Complexity and capacity bounds for quantum channels

https://doi.org/10.1109/TIT.2018.2833466

Published in:
IEEE Transactions on Information Theory

Document Version:
Peer reviewed version

Queen's University Belfast - Research Portal:
Link to publication record in Queen's University Belfast Research Portal

Publisher rights
© 2018 IEEE
This work is made available online in accordance with the publisher’s policies. Please refer to any applicable terms of use of the publisher.

General rights
Copyright for the publications made accessible via the Queen's University Belfast Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy
The Research Portal is Queen's institutional repository that provides access to Queen's research output. Every effort has been made to ensure that content in the Research Portal does not infringe any person’s rights, or applicable UK laws. If you discover content in the Research Portal that you believe breaches copyright or violates any law, please contact openaccess@qub.ac.uk.
Abstract—We generalise some well-known graph parameters to operator systems by considering their underlying quantum channels. In particular, we introduce the quantum complexity as the dimension of the smallest co-domain Hilbert space a quantum channel requires to realise a given operator system as its non-commutative confusability graph. We describe quantum complexity as a generalised minimum semi-definite rank and, in the case of a graph operator system, as a quantum intersection number. The quantum complexity and a closely related quantum version of orthogonal rank turn out to be upper bounds for the Shannon zero-error capacity of a quantum channel, and we construct examples for which these bounds beat the best previously known general upper bound for the capacity of quantum channels, given by the quantum Lovász theta number.

Thus, we are lead to define the quantum complexity $\gamma(S)$ of an operator subsystem $S$ of $M_n$ as the least positive integer $k$ for which there exists a quantum channel $\Phi : M_n \to M_k$ such that $S = S_k$.

The goal of this paper is to study this and other, closely related, measures of complexity and to derive their relationships with various measures of capacity for classical and quantum channels. We will show, in particular, that the measures of complexity we introduce give upper bounds on the zero-error capacity of a quantum channel.

One of the most useful general bounds on the Shannon capacity of a classical channel comes from $\vartheta$, the Lovász theta function $[Lo79]$. While, for classical channels, the complexity based bound is outperformed by the Lovász number (see $[Lo79$, Theorem 11]), we will show that there exist quantum channels for which the quantum complexity bound on capacity we suggest is better than the bounds arising from the non-commutative analogue $\vartheta$ of the Lovász number introduced in $[DSW13]$. In fact, we will show that there exist quantum channels $\Phi_k$ for $k \in \mathbb{N}$ for which the ratio of the quantum Lovász theta number $\vartheta(\Phi_k)$ to the quantum complexity $\gamma(\Phi_k)$ introduced herein is arbitrarily large, while the upper bound $\gamma(\Phi_k)$ for the quantum Shannon zero-error capacity $\Theta(\Phi_k)$ is accurate to within a factor of two (see Corollary V.3).

We will see that the classical complexity of a graph $G$ is a familiar parameter which coincides with its intersection number (provided $G$ lacks isolated vertices). For operator systems, the measure of quantum complexity we propose has not been previously studied. We will characterise it in several different ways. Since every graph $G$ gives rise to a canonical operator system $S_G$, it can in addition be endowed with a quantum complexity, which can be strictly smaller than its classical counterpart, and can be equivalently characterised as a quantum intersection number of $G$.

The paper is organised as follows. In Section II, we begin by recalling the graph theoretic parameters needed in the sequel and show that our measure of the classical complexity of a graph coincides with its intersection number. In Section III, we turn to the quantum complexity of a graph, and show that it coincides with its minimum semi-definite rank (modulo any isolated vertices). In Section IV, we achieve a parallel development for operator systems, considering simultaneously a closely related notion of subcomplexity that coincides with
the quantum chromatic number introduced in [Sta16]. We show that our operator system parameters are genuine exten-
sions of the graph theoretic ones (Theorem IV.10) and explore similarities and differences between their behaviour
on commutative and non-commutative graphs. In Section V, we establish the bounds on capacities in terms of complexities
(Theorem V.1) and show by example that these bounds can improve dramatically on the Lovász $\vartheta$ bound. Finally, in
Appendix A we establish the partial ordering among various bounds on the quantum Shannon zero-error capacity, from this
paper and elsewhere.

In the sequel, we employ standard notation from linear algebra: we denote by $M_{k,n}$ the space of all $k$ by $n$ matrices
with complex entries, and set $M_n = M_{1,n}$. We let $\|X\|$ be the operator norm of a matrix $X \in M_{k,n}$, so that $\|X\|_2$ is
the largest eigenvalue of $X^*X$. We equip $M_n$ with the inner product given by $\langle X, Y \rangle = \text{tr}(Y^*X)$, where $\text{tr}(Z)$ is the trace of a matrix $X \in M_n$. We write $I_n$ (or simply $I$) for the identity
matrix in $M_n$. The positive cone of $M_n$ (that is, the set of all positive semi-definite $n \times n$ matrices) will be denoted
by $M_n^+$; if $S \subseteq M_n$, we let $S^+ = S \cap M_n^+$. We write $\mathbb{R}_k^+$ for the cone of all vectors in $\mathbb{R}_k$ with non-negative entries, and let $(e_i)_{i=1}^k$ be the standard basis of $\mathbb{C}^k$. If $v, w \in \mathbb{C}^n$, we denote by $vw^*$ the rank one operator on $\mathbb{C}^n$ given by $(vw^*)(z) = \langle z, w \rangle v$, $z \in \mathbb{C}^n$. The cardinality of a set $S$ will be denoted by $|S|$.

II. GRAPH PARAMETERS

In this section, we recall some graph theoretic parameters and point out their connection with Shannon’s confusability
graphs and channel capacities. We start by establishing notation and terminology. Unless otherwise stated, all graphs in
this paper will be simple graphs: undirected graphs without loops and at most one edge between any pair of vertices. Let
$n \in \mathbb{N}$ and let $G$ be a graph with vertex set $[n] := \{1, \ldots, n\}$. For $i, j \in [n]$ we write $i \sim j$ or $i \sim_G j$ to denote non-strict adjacency: either $i = j$, or $G$ contains the edge $ij$. We denote by $G^c$ the complement of the graph $G$; by definition, $G^c$ has vertex set $[n]$ and, for distinct $i, j \in [n]$, we have $i \sim_G j$ if and only if $i \not\sim_G j$. For graphs $H, G$ with vertex set $[n]$, we write $H \subseteq G$, and say that $H$ is a subgraph of $G$, if every edge of $H$ is an edge of $G$. An independent set in $G$ is a subset of its vertices between which there are no edges of $G$. Let $k \in \mathbb{N}$, and for an $n$-tuple $x = (x_1, \ldots, x_n)$, where each $x_i \in \mathbb{C}^k$ is a non-zero vector, we define $G(x)$ to be the non-orthogonality graph of $x$, with vertex set $[n]$ and adjacency relation given by

$$i \sim_{G(x)} j \iff \langle x_j, x_i \rangle \neq 0.$$

Let $G$ be a graph with vertex set $[n]$. Consider the following graph parameters:

(a) the independence number $\alpha(G)$ of $G$, given by

$$\alpha(G) = \max\{|S| : S \text{ is independent in } G\};$$

(b) the quantum complexity

$$\gamma(G) = \min\{k \in \mathbb{N} : G(x) = G \text{ for some } x \in (\mathbb{C}^k \setminus \{0\})^n\}$$

and the quantum subcomplexity

$$\beta(G) = \min\{k \in \mathbb{N} : G(x) \subseteq G \text{ for some } x \in (\mathbb{C}^k \setminus \{0\})^n\} = \min\{\gamma(H) : H \text{ is a subgraph of } G\};$$

(c) the intersection number $\text{int}(G)$ of $G$, given by

$$\text{int}(G) = \min\{k \in \mathbb{N} : G(x) = G \text{ for some } x \in (\mathbb{R}_+^k \setminus \{0\})^n\}.$$
the pair \( xx' \) is an edge if and only if there exists \( y \in Y \) such that 
\[
p(y|x)p(y|x') > 0.
\]
Equivalently, \( x \sim_{G_N} x' \) if and only if there exists \( y \in Y \) such that the symbols \( x \) and \( x' \) can be transformed into the same \( y \) via \( N \) and hence confused.

The one-shot zero-error capacity of \( N \), denoted \( \alpha(N) \), is defined to be the cardinality of the largest subset \( X_1 \) of \( X \) such that, whenever an element of \( X_1 \) is sent via \( N \), no matter which element of \( Y \) is received, the receiver can determine with certainty the input element from \( X_1 \). It is straightforward that \( \alpha(N) = \alpha(G_N) \).

**Definition II.2.** Let \( G \) be a graph with vertex set \([n]\). The complexity \( \text{plex}(G) \) of \( G \) is the minimal cardinality of a set \( Y \) such that \( G = G_N \) for some channel \( N : [n] \to Y \).

If \( N : X \to Y \) is a channel, we set \( \text{plex}(N) = \text{plex}(G_N) \) and call it this parameter the complexity of \( N \).

**Proposition II.3.** Let \( G \) be a graph with vertex set \([n]\). Then
\[
\text{plex}(G) = \text{int}(G).
\]
In other words, if \( N : X \to Y \) is a channel then \( \text{plex}(N) = \text{int}(G_N) \).

