow Y$ to indicate the matrix $(p(y|x))$, and refer to such matrices as (*classical*) *channels*. The *confusability graph* of $\mathcal{N}$ is the graph $G_{\mathcal{N}}$ with vertex set X for which, given two distinct $x, x' \in X$,

the pair xx' is an edge if and only if there exists $y \in Y$ such that $p(y|x)p(y|x') > 0$. Equivalently, $x \simeq_{G_N} x'$ if and only if there exists $y \in Y$ such that the symbols x and x' can be transformed into the same y via N and hence confused.

The *one-shot zero-error capacity* of N , denoted $\alpha(N)$, is defined to be the cardinality of the largest subset X_1 of X such that, whenever an element of X_1 is sent via N , no matter which element of Y is received, the receiver can determine with certainty the input element from X_1 . It is straightforward that $\alpha(N) = \alpha(G_N)$.

Definition II.2. Let G be a graph with vertex set $[n]$. The *complexity* $\text{plex}(G)$ of G is the minimal cardinality of a set Y such that $G = G_N$ for some channel $N : [n] \rightarrow Y$.

If $N : X \rightarrow Y$ is a channel, we set $\text{plex}(N) = \text{plex}(G_N)$ and call it this parameter the *complexity* of N .

Proposition II.3. Let G be a graph with vertex set $[n]$. Then $\text{plex}(G) = \text{int}(G)$. In other words, if $N : X \rightarrow Y$ is a channel then $\text{plex}(N) = \text{int}(G_N)$.

Proof. Let $N : [n] \rightarrow Y$ be a channel so that $G = G_N$. For $1 \leq i \leq n$ set $R_i = \{y \in Y : p(y|i) \neq 0\}$. Then R_i is non-empty for each i and $i \simeq_G j \iff R_i \cap R_j \neq \emptyset$. This shows that $\text{int}(G)$ is a lower bound for the complexity of G .

Conversely, suppose that $R_1, \dots, R_n$ are non-empty subsets of $[k]$ such that $i \simeq_G j$ if and only if $R_i \cap R_j \neq \emptyset$. Set

$$p(y|i) = \begin{cases} 1/|R_i|, & y \in R_i; \\ 0, & y \notin R_i; \end{cases}$$

then $N = (p(y|i))$ is a channel from $[n]$ to $[k]$ with $G_N = G$. This shows that $\text{int}(G)$ is an upper bound for the complexity of G . $\square$

Remark II.4. (i) We will discuss later (Remark IV.11) the natural way to view a classical channel N as a quantum channel; we will see that the quantum complexity of N , studied in Section IV, coincides with $\gamma(G_N)$.

(ii) It is well-known that $\beta(G) \leq \chi(G^c)$ for any graph G (here $\chi(H)$ denotes the chromatic number of a graph H [GR01]), so it is natural to ask if $\chi(G^c)$ fits into chain of inequalities (1). In fact, it does not: one can check using a computer program that $\chi(G^c) \leq \gamma(G)$ for all graphs on 7 or fewer vertices, but this inequality fails in general, for example if x is a Kochen-Specker set and $G = G(x)$ (see [HPSWM11, Section 1.2]).

(iii) For each $\pi \in \{\alpha, \beta, \gamma, \text{int}\}$, we have that $\pi(G) = 1$ if and only if G is a complete graph. Indeed, if G is a complete graph, then $\pi(G) \leq \text{int}(G) = 1$, so $\pi(G) = 1$; and if G is not a complete graph, then G^c contains at least one edge, so $\pi(G) \geq \alpha(G) > 1$.

III. THE QUANTUM INTERSECTION NUMBER

In this section we show that the graph parameter γ , discussed in Section II, has a reformulation in terms of projective colourings of the graph G , which leads to a parameter that we call the quantum intersection number of G . This will allow a key step in the proof of Theorem IV.10, where we show that γ has a natural operator system generalisation.

Fix $n \in \mathbb{N}$. Given $k \in \mathbb{N}$, we write $\mathcal{P}(k)$ for the set of n -tuples $P = (P_1, \dots, P_n)$ where each P_i is a non-zero projection in M_k . Let $\mathcal{P}_c(k)$ denote the subset of $\mathcal{P}(k)$ consisting of the elements $P = (P_1, \dots, P_n)$ with commuting entries: $P_i P_j = P_j P_i$ for all i, j . To any $P \in \mathcal{P}(k)$ we associate the *non-orthogonality graph* $G(P)$ with vertex set $[n]$ and edges defined by the relation

$$i \simeq_{G(P)} j \iff P_i P_j \neq 0.$$

We define the *quantum intersection number* $\text{qint}(G)$ of a graph G with vertex set $[n]$ by letting

$$\text{qint}(G) = \min \{k \in \mathbb{N} : G(P) = G \text{ for some } P \in \mathcal{P}(k)\}.$$

The next proposition explains the choice of terminology.

Proposition III.1. Let G be a graph with vertex set $[n]$. Then

$$\text{int}(G) = \min \{k \in \mathbb{N} : G(P) = G \text{ for some } P \in \mathcal{P}_c(k)\}. \quad (2)$$

Proof. Let l be the minimum on the right hand side of (2), and suppose that $G(x) = G$ for some n -tuple x of non-zero vectors in $\mathbb{R}_+^k$. Letting P_i be the orthogonal projection onto the linear span of $\{e_r : \langle x_i, e_r \rangle \neq 0\}$ yields a tuple $P = (P_1, \dots, P_n) \in \mathcal{P}_c(k)$ with $G(P) = G$; thus, $l \leq \text{int}(G)$.

Conversely, suppose that $P = (P_1, \dots, P_n) \in \mathcal{P}_c(l)$ is such that $G(P) = G$. Simultaneously diagonalising the P_i 's with respect to a basis $\{b_r : r \in [l]\}$ and defining $R_i = \{r \in [l] : P_i b_r \neq 0\}$, $i \in [n]$, we see that $i \simeq_G j$ if and only if $R_i \cap R_j \neq \emptyset$, and it follows that $\text{int}(G) \leq l$. $\square$

Let $t = (t_1, \dots, t_n) \in \mathbb{N}^n$, and write $|t| = \sum_{i=1}^n t_i$. Extending ideas from [HPRS15], let

$$\mathcal{P}(k, t) = \{(P_1, \dots, P_n) \in \mathcal{P}(k) : \text{rank } P_i = t_i, i \in [n]\}$$

and define

$$\text{m}_t^+(G) = \min \{k \in \mathbb{N} : G(P) = G \text{ for some } P \in \mathcal{P}(k, t)\}.$$

Consider each element $A \in M_{|t|}$ as a block matrix $A = [A_{i,j}]_{i,j \in [n]}$ where $A_{i,j} \in M_{t_i, t_j}$. We define

$$\mathcal{F}_t^+(G) = \left\{ A \in M_{|t|}^+ : A_{i,j} \neq 0 \iff i \simeq j, \right. \\ \left. \text{and } \text{rank}(A_{i,i}) = t_i \text{ for each } i \in [n] \right\}$$

and

$$\mathcal{H}_t^+(G) = \{A \in \mathcal{F}_t^+(G) : A_{i,i} = I_{t_i} \text{ for each } i \in [n]\}.$$

We write $\mathcal{F}^+(G) = \mathcal{F}_1^+(G)$ and $\mathcal{H}^+(G) = \mathcal{H}_1^+(G)$, where $1 = (1, 1, \dots, 1)$. Note that in [HPRS15], $\text{m}_t^+(G)$ and $\mathcal{H}_t^+(G)$ were defined in the special case where $t = (r, r, \dots, r)$ for some $r \in \mathbb{N}$.

Proposition III.2. Let G be a graph with vertex set $[n]$ and let $t \in \mathbb{N}^n$. Then

$$\text{m}_t^+(G) = \min \{\text{rank } A : A \in \mathcal{F}_t^+(G)\} \\ = \min \{\text{rank } B : B \in \mathcal{H}_t^+(G)\}.$$

Proof. The proof is an adaptation of [HPRS15, Theorem 3.10]. Suppose first that $A \in \mathcal{F}_t^+(G)$ and $\text{rank } A = k$. Then there

exists a matrix $X \in M_{k,|t|}$ such that $A = X^*X$. Write $X = [X_1 \ X_2 \ \cdots \ X_n]$, where $X_i \in M_{k,t_i}$, $i = 1, \dots, n$. We have

$$\text{rank } X_i = \text{rank}(X_i^* X_i) = \text{rank}(A_{ii}) = t_i, \quad i \in [n].$$

Writing $P = (P_1, \dots, P_n)$ where $P_i \in M_k$ is the orthogonal projection onto the range of X_i , we have $P \in \mathcal{P}(k, t)$. Additionally,

$$P_i P_j \neq 0 \iff X_i^* X_j \neq 0 \iff A_{i,j} \neq 0 \iff i \simeq j,$$

so $G(P) = G$. Hence

$$m_t^+(G) \leq \min \{ \text{rank } A : A \in \mathcal{F}_t^+(G) \}.$$

Since $\mathcal{H}_t^+(G) \subseteq \mathcal{F}_t^+(G)$, the inequality

$$\min \{ \text{rank } A : A \in \mathcal{F}_t^+(G) \} \leq \min \{ \text{rank } B : B \in \mathcal{H}_t^+(G) \}$$

holds trivially.

