**QUEEN'S
UNIVERSITY
BELFAST**

Pseudomodes for Schrödinger operators with complex potentials

Krejčířík, D., & Siegl, P. (2018). Pseudomodes for Schrödinger operators with complex potentials. *Journal of Functional Analysis*, 276(9), 2856. <https://doi.org/10.1016/j.jfa.2018.10.004>

Published in:
Journal of Functional Analysis

Document Version:
Peer reviewed version

Queen's University Belfast - Research Portal:
[Link to publication record in Queen's University Belfast Research Portal](#)

Publisher rights

© 2018 Elsevier Inc. All rights reserved.

This manuscript version is made available under the CC-BY-NC-ND 4.0 license <http://creativecommons.org/licenses/by-nc-nd/4.0/>, which permits distribution and reproduction for noncommercial purposes, provided the author and source are cited

General rights

Copyright for the publications made accessible via the Queen's University Belfast Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The Research Portal is Queen's institutional repository that provides access to Queen's research output. Every effort has been made to ensure that content in the Research Portal does not infringe any person's rights, or applicable UK laws. If you discover content in the Research Portal that you believe breaches copyright or violates any law, please contact openaccess@qub.ac.uk.

PSEUDOMODES FOR SCHRÖDINGER OPERATORS WITH COMPLEX POTENTIALS

DAVID KREJČÍŘÍK AND PETR SIEGL

ABSTRACT. For one-dimensional Schrödinger operators with complex-valued potentials, we construct pseudomodes corresponding to large pseudoeigenvalues. We develop a first systematic non-semi-classical approach, which results in a substantial progress in achieving optimal conditions and conclusions as well as in covering a wide class of previously inaccessible potentials, including discontinuous ones. Applications of the present results to higher-dimensional Schrödinger operators are also discussed.

1. INTRODUCTION

While the spectral theorem reduces the study of self-adjoint operators to determining the individual components of the spectrum and the corresponding spectral measures, it is well known that the spectrum of a non-normal operator provides by far insufficient information about its properties. It is not the spectrum that determines the decay of the associated heat semigroup and the behaviour of eigenvalues under small perturbations, but rather the *pseudospectrum*, which measures the largeness of the resolvent, see *e.g.* [21, 6, 11].

The ε -pseudospectrum of a closed operator H consists of the union of its spectrum and complex points λ satisfying $\|(H - \lambda)f\| < \varepsilon \|f\|$ for some vector f from the domain of H . The number λ and the vector f are respectively called the *pseudoeigenvalue* (or *approximate eigenvalue*) and *pseudoeigenvector* (or *pseudomode*) of H . The pseudoeigenvalues of H may be turned into genuine eigenvalues of a perturbed operator $H + L$ with $\|L\| < \varepsilon$ and they can lie outside (in fact “very far” from) the ε -neighbourhood of the spectrum of H if the operator is not normal. This is the well-known *spectral instability* of non-normal operators under small perturbations.

This paper is concerned with a study, in several aspects complete, of approximate eigenvalues and pseudomodes of the one-dimensional Schrödinger operators

$$H_V := -\frac{d^2}{dx^2} + V(x), \quad (1.1)$$

where V is a *complex*-valued function. We consider L^2 -realisations of H_V on the whole line $\mathbb{R}$ or the semi-axis $\mathbb{R}_+$, the latter having immediate consequences for multi-dimensional operators with radial potentials and their perturbations. Thus our objective is to construct a λ -dependent family of pseudomodes f_λ such that

$$\|(H_V - \lambda)f_\lambda\| = o(1) \|f_\lambda\| \quad \text{as} \quad \lambda \rightarrow \infty \quad \text{in} \quad \Omega \subset \mathbb{C}. \quad (1.2)$$

The abstract self-adjoint theory yields immediately that real-valued potentials V are irrelevant here, since then (1.2) may hold only when λ approaches the spectrum of H_V . On the other hand, the by now well-known examples of potentials for which (1.2) holds in vast complex regions Ω are just purely imaginary monomials

Date: August 15, 2018.

2010 Mathematics Subject Classification. 34E20, 34L40, 35P20, 47A10, 81Q12, 81Q20.

Key words and phrases. pseudospectrum, Schrödinger operators, complex potential, WKB.

$V(x) := ix^n$ and their perturbations, see *e.g.* [5, 4, 19, 20, 12, 16]. Hence, the state of the art of the current research in construction of the “large-energy” pseudomodes for (non-semiclassical) Schrödinger operators is by far incomplete and the objective of this paper is to fill up the gap. In fact, all known cases (as well as all semiclassical ones) represent the simplest illustrations of our results, see Examples 3.8, 5.3 and 5.4.

The fundamental questions that we address here read as follows:

- For which potentials does there exist a non-trivial region $\Omega \subset \mathbb{C}$ where (1.2) holds?
- Comparing to $\operatorname{Im} V$, how large can $\operatorname{Re} V$ be so that (1.2) is preserved?
- Depending on V , what is the shape of Ω ?
- Is the polynomial-like character of the so far studied operators important?
- What is the role of the regularity of V ?

The main results of this paper giving answers to the raised questions are Theorems 3.7 (on pseudomodes for $\lambda \rightarrow +\infty$), Theorem 4.9 (on pseudomodes for $\lambda \rightarrow +\infty$ and potentials of low regularity) and Theorem 5.2 (on pseudomodes for $\lambda \rightarrow \infty$ on general curves in $\mathbb{C}$). These statements proved under technical Assumptions I, II and III are applied to more concrete classes of potentials in Sections 3.4, 4.4 and 5.3.