**Proof.** Let \( N : [n] \to Y \) be a channel so that \( G = G_N \). For \( 1 \leq i \leq n \) set \( R_i = \{ y \in Y : p(y|i) \neq 0 \} \). Then \( R_i \) is non-empty for each \( i \) and \( i \sim_{G} j \iff R_i \cap R_j \neq \emptyset \). This shows that \( \text{int}(G) \) is a lower bound for the complexity of \( G \).

Conversely, suppose that \( R_1, \ldots, R_n \) are non-empty subsets of \([k]\) such that \( i \sim_{G} j \) if and only if \( R_i \cap R_j \neq \emptyset \). Set
\[
p(y|i) = \begin{cases} 1/|R_i|, & y \in R_i \\ 0, & y \notin R_i \end{cases};
\]
then \( N = (p(y|i)) \) is a channel from \([n]\) to \([k]\) with \( G_N = G \). This shows that \( \text{int}(G) \) is an upper bound for the complexity of \( G \).

**Remark II.4.** (i) We will discuss later (Remark IV.11) the natural way to view a classical channel \( N \) as a quantum channel; we will see that the quantum complexity of \( N \), studied in Section IV, coincides with \( \gamma(G_N) \).

(ii) It is well-known that \( \beta(G) \leq \chi(G^c) \) for any graph \( G \) (here \( \chi(G) \) denotes the chromatic number of a graph \( G \) [GR01]), so it is natural to ask if \( \chi(G^c) \) fits into chain of inequalities (1). In fact, it does not: one can check using a computer program that \( \chi(G^c) \leq \gamma(G) \) for all graphs on 7 or fewer vertices, but this inequality fails in general, for example for \( x \) is a Kochen-Specker set and \( G = G(x) \) (see [HPSWM11, Section 1.2]).

(iii) For each \( \pi \in \{\alpha, \beta, \gamma, \int \} \), we have that \( \pi(G) = 1 \) if and only if \( G \) is a complete graph. Indeed, if \( G \) is a complete graph, then \( \pi(G) \leq \text{int}(G) = 1 \), so \( \pi(G) = 1 \); and if \( G \) is not a complete graph, then \( G^c \) contains at least one edge, so \( \pi(G) \geq \alpha(G) > 1 \).

III. THE QUANTUM INTERSECTION NUMBER

In this section we show that the graph parameter \( \gamma \), discussed in Section II, has a reformulation in terms of projective colourings of the graph \( G \), which leads to a parameter that we call the quantum intersection number of \( G \). This will allow us to make a key step in the proof of Theorem IV.10, where we show that \( \gamma \) has a natural operator system generalisation.

Fix \( n \in \mathbb{N} \). Given \( k \in \mathbb{N} \), we write \( P(k) \) for the set of \( n \)-tuples \( P = (P_1, \ldots, P_n) \) where each \( P_i \) is a non-zero projection in \( M_k \). Let \( P_c(k) \) denote the subset of \( P(k) \) consisting of the elements \( P = (P_1, \ldots, P_n) \) with commuting entries: \( P_iP_j = P_jP_i \) for all \( i, j \). To any \( P \in P(k) \) we associate the non-orthogonality graph \( G(P) \) with vertex set \([n]\) and edges defined by the relation \( i \sim_{G(P)} j \iff P_iP_j \neq 0 \).

We define the quantum intersection number \( \text{qint}(G) \) of a graph \( G \) with vertex set \([n]\) by letting
\[
\text{qint}(G) = \min \{ k \in \mathbb{N} : G(P) = G \text{ for some } P \in P_c(k) \}.
\]

The next proposition explains the choice of terminology.

**Proposition III.1.** Let \( G \) be a graph with vertex set \([n]\). Then
\[
\text{int}(G) = \min \{ k \in \mathbb{N} : G(P) = G \text{ for some } P \in P_c(k) \}.
\]

**Proof.** Let \( l \) be the minimum on the right hand side of (2), and suppose that \( G(x) = G \) for some \( n \)-tuple \( x \) of non-zero vectors in \( \mathbb{R}^k \). Letting \( P_i \) be the orthogonal projection onto the linear span of \( \{ r_i : (x_i, r_i) \neq 0 \} \) yields a tuple \( P = (P_1, \ldots, P_n) \) in \( P_c(k) \) with \( G(P) = G \); thus, \( l \leq \text{int}(G) \).

Conversely, suppose that \( P = (P_1, \ldots, P_n) \in P_c(l) \) is such that \( G(P) = G \). Simultaneously diagonalising the \( P_i \)’s with respect to a basis \( \{ b_r : r \in [l] \} \) and defining \( R_i = \{ r \in [l] : P_i b_r \neq 0 \} \), \( i \in [n] \), we see that \( i \sim_{G} j \) if and only if \( R_i \cap R_j \neq \emptyset \), and it follows that \( \text{int}(G) \leq l \).

Let \( t = (t_1, \ldots, t_n) \in \mathbb{N}^n \) and write \( |t| = \sum_{i=1}^n t_i \).

Extending ideas from [HPRS15], let
\[
P(k, t) = \{ (P_1, \ldots, P_n) \in P(k) : \text{rank } P_i = t_i, \ i \in [n] \}
\]
and define
\[
m^+_t(G) = \min \{ k \in \mathbb{N} : G(P) = G \text{ for some } P \in P(k, t) \}.
\]
Consider each element \( A \in M_n \) as a block matrix \( A = [A_{i,j}]_{i,j \in [n]} \) where \( A_{i,j} \in M_{t_i,t_j} \). We define
\[
\mathcal{F}^+_t(G) = \left\{ A \in M_n : A_{i,j} \neq 0 \iff i \neq j, \text{ and rank}(A_{i,i}) = t_i \text{ for each } i \in [n] \right\}
\]
and
\[
\mathcal{H}^+_t(G) = \{ A \in \mathcal{F}^+_t(G) : A_{i,i} = I_{t_i} \text{ for each } i \in [n] \}.
\]
We write \( \mathcal{F}^+(G) = \mathcal{F}^+_1(G) \) and \( \mathcal{H}^+(G) = \mathcal{H}^+_1(G) \), where \( \mathbb{I} = (1,1, \ldots, 1) \). Note that in [HPRS15], \( m^+_1(G) \) and \( \mathcal{H}^+_1(G) \) were defined in the special case where \( t = (r, r, \ldots, r) \) for some \( r \in \mathbb{N} \).

**Proposition III.2.** Let \( G \) be a graph with vertex set \([n]\) and let \( t \in \mathbb{N}^n \). Then
\[
m^+_t(G) = \min \{ \text{rank } A : A \in \mathcal{F}^+_t(G) \} = \min \{ \text{rank } B : B \in \mathcal{H}^+_t(G) \}.
\]

**Proof.** The proof is an adaptation of [HPRS15, Theorem 3.10]. Suppose first that \( A \in \mathcal{F}^+_t(G) \) and \( \text{rank } A = k \). Then there
exists a matrix $X \in M_{k,|t|}$ such that $A = X^*X$. Write $X = [X_1 \ X_2 \cdots \ X_n]$, where $X_i \in M_{k,t_i}$, $i = 1, \ldots, n$. We have

$$\text{rank } X_i = \text{rank} (X_i^* X_i) = \text{rank} (A_{ii}) = t_i, \quad i \in [n].$$

Writing $P = (P_1, \ldots, P_n)$ where $P_i \in M_k$ is the orthogonal projection onto the range of $X_i$, we have $P \in \mathcal{P}(k, t)$. Additionally,

$$P_i P_j \neq 0 \iff X_i^* X_j \neq 0 \iff A_{i,j} \neq 0 \iff \epsilon \approx j,$$

so $G(P) = G$. Hence

$$m_i^+(G) \leq \min \{ \text{rank } A : A \in \mathcal{F}_i^+(G) \}.$$

Since $\mathcal{H}_i^+(G) \subseteq \mathcal{F}_i^+(G)$, the inequality

$$\min \{ \text{rank } A : A \in \mathcal{F}_i^+(G) \} \leq \min \{ \text{rank } B : B \in \mathcal{H}_i^+(G) \}$$

holds trivially. Now suppose that $P = (P_1, \ldots, P_n) \in \mathcal{P}(k, t)$ with $G(P) = G$, and for each $i \in [n]$ let $X_i \in M_{k,t_i}$ be a matrix whose columns form an orthonormal basis for the range of $P_i$. Define $X = [X_1 \ X_2 \cdots \ X_n] \in M_{k,|t|}$ and let $B = X^*X \in M_{k,|t|}^+$ so that $\text{rank } B = \text{rank } X \leq k$. Note that the $t_i \times t_i$ block $B_{i,i}$ coincides with $X_i^* X_i$. Since $i \approx j \iff P_i P_j \neq 0 \iff X_i^* X_j \neq 0$, we have $B_{i,j} = 0 \iff i \approx j$. Moreover, the condition on the columns of $X$ implies that $B_{i,i} = I_{t_i}$ for each $i$. So $B \in \mathcal{H}_i^+(G)$, hence

$$\min \{ \text{rank } B : B \in \mathcal{H}_i^+(G) \} \leq m_i^+(G). \quad \square$$

**Theorem III.3.** For any graph $G$, we have $\text{qint}(G) = \gamma(G)$.

**Proof.** Directly from the definitions, we have

$$\text{qint}(G) = \min_{t \in \mathbb{N}^n} m_t^+(G) = m_i^+(G) = \gamma(G).$$

Let $t \in \mathbb{N}^n$. We claim that if $t_1 \geq 2$ and $s = (t_1 - 1, t_2, t_3, \ldots, t_n)$, then $m_s^+(G) \leq m_t^+(G)$. By symmetry and induction, this yields $\gamma(G) = m_1^+(G) \leq m_t^+(G)$ for any $t \in \mathbb{N}^n$, hence $\text{qint}(G) = \gamma(G)$.