Now suppose that $P = (P_1, \dots, P_n) \in \mathcal{P}(k, t)$ with $G(P) = G$, and for each $i \in [n]$ let $X_i \in M_{k,t_i}$ be a matrix whose columns form an orthonormal basis for the range of P_i . Define $X = [X_1 \ X_2 \ \cdots \ X_n] \in M_{k,|t|}$ and let $B = X^*X \in M_{|t|}^+$, so that $\text{rank } B = \text{rank } X \leq k$. Note that the $t_i \times t_j$ block $B_{i,j}$ coincides with $X_i^* X_j$. Since $i \simeq j \iff P_i P_j \neq 0 \iff X_i^* X_j \neq 0$, we have $B_{i,j} \neq 0 \iff i \simeq j$. Moreover, the condition on the columns of X_i implies that $B_{i,i} = I_{t_i}$ for each i . So $B \in \mathcal{H}_t^+(G)$, hence

$$\min \{ \text{rank } B : B \in \mathcal{H}_t^+(G) \} \leq m_t^+(G). \quad \square$$

Theorem III.3. *For any graph G , we have $\text{qint}(G) = \gamma(G)$.*

Proof. Directly from the definitions, we have

$$\text{qint}(G) = \min_{t \in \mathbb{N}^n} m_t^+(G) \leq m_{\mathbb{1}}^+(G) = \gamma(G).$$

Let $t \in \mathbb{N}^n$. We claim that if $t_1 \geq 2$ and $s = (t_1 - 1, t_2, t_3, \dots, t_n)$, then $m_s^+(G) \leq m_t^+(G)$. By symmetry and induction, this yields $\gamma(G) = m_{\mathbb{1}}^+(G) \leq m_t^+(G)$ for any $t \in \mathbb{N}^n$, hence $\text{qint}(G) = \gamma(G)$.

To establish the claim, suppose that $t_1 \geq 2$, let $k = m_t^+(G)$, and use Proposition III.2 to choose $B \in \mathcal{H}_t^+(G)$ with $\text{rank } B = k$. We may write $B = X^*X$ where $X \in M_{k,|t|}$. Write $X = [X_1 \ X_2 \ \cdots \ X_n]$ where $X_i \in M_{k,t_i}$, let $Y \in M_{k,|t|-1}$ be X with the first column deleted and let $Z_1 \in M_{k,t_1-1}$ be the matrix with every column equal to the first column of X . Let $Z = [Z_1 \ 0 \ \dots \ 0] \in M_{k,|t|-1}$.

Let $Y_1 \in M_{k,t_1-1}$ consist of the first $(t_1 - 1)$ columns of Y . Note that X^*Y_1 contains $X_1^*Y_1$ as a submatrix, which is equal to I_{t_1} with the first column removed. In particular, $X^*Y_1 \neq 0$. Similarly, the first column of $X_1^*Z_1$ is the first column of I_{t_1} , so $X^*Z_1 \neq 0$. Define

$$a = \min \{ |w| : w \neq 0, w \text{ is an entry of } X^*Y_1 \}$$

$$b = \max \{ |w| : w \text{ is an entry of } X^*Z_1 \}$$

and let $\varepsilon \in (0, \frac{1}{2}ab^{-1})$. Define

$$W_\varepsilon = Y + \varepsilon Z \in M_{k,|t|-1}$$

and

$$A = A_\varepsilon = W_\varepsilon^* W_\varepsilon \in M_{|t|-1}^+.$$

Note that $\text{rank } A = \text{rank } W_\varepsilon \leq k$. Let us write $W_\varepsilon = [W_1 \ W_2 \ \cdots \ W_n]$ and $A = [A_{i,j}]_{i,j \in [n]}$, with block sizes given by $s = (t_1 - 1, t_2, \dots, t_n)$. Note that if $2 \leq i, j \leq n$, then $W_i = X_i$ and $W_j = X_j$, hence $A_{i,j} = W_i^* W_j = X_i^* X_j = B_{i,j} \neq 0 \iff i \simeq j$. For $2 \leq i \leq n$, we have

$$A_{i,1} = X_i^*(Y_1 + \varepsilon Z_1) = X_i^*Y_1 + \varepsilon X_i^*Z_1. \quad (3)$$

If $i \not\simeq 1$, then $X_i^*Y_1$ is a submatrix of $X_i^*X_1 = 0$, and $X_i^*Z_1$ is the matrix with every column equal to the first column of $X_i^*X_1 = 0$. Hence if $i \not\simeq 1$, then $A_{i,1} = 0 = A_{1,i}$. Now suppose that $i \simeq 1$. Since $X_i^*Y_1$ and $X_i^*Z_1$ are submatrices of X^*Y_1 and X^*Z_1 , respectively, by (3) and our choice of ε , if $X_i^*Y_1$ has any non-zero entry, then the corresponding entry of $A_{i,1}$ is also non-zero. On the other hand, if $X_i^*Y_1 = 0$, then since $i \simeq 1$ yields $X_i^*X_1 \neq 0$, we must have $X_i^*Z_1 \neq 0$, hence $A_{i,1} \neq 0$; since $A = A^*$, we also have $A_{1,i} \neq 0$. This shows that for $\varepsilon \in (0, \frac{1}{2}ab^{-1})$, the matrix $A = A_\varepsilon$ satisfies

$$A_{i,j} \neq 0 \iff i \simeq j, \quad \text{for any } (i, j) \in [n] \times [n] \setminus \{(1, 1)\}.$$

Since $W_\varepsilon \rightarrow Y$ as $\varepsilon \rightarrow 0$, we see that A_ε converges to the matrix B with the first row and column removed; in particular, the top left $(t_1 - 1) \times (t_1 - 1)$ block of A_ε converges to I_{t_1-1} . Hence by choosing $\varepsilon \in (0, \frac{1}{2}ab^{-1})$ sufficiently small, we may ensure that $A_{1,1}$ has rank $t_1 - 1$, hence $A \in \mathcal{F}_s^+(G)$. By Proposition III.2, we have $m_s^+(G) \leq \text{rank } A \leq k = m_t^+(G)$. $\square$

IV. QUANTUM CHANNELS AND OPERATOR SYSTEM PARAMETERS

We recall that an *operator subsystem* of M_n is a subspace $\mathcal{S} \subseteq M_n$ such that $I \in \mathcal{S}$ and $X \in \mathcal{S} \implies X^* \in \mathcal{S}$. In this paper we will sometimes refer to such a self-adjoint unital subspace $\mathcal{S} \subseteq M_n$ simply as an *operator system*; we refer the reader to [Pau02] for the general theory of operator systems and completely bounded maps. A linear map $\Phi : M_n \rightarrow M_k$ is called a *quantum channel* if it is completely positive and trace preserving. By theorems of Choi and Kraus, Φ is a quantum channel if and only if there exist $m \in \mathbb{N}$ and matrices $A_1, \dots, A_m \in M_{k,n}$, satisfying $\sum_{i=1}^m A_i^* A_i = I_n$, so that

$$\Phi(X) = \sum_{i=1}^m A_i X A_i^*, \quad X \in M_n.$$

This realisation of Φ is called a *Choi-Kraus representation* and the matrices A_i are called its *Kraus operators*. The Choi-Kraus representation is far from unique, but it was shown in [DSW13] that the subspace of M_n spanned by the set $\{A_i^* A_j : 1 \leq i, j \leq m\}$ is independent of it. Consequently, [DSW13] set

$$\mathcal{S}_\Phi := \text{span} \{A_i^* A_j : 1 \leq i, j \leq m\},$$

where $\Phi(X) = \sum_{i=1}^m A_i X A_i^*$ is any Choi-Kraus representation of Φ . This space, easily seen to be an operator system, is called the *non-commutative confusability graph* of Φ .

Regarding operator subsystems of M_n as non-commutative confusability graphs, we wish to define operator system analogues of the graph parameters considered in Section II. Just as every graph is the confusability graph of some classical

channel, [DSW13] showed that the map $\Phi \mapsto \mathcal{S}_\Phi$ from quantum channels with domain M_n to operator subsystems of M_n , is surjective. A more detailed statement, including an estimate on the dimension of the target Hilbert space of such Φ , was provided in [SS15]. Here we include yet another argument for the convenience of the reader.

Proposition IV.1. *Let $n \in \mathbb{N}$. If $\mathcal{S} \subseteq M_n$ is an operator system, then there exists $k \in \mathbb{N}$ and a quantum channel $\Phi: M_n \rightarrow M_k$ such that $\mathcal{S} = \mathcal{S}_\Phi$. In fact, if $m \in \mathbb{N}$ is such that $\binom{m}{2} \geq \dim \mathcal{S} - 1$, then we can take $k = mn$.*

Proof. Let $d = \dim \mathcal{S}$ and let $I_n, S_1, \dots, S_{d-1}$ be a basis of $\mathcal{S}$. Suppose that $\binom{m}{2} \geq \dim \mathcal{S} - 1$. Then we can form a hermitian $m \times m$ block matrix $H = [H_{i,j}]_{i,j \in [m]} \in M_m(M_n)$ so that

$$H_{i,i} = 0 \text{ for } i \in [m]$$

and

$$\{S_1, \dots, S_{d-1}\} = \{H_{i,j}: 1 \leq i < j \leq m\}.$$

For sufficiently small $\varepsilon > 0$, the hermitian matrix $X = m^{-1}(I_{mn} + \varepsilon H)$ is positive semi-definite (indeed, its spectrum is contained in the interval $[m^{-1}(1 - \varepsilon r), m^{-1}(1 + \varepsilon r)]$ where r is the spectral radius of H) hence $X = C^*C$ for some $C \in M_m(M_n)$. Note that the block entries $X_{i,j}$ of X span $\mathcal{S}$, and $X_{i,i} = m^{-1}I_n$ for $i \in [m]$. The $mn \times n$ block columns $C_1, \dots, C_m$ of C satisfy $C_i^*C_j = X_{i,j}$, so in particular, $\sum_{i=1}^m C_i^*C_i = \sum_{i=1}^m X_{i,i} = I_n$. Hence $\{C_1, \dots, C_m\}$ are Kraus operators for a quantum channel $\Phi: M_n \rightarrow M_{mn}$ for which $\mathcal{S}_\Phi$ is spanned by the entries of X , so $\mathcal{S}_\Phi = \mathcal{S}$. $\square$