To establish the claim, suppose that $t_1 \geq 2$, let $k = m_t^+(G)$, and use Proposition III.2 to choose $B \in \mathcal{H}_1^+(G)$ with $\text{rank } B = k$. We may write $B = X^*X$ where $X \in M_{k,|t|}$. Write $X = [X_1 \ X_2 \cdots \ X_n]$, where $X_i \in M_{k,t_i}$, let $Y \in M_{k,t_1-1}$ be $X$ with the first column deleted and let $Z_i \in M_{k,t_i-1}$ be the matrix with every column equal to the first column of $X_i$. Let $Z = [Z_1 \ 0 \cdots \ 0] \in M_{k,|t|-1}$.

Let $Y_1 \in M_{k,t_1-1}$ consist of the first $(t_1 - 1)$ columns of $Y$. Note that $X^*Y_1$ contains $X_1^*Y_1$ as a submatrix, which is equal to $I_{t_1}$ with the first column removed. In particular, $X^*Y_1 \neq 0$. Similarly, the first column of $X_i^*Z_i$ is the first column of $I_{t_i}$, so $X^*Z_i \neq 0$. Define

$$a = \min \{|w| : w \neq 0, w \text{ is an entry of } X^*Y_1\}$$

and let $\varepsilon \in (0, \frac{1}{2}ab^{-1})$. Define

$$W_\varepsilon = Y + \varepsilon Z \in M_{k,|t|-1}$$

and

$$A_\varepsilon = W_\varepsilon^* W_\varepsilon \in M_{|t|-1}^+.$$
channel, [DSW13] showed that the map $\Phi \mapsto S_\Phi$ from quantum channels with domain $M_n$ to operator subsystems of $M_m$, is surjective. A more detailed statement, including an estimate on the dimension of the target Hilbert space of such $\Phi$, was provided in [SS15]. Here we include yet another argument for the convenience of the reader.

**Proposition IV.1.** Let $n \in \mathbb{N}$. If $S \subseteq M_n$ is an operator system, then there exists $k \in \mathbb{N}$ and a quantum channel $\Phi: M_n \to M_k$ such that $S = S_\Phi$. In fact, if $m \in \mathbb{N}$ is such that $\binom{m}{2} \geq \dim S - 1$, then we can take $k = mn$.

**Proof.** Let $d = \dim S$ and let $I_{n1}, S_1, \ldots, S_{d-1}$ be a basis of $S$. Suppose that $\binom{m}{2} \geq \dim S - 1$. Then we can form a hermitian $m \times m$ block matrix $H = [H_{ij}]_{i,j\in[m]} \in M_m(M_n)$ so that

$$H_{ij} = 0 \text{ for } i \in [m]$$

and

$$\{S_1, \ldots, S_{d-1}\} = \{H_{ij} : 1 \leq i < j \leq m\}.$$

For sufficiently small $\varepsilon > 0$, the hermitian matrix $X = m^{-1}(I_m + \varepsilon H)$ is positive semi-definite (indeed, its spectrum is contained in the interval $[m^{-1}(1-\varepsilon r), m^{-1}(1+\varepsilon r)]$ where $r$ is the spectral radius of $H$) hence $X = C^*C$ for some $C \in M_m(M_n)$. Note that the block entries $X_{ij}$ of $X$ span $S$, and $X_{ij} = m^{-1}I_n$ for $i \in [m]$. The $mn \times n$ block columns $C_1, \ldots, C_m$ of $C$ satisfy $C_i^*C_j = X_{ij}$, so in particular, $\sum_{i=1}^m C_i^*C_i = \sum_{i=1}^m X_{ii} = I_n$. Hence $\{C_1, \ldots, C_m\}$ are Kraus operators for a quantum channel $\Phi: M_n \to M_m$ for which $S_\Phi$ is spanned by the entries of $X$, so $S_\Phi = S$. \qed

We now define parameters of operator systems which, as we will shortly see, generalise the graph parameters above. Let $S \subseteq M_n$ be an operator system. As usual, we write

$$S^\perp = \{A \in M_n : tr(A^*S) = 0 \text{ for all } S \in S\}.$$

(a) Let $S \subseteq M_n$ be an operator system. Recall [DSW13] that an $S$-independent set of size $m$ is an $m$-tuple $x = (x_1, \ldots, x_m)$ with each $x_i$ a non-zero vector in $\mathbb{C}^n$, so that $x_px_q^* \in S^\perp$ whenever $p, q \in [m]$ with $p \neq q$. The independence number $\alpha(S)$ is then defined by letting $\alpha(S) = \max\{m \in \mathbb{N} : \exists \text{ an } S\text{-independent set of size } m\}$.

(b) We define the quantum complexity $\gamma(S)$ by letting

$$\gamma(S) = \min\{k \in \mathbb{N} : S_\Phi = S \text{ for some quantum channel } \Phi: M_n \to M_k\}$$

and the quantum subcomplexity $\beta(S)$ by letting

$$\beta(S) = \min\{k \in \mathbb{N} : S_\Phi \subseteq S \text{ for some quantum channel } \Phi: M_n \to M_k\}$$

$$= \min\{\gamma(T) : T \subseteq S \text{ is an operator subsystem}\}.$$

(c) A quantum channel $\Phi$ which has a set of Kraus operators each of which is of the form $AD$ for some entrywise non-negative matrix $A$ and an invertible diagonal matrix $D$ will be said to be a non-cancelling. We define

$$\text{int}(S) = \inf\{k \in \mathbb{N} : S_\Phi = S \text{ for some non-cancelling quantum channel } \Phi: M_n \to M_k\}.$$

**Corollary IV.2.** Let $S \subseteq M_n$ be an operator system. Then $\gamma(S) \leq 2n^2$.

**Proof.** Since $\dim S \leq n^2$, we can take $m = 2n$ in Proposition IV.1. \qed

We will refer to $\gamma(\Phi)$ as the quantum complexity of $\Phi$ and $\beta(\Phi)$ as the quantum subcomplexity of $\Phi$. Given a channel $\Phi$, we set $\pi(\Phi) = \pi(S_\Phi)$ for $\pi \in \{\beta, \gamma, \text{int}\}$.

**Remark IV.3.** (i) A set of quantum states can be perfectly distinguished by a measurement system if and only if they are orthogonal. Consequently, [DSW13] defined the one-shot zero-error capacity $\alpha(\Phi)$ of a quantum channel $\Phi: M_n \to M_k$ to be the maximum cardinality of a set $\{v_1, \ldots, v_p\} \subseteq \mathbb{C}^n$ orthogonal unit vectors, such that

$$tr(\Phi(v_i^*v_i^*)\Phi(v_jv_j^*)) = 0, \quad i \neq j.$$

It was shown in [DSW13] (see also [Pau16]) that $\alpha(\Phi) = \alpha(S_\Phi)$.

(ii) Let $S$ be an operator system. The quantum chromatic number $\chi_q(S^\perp)$ of the orthogonal complement $S^\perp$ of $S$ was introduced by D. Stahlke in [Sta16]. It is straightforward that $\beta(S) = \chi_q(S^\perp)$.

(iii) For an operator system $S \subseteq M_n$ and $\pi \in \{\beta, \gamma, \text{int}\}$, we have $\pi(S) = 1 \iff S = M_n$. Indeed, the trace $tr: M_n \to \mathbb{C}$ is a non-cancelling quantum channel since it has the entry-wise non-negative Kraus operators $e_1^*, \ldots, e_n^*$ (where $e_i^*$ is the functional corresponding to the vector $e_i$), so $1 \leq tr(M_n) \leq \text{int}(M_n) = 1$ and hence $\pi(M_n) = 1$. Conversely, $\pi(S) = 1$ implies that $\beta(S) = 1$; the trace is the only scalar-valued quantum channel on $M_n$, so $S = S_{tr} \subseteq S \subseteq M_n$, that is, $S = M_n$.

Note that, in contrast with Remark II.4 (iii), it is not true that $M_n$ is the only operator system $S$ with $\alpha(S) = 1$; see Proposition IV.12.

(iv) We claim that

$$\alpha(CI_n) = \beta(CI_n) = \gamma(CI_n) = \text{int}(CI_n) = n.$$

Indeed, one sees immediately that $\alpha(CI_n) \geq n$ by considering the $CI_n$-independent set $(e_1, \ldots, e_n)$, and since the identity channel $M_n \to M_n$ is non-cancelling, we have that $\text{int}(CI_n) \leq n$, so an appeal to Theorem IV.4 below establishes the claim.

(v) Let $S \subseteq M_n$. Using (iv), we have

$$\beta(S) = \min\{\gamma(T) : T \subseteq S\} \leq \gamma(CI_n) = n.$$

On the other hand, $\gamma(S)$ may exceed $n$, even for $n = 2$ (see Proposition IV.12).

(vi) There exist operator systems $S \subseteq M_n$ with $\text{int}(S) = \infty$, so the infimum in the definition of $\text{int}(S)$ cannot be replaced with a minimum. For example, it is not difficult to see that this is the case for the two-dimensional operator system $S \subseteq M_4$ spanned by the identity and $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, where $X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

**Theorem IV.4.** Let $S \subseteq M_n$ be an operator system. Then

$$\alpha(S) \leq \beta(S) \leq \gamma(S) \leq \text{int}(S).$$
Proof. Suppose that $\Phi : M_n \to M_k$ is a quantum channel with $S_{p_2} \subseteq S$; let $A_i \in M_{k,n}$, $i = 1, \ldots, d$, be its Kraus operators. Let $(x_p)_p^{m=1}$ be an $S$-independent set of size $m$. For each $p \in [m]$, let $E_p$ be the projection in $M_k$ onto the span of $\{A_i x_p: i = 1, \ldots, d\}$. Since
\[
\sum_{i=1}^{d} ||A_i x_p||^2 = \left(\sum_{i=1}^{d} A_i^* A_i\right) x_p x_p = 1,
\]
we have that $E_p \neq 0$ for each $p$. On the other hand, since $A_i^* A_i \in S$ for all $i, j = 1, \ldots, d$ and $(x_p)_p^{m=1}$ is $S$-independent, we have that
\[
\langle A_i x_p, A_j x_q \rangle = \langle A_i^* A_j x_p x_q \rangle = 0, \quad p \neq q, \quad i, j = 1, \ldots, d.
\]
Thus, $E_1, \ldots, E_m$ are pairwise orthogonal projections in $M_k$; it follows that $m \leq k$ and hence $\alpha(S) \leq \beta(S)$.

The inequalities $\beta(S) \leq \gamma(S) \leq \text{int}(S)$ hold trivially. \hspace{1cm} \square

In the next proposition, we collect some properties of the operator system parameters introduced above.

**Proposition IV.5.** Let $S \subseteq M_n$ and $S_1 \subseteq M_{n_1}, \ i = 1, 2$ be operator systems.

(i) If $\pi \in \{\alpha, \beta, \gamma\}$ and $U \in M_n$ is unitary, then $\pi(U^* S U) = \pi(S)$;

(ii) If $\pi \in \{\alpha, \beta, \gamma\}$ and $P \in M_n$ is a projection of rank $r$, then, viewing PSP as an operator subsystem of $M_r$, we have $\pi(P S P) \leq \pi(S)$;

(iii) If $\pi \in \{\beta, \gamma\}$, $n = n_1 n_2$ and $S = S_1 \otimes S_2$, then $\max\{\pi(S_1), \pi(S_2)\} \leq \pi(S) \leq \pi(S_1) \pi(S_2)$;

(iv) If $\pi \in \{\beta, \gamma, \text{int}\}$, $n = n_1 n_2$ and $S = \text{span}(S_1 \cup S_2)$, then $\pi(S) \leq \pi(S_1) + \pi(S_2)$.

(v) If $\pi \in \{\beta, \gamma\}$, then $\pi(S_1 \otimes S_2) = \pi(S_1) + \pi(S_2)$.

**Proof.** The proofs for $\pi = \alpha$ are easy and are left to the reader. We give the proofs for $\pi = \gamma$; the other proofs follow identical patterns.

(i) If $(A_p)_p^{m=1}$ are Kraus operators in $M_{k,n}$ for which $\text{span}\{(A_p U)^* A_p U\}_p^{m=1} = S$, then $(A_p U)_p^{m=1}$ are Kraus operators in $M_{k,n}$ such that
\[
\text{span}\{(A_p U)^* A_p U\}_p^{m=1} = U^* \text{span}\{A_p^* A_p\}_p^{m=1} U = U^* S U,
\]
so $\gamma(U^* S U) \leq \gamma(S)$; the reverse inequality follows by symmetry.

(ii) If $(A_p)_p^{m=1}$ are Kraus operators in $M_{k,n}$ for which $\text{span}\{A_p^* A_p\}_p^{m=1} = S$, then after identifying the range of $P$ with $C^n$, we see that $(A_p P)_p^{m=1}$ are Kraus operators in $M_{k,r}$ with
\[
\text{span}\{(A_p P)^* A_p P\}_p^{m=1} = P \text{span}\{A_p^* A_p\}_p^{m=1} P = PSP,
\]
so $\gamma(PSP) \leq \gamma(S)$.

(iii) Suppose that $(A_{p,i}: p \in [m_1]) \subseteq M_{k_{i,n}}$ is a family of Kraus operators for $i = 1, 2$, so that $S_i = \text{span}\{A_{p,i} A_{q,i}: p, q \in [m_1]\}, \ i = 1, 2$. Set $B_{p,r} := A_{p,1} \otimes A_{r,2}$; then $(B_{p,r}: p \in [m_1], r \in [m_2])$ is a family of Kraus operators in $M_{k_1+k_2,n}$ with $S = \text{span}\{B_{p,r}^* B_{p,s} \mid p, q \in [m_1], r, s \in [m_2]\}$. It follows that $\gamma(S) \leq \gamma(S_1) + \gamma(S_2)$.

If we set $P_1 = I_{n_1} \otimes Q$ where $Q \in M_{n_2}$ is a rank one projection, then we have that $P_1(S_1 \otimes S_2)P_1 = S_1 \otimes \mathbb{C} \equiv S_1$. Hence by (ii), $\pi(S_1) \leq \pi(S_1 \otimes S_2)$ and the lower bound follows.

(iv) Let $(A_{p,i}: p \in [m_1])$ be a family of Kraus operators in $M_{k_{i,n}}$ with $S_i = \text{span}\{A_{p,i} A_{q,i} \mid p, q \in [m_1]\}, \ i = 1, 2$. Set $B_{p,1} = [A_{p,1}^*]$ and $B_{p,2} = [A_{p,2}]^*$, viewed as elements of $M_{k_1+k_2,n}$. Then
\[
\frac{1}{2} B_{p,i} : i = 1, 2, p \in [m_1]
\]
is a family of Kraus operators with
\[
\text{span}\{B_{p,1}^* B_{q,2}\}_p^{m=1} = \text{span}\{B_{p,i}^* B_{q,i}\}_p^{m=1} = S,
\]
so $\gamma(S) \leq k_1 + k_2$. Hence, $\gamma(S) \leq (\gamma(S_1) + \gamma(S_2))$.

(v) If $S = S_1 \otimes S_2$, then $n = n_1 n_2$. Suppose that $\gamma(S) = k$, so that there exists a family $\{C_p \mid p \in [m]\}$ of $k \times n$-Kraus operators, where $A_p \in M_{k,n_1}$ and $B_p \in M_{k,n_2}$, with $S = \text{span}\{(C_p C_q^*)_p^{m=1} \mid p, q \in [m]\}$. Since $C_p^* C_q = \frac{1}{2} \{B_{p,i}^* B_{q,i} \mid p, q \in [m]\}$, $A_p$ and $B_p$ are orthogonal for every $p, q$. The projections $P_1$ and $P_2$ onto the linear span of the ranges of $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ are therefore orthogonal, so if $k_1 = \text{rank } P_1$ and $k_2 = \text{rank } P_1$, then there is a unitary $U : C^k \to C^{k_1} \oplus C^{k_2}$ for which $U A_p = [A_{p,0}^*]$, for some $k_1 \times n$ matrices $A_{p,0}$, and $U B_p = [B_{p,0}]^*$ for some $k_2 \times n$ matrices $B_{p,0}$. Now $C_p^* C_q = (U C_{p,0}^*)^* (U C_{q,0})$, namely, the minimum semidefinite rank, $\gamma(S_1) + \gamma(S_2) \leq k_1 + k_2 = k = \gamma(S)$. Combined with (iv), this shows that $\gamma(S_1 \otimes S_2) = \gamma(S_1) + \gamma(S_2)$. \hspace{1cm} \square

**Remark IV.6.** (i) Let $\pi \in \{\alpha, \beta, \gamma\}$ and $d \in \mathbb{N}$. Then $\pi(M_d(S)) = \pi(S)$. Indeed, by Proposition IV.5 (ii), we have $\pi(S) \leq \pi(M_d(S))$, and the reverse inequality for $\pi \in \{\beta, \gamma\}$ follows from Proposition IV.5 (iii) and Remark IV.3 (ii). To see the corresponding result for $\pi = \alpha$, suppose that $\{\xi_{p}^{m}_{p=1}\}$ is an independent set for $S \otimes M_d$. Then for $X, Y \in M_d, A \in S$ and $p \neq q$, we have
\[
\langle (A \otimes I)(I \otimes X)\xi_{p}^{m}_{p=1}, (I \otimes Y)\xi_{q}^{m}_{q=1}\rangle = 0 \tag{4}
\]
Let $Q_p$ be the projection onto span $\{(I \otimes X)\xi_{p}^{m}_{p=1}: X \in M_d\}$; then $Q_p = E_p \otimes I_d$ for some non-zero projection $E_p$ on $C^n$, and (4) implies that $E_p S E_p = \{0\}$ provided $p \neq q$. If $v_{p}$ is a unit vector with $E_{p} v_{p} = v_{p}, p \in [m]$, we therefore have that $\{v_{p}\}_{p=1}^{m}$ is an independent set for $S$. It follows that $\alpha(S) \geq \alpha(M_d(S))$, and hence we have equality.