We now define parameters of operator systems which, as we will shortly see, generalise the graph parameters above. Let $\mathcal{S} \subseteq M_n$ be an operator system. As usual, we write

$$\mathcal{S}^\perp = \{A \in M_n: \text{tr}(A^*S) = 0 \text{ for all } S \in \mathcal{S}\}.$$

- (a) Let $\mathcal{S} \subseteq M_n$ be an operator system. Recall [DSW13] that an $\mathcal{S}$ -independent set of size m is an m -tuple $x = (x_1, \dots, x_m)$ with each x_i a non-zero vector in $\mathbb{C}^n$, so that $x_p x_q^* \in \mathcal{S}^\perp$ whenever $p, q \in [m]$ with $p \neq q$. The independence number $\alpha(\mathcal{S})$ is then defined by letting

$$\alpha(\mathcal{S}) = \max\{m \in \mathbb{N}: \exists \text{ an } \mathcal{S}\text{-independent set of size } m\}.$$

- (b) We define the *quantum complexity* $\gamma(\mathcal{S})$ by letting

$$\gamma(\mathcal{S}) = \min\{k \in \mathbb{N}: \mathcal{S}_\Phi = \mathcal{S} \text{ for some quantum channel } \Phi: M_n \rightarrow M_k\}$$

and the *quantum subcomplexity* $\beta(\mathcal{S})$ by letting

$$\begin{aligned} \beta(\mathcal{S}) &= \min\{k \in \mathbb{N}: \mathcal{S}_\Phi \subseteq \mathcal{S} \text{ for some quantum channel } \Phi: M_n \rightarrow M_k\} \\ &= \min\{\gamma(\mathcal{T}): \mathcal{T} \subseteq \mathcal{S} \text{ is an operator subsystem}\}. \end{aligned}$$

- (c) A quantum channel Φ which has a set of Kraus operators each of which is of the form AD for some entrywise non-negative matrix A and an invertible diagonal matrix D will be said to be a *non-cancelling*. We define

$$\text{int}(\mathcal{S}) = \inf\{k \in \mathbb{N}: \mathcal{S}_\Phi = \mathcal{S} \text{ for some non-cancelling quantum channel } \Phi: M_n \rightarrow M_k\}.$$

Corollary IV.2. *Let $\mathcal{S} \subseteq M_n$ be an operator system. Then $\gamma(\mathcal{S}) \leq 2n^2$.*

Proof. Since $\dim \mathcal{S} \leq n^2$, we can take $m = 2n$ in Proposition IV.1. $\square$

We will refer to $\gamma(\Phi)$ as the *quantum complexity* of Φ and $\beta(\Phi)$ as the *quantum subcomplexity* of Φ . Given a channel Φ , we set $\pi(\Phi) = \pi(\mathcal{S}_\Phi)$ for $\pi \in \{\beta, \gamma, \text{int}\}$.

Remark IV.3. (i) A set of quantum states can be perfectly distinguished by a measurement system if and only if they are orthogonal. Consequently, [DSW13] defined the *one-shot zero-error capacity* $\alpha(\Phi)$ of a quantum channel $\Phi: M_n \rightarrow M_k$ to be the maximum cardinality of a set $\{v_1, \dots, v_p\} \subseteq \mathbb{C}^n$ orthogonal unit vectors, such that

$$\text{tr}(\Phi(v_i v_i^*) \Phi(v_j v_j^*)) = 0, \quad i \neq j.$$

It was shown in [DSW13] (see also [Pau16]) that $\alpha(\Phi) = \alpha(\mathcal{S}_\Phi)$.

(ii) Let $\mathcal{S}$ be an operator system. The *quantum chromatic number* $\chi_q(\mathcal{S}^\perp)$ of the orthogonal complement $\mathcal{S}^\perp$ of $\mathcal{S}$ was introduced by D. Stahlke in [Sta16]. It is straightforward that $\beta(\mathcal{S}) = \chi_q(\mathcal{S}^\perp)$.

(iii) For an operator system $\mathcal{S} \subseteq M_n$ and $\pi \in \{\beta, \gamma, \text{int}\}$, we have $\pi(\mathcal{S}) = 1 \iff \mathcal{S} = M_n$. Indeed, the trace $\text{tr}: M_n \rightarrow \mathbb{C}$ is a non-cancelling quantum channel since it has the entrywise non-negative Kraus operators $e_1^*, \dots, e_n^*$ (where e_i^* is the functional corresponding to the vector e_i), so $1 \leq \pi(M_n) \leq \text{int}(M_n) = 1$ and hence $\pi(M_n) = 1$. Conversely, $\pi(\mathcal{S}) = 1$ implies that $\beta(\mathcal{S}) = 1$; the trace is the only scalar-valued quantum channel on M_n , so $M_n = \mathcal{S}_{\text{tr}} \subseteq \mathcal{S} \subseteq M_n$, that is, $\mathcal{S} = M_n$.

Note that, in contrast with Remark II.4 (iii), it is not true that M_n is the only operator system $\mathcal{S}$ with $\alpha(\mathcal{S}) = 1$; see Proposition IV.12.

(iv) We claim that

$$\alpha(\mathbb{C}I_n) = \beta(\mathbb{C}I_n) = \gamma(\mathbb{C}I_n) = \text{int}(\mathbb{C}I_n) = n.$$

Indeed, one sees immediately that $\alpha(\mathbb{C}I_n) \geq n$ by considering the $\mathbb{C}I_n$ -independent set $(e_1, \dots, e_n)$, and since the identity channel $M_n \rightarrow M_n$ is non-cancelling, we have that $\text{int}(\mathbb{C}I_n) \leq n$, so an appeal to Theorem IV.4 below establishes the claim.

(v) Let $\mathcal{S} \subseteq M_n$. Using (iv), we have

$$\beta(\mathcal{S}) = \min\{\gamma(\mathcal{T}): \mathcal{T} \subseteq \mathcal{S}\} \leq \gamma(\mathbb{C}I_n) = n.$$

On the other hand, $\gamma(\mathcal{S})$ may exceed n , even for $n = 2$ (see Proposition IV.12).

(vi) There exist operator systems $\mathcal{S} \subseteq M_n$ with $\text{int}(\mathcal{S}) = \infty$, so the infimum in the definition of $\text{int}(\mathcal{S})$ cannot be replaced with a minimum. For example, it is not difficult to see that this is the case for the two-dimensional operator system $\mathcal{S} \subseteq M_4$ spanned by the identity and $H = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}$ where $X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Theorem IV.4. *Let $\mathcal{S} \subseteq M_n$ be an operator system. Then*

$$\alpha(\mathcal{S}) \leq \beta(\mathcal{S}) \leq \gamma(\mathcal{S}) \leq \text{int}(\mathcal{S}).$$

Proof. Suppose that $\Phi : M_n \rightarrow M_k$ is a quantum channel with $\mathcal{S}_\Phi \subseteq \mathcal{S}$; let $A_i \in M_{k,n}$, $i = 1, \dots, d$, be its Kraus operators. Let $(x_p)_{p=1}^m$ be an $\mathcal{S}$ -independent set of size m . For each $p \in [m]$, let E_p be the projection in M_k onto the span of $\{A_i x_p : i = 1, \dots, d\}$. Since

$$\sum_{i=1}^d \|A_i x_p\|^2 = \left\langle \left(\sum_{i=1}^d A_i^* A_i \right) x_p, x_p \right\rangle = 1,$$

we have that $E_p \neq 0$ for each p . On the other hand, since $A_j^* A_i \in \mathcal{S}$ for all $i, j = 1, \dots, d$ and $(x_p)_{p=1}^m$ is $\mathcal{S}$ -independent, we have that

$$\langle A_i x_p, A_j x_q \rangle = \langle A_j^* A_i x_p, x_q \rangle = 0, \quad p \neq q, \quad i, j = 1, \dots, d.$$

Thus, $E_1, \dots, E_m$ are pairwise orthogonal projections in M_k ; it follows that $m \leq k$ and hence $\alpha(\mathcal{S}) \leq \beta(\mathcal{S})$.

The inequalities $\beta(\mathcal{S}) \leq \gamma(\mathcal{S}) \leq \text{int}(\mathcal{S})$ hold trivially. $\square$

In the next proposition, we collect some properties of the operator system parameters introduced above.

Proposition IV.5. *Let $\mathcal{S} \subseteq M_n$ and $\mathcal{S}_i \subseteq M_{n_i}$, $i = 1, 2$ be operator systems.*

- (i) *If $\pi \in \{\alpha, \beta, \gamma\}$ and $U \in M_n$ is unitary, then $\pi(U^* \mathcal{S} U) = \pi(\mathcal{S})$;*
- (ii) *If $\pi \in \{\alpha, \beta, \gamma\}$ and $P \in M_n$ is a projection of rank r , then, viewing $P\mathcal{S}P$ as an operator subsystem of M_r , we have $\pi(P\mathcal{S}P) \leq \pi(\mathcal{S})$;*
- (iii) *If $\pi \in \{\beta, \gamma\}$, $n = n_1 n_2$ and $\mathcal{S} = \mathcal{S}_1 \otimes \mathcal{S}_2$, then*

$$\max\{\pi(\mathcal{S}_1), \pi(\mathcal{S}_2)\} \leq \pi(\mathcal{S}) \leq \pi(\mathcal{S}_1) \pi(\mathcal{S}_2);$$

- (iv) *If $\pi \in \{\beta, \gamma, \text{int}\}$, $n = n_1 = n_2$ and $\mathcal{S} = \text{span}(\mathcal{S}_1 \cup \mathcal{S}_2)$, then*

$$\pi(\mathcal{S}) \leq \pi(\mathcal{S}_1) + \pi(\mathcal{S}_2).$$

- (v) *If $\pi \in \{\beta, \gamma\}$, then $\pi(\mathcal{S}_1 \oplus \mathcal{S}_2) = \pi(\mathcal{S}_1) + \pi(\mathcal{S}_2)$.*

Proof. The proofs for $\pi = \alpha$ are easy and are left to the reader. We give the proofs for $\pi = \gamma$; the other proofs follow identical patterns.