(ii) The parameter $\gamma$ is neither order-preserving nor order-reversing for inclusion. For example, $\mathbb{C} I_2 \subseteq S \subseteq M_2$ where $S$ is the operator system of Proposition IV.12, and these operator systems have $\gamma$-values $2, 3, 1$, respectively.

We will now show that like its graph-theoretic counterpart, namely, the minimum semidefinite rank, $\gamma(S)$ is the solution to a rank minimisation problem.
Proposition IV.7. For any operator system $S \subseteq M_n$, we have

$$\gamma(S) = \min_{m \in \mathbb{N}} \left\{ \text{rank } B : B = [B_{i,j}] \in M_m(S)^+ \text{ with } \text{span}\{B_{i,j} : i,j \in [m]\} = S \text{ and } \sum_{i=1}^m B_{i,i} = I_n \right\}$$

and

$$\beta(S) = \min_{m \in \mathbb{N}} \left\{ \text{rank } B : B = [B_{i,j}] \in M_m(S)^+ \text{ and } \sum_{i=1}^m B_{i,i} = I_n \right\}.$$

Moreover, the minima on the right hand sides are achieved for $m$ not exceeding $2n^3$.

Proof. Suppose that $m \in \mathbb{N}$ and that $B = [B_{i,j}] \in M_m(S)^+$ satisfies the relations $\text{span}\{B_{i,j} \}_{i,j=1}^m = S$ and $\sum_{i=1}^m B_{i,i} = I_n$. Then $B = A^*A$ for some $A = [A_1 \ldots A_m] \in M_{1,m}(M_{n,n})$ where $k = \text{rank } B$. Since $B_{i,j} = A_i^*A_j$, we see that $\{A_1, \ldots, A_m\}$ are Kraus operators for a quantum channel $\Phi$ with $S_\Phi = S$; thus, $\gamma(S) \leq k$.

Conversely, let $k = \gamma(S)$, $m \in \mathbb{N}$ and $A_1, \ldots, A_m \in M_{k,n}$ be Kraus operators for a quantum channel $\Phi$ with $S_\Phi = S$. Set $B := [A_i^*A_j] \in M_m(S)^+$; we have $\text{span}\{A_i^*A_j : i,j \in [m]\} = S$, $\sum_{i=1}^m A_i^*A_i = I_n$ and $\text{rank } B = \text{rank}\{A_1 \ldots A_m\} \leq k$. Hence the minimum rank in the first expression is no greater than $\gamma(S)$.

To see that some $m \leq 2n^3$ attains this minimum, set $k = \gamma(S)$. Then there exists a quantum channel $\Phi : M_n \to M_k$ with $S_\Phi = S$ and, by Corollary IV.2, $k \leq 2n^2$. By [Ch75, Remark 6], the channel $\Phi$ can be realised using at most $nk \leq 2n^3$ Kraus operators. Since $m$ is precisely the number of Kraus operators in the preceding argument, we see that the minimum in the expression for $\gamma(S)$ is attained for some $m \leq 2n^3$.

The expression for $\beta(S)$ follows from the fact that $\beta(S) = \min\{\gamma(T) : T \subseteq S\}$. Since the bound on $m$ for $\gamma$, namely $2n^3$, is independent of the operator system $S \subseteq M_n$, this fact shows that here we may also take $m \leq 2n^3$.

Our next task is to show that the operator system parameters just defined generalise the graph parameters of Section II. Recall that if $G$ is a graph with vertex set $[n]$, we let

$$S_G = \text{span}\{E_{i,j} : i \simeq j\}$$

be the associated operator subsystem of $M_n$.

Lemma IV.8. Let $n,k \in \mathbb{N}$, and let $\Delta$ be the group of diagonal $n \times n$ matrices whose diagonal entries are each either 1 or $-1$. Let $x = (x_1, \ldots, x_n)$ be an $n$-tuple of non-zero vectors in $\mathbb{C}^k$, and let $A = [\hat{x}_1 \cdots \hat{x}_n]$ be the $k \times n$ matrix whose $i$-th column is the unit vector $\hat{x}_i = ||x_i||^{-1}x_i$. Then the map

$$\Delta_x : M_n \to M_k, \quad \Delta_x(X) = 2^{-n} \sum_{D \in \Delta} ADXDA^*,$$

is a quantum channel. Moreover, if $x_i \in \mathbb{R}^k_+ \setminus \{0\}$, $i = 1, \ldots, n$, then $\Delta_x$ is non-cancelling.

Proof. For $D \in \Delta$, let $d_i \in \{1, -1\}$ be the $i$-th diagonal entry of $D$. We have

$$2^{-n} \sum_{D \in \Delta} (DA^*AD)_{ij} = 2^{-n} \langle \hat{x}_j, \hat{x}_i \rangle \sum_{D \in \Delta} d_id_j = \delta_{ij},$$

since if $i \neq j$ then the sum reduces to 0 by symmetry, whereas if $i = j$ then every term in the sum is 1. Since each $\hat{x}_i$ is a unit vector, we obtain $2^{-n} \sum_{D \in \Delta} DA^*AD = I_n$, so $\Delta_x$ is a quantum channel. The assertion about non-cancelling channels follows trivially.

Proposition IV.9. Let $n,k \in \mathbb{N}$, $x_i$ be a non-zero vector in $\mathbb{C}^k$, $i = 1, \ldots, n$, and $x = (x_1, \ldots, x_n)$. Then $S_G(x) = S_{\Delta_x}$.

Proof. Let $S = S_{G(x)}$ and $T = S_{\Delta_x}$. Set $\hat{x}_i = ||x_i||^{-1}x_i$ and $A = [\hat{x}_1 \cdots \hat{x}_n]$, and note that $T$ is spanned by the operators $DA^*AD'$ for $D = \text{diag}(d_1, \ldots, d_n)$ and $D' = \text{diag}(d_1', \ldots, d_n')$ in $\Delta$. For $i,j \in [n]$, we have

$$\sum_{D,D' \in \Delta} DA^*AD' = \left( \sum_{D \in \Delta} D \right)A^*A\left( \sum_{D' \in \Delta} D' \right) = 4^{n-1}E_{i,j}A^*AE_{i,j} \equiv 4^{n-1}E_{i,j}.$$

If $E_{i,j} \in S$, then $i \neq G(x) j$, so $\langle \hat{x}_j, \hat{x}_i \rangle \neq 0$, hence $E_{i,j} \in T$. Thus $S \subseteq T$. On the other hand, $S \subseteq T$ is spanned by the matrix units $E_{i,j}$ with $i \neq G(x) j$. For such $i,j$ and any $D,D' \in \Delta$, we have $\langle DA^*AD'_{ij} \rangle = d_i d_j \langle \hat{x}_i, \hat{x}_j \rangle d_i d_j = 0$, so $E_{i,j} \in T$. Hence $S \subseteq T$ as required.

Theorem IV.10. For any graph $G$ with vertex set $[n]$ and $\pi = \{\alpha, \beta, \gamma, \int\}$, we have $\pi(S_G) = \pi(G)$.

Proof. The case $\pi = \alpha$ is known [DSW13].

We next consider the case $\pi = \gamma$. If $k = \gamma(G)$, then there exists $x = (x_1, \ldots, x_n)$, where $x_i \in \mathbb{C}^k \setminus \{0\}$, $i = 1, \ldots, n$, so that $G(x) = G$, and hence $S_G(x) = S_G$. By Proposition IV.9, $S_G(x)$ is the operator system of a quantum channel $M_n \to M_k$, hence $\gamma(S_G) \leq k = \gamma(G)$.

Now let $k = \gamma(S_G)$, so that there are Kraus operators $A_1, \ldots, A_m \in M_{k,n}$ for a quantum channel $\Phi : M_n \to M_k$ with $S_\Phi = \text{span}\{A_p^*A_p : p,q \in [m]\} = S_G$. Since the column operator with entries $A_1, \ldots, A_m$ is an isometry, for each $i \in [n]$ we have $\sum_{p=1}^m ||A_pe_i||^2 = ||e_i|| = 1$. In particular, $A_pe_i \neq 0$ for at least one $p \in [m]$. Thus, the projection $P_i \in M_k$ onto the span of $\{A_pe_i : p \in [m]\}$ is non-zero.

Consider the tuple $P = (P_1, \ldots, P_n) \in P(k)$. Since $S_\Phi = S_G$, for $i,j \in [n]$ we have

$$i \neq G(P) j \iff P_i P_j = 0 \iff \langle A_pe_i, A_qe_j \rangle = \text{tr}(E_{i,j}A_q^*A_p) = 0, \quad \text{for all } p,q \in [m] \iff E_{i,j} \in S_G^+ = S_G^\perp \iff i \neq G j.$$

Hence $G(P) = G$. Using Theorem III.3, we obtain $\gamma(G) = \text{qint}(G) \leq k = \gamma(S_G)$.

The case $\pi = \beta$ is similar to the case $\pi = \gamma$; alternatively, see [Sta16, Theorem 12].

Let $\pi = \int$. Set $K = \text{int}(G)$; then there exists a tuple $x = (x_1, \ldots, x_n) \in (\mathbb{R}^k_+ \setminus \{0\})^n$ with $G(x) = G$. By
Proposition IV.9. $S_{\Delta_n} = S_G(x) = S_G$. By Lemma IV.8, $\Delta_x$ is a non-cancelling quantum channel; therefore, $\text{int}(S_G) \leq k$.

In particular, $\text{int}(S_G) < \infty$. Now if $k = \text{int}(S_G)$, let $\Phi : M_n \to M_k$ be a non-cancelling quantum channel such that $S_{\Phi} = S_G$. Fix Kraus operators $A_1 D_1, \ldots, A_m D_m$ for $\Phi$ where each $A_g \in M_{k,n}$ is entrywise non-negative and each $D_g$ is an invertible diagonal $n \times n$ matrix. For $i \in [n]$, define

$$R_i = \{ r \in [k] : (A_p e_i, e_r) \neq 0 \text{ for some } p \in [m] \}.$$ 

Since the column operator with entries $A_1 D_1, \ldots, A_m D_m$ is an isometry, we have $\sum_{p=1}^{m} \|A_p D_p e_i\|^2 = 1$, and so $R_i$ is non-empty for all $i \in [n]$. Let $x = (x_1, \ldots, x_n)$ where $x_i = \sum_{r \in R_i} e_r$. For $i, j \in [n]$, we have

$$i \simeq_G j \iff E_{ij} S_G E_{i,j} \neq 0 \iff \exists p, q \in [m] \text{ such that } (A_p D_p e_i, A_q D_q e_j) \neq 0 \iff \exists q \in [m] \text{ such that } (A_p e_i, A_q e_j) \neq 0.$$ 

Now

$$(A_p e_i, A_q e_j) = \sum_{r \in [k]} (A_p e_i, e_r) (e_r, A_q e_j)$$

and every term in the latter sum is non-negative. It follows that

$$i \simeq_G j \iff \exists q \in [m], r \in [k] \text{ with } (A_p e_i, e_r) \neq 0 \text{ and } (A_q e_j, e_r) \neq 0 \iff R_i \cap R_j \neq \emptyset \iff \langle x_j, x_i \rangle \neq 0 \iff i \simeq_{G(x)} j.$$ 

Hence $G = G(x)$, so $\text{int}(G) \leq k = \text{int}(S_G)$. \hfill $\square$

Remark IV.11. Let $(p(y|x))$ be a $k \times n$ non-negative column-stochastic matrix defining a classical channel $N : [n] \to [k]$ with confusability graph $G = \mathcal{N}_N$. The canonical quantum channel $N' : M_n \to M_k$ associated with $N$ is defined by setting $N'_p(E_{x,y}) = \sum_{y \in [k]} p(y|x) E_{y,y}$ and $N'_q(E_{x',y'}) = 0$ if $x \neq x'$. We have that $N'_N = S_G$ [DSW13] (see also [Paul16]).

So we see that $\gamma(G) = \gamma(S_G)$ is the quantum complexity of the classical channel $N$ when viewed as a quantum channel.

Let $G \boxtimes H$ denote the strong product of the graphs $G$ and $H$ [Sa60], in which $(x, y) \simeq_{G \boxtimes H} (x', y')$ if and only if $x \simeq G x'$ and $y \simeq H y'$. Note that $S_{G \boxtimes H} = S_G \otimes S_H$. If $G, H$ are graphs and $n$ is the number of vertices of $G$, then

(i) $\alpha(G) = 1$ if and only if $G = K_n$, i.e., if and only if $S_G = M_n$;

(ii) $\gamma(G) \leq n$; and

(iii) $\gamma(G \boxtimes H) \leq \gamma(G) \gamma(H)$, but it is unknown whether strict inequality can occur.

The following proposition shows that the parameters for general operator systems $S \subseteq M_n$ behave quite differently, with respect to the latter properties, than their graph theoretic counterparts.

Proposition IV.12. Let $S = \{ \lambda \frac{a \otimes b}{\lambda} : a, b, \lambda \in \mathbb{C} \}$. Then $\alpha(S) = 1$, $\beta(S) = 2$, $\gamma(S) = \text{int}(S) = 3$ and $\gamma(S \otimes S) < \gamma(S)^2$.

Proof. The Kraus operators $A_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ yield a non-cancelling quantum channel with operator system $S$, so $\gamma(S) \leq \text{int}(S) \leq 3$. Since $S^\perp$ is spanned by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, it contains no rank one operators, and hence $\alpha(S) = 1$. By Remark IV.3 (v), $\beta(S) \leq 2$, while by Remark IV.3 (iii), $\beta(S) \neq 1$; thus, $\beta(S) = 2$.

If $\gamma(S) \leq 2$, then there are $A_j = [v_j w_j] \in M_{8}$ for some $v_j, w_j \in \mathbb{C}^2$, $j = 1, \ldots, m$ which are the Kraus operators of a quantum channel with operator system $S$. We have $\langle v_j, v_i \rangle = (A_j^* A_j)_{11} = (A_j^* A_j)_{22} = \langle w_j, w_i \rangle$ for each $j, i$. It follows that there is a $2 \times 2$ unitary $U$ with $U v_j = w_j$, $j = 1, \ldots, m$.

Recall that $\sum_{i=1}^{m} A_i^* A_i = I_2$. This is equivalent to

$$\sum_{i=1}^{m} \|v_i\|^2 = \sum_{i=1}^{m} \|u_i\|^2 = 1$$

and

$$\sum_{i=1}^{m} \langle w_i, v_i \rangle = \sum_{i=1}^{m} (U v_i, v_i) = 0.$$ 

In particular, 0 lies in the numerical range of $U$. However, the numerical range of a normal matrix is the convex hull of its spectrum, and hence $\sigma(U) = \{ \alpha, -\alpha \}$ for some $\alpha \in \mathbb{T}$. Thus $\overline{\sigma} U$ is hermitian, and so $U = \alpha^2 U^*$. Now

$$A_j^* A_j = \left( \begin{array}{cc} \langle v_j, v_j \rangle & \langle U^* v_j, v_i \rangle \\ \langle U v_j, v_i \rangle & \langle U^* v_j, v_i \rangle \end{array} \right) = \langle v_j, v_i \rangle I + \langle U^* v_j, v_i \rangle \begin{bmatrix} 0 & 0 \\ 0 & \alpha^2 \end{bmatrix},$$ 

hence $3 = \dim S = \dim \text{span} \{ A_j^* A_j \} \leq 2$, a contradiction.

To see that $\gamma(S \otimes S) < \gamma(S)^2 = 9$, consider the isometries $V_i \in M_{8,4}$ given by

$$V_1 = [e_1 e_2 e_3 e_4], \quad V_2 = [e_2 e_5 e_4 e_6],$$

$$V_3 = [e_3 e_4 e_7 e_8], \quad V_4 = [e_6 e_7 e_5 e_1].$$

Let $W = \frac{1}{2} [V_1 V_2 V_3 V_4] \in M_{8,16}$; then

$$W^* W = \frac{1}{4} \begin{bmatrix} I_4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{bmatrix} = I_4$$

Writing $B_{i,j}$ for the $(i,j)$-th $4 \times 4$ block of $W^* W$, we observe that

$$\text{span} \{ B_{i,j} : i, j \in [4] \} = S \otimes S \quad \text{and} \quad \sum_{i=1}^{4} B_{i,i} = I_4.$$ 

By Proposition IV.7,

$$\gamma(S \otimes S) \leq \text{rank} W^* W = \text{rank} W = 8.$$ 

$\square$
V. APPLICATIONS TO CAPACITY

Let \( \mathcal{N} : X \to Y \) be a classical information channel with confusability graph \( G \). Its parallel use \( r \) times can be expressed as a channel \( \mathcal{N}^{\times r} : X^r \to Y^r \), for which

\[
p((y_s)_{s=1}^r | (x_s)_{s=1}^r) = \prod_{s=1}^r p(y_s | x_s),
\]

for \( x_s \in X \), \( y_s \in Y \) and \( s = 1, \ldots, r \). Note that

\[
G_{\mathcal{N}^{\times r}} = G_{\mathcal{N}} \boxtimes \cdots \boxtimes G_{\mathcal{N}}.
\]

The Shannon capacity of the channel \( \mathcal{N} : X \to Y \) (or equivalently of the graph \( G \)) is the quantity

\[
\Theta(\mathcal{N}) = \Theta(G) = \lim_{r \to \infty} \sqrt[r]{\alpha(G^{\times r})} = \lim_{r \to \infty} \sqrt[r]{\alpha(G_{\mathcal{N}^{\times r}})}.
\]

(Some authors prefer to use the logarithm of the quantities defined above.)