- (i) If $\{A_p\}_{p=1}^m$ are Kraus operators in $M_{k,n}$ for which $\text{span}\{A_p^* A_q\}_{p,q=1}^m = \mathcal{S}$, then $\{A_p U\}_{p=1}^m$ are Kraus operators in $M_{k,n}$ such that

$$\begin{aligned} \text{span}\{(A_p U)^* A_q U\}_{p,q=1}^m &= U^* \text{span}\{A_p^* A_q\}_{p,q=1}^m U \\ &= U^* \mathcal{S} U, \end{aligned}$$

so $\gamma(U^* \mathcal{S} U) \leq \gamma(\mathcal{S})$; the reverse inequality follows by symmetry.

- (ii) If $\{A_p\}_{p=1}^m$ are Kraus operators in $M_{k,n}$ for which $\text{span}\{A_p^* A_q\}_{p,q=1}^m = \mathcal{S}$, then after identifying the range of P with $\mathbb{C}^r$, we see that $\{A_p P\}_{p=1}^m$ are Kraus operators in $M_{k,r}$ with

$$\text{span}\{(A_p P)^* A_q P\}_{p,q=1}^m = P \text{span}\{A_p^* A_q\}_{p,q=1}^m P = P\mathcal{S}P,$$

so $\gamma(P\mathcal{S}P) \leq \gamma(\mathcal{S})$.

- (iii) Suppose that $\{A_{p,i} : p \in [m_i]\} \subseteq M_{k_i, n_i}$ is a family of Kraus operators for $i = 1, 2$, so that $\mathcal{S}_i = \text{span}\{A_{p,i}^* A_{q,i} : p, q \in [m_i]\}$, $i = 1, 2$. Set $B_{p,1} := A_{p,1} \otimes A_{r,2}$; then $\{B_{p,r} : p \in [m_1], r \in [m_2]\}$ is a family of

Kraus operators in $M_{k_1 k_2, n}$ with $\mathcal{S} = \text{span}\{B_{p,r}^* B_{q,s} : p, q \in [m_1], r, s \in [m_2]\}$. It follows that $\gamma(\mathcal{S}) \leq \gamma(\mathcal{S}_1) \gamma(\mathcal{S}_2)$.

If we set $P_1 = I_{n_1} \otimes Q$ where $Q \in M_{n_2}$ is a rank one projection, then we have that $P_1(\mathcal{S}_1 \otimes \mathcal{S}_2)P_1 = \mathcal{S}_1 \otimes \mathbb{C} \equiv \mathcal{S}_1$. Hence by (ii), $\pi(\mathcal{S}_1) \leq \pi(\mathcal{S}_1 \otimes \mathcal{S}_2)$ and the lower bound follows.

- (iv) Let $\{A_{p,i} : p \in [m_i]\}$ be a family of Kraus operators in $M_{k_i, n}$ with $\mathcal{S}_i = \text{span}\{A_{p,i}^* A_{q,i} : p, q \in [m_i]\}$, $i = 1, 2$. Set $B_{p,1} = \begin{bmatrix} A_{p,1} \\ 0 \end{bmatrix}$ and $B_{p,2} = \begin{bmatrix} 0 \\ A_{p,2} \end{bmatrix}$, viewed as elements of $M_{k_1 + k_2, n}$. Then

$$\left\{ \frac{1}{\sqrt{2}} B_{p,i} : i = 1, 2, p \in [m_i] \right\}$$

is a family of Kraus operators with

$$\text{span}\{B_{p,i}^* B_{q,j}\}_{p,q,i,j} = \text{span}\{B_{p,i}^* B_{q,i}\}_{p,q,i} = \mathcal{S},$$

so $\gamma(\mathcal{S}) \leq k_1 + k_2$. Hence, $\gamma(\mathcal{S}) \leq \gamma(\mathcal{S}_1) + \gamma(\mathcal{S}_2)$.

- (v) If $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$, then $n = n_1 + n_2$. Suppose that $\gamma(\mathcal{S}) = k$, so that there exists a family $\{C_p = [A_p \ B_p] : p \in [m]\}$ of $k \times n$ -Kraus operators, where $A_p \in M_{k, n_1}$ and $B_p \in M_{k, n_2}$, with $\mathcal{S} = \text{span}\{C_p^* C_q\}_{p,q=1}^m$. Since $C_p^* C_q = \begin{bmatrix} A_p^* A_q & A_p^* B_q \\ B_p^* A_q & B_p^* B_q \end{bmatrix} \in \mathcal{S}_1 \oplus \mathcal{S}_2$, we have $A_p^* B_q = 0$, hence the ranges of A_p and B_q are orthogonal for every p, q . The projections P_1 and P_2 onto the linear span of the ranges of $A_1, \dots, A_m$ and $B_1, \dots, B_m$ are therefore orthogonal, so if $k_1 = \text{rank } P_1$ and $k_2 = k - \text{rank } P_1$, then there is a unitary $U : \mathbb{C}^k \rightarrow \mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}$ for which $U A_p = \begin{bmatrix} A'_p \\ 0 \end{bmatrix}$ for some $k_1 \times n$ matrices A'_p , and $U B_p = \begin{bmatrix} 0 \\ B'_p \end{bmatrix}$ for some $k_2 \times n$ matrices B'_p . Now $C_p^* C_q = (U C_p)^* (U C_q) = \begin{bmatrix} A_p'^* A_q' & 0 \\ 0 & B_p'^* B_q' \end{bmatrix}$, and it follows that $\gamma(\mathcal{S}_i) \leq k_i$ for $i = 1, 2$. Hence, $\gamma(\mathcal{S}_1) + \gamma(\mathcal{S}_2) \leq k_1 + k_2 = k = \gamma(\mathcal{S})$. Combined with (iv), this shows that $\gamma(\mathcal{S}_1 \oplus \mathcal{S}_2) = \gamma(\mathcal{S}_1) + \gamma(\mathcal{S}_2)$. $\square$

Remark IV.6. (i) Let $\pi \in \{\alpha, \beta, \gamma\}$ and $d \in \mathbb{N}$. Then $\pi(M_d(\mathcal{S})) = \pi(\mathcal{S})$. Indeed, by Proposition IV.5 (ii), we have $\pi(\mathcal{S}) \leq \pi(M_d(\mathcal{S}))$, and the reverse inequality for $\pi \in \{\beta, \gamma\}$ follows from Proposition IV.5 (iii) and Remark IV.3 (iii). To see the corresponding result for $\pi = \alpha$, suppose that $\{\xi_p\}_{p=1}^m$ is an independent set for $\mathcal{S} \otimes M_d$. Then for $X, Y \in M_d$, $A \in \mathcal{S}$ and $p \neq q$, we have

$$\langle (A \otimes I)(I \otimes X) \xi_p, (I \otimes Y) \xi_q \rangle = 0 \quad (4)$$

Let Q_p be the projection onto $\text{span}\{(I \otimes X) \xi_p : X \in M_d\}$; then $Q_p = E_p \otimes I_d$ for some non-zero projection E_p on $\mathbb{C}^n$, and (4) implies that $E_q \mathcal{S} E_p = \{0\}$ provided $p \neq q$. If v_p is a unit vector with $E_p v_p = v_p$, $p \in [m]$, we therefore have that $\{v_p\}_{p=1}^m$ is an independent set for $\mathcal{S}$. It follows that $\alpha(\mathcal{S}) \geq \alpha(M_d(\mathcal{S}))$ and hence we have equality.

- (ii) The parameter γ is neither order-preserving nor order-reversing for inclusion. For example, $\mathbb{C} I_2 \subseteq \mathcal{S} \subseteq M_2$ where $\mathcal{S}$ is the operator system of Proposition IV.12, and these operator systems have γ -values 2, 3, 1, respectively.

We will now show that like its graph-theoretic counterpart, namely, the minimum semidefinite rank, $\gamma(\mathcal{S})$ is the solution to a rank minimisation problem.

Proposition IV.7. For any operator system $\mathcal{S} \subseteq M_n$, we have

$$\gamma(\mathcal{S}) = \min_{m \in \mathbb{N}} \left\{ \text{rank } B : B = [B_{i,j}] \in M_m(\mathcal{S})^+ \text{ with } \right. \\ \left. \text{span}\{B_{i,j} : i, j \in [m]\} = \mathcal{S} \text{ and } \sum_{i=1}^m B_{i,i} = I_n \right\}$$

and

$$\beta(\mathcal{S}) = \min_{m \in \mathbb{N}} \left\{ \text{rank } B : B = [B_{i,j}] \in M_m(\mathcal{S})^+ \right. \\ \left. \text{and } \sum_{i=1}^m B_{i,i} = I_n \right\}.$$

Moreover, the minima on the right hand sides are achieved for m not exceeding $2n^3$.