Similarly, if \( \Phi : M_n \to M_k \) is a quantum channel, letting \( \Phi^{\otimes r} : M_n^{\otimes r} \to M_k^{\otimes r} \) be its \( r \)-th power, we find

\[
S_{\Phi^{\otimes r}} = S_{\Phi} \otimes \cdots \otimes S_{\Phi}.
\]

The analogue of the Shannon capacity of a quantum channel introduced in [DSW13] is the parameter

\[
\Theta(\Phi) = \lim_{r \to \infty} \sqrt[r]{\alpha(\Phi^{\otimes r})}.
\]

Lovász [Lo79] introduced his famous \( \vartheta \)-parameter of a graph and proved that \( \alpha(G) \leq \vartheta(G) \) and that \( \vartheta \) is multiplicative for strong graph product; hence,

\[
\Theta(\mathcal{N}) = \Theta(G_{\mathcal{N}}) \leq \vartheta(G),
\]

for any classical channel, thus giving a bound on the Shannon capacity of classical channels. He also proved [Lo79, Theorem 11] that

\[
\vartheta(G) \leq \beta(G),
\]

so that his \( \vartheta \)-bound is a better bound on the capacity of classical channels than any of the bounds that we derived from complexity considerations. However, as we will shortly show, for quantum channels, \( \beta \) yields a bound on capacity that can outperform \( \vartheta \). We note that a different bound on \( \Theta(G) \), based on ranks of Hermitian matrices in the operator system \( S_{\mathcal{G}}^{\otimes r} \), was introduced by Haemers in [Hae81]. It is an interesting open question to formulate general non-commutative analogues of Haemers’ parameter.

Lovász gave many characterisations of his parameter, but the most useful for our purposes is the expression

\[
\vartheta(G) = \max \left\{ ||I + K|| : I + K \in M_n^+, K \in S_{\mathcal{G}}^+ \right\}.
\]

The latter formula motivated [DSW13] to define, for any operator subsystem \( S \) of \( M_n \),

\[
\vartheta(S) = \max \left\{ ||I + K|| : I + K \in M_n^+, K \in S^+ \right\};
\]

note that \( \vartheta(G) = \vartheta(S_G) \). It was shown in [DSW13] that, for any quantum channel \( \Phi \), one has

\[
\alpha(\Phi) = \alpha(S_{\Phi}) \leq \vartheta(S_{\Phi}).
\]

However, \( \vartheta \) is only supermultiplicative for tensor products of general operator systems. This motivated [DSW13] to introduce a “complete” version, denoted \( \tilde{\vartheta} \), which is multiplicative for tensor products of operator systems and satisfies \( \tilde{\vartheta}(S) \leq \vartheta(S) \). This allowed them to bound the quantum capacity of a quantum channel, since

\[
\Theta(\Phi) = \lim_{r \to \infty} \sqrt[r]{\alpha(S_{\Phi}^{\otimes r})} \leq \lim_{r \to \infty} \sqrt[r]{\tilde{\vartheta}(S_{\Phi}^{\otimes r})} \leq \lim_{r \to \infty} \sqrt[r]{\tilde{\vartheta}(S_{\Phi}^{\otimes r})} = \tilde{\vartheta}(S_{\Phi}).
\]

These bounds are often difficult to compute. The quantity \( \lim_{r \to \infty} \sqrt[r]{\tilde{\vartheta}(S_{\Phi}^{\otimes r})} \) requires evaluation of a limit, each term of which may be intractable, and the possibly larger bound \( \vartheta(S_{\Phi}) = \sup_{\alpha \in \mathbb{N}} \tilde{\vartheta}(S_{\Phi} \otimes M_n) \) requires the evaluation of a supremum, although this parameter has the advantage of possessing a reformulation as a semidefinite program [DSW13].

**Theorem V.1.** For any quantum channel \( \Phi \), we have

\[
\alpha(\Phi) \leq \Theta(\Phi) \leq \beta(S_{\Phi}).
\]

**Proof.** Let \( \Phi \) be a quantum channel and \( S = S_{\Phi} \). The inequality \( \alpha \leq \vartheta \leq \beta \) holds immediately from the supermultiplicative property of \( \alpha \). Since \( \beta \) is submultiplicative for tensor products (Proposition IV.5 (iii)) and \( \alpha \) is dominated by \( \beta \) (Theorem IV.4), we have

\[
\Theta(\Phi) = \Theta(S) = \lim_{r \to \infty} \sqrt[r]{\alpha(S^{\otimes r})} \leq \beta(S).
\]

In the remainder of the section, we will exhibit operator systems for which \( \beta(S) \ll \vartheta(S) \). For \( k \in \mathbb{N} \), let

\[
S_k = \{(a_{i,j})_{i,j=1}^k \in M_k : a_{1,1} = a_{2,2} = \cdots = a_{k,k}\}.
\]

It is easy to show directly that

\[
\vartheta(S_k) = k.
\]

For any \( m \in \mathbb{N} \), applying the canonical shuffle which identifies \( M_k \otimes M_m \) with \( M_m \otimes M_k \), we have

\[
S_k \otimes S_m = \{(A_{i,j})_{i,j=1}^k \in M_k(M_m) : A_{1,1} = A_{2,2} = \cdots = A_{k,k}\}.
\]

Thus, for any operator system \( S \subseteq M_n \), we have

\[
S_k \otimes S = \{(A_{i,j})_{i,j=1}^k \in M_k(S) : A_{1,1} = A_{2,2} = \cdots = A_{k,k}\}.
\]

**Theorem V.2.** We have

\[
\beta(S_k \otimes S_{k^2}) \leq k^2 < k^3 \leq \vartheta(S_k \otimes S_{k^2}).
\]

**Proof.** Let \( \omega \) be a primitive \( k \)-th root of unity. Let \( S \subseteq M_k \) be given by \( S_{\epsilon_i} = \epsilon_{i+1}, i = 1, \ldots, k^2 \), where addition is modulo \( k^2 \), while \( D \in M_{k^2} \) be the diagonal matrix with diagonal \( (1, \omega, \omega^2, \ldots, \omega^{k^2-1}) \). Note that, if \( D_0 \in M_k \) is the diagonal matrix with diagonal \( (1, \omega, \omega^2, \ldots, \omega^{k-1}) \), then \( D = D_0 \oplus \cdots \oplus D_0 \). We have that \( D_i S = \omega^{ij} S D_j \) for any \( i, j \in \mathbb{Z} \), and hence

\[
D_i S^i = \omega^{ij} S^i D_j, \quad i, j \in \mathbb{Z}.
\]
Any element $A$ of $M_k(S_k \otimes S_{k_2})$ has the form $A = (C_{r,s})_{r,s=0}^{k-1}$, where $C_{r,s} \in S_k \otimes S_{k_2}$ for all $r, s = 0, \ldots, k-1$. In view of the remarks before the statement of the theorem, we may write $C_{r,s} = (A_{kr+i,ks+j})_{i,j=0}^{k-1}$, where $A_{kr+i,ks+j} \in S_{k_2}$ for all $r, s, i, j = 0, \ldots, k-1$, and

$$A_{kr,ks} = A_{kr+1,ks+1} = A_{kr+2,ks+2} = \cdots = A_{kr+k-1,ks+k-1},$$

for all $r, s = 0, \ldots, k$.

Let

$$u_{kr+i} = S^{kr+i}D^r, \quad r, i = 0, \ldots, k-1,$n

and

$$B = (u_0, u_1, u_2, \ldots, u_{k-1}) \in M_{k_2,k^*}.$n

Set $B_{r,s} = (u_{kr+i}u_{ks+j})_{i,j=0}^{k-1}$; then the matrix

$$B^*B = (B_{r,s})_{r,s=0}^{k-1} = (u_{kr+i}u_{ks+j})_{r,s,i,j}$$

is positive and has rank at most $k^2$. We will show the following:

(i) $u_{kr+i}u_{ks+j} \in S_{k^2}$, for all $r, s, i, j$, $0 \leq i, j \leq 1$, $k - 1$;
(ii) $u_{kr+s}u_{kr+s} = u_{kr+i}u_{kr+i+1} = u_{kr+i}u_{kr+i+2} = \cdots = u_{kr+k-i}u_{kr+k-i}$, for all $r, s, i, j$;
(iii) $\sum_{r=1}^k B_{r,r} = kI$,

which will imply that $\beta(S_k \otimes S_{k_2}) \leq k^2$.

To show (i), note that

$$u_{kr+i}u_{ks+j} = D^{-r}S^{-kr+i}S^{ks+j}D^r = D^{-r}S^{ks-kr+j-i}D^r.$$

If $ks - kr + j - i \neq 0$, then $u_{kr+i}u_{ks+j}$ has zero diagonal and thus belongs to $S_{k_2}$. Suppose that $ks - kr + j - i = 0$. Then $k(i-j)$ and hence $i = j$. If, in addition, $r \neq s$ then $u_{kr+i}u_{ks+j}$ has zero diagonal and therefore belongs to $S_{k_2}$; if, on the other hand, $r = s$, then $u_{kr+i}u_{ks+j} = I$ and hence again belongs to $S_{k_2}$.