Proof. Suppose that $m \in \mathbb{N}$ and that $B = [B_{i,j}] \in M_m(\mathcal{S})^+$ satisfies the relations $\text{span}\{B_{i,j}\}_{i,j=1}^m = \mathcal{S}$ and $\sum_{i=1}^m B_{i,i} = I_n$. Then $B = A^*A$ for some $A = [A_1 \dots A_m] \in M_{1,m}(M_{k,n})$ where $k = \text{rank } B$. Since $B_{i,j} = A_i^*A_j$, we see that $\{A_1, \dots, A_m\}$ are Kraus operators for a quantum channel Φ with $\mathcal{S}_\Phi = \mathcal{S}$; thus, $\gamma(\mathcal{S}) \leq k$.

Conversely, let $k = \gamma(\mathcal{S})$, $m \in \mathbb{N}$ and $A_1, \dots, A_m \in M_{k,n}$ be Kraus operators for a quantum channel Φ with $\mathcal{S}_\Phi = \mathcal{S}$. Set $B := [A_i^*A_j] \in M_m(\mathcal{S})^+$; we have $\text{span}\{A_i^*A_j : i, j \in [m]\} = \mathcal{S}$, $\sum_{i=1}^m A_i^*A_i = I_n$ and $\text{rank } B = \text{rank}[A_1 \dots A_m] \leq k$. Hence the minimum rank in the first expression is no greater than $\gamma(\mathcal{S})$.

To see that some $m \leq 2n^3$ attains this minimum, set $k = \gamma(\mathcal{S})$. Then there exists a quantum channel $\Phi : M_n \rightarrow M_k$ with $\mathcal{S}_\Phi = \mathcal{S}$ and, by Corollary IV.2, $k \leq 2n^2$. By [Ch75, Remark 6], the channel Φ can be realised using at most $nk \leq 2n^3$ Kraus operators. Since m is precisely the number of Kraus operators in the preceding argument, we see that the minimum in the expression for $\gamma(\mathcal{S})$ is attained for some $m \leq 2n^3$.

The expression for $\beta(\mathcal{S})$ follows from the fact that $\beta(\mathcal{S}) = \min\{\gamma(\mathcal{T}) : \mathcal{T} \subseteq \mathcal{S}\}$. Since the bound on m for γ , namely $2n^3$, is independent of the operator system $\mathcal{S} \subseteq M_n$, this fact shows that here we may also take $m \leq 2n^3$. $\square$

Our next task is to show that the operator system parameters just defined generalise the graph parameters of Section II. Recall that if G is a graph with vertex set $[n]$, we let

$$\mathcal{S}_G = \text{span}\{E_{i,j} : i \simeq j\}$$

be the associated operator subsystem of M_n .

Lemma IV.8. Let $n, k \in \mathbb{N}$, and let Δ be the group of diagonal $n \times n$ matrices whose diagonal entries are each either 1 or -1 . Let $x = (x_1, \dots, x_n)$ be an n -tuple of non-zero vectors in $\mathbb{C}^k$, and let $A = [\hat{x}_1 \dots \hat{x}_n]$ be the $k \times n$ matrix whose i -th column is the unit vector $\hat{x}_i = \|x_i\|^{-1}x_i$. Then the map

$$\Delta_x : M_n \rightarrow M_k, \quad \Delta_x(X) = 2^{-n} \sum_{D \in \Delta} ADXD^*,$$

is a quantum channel. Moreover, if $x_i \in \mathbb{R}_+^k \setminus \{0\}$, $i = 1, \dots, n$, then Δ_x is non-cancelling.

Proof. For $D \in \Delta$, let $d_i \in \{1, -1\}$ be the i -th diagonal entry of D . We have

$$2^{-n} \sum_{D \in \Delta} (DA^*AD)_{ij} = 2^{-n} \langle \hat{x}_j, \hat{x}_i \rangle \sum_{D \in \Delta} d_i d_j = \delta_{ij},$$

since if $i \neq j$ then the sum reduces to 0 by symmetry, whereas if $i = j$ then every term in the sum is 1. Since each $\hat{x}_i$ is a unit vector, we obtain $2^{-n} \sum_{D \in \Delta} DA^*AD = I_n$, so Δ_x is a quantum channel. The assertion about non-cancelling channels follows trivially. $\square$

Proposition IV.9. Let $n, k \in \mathbb{N}$, x_i be a non-zero vector in $\mathbb{C}^k$, $i = 1, \dots, n$, and $x = (x_1, \dots, x_n)$. Then $\mathcal{S}_{G(x)} = \mathcal{S}_{\Delta_x}$.

Proof. Let $\mathcal{S} = \mathcal{S}_{G(x)}$ and $\mathcal{T} = \mathcal{S}_{\Delta_x}$. Set $\hat{x}_i = \|x_i\|^{-1}x_i$ and $A = [\hat{x}_1 \dots \hat{x}_n]$, and note that $\mathcal{T}$ is spanned by the operators DA^*AD' for $D = \text{diag}(d_1, \dots, d_n)$ and $D' = \text{diag}(d'_1, \dots, d'_n)$ in Δ . For $i, j \in [n]$, we have

$$\begin{aligned} \sum_{\substack{D, D' \in \Delta, \\ d_i = d'_i = 1}} DA^*AD' &= \left(\sum_{\substack{D \in \Delta, \\ d_i = 1}} D \right) A^* A \left(\sum_{\substack{D' \in \Delta, \\ d'_j = 1}} D' \right) \\ &= 4^{n-1} E_{i,i} A^* A E_{j,j} \\ &= 4^{n-1} \langle \hat{x}_j, \hat{x}_i \rangle E_{i,j}. \end{aligned}$$

If $E_{i,j} \in \mathcal{S}$, then $i \simeq_{G(x)} j$, so $\langle \hat{x}_j, \hat{x}_i \rangle \neq 0$, hence $E_{i,j} \in \mathcal{T}$. Thus $\mathcal{S} \subseteq \mathcal{T}$. On the other hand, $\mathcal{S}^\perp$ is spanned by the matrix units $E_{i,j}$ with $i \not\simeq_{G(x)} j$. For such i, j and any $D, D' \in \Delta$, we have $(DA^*AD')_{ji} = d_j \langle \hat{x}_i, \hat{x}_j \rangle d'_i = 0$, so $E_{i,j} \in \mathcal{T}^\perp$. Hence $\mathcal{S}^\perp \subseteq \mathcal{T}^\perp$ as required. $\square$

Theorem IV.10. For any graph G with vertex set $[n]$ and $\pi \in \{\alpha, \beta, \gamma, \text{int}\}$, we have $\pi(\mathcal{S}_G) = \pi(G)$.

Proof. The case $\pi = \alpha$ is known [DSW13].

We next consider the case $\pi = \gamma$. If $k = \gamma(G)$, then there exists $x = (x_1, \dots, x_n)$, where $x_i \in \mathbb{C}^k \setminus \{0\}$, $i = 1, \dots, n$, so that $G(x) = G$, and hence $\mathcal{S}_{G(x)} = \mathcal{S}_G$. By Proposition IV.9, $\mathcal{S}_{G(x)}$ is the operator system of a quantum channel $M_n \rightarrow M_k$, hence $\gamma(\mathcal{S}_G) \leq k = \gamma(G)$.

Now let $k = \gamma(\mathcal{S}_G)$, so that there are Kraus operators $A_1, \dots, A_m \in M_{k,n}$ for a quantum channel $\Phi : M_n \rightarrow M_k$ with $\mathcal{S}_\Phi = \text{span}\{A_q^*A_p : p, q \in [m]\} = \mathcal{S}_G$. Since the column operator with entries $A_1, \dots, A_m$ is an isometry, for each $i \in [n]$ we have $\sum_{p=1}^m \|A_p e_i\|^2 = \|e_i\|^2 = 1$. In particular, $A_p e_i \neq 0$ for at least one $p \in [m]$. Thus, the projection $P_i \in M_k$ onto the span of $\{A_p e_i : p \in [m]\}$ is non-zero.

Consider the tuple $P = (P_1, \dots, P_n) \in \mathcal{P}(k)$. Since $\mathcal{S}_\Phi = \mathcal{S}_G$, for $i, j \in [n]$ we have

$$\begin{aligned} i \not\simeq_{G(P)} j &\iff P_i P_j = 0 \\ &\iff \langle A_p e_i, A_q e_j \rangle = \text{tr}(E_{i,j} A_q^* A_p) = 0, \\ &\quad \text{for all } p, q \in [m] \\ &\iff E_{j,i} \in \mathcal{S}_\Phi^\perp = \mathcal{S}_G^\perp \iff i \not\simeq_G j. \end{aligned}$$

Hence $G(P) = G$. Using Theorem III.3, we obtain $\gamma(G) = \text{qint}(G) \leq k = \gamma(\mathcal{S}_G)$.

The case $\pi = \beta$ is similar to the case $\pi = \gamma$; alternatively, see [Sta16, Theorem 12].

Let $\pi = \text{int}$. Set $k = \text{int}(G)$; then there exists a tuple $x = (x_1, \dots, x_n) \in (\mathbb{R}_+^k \setminus \{0\})^n$ with $G(x) = G$. By

Proposition IV.9, $\mathcal{S}_{\Delta_x} = \mathcal{S}_{G(x)} = \mathcal{S}_G$. By Lemma IV.8, Δ_x is a non-cancelling quantum channel; therefore, $\text{int}(\mathcal{S}_G) \leq k$. In particular, $\text{int}(\mathcal{S}_G) < \infty$. Now if $k = \text{int}(\mathcal{S}_G)$, let $\Phi: M_n \rightarrow M_k$ be a non-cancelling quantum channel such that $\mathcal{S}_\Phi = \mathcal{S}_G$. Fix Kraus operators $A_1 D_1, \dots, A_m D_m$ for Φ where each $A_p \in M_{k,n}$ is entrywise non-negative and each D_p is an invertible diagonal $n \times n$ matrix. For $i \in [n]$, define

$$R_i = \{r \in [k]: \langle A_p e_i, e_r \rangle \neq 0 \text{ for some } p \in [m]\}.$$

Since the column operator with entries $A_1 D_1, \dots, A_m D_m$ is an isometry, we have $\sum_{p=1}^m \|A_p D_p e_i\|^2 = 1$, and so R_i is non-empty for all $i \in [n]$. Let $x = (x_1, \dots, x_n)$ where $x_i = \sum_{r \in R_i} e_r$. For $i, j \in [n]$, we have

$$\begin{aligned} i \simeq_G j &\iff E_{j,j} \mathcal{S}_G E_{i,i} \neq \{0\} \\ &\iff \exists p, q \in [m] \text{ such that } \langle A_p D_p e_i, A_q D_q e_j \rangle \neq 0 \\ &\iff \exists p, q \in [m] \text{ such that } \langle A_p e_i, A_q e_j \rangle \neq 0. \end{aligned}$$

Now

$$\langle A_p e_i, A_q e_j \rangle = \sum_{r \in [k]} \langle A_p e_i, e_r \rangle \langle e_r, A_q e_j \rangle$$

and every term in the latter sum is non-negative. It follows that

$$\begin{aligned} i \simeq_G j &\iff \exists p, q \in [m], r \in [k] \text{ with } \langle A_p e_i, e_r \rangle \neq 0 \\ &\quad \text{and } \langle A_q e_j, e_r \rangle \neq 0 \\ &\iff R_i \cap R_j \neq \emptyset \iff \langle x_j, x_i \rangle \neq 0 \\ &\iff i \simeq_{G(x)} j. \end{aligned}$$

Hence $G = G(x)$, so $\text{int}(G) \leq k = \text{int}(\mathcal{S}_G)$. $\square$

Remark IV.11. Let $(p(y|x))$ be a $k \times n$ non-negative column-stochastic matrix defining a classical channel $\mathcal{N}: [n] \rightarrow [k]$ with confusability graph $G = G_{\mathcal{N}}$. The canonical quantum channel $\mathcal{N}_q: M_n \rightarrow M_k$ associated with $\mathcal{N}$ is defined by setting $\mathcal{N}_q(E_{x,x}) = \sum_{y \in [k]} p(y|x) E_{y,y}$ and $\mathcal{N}_q(E_{x,x'}) = 0$ if $x \neq x'$. We have that $\mathcal{S}_{\mathcal{N}_q} = \mathcal{S}_G$ [DSW13] (see also [Pau16]). So we see that $\gamma(G) = \gamma(\mathcal{S}_G)$ is the quantum complexity of the classical channel $\mathcal{N}$ when viewed as a quantum channel.

Let $G \boxtimes H$ denote the *strong product* of the graphs G and H [Sa60], in which $(x, y) \simeq_{G \boxtimes H} (x', y')$ if and only if $x \simeq_G x'$ and $y \simeq_H y'$. Note that $\mathcal{S}_{G \boxtimes H} = \mathcal{S}_G \otimes \mathcal{S}_H$. If G, H are graphs and n is the number of vertices of G , then

- (i) $\alpha(G) = 1$ if and only if $G = K_n$, i.e., if and only if $\mathcal{S}_G = M_n$;
- (ii) $\gamma(G) \leq n$; and
- (iii) $\gamma(G \boxtimes H) \leq \gamma(G)\gamma(H)$, but it is unknown whether strict inequality can occur.

The following proposition shows that the parameters for general operator systems $\mathcal{S} \subseteq M_n$ behave quite differently, with respect to the latter properties, than their graph theoretic counterparts.

Proposition IV.12. *Let $\mathcal{S} = \left\{ \begin{bmatrix} \lambda & a \\ b & \lambda \end{bmatrix} : a, b, \lambda \in \mathbb{C} \right\}$. Then $\alpha(\mathcal{S}) = 1$, $\beta(\mathcal{S}) = 2$, $\gamma(\mathcal{S}) = \text{int}(\mathcal{S}) = 3$ and $\gamma(\mathcal{S} \otimes \mathcal{S}) < \gamma(\mathcal{S})^2$.*

Proof. The Kraus operators $A_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ yield a non-cancelling quantum channel with operator system $\mathcal{S}$, so $\gamma(\mathcal{S}) \leq \text{int}(\mathcal{S}) \leq 3$. Since $\mathcal{S}^\perp$ is spanned by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, it contains no rank one operators, and hence $\alpha(\mathcal{S}) = 1$. By Remark IV.3 (v), $\beta(\mathcal{S}) \leq 2$ while, by Remark IV.3 (iii), $\beta(\mathcal{S}) \neq 1$; thus, $\beta(\mathcal{S}) = 2$.

If $\gamma(\mathcal{S}) \leq 2$, then there are $A_j = [v_j \ w_j] \in M_2$ for some $v_j, w_j \in \mathbb{C}^2$, $j = 1, \dots, m$ which are the Kraus operators of a quantum channel with operator system $\mathcal{S}$. We have $\langle v_j, v_i \rangle = (A_i^* A_j)_{11} = (A_i^* A_j)_{22} = \langle w_j, w_i \rangle$ for each i, j . It follows that there is a 2×2 unitary U with $U v_j = w_j$, $j = 1, \dots, m$. Recall that $\sum_{i=1}^m A_i^* A_i = I_2$. This is equivalent to

$$\sum_{i=1}^m \|v_i\|^2 = \sum_{i=1}^m \|w_i\|^2 = 1$$

and

$$\sum_{i=1}^m \langle w_i, v_i \rangle = \sum_{i=1}^m \langle U v_i, v_i \rangle = 0.$$

In particular, 0 lies in the numerical range of U . However, the numerical range of a normal matrix is the convex hull of its spectrum, and hence $\sigma(U) = \{\alpha, -\alpha\}$ for some $\alpha \in \mathbb{T}$. Thus $\bar{\alpha}U$ is hermitian, and so $U = \alpha^2 U^*$. Now

$$\begin{aligned} A_i^* A_j &= \begin{pmatrix} \langle v_j, v_i \rangle & \langle U v_j, v_i \rangle \\ \langle U^* v_j, v_i \rangle & \langle U v_j, U v_i \rangle \end{pmatrix} \\ &= \langle v_j, v_i \rangle I + \langle U^* v_j, v_i \rangle \begin{pmatrix} 0 & \alpha^2 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

hence $3 = \dim \mathcal{S} = \dim \text{span}\{A_i^* A_j\} \leq 2$, a contradiction.

To see that $\gamma(\mathcal{S} \otimes \mathcal{S}) < \gamma(\mathcal{S})^2 = 9$, consider the isometries $V_i \in M_{8,4}$ given by

$$\begin{aligned} V_1 &= [e_1 \ e_2 \ e_3 \ e_4], & V_2 &= [e_2 \ e_5 \ e_4 \ e_6], \\ V_3 &= [e_3 \ e_4 \ e_7 \ e_8], & V_4 &= [e_6 \ e_7 \ e_5 \ e_1]. \end{aligned}$$

Let $W = \frac{1}{2}[V_1 \ V_2 \ V_3 \ V_4] \in M_{8,16}$; then

$$W^* W = \frac{1}{4} \begin{bmatrix} I_4 & \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} & I_4 & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & I_4 & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & I_4 \end{bmatrix}$$

Writing $B_{i,j}$ for the (i, j) -th 4×4 block of $W^* W$, we observe that

$$\text{span}\{B_{i,j}: i, j \in [4]\} = \mathcal{S} \otimes \mathcal{S} \quad \text{and} \quad \sum_{i=1}^4 B_{i,i} = I_4.$$

By Proposition IV.7,

$$\gamma(\mathcal{S} \otimes \mathcal{S}) \leq \text{rank } W^* W = \text{rank } W = 8. \quad \square$$

V. APPLICATIONS TO CAPACITY

Let $\mathcal{N} : X \rightarrow Y$ be a classical information channel with confusability graph G . Its parallel use r times can be expressed as a channel $\mathcal{N}^{\otimes r} : X^r \rightarrow Y^r$, for which

$$p((y_s)_{s=1}^r | (x_s)_{s=1}^r) = \prod_{s=1}^r p(y_s | x_s),$$

for $x_s \in X$, $y_s \in Y$ and $s = 1, \dots, r$. Note that

$$G_{\mathcal{N}^{\otimes r}} = \underbrace{G_{\mathcal{N}} \boxtimes \cdots \boxtimes G_{\mathcal{N}}}_{r \text{ times}}.$$

The *Shannon capacity* of the channel $\mathcal{N} : X \rightarrow Y$ (or equivalently of the graph G) is the quantity

$$\Theta(\mathcal{N}) = \Theta(G) = \lim_{r \rightarrow \infty} \sqrt[r]{\alpha(\mathcal{N}^{\otimes r})} = \lim_{r \rightarrow \infty} \sqrt[r]{\alpha(G_{\mathcal{N}}^{\boxtimes r})}.$$

(Some authors prefer to use the logarithm of the quantities defined above.)