To show (ii), note that for $i = 0, \ldots, k - 1$, we have

$$u_{kr+i}u_{kr+i} = D^{-r}S^{-kr-i}S^{kr+i}D^r = D^{-r}S^{k(s-r)}D^r.$$

In order to show (iii), suppose that $i, j \in \{0, \ldots, k - 1\}$ with $i \neq j$. and, using (5), note that

$$\sum_{r=0}^{k-1} u_{kr+i}u_{kr+j} = \sum_{r=0}^{k-1} D^{-r}S^{-kr-i}S^{kr+j}D^r = \sum_{r=0}^{k-1} D^{-r}S^{j-i}D^r = \left(\sum_{r=0}^{k-1} \omega^{-r(j-i)}\right)S^{j-i}.$$n

Since $\omega$ is a primitive root of unity, so is $\omega^{-1}$. Thus, $\omega^{-r(j-i)}$ is a $k$-th root of unity with $\omega^{-r(j-i)} \neq 1$. It follows that $\sum_{r=0}^{k-1} \omega^{-r(j-i)} = 0$. Thus, $\sum_{r=0}^{k-1} u_{kr+i}u_{kr+j} = 0$ whenever $i \neq j$. On the other hand,

$$\sum_{r=0}^{k-1} u_{kr+i}u_{kr+i} = \sum_{r=0}^{k-1} I = kI,$n

and (iii) is proved.

By [DSW13, Lemma 4],

$$\vartheta(S_k \otimes S_{k_2}) \geq \vartheta(S_k)\vartheta(S_{k_2}) = k^3,$n

and the proof is complete.

**Corollary V.3.** (i) The ratios $\vartheta(S)/\beta(S)$ and $\vartheta(S)/\gamma(S)$ can be arbitrarily large, as $S$ varies over all non-commutative graphs.

(ii) For $\pi \in \{\beta, \gamma\}$, the ratio $\vartheta(S)/\pi(S)$ can be arbitrarily large, as $S$ varies over all non-commutative graphs with $\frac{1}{2} \pi(S) \leq \Theta(S) \leq \pi(S)$.

Moreover, these statements hold if throughout we replace $\vartheta$ by $\tilde{\vartheta}$, the quantum Lovász theta number.

**Proof.** Since $\vartheta \leq \tilde{\vartheta}$, it suffices to prove these statements for $\vartheta$.

(i) For the first ratio, consider $S = T_\beta := S_k \otimes S_{k_2}$ and apply Theorem V.2. For the second ratio, let $T_\pi$ be the span of the matrices $B_{r,s} \in S_k \otimes S_{k_2}$ appearing in the proof of Theorem V.2. Then $T_\pi \subseteq S_k \otimes S_{k_2}$ is an operator system and the set $\{B_{r,s}\}$ is one of the terms that appear in the minimum that defines $\gamma(T_\pi)$. Hence,

$$\gamma(T_\pi) \leq \text{rank}((B_{r,s})) \leq k^2 < k^3 \leq \vartheta(S_k \otimes S_{k_2}) \leq \vartheta(T_\pi).$$

(ii) For $\pi \in \{\beta, \gamma\}$, consider $R_\pi := T_\pi \oplus CI_{k_2} \subseteq M_{k+2k_2}$. For $k > 3$, by Proposition IV.5 (v) and Remark IV.3 (iv), we have

$$\pi(R_\pi) = \pi(T_\pi) + \pi(CI_{k_2}) \leq k^2 + k^2 = 2k^2 = 2\alpha(CI_{k_2}) \leq 2\alpha(R_\pi) \leq 2\Theta(R_\pi).$$

Since $\vartheta$ is order-reversing for inclusion of operator systems and $\text{ran} \subseteq (S_k \otimes S_{k_2}) \oplus CI_{k_2} \subseteq (S_k \oplus \mathbb{C}) \otimes S_{k_2}$ (the second inclusion holds up to a unitary shuffle equivalence) and it is easy to see that $\vartheta(S \oplus T) \geq \vartheta(S)$ for any operator systems $S$ and $T$, we have

$$\vartheta(R_\pi) \geq \vartheta(S_k \oplus \mathbb{C})\vartheta(S_{k_2}) \geq k^3.$$

**APPENDIX**

In this appendix, we briefly summarise the order relationships between various bounds on the quantum Shannon zero-error capacity. These may be succinctly described by the directed graph in Figure 1. The parameters $\alpha_q$, $\chi_q$ and $\chi$ are defined in [DSW13], and [Sta16, Definition 11]; the reader should swap $S$ and $S^\perp$ when translating between our non-commutative graphs and Stahlke’s “trace-free non-commutative graphs”.

Let $S \subseteq M_n$ be an operator system. As observed in Remark IV.3 (ii), we have $\chi_q(S^\perp) = \beta(S)$. The first two inequalities in the chain

$$\alpha_q(S) \leq \sqrt{\vartheta(S)} \leq \chi_q(S^\perp) \leq \chi(S^\perp)$$

appear in [Sta16], following Corollary 20, and the third inequality is a simple consequence of his Proposition 9. The inequality $\alpha_q(S) \leq \alpha(S)$ is immediate from the definitions (and appears in [DSW13, Proposition 2]), and we have seen in Theorems IV.4 and V.1 that $\alpha \leq \Theta \leq \beta \leq \gamma \leq \text{int}.\alpha.$
The inequality \( \Theta \leq \tilde{\vartheta} \) follows immediately from [DSW13, Proposition 2 and Corollary 10], noting that in the notation of that paper, \( \log_2 \Theta = C_0 \leq C_{0E} \); and \( \sqrt[\alpha]{\vartheta} \leq \tilde{\vartheta} \) is trivial.

It only remains to prove the incomparability assertions of Figure 1. These follow from the inequalities already established and the examples below.

- Let \( G = C_5 \) be the 5-cycle, and let \( S = S_G \). Lovász has shown [Lo79] that \( \vartheta(G) = \sqrt{5} \) while, for graph operator systems, as pointed out in [DSW13], we have \( \tilde{\vartheta}(S_G) = \vartheta(G) \). It is not difficult to see that
  \[
  \alpha(S_G) = \alpha(G) = 2 < \beta(G) = \beta(S_G).
  \]
So, in this example,
  \[
  \sqrt{\tilde{\vartheta}(S)} < \alpha(S) \quad \text{and} \quad \tilde{\vartheta}(S) < \beta(S).
  \]

- Consider \( G = C_6 \), the complement of the 6-cycle, and \( S = S_G \). It is easy to see directly that \( \gamma(S) = \gamma(G) > 2 \), and \( \chi(S^\perp) = \chi(C_6) = 2 \), so in this case,
  \[
  \chi(S^\perp) < \gamma(S).
  \]

- Let \( S \) be the operator system of Proposition IV.12 (i.e., in the notation of Section V, \( S = S_Z \)). Note that \( \alpha(S) = 1 \). We claim that if \( \mathcal{T} \) is any operator system with \( \alpha(\mathcal{T}) = 1 \), then \( \alpha(S \otimes \mathcal{T}) = 1 \). Indeed, \( S \otimes \mathcal{T} \) may be identified with all \( 2 \times 2 \) block matrices of the form \( \begin{bmatrix} T & A \\ B & T \end{bmatrix} \) for \( T, A, B \in \mathcal{T} \), and if \( x, y \) are non-zero vectors with \( xy^* \in (S \otimes \mathcal{T})^\perp \), then writing \( x = [z_1] \) and \( y = [y_1] \), we obtain \( xy^* = (x_1 y_1^*)_{i,j=1,2} \in (S \otimes \mathcal{T})^\perp \). By considering the off-diagonal entries and the condition \( \alpha(\mathcal{T}) = 1 \), it readily follows that \( x_1 = 0 \) or \( y_1 = 0 \), and \( x_2 = 0 \) or \( y_2 = 0 \).

If \( x_1 = 0 \), then \( y_1 = 0 \); hence, \( xy^* = 0 \oplus x_2 y_2^* \), so \( x_2 y_2^* \notin \mathcal{T}^\perp \), so \( x = y = 0 \), a contradiction. The other case proceeds to a similar contradiction, so \( \alpha(S \otimes \mathcal{T}) = 1 \). Hence, in particular, \( \Theta(S) = 1 \). On the other hand, \( \tilde{\vartheta}(S) = 2 \) by [DSW13, p. 1172]; thus, in this case we have
  \[
  \Theta(S) < \sqrt{\tilde{\vartheta}(S)}.
  \]

- Finally, let \( S = \mathbb{C}I_2 \) to obtain an example for which
  \[
  \alpha(S) < \beta(S),
  \]
since the left hand side is 2 by Remark IV.3 (iv), and, as observed in [DSW13], the right hand side is 4.

Acknowledgements

The authors are grateful to the Fields Institute and the Institut Henri Poincaré for financial support to attend the Workshop on Operator Systems in Quantum Information and the Workshop on Operator Algebras and Quantum Information Theory, respectively, greatly facilitating our work on this project. The first named author also wishes to thank Helena Šmigoc and Polona Oblak for stimulating discussion of the minimum semidefinite rank.

References