Similarly, if $\Phi : M_n \rightarrow M_k$ is a quantum channel, letting $\Phi^{\otimes r} : M_n^{\otimes r} \rightarrow M_k^{\otimes r}$ be its r -th power, we find

$$\mathcal{S}_{\Phi^{\otimes r}} = \underbrace{\mathcal{S}_{\Phi} \otimes \cdots \otimes \mathcal{S}_{\Phi}}_{r \text{ times}}.$$

The analogue of the Shannon capacity of a quantum channel introduced in [DSW13] is the parameter

$$\Theta(\Phi) = \lim_{r \rightarrow \infty} \sqrt[r]{\alpha(\Phi^{\otimes r})}.$$

Lovász [Lo79] introduced his famous ϑ -parameter of a graph and proved that $\alpha(\mathcal{N}) \leq \vartheta(G)$ and that ϑ is multiplicative for strong graph product; hence,

$$\Theta(\mathcal{N}) = \Theta(G_{\mathcal{N}}) \leq \vartheta(G),$$

for any classical channel, thus giving a bound on the Shannon capacity of classical channels. He also proved [Lo79, Theorem 11] that

$$\vartheta(G) \leq \beta(G),$$

so that his ϑ -bound is a better bound on the capacity of classical channels than any of the bounds that we derived from complexity considerations. However, as we will shortly show, for quantum channels, β yields a bound on capacity that can outperform ϑ . We note that a different bound on $\Theta(G)$, based on ranks of Hermitian matrices in the operator system $\mathcal{S}_{G^c}$, was introduced by Haemers in [Hae81]. It is an interesting open question to formulate general non-commutative analogues of Haemers' parameter.

Lovász gave many characterisations of his parameter, but the most useful for our purposes is the expression

$$\vartheta(G) = \max \{ \|I + K\| : I + K \in M_n^+, K \in \mathcal{S}_G^\perp \}.$$

The latter formula motivated [DSW13] to define, for any operator subsystem $\mathcal{S}$ of M_n ,

$$\vartheta(\mathcal{S}) = \max \{ \|I + K\| : I + K \in M_n^+, K \in \mathcal{S}^\perp \};$$

note that $\vartheta(G) = \vartheta(\mathcal{S}_G)$. It was shown in [DSW13] that, for any quantum channel Φ , one has

$$\alpha(\Phi) = \alpha(\mathcal{S}_\Phi) \leq \vartheta(\mathcal{S}_\Phi).$$

However, ϑ is only supermultiplicative for tensor products of general operator systems. This motivated [DSW13] to introduce a “complete” version, denoted $\tilde{\vartheta}$, which is multiplicative for tensor products of operator systems and satisfies $\vartheta(\mathcal{S}) \leq \tilde{\vartheta}(\mathcal{S})$. This allowed them to bound the quantum capacity of a quantum channel, since

$$\begin{aligned} \Theta(\Phi) &= \lim_{r \rightarrow \infty} \sqrt[r]{\alpha(\mathcal{S}_\Phi^{\otimes r})} \leq \lim_{r \rightarrow \infty} \sqrt[r]{\vartheta(\mathcal{S}_\Phi^{\otimes r})} \\ &\leq \lim_{r \rightarrow \infty} \sqrt[r]{\tilde{\vartheta}(\mathcal{S}_\Phi^{\otimes r})} = \tilde{\vartheta}(\mathcal{S}_\Phi). \end{aligned}$$

These bounds are often difficult to compute. The quantity $\lim_{r \rightarrow \infty} \sqrt[r]{\vartheta(\mathcal{S}_\Phi^{\otimes r})}$ requires evaluation of a limit, each term of which may be intractable, and the possibly larger bound $\tilde{\vartheta}(\mathcal{S}_\Phi) = \sup_{n \in \mathbb{N}} \vartheta(\mathcal{S}_\Phi \otimes M_n)$ requires the evaluation of a supremum, although this parameter has the advantage of possessing a reformulation as a semidefinite program [DSW13].

Theorem V.1. *For any quantum channel Φ , we have*

$$\alpha(\Phi) \leq \Theta(\Phi) \leq \beta(\mathcal{S}_\Phi).$$

Proof. Let Φ be a quantum channel and $\mathcal{S} = \mathcal{S}_\Phi$. The inequality $\alpha \leq \Theta$ is well known, and follows immediately from the supermultiplicative property of α . Since β is submultiplicative for tensor products (Proposition IV.5 (iii)) and α is dominated by β (Theorem IV.4), we have

$$\Theta(\Phi) = \Theta(\mathcal{S}) = \lim_{r \rightarrow \infty} \sqrt[r]{\alpha(\mathcal{S}^{\otimes r})} \leq \beta(\mathcal{S}). \quad \square$$

In the remainder of the section, we will exhibit operator systems for which $\beta(\mathcal{S}) \ll \vartheta(\mathcal{S})$. For $k \in \mathbb{N}$, let

$$\mathcal{S}_k = \{(a_{i,j})_{i,j=1}^k \in M_k : a_{1,1} = a_{2,2} = \cdots = a_{k,k}\}.$$

It is easy to show directly that

$$\vartheta(\mathcal{S}_k) = k.$$

For any $m \in \mathbb{N}$, applying the canonical shuffle which identifies $M_k \otimes M_m$ with $M_m \otimes M_k$, we have

$$\begin{aligned} \mathcal{S}_k \otimes M_m &= \{(A_{i,j})_{i,j=1}^k \in M_k(M_m) : \\ &\quad A_{1,1} = A_{2,2} = \cdots = A_{k,k}\}. \end{aligned}$$

Thus, for any operator system $\mathcal{S} \subseteq M_m$, we have

$$\begin{aligned} \mathcal{S}_k \otimes \mathcal{S} &= \{(A_{i,j})_{i,j=1}^k \in M_k(\mathcal{S}) : \\ &\quad A_{1,1} = A_{2,2} = \cdots = A_{k,k}\}. \end{aligned}$$

Theorem V.2. *We have*

$$\beta(\mathcal{S}_k \otimes \mathcal{S}_{k^2}) \leq k^2 < k^3 \leq \vartheta(\mathcal{S}_k \otimes \mathcal{S}_{k^2}).$$

Proof. Let ω be a primitive k -th root of unity. Let $\mathcal{S} \in M_{k^2}$ be given by $Se_i = e_{i+1}$, $i = 1, \dots, k^2$, where addition is modulo k^2 , while $D \in M_{k^2}$ be the diagonal matrix with diagonal $(1, \omega, \omega^2, \dots, \omega^{k^2-1})$. Note that, if $D_0 \in M_k$ is the diagonal matrix with diagonal $(1, \omega, \omega^2, \dots, \omega^{k-1})$, then $D = \underbrace{D_0 \oplus \cdots \oplus D_0}_{k \text{ times}}$. We have that $D^j \mathcal{S} = \omega^j \mathcal{S} D^j$ for any $j \in \mathbb{Z}$, and hence

$$D^j \mathcal{S}^i = \omega^{ij} \mathcal{S}^i D^j, \quad i, j \in \mathbb{Z}. \quad (5)$$

Any element A of $M_k(\mathcal{S}_k \otimes \mathcal{S}_{k^2})$ has the form $A = (C_{r,s})_{r,s=0}^{k-1}$, where $C_{r,s} \in \mathcal{S}_k \otimes \mathcal{S}_{k^2}$ for all $r, s = 0, \dots, k-1$. In view of the remarks before the statement of the theorem, we may write $C_{r,s} = (A_{kr+i,ks+j})_{i,j=0}^{k-1}$, where $A_{kr+i,ks+j} \in \mathcal{S}_{k^2}$ for all $r, s, i, j = 0, \dots, k-1$, and

$$A_{kr,ks} = A_{kr+1,ks+1} = A_{kr+2,ks+2} = \dots = A_{kr+k-1,ks+k-1},$$

for all $r, s = 0, \dots, k$.

Let

$$u_{kr+i} = S^{kr+i} D^r, \quad r, i = 0, \dots, k-1,$$

and

$$B = (u_0, u_1, u_2, \dots, u_{k^2-1}) \in M_{k^2, k^4}.$$

Set $B_{r,s} = (u_{kr+i}^* u_{ks+j})_{i,j=0}^{k-1}$; then the matrix

$$B^* B = (B_{r,s})_{r,s=0}^{k-1} = (u_{kr+i}^* u_{ks+j})_{r,s,i,j}$$

is positive and has rank at most k^2 . We will show the following:

- (i) $u_{kr+i}^* u_{ks+j} \in \mathcal{S}_{k^2}$, for all $r, s, i, j = 0, 1, \dots, k-1$;
- (ii) $u_{kr}^* u_{ks} = u_{kr+1}^* u_{ks+1} = u_{kr+2}^* u_{ks+2} = \dots = u_{kr+k-1}^* u_{ks+k-1}$, for all $r, s = 0, \dots, k-1$, and
- (iii) $\sum_{r=0}^{k-1} B_{r,r} = kI$,

which will imply that $\beta(\mathcal{S}_k \otimes \mathcal{S}_{k^2}) \leq k^2$.

To show (i), note that

$$u_{kr+i}^* u_{ks+j} = D^{-r} S^{-kr-i} S^{ks+j} D^s = D^{-r} S^{ks-kr+j-i} D^s.$$

If $ks - kr + j - i \neq 0$, then $u_{kr+i}^* u_{ks+j}$ has zero diagonal and thus belongs to $\mathcal{S}_{k^2}$. Suppose that $ks - kr + j - i = 0$. Then $k|(i - j)$ and hence $i = j$. If, in addition, $r \neq s$ then $u_{kr+i}^* u_{ks+j}$ has zero diagonal and therefore belongs to $\mathcal{S}_{k^2}$; if, on the other hand, $r = s$, then $u_{kr+i}^* u_{ks+j} = I$ and hence again belongs to $\mathcal{S}_{k^2}$.

To show (ii), note that for $i = 0, \dots, k-1$, we have

$$u_{kr+i}^* u_{ks+i} = D^{-r} S^{-kr-i} S^{ks+i} D^s = D^{-r} S^{k(s-r)} D^s.$$

In order to show (iii), suppose that $i, j \in \{0, \dots, k-1\}$ with $i \neq j$, and, using (5), note that

$$\begin{aligned} \sum_{r=0}^{k-1} u_{kr+i}^* u_{kr+j} &= \sum_{r=0}^{k-1} D^{-r} S^{-kr-i} S^{kr+j} D^r \\ &= \sum_{r=0}^{k-1} D^{-r} S^{j-i} D^r \\ &= \left(\sum_{r=0}^{k-1} \omega^{-r(j-i)} \right) S^{j-i}. \end{aligned}$$

Since ω is a primitive root of unity, so is ω^{-1} . Thus, $\omega^{-(j-i)}$ is a k -th root of unity with $\omega^{-(j-i)} \neq 1$. It follows that $\sum_{r=0}^{k-1} \omega^{-r(j-i)} = 0$. Thus, $\sum_{r=0}^{k-1} u_{kr+i}^* u_{kr+j} = 0$ whenever $i \neq j$. On the other hand,

$$\sum_{r=0}^{k-1} u_{kr+i}^* u_{kr+i} = \sum_{r=0}^{k-1} I = kI,$$

and (iii) is proved.

By [DSW13, Lemma 4],

$$\vartheta(\mathcal{S}_k \otimes \mathcal{S}_{k^2}) \geq \vartheta(\mathcal{S}_k) \vartheta(\mathcal{S}_{k^2}) = k^3,$$

and the proof is complete. $\square$

Corollary V.3. (i) *The ratios*

$$\vartheta(\mathcal{S})/\beta(\mathcal{S}) \text{ and } \vartheta(\mathcal{S})/\gamma(\mathcal{S})$$

can be arbitrarily large, as $\mathcal{S}$ varies over all non-commutative graphs.

- (ii) *For $\pi \in \{\beta, \gamma\}$, the ratio $\vartheta(\mathcal{S})/\pi(\mathcal{S})$ can be arbitrarily large, as $\mathcal{S}$ varies over all non-commutative graphs with $\frac{1}{2}\pi(\mathcal{S}) \leq \Theta(\mathcal{S}) \leq \pi(\mathcal{S})$.*

Moreover, these statements hold if throughout we replace ϑ by $\tilde{\vartheta}$, the quantum Lovász theta number.

Proof. Since $\vartheta \leq \tilde{\vartheta}$, it suffices to prove these statements for ϑ .

(i) For the first ratio, consider $\mathcal{S} = \mathcal{T}_\beta := \mathcal{S}_k \otimes \mathcal{S}_{k^2}$ and apply Theorem V.2. For the second ratio, let $\mathcal{T}_\gamma$ be the span of the matrices $B_{r,s} \in \mathcal{S}_k \otimes \mathcal{S}_{k^2}$ appearing in the proof of Theorem V.2. Then $\mathcal{T}_\gamma \subseteq \mathcal{S}_k \otimes \mathcal{S}_{k^2}$ is an operator system and the set $\{B_{r,s}\}$ is one of the terms that appear in the minimum that defines $\gamma(\mathcal{T}_\gamma)$. Hence,

$$\gamma(\mathcal{T}_\gamma) \leq \text{rank}((B_{r,s})) \leq k^2 < k^3 \leq \vartheta(\mathcal{S}_k \otimes \mathcal{S}_{k^2}) \leq \vartheta(\mathcal{T}_\gamma).$$

(ii) For $\pi \in \{\beta, \gamma\}$, consider $\mathcal{R}_\pi := \mathcal{T}_\pi \oplus \mathbb{C}I_{k^2} \subseteq M_{k^3+k^2}$. For $k > 3$, by Proposition IV.5 (v) and Remark IV.3 (iv), we have

$$\begin{aligned} \pi(\mathcal{R}_\pi) &= \pi(\mathcal{T}_\pi) + \pi(\mathbb{C}I_{k^2}) \leq k^2 + k^2 = 2k^2 = 2\alpha(\mathbb{C}I_{k^2}) \\ &\leq 2\alpha(\mathcal{R}_\pi) \leq 2\Theta(\mathcal{R}_\pi). \end{aligned}$$

Since ϑ is order-reversing for inclusion of operator systems and $\mathcal{R}_\pi \subseteq (\mathcal{S}_k \otimes \mathcal{S}_{k^2}) \oplus \mathbb{C}I_{k^2} \subseteq (\mathcal{S}_k \oplus \mathbb{C}) \otimes \mathcal{S}_{k^2}$ (the second inclusion holds up to a unitary shuffle equivalence) and it is easy to see that $\vartheta(\mathcal{S} \oplus \mathcal{T}) \geq \vartheta(\mathcal{S})$ for any operator systems $\mathcal{S}$ and $\mathcal{T}$, we have

$$\vartheta(\mathcal{R}_\pi) \geq \vartheta(\mathcal{S}_k \oplus \mathbb{C}) \vartheta(\mathcal{S}_{k^2}) \geq k^3. \quad \square$$

APPENDIX

In this appendix, we briefly summarise the order relationships between various bounds on the quantum Shannon zero-error capacity. These may be succinctly described by the directed graph in Figure 1. The parameters α_q , χ_q and χ are defined in [DSW13], and [Sta16, Definition 11]; the reader should swap $\mathcal{S}$ and $\mathcal{S}^\perp$ when translating between our non-commutative graphs and Stahlke's "trace-free non-commutative graphs".

Let $\mathcal{S} \subseteq M_n$ be an operator system. As observed in Remark IV.3 (ii), we have $\chi_q(\mathcal{S}^\perp) = \beta(\mathcal{S})$. The first two inequalities in the chain

$$\alpha_q(\mathcal{S}) \leq \sqrt{\tilde{\vartheta}(\mathcal{S})} \leq \chi_q(\mathcal{S}^\perp) \leq \chi(\mathcal{S}^\perp)$$

appear in [Sta16], following Corollary 20, and the third inequality is a simple consequence of his Proposition 9. The inequality $\alpha_q(\mathcal{S}) \leq \alpha(\mathcal{S})$ is immediate from the definitions (and appears in [DSW13, Proposition 2]), and we have seen in Theorems IV.4 and V.1 that $\alpha \leq \Theta \leq \beta \leq \gamma \leq \text{int}$.

Fig. 1. A directed graph showing the partial order among various parameters bounding the quantum Shannon zero-error capacity $\Theta(S)$ of a non-commutative graph S . The ordering $\pi_1(S) \leq \pi_2(S)$ for every operator system $S \subseteq M_n$ is indicated by placing $\pi_1(S)$ below $\pi_2(S)$, joined with a path directed towards $\pi_1(S)$; the absence of a directed path between a pair of vertices indicates that the corresponding parameters are incomparable.

The inequality $\Theta \leq \tilde{\vartheta}$ follows immediately from [DSW13, Proposition 2 and Corollary 10], noting that in the notation of that paper, $\log_2 \Theta = C_0 \leq C_{0E}$; and $\sqrt{\tilde{\vartheta}} \leq \tilde{\vartheta}$ is trivial.

It only remains to prove the incomparability assertions of Figure 1. These follow from the inequalities already established and the examples below.

- Let $G = C_5$ be the 5-cycle, and let $S = S_G$. Lovász has shown [Lo79] that $\vartheta(G) = \sqrt{5}$ while, for graph operator systems, as pointed out in [DSW13], we have $\tilde{\vartheta}(S_G) = \vartheta(G)$. It is not difficult to see that

$$\alpha(S_G) = \alpha(G) = 2 < \beta(G) = \beta(S_G).$$

So, in this example,

$$\sqrt{\tilde{\vartheta}(S)} < \alpha(S) \quad \text{and} \quad \tilde{\vartheta}(S) < \beta(S).$$

- Consider $G = C_6^c$, the complement of the 6-cycle, and $S = S_G$. It is easy to see directly that $\gamma(S) = \gamma(G) > 2$, and $\chi(S^\perp) = \chi(C_6) = 2$, so in this case,

$$\chi(S^\perp) < \gamma(S).$$

- Let S be the operator system of Proposition IV.12 (i.e., in the notation of Section V, $S = S_2$). Note that $\alpha(S) = 1$. We claim that if $\mathcal{T}$ is any operator system with $\alpha(\mathcal{T}) = 1$, then $\alpha(S \otimes \mathcal{T}) = 1$. Indeed, $S \otimes \mathcal{T}$ may be identified with all 2×2 block matrices of the form $\begin{bmatrix} T & A \\ B & T \end{bmatrix}$ for $T, A, B \in \mathcal{T}$, and if x, y are non-zero vectors with $xy^* \in (S \otimes \mathcal{T})^\perp$, then writing $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we obtain $xy^* = (x_i y_j^*)_{i,j=1,2} \in (S \otimes \mathcal{T})^\perp$. By considering the off-diagonal entries and the condition $\alpha(\mathcal{T}) = 1$, it readily follows that $x_1 = 0$ or $y_2 = 0$, and $x_2 = 0$ or $y_1 = 0$.

If $x_1 = 0$, then $y_1 = 0$; hence, $xy^* = 0 \oplus x_2 y_2^*$, so $x_2 y_2^* \in \mathcal{T}^\perp$, so $x = y = 0$, a contradiction. The other case proceeds to a similar contradiction, so $\alpha(S \otimes \mathcal{T}) = 1$. Hence, in particular, $\Theta(S) = 1$. On the other hand, $\tilde{\vartheta}(S) = 2$ by [DSW13, p. 1172]; thus, in this case we have

$$\Theta(S) < \sqrt{\tilde{\vartheta}(S)}.$$

- Finally, let $S = \mathbb{C}I_2$ to obtain an example for which

$$\text{int}(S) < \tilde{\vartheta}(S),$$

since the left hand side is 2 by Remark IV.3 (iv), and, as observed in [DSW13], the right hand side is 4.

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