Basically all available results on non-trivial pseudospectra of Schrödinger operators are deduced by scaling from *semiclassical pseudomodes*, where a small parameter h^2 is added in front of the second-derivative in (1.2), see *e.g.*, [5, 7]. However, such an approach has several drawbacks. First of all, only very specific (homogeneous or their perturbations) potentials can be treated and unboundedness of $\operatorname{Im} V$ at infinity may seem to be crucial due to the scaling. Second, the artificial transition to the new parameter h , related in various ways to λ , complicates the natural interpretation of the results as well as the main points in the proofs. Finally, with the exception of the *imaginary shifted harmonic oscillator* $V(x) := (x+i)^2$ treated in [12], no claims seem to be available when $\operatorname{Re} V$ is larger than $\operatorname{Im} V$ at infinity. For these reasons, in this paper we attack the problem directly (without introducing the semiclassical parameter h).

The present results also have a connection to some open problems posed during the 2015 AIM workshop [1]. In particular, we would like to emphasise the following insights provided by this paper.

The semiclassical setting as a consequence. From our approach the known claims in the semiclassical setting follow immediately. In particular, the Davies’ condition [5] $\operatorname{Im} V' \neq 0$ or its (weaker) versions (see [22, 18]) can be easily generalised, see Example 5.4. It is also worth noting that our general non-semiclassical pseudomodes do not always localise, instead their support may extend.

Optimality of potentials. Our assumption (3.3) on the allowed size of $\operatorname{Re} V$ is optimal, at least for polynomial-like potentials (with $\nu_{\pm} = -1$ in assumption (3.2)). Indeed, by completely different methods, it has been established in [14, 15] that *e.g.* for potentials V satisfying $\operatorname{Re} V(x) = |x|^{\beta}$ with $\beta \geq 1$ and

$$\exists \epsilon > 0, \quad |\operatorname{Im} V(x)|^2 = \mathcal{O}(|x|^{\beta-2-\epsilon}), \quad |x| \rightarrow \infty, \quad (1.3)$$

the eigensystem of H_V contains a Riesz basis (and there are possibly only finitely many degenerate eigenvalues) and hence the only non-trivial pseudomodes exist for λ close to the eigenvalues of H_V (with known asymptotics, see [14]). In turn, the current results suggest that the condition (1.3) is optimal with respect to the Riesz basis property of H_V (which can be indeed concluded if more information about the position of eigenvalues of H_V is available) and confirms that the borderline

case (potentials with $\epsilon = 0$ in (1.3)) is the most challenging one, see [1, Open Problem 15.1]. Moreover, the assumption (3.3) has a very natural interpretation, namely, the pseudomodes loose their exponential decay if (3.3) is not satisfied, see Remark 3.3.

Optimality of pseudospectral regions. Our restrictions on the set Ω in (5.5), expressed in terms of conditions on $a := \operatorname{Re} \lambda$ and $b := \operatorname{Im} \lambda$, seem to be optimal. The optimality for the *rotated harmonic oscillator* $V(x) := ix^2$ follows by Boulton’s conjecture [4] solved by Pravda-Starov [19], see Example 5.3. The lower bound of (5.16) is also known to be optimal for the *imaginary cubic oscillator* $V(x) := ix^3$, see [3, Sec. 4.1]. The study of optimality of our estimates on the region Ω in general cases constitutes an interesting open problem.

Generality. We are able to treat a wide class of potentials being far beyond polynomial or scalable ones (we also allow a large $\operatorname{Re} V$ without restricting its sign). The method can be further straightforwardly generalised for even wilder potentials than already a quite wide range covered here (from bounded or even decaying, see Section 3.5, to super-exponential ones). For instance, the previously inaccessible (non-scalable) cases like $V(x) := i \sinh(x)$ or $V(x) := i \arctan(x)$ are included, see Examples 3.9 and 3.10. It is also important to stress that for realisations in $L^2(\mathbb{R})$, just the different asymptotic behaviour of $\operatorname{Im} V$ at $\pm\infty$:

$$\lim_{x \rightarrow -\infty} \operatorname{Im} V(x) \cdot \lim_{x \rightarrow +\infty} \operatorname{Im} V(x) < 0 \quad (1.4)$$

(see also (3.1) for a slight generalisation) is crucial to ensure the “significant non-self-adjointness” of H_V and thus the validity of (1.2) for $\lambda \rightarrow +\infty$. For decaying but non-integrable potentials V , condition (1.4) can be further weakened by requiring that $\operatorname{Im} V$ approaches 0 from opposite sides at $\pm\infty$, see Section 3.5. The various conditions of the type (1.4) can be viewed as a global version of the local Davies’ condition $\operatorname{Im} V' \neq 0$ or its weaker versions mentioned above.

Rough potentials. In fact, we cover even *discontinuous* potentials, which were previously inaccessible to semiclassical techniques. This is achieved by developing a robust method of λ -dependent mollifications of the potential. This new idea enables us to eventually solve an open problem raised during the AIM workshop [1, Open Problem 10.1].

The regularity of potentials and decay rates of pseudomodes. We explicitly demonstrate the crucial influence of the regularity (or local deformations) of V on the best possible rates in (1.2). The existing results suggest a difference in the rates for analytic and smooth potentials (exponential versus “faster than any power” rates), see *e.g.* [5, 7]. However, the optimal upper bounds for the resolvent norm are usually not available and so such observations are not always proved. In this paper we stress (and prove) the difference in rates for various step-like potentials of the type (arctan may be replaced by any “regularisation” of sgn)

$$V_1(x) := i \operatorname{sgn}(x) \quad \text{versus} \quad V_2(x) := i \arctan(x). \quad (1.5)$$

Here the best possible rate is linear in the first case (as proved in [9] by a careful analysis of the resolvent kernel) versus the “faster than any power” rate in the second case, see Example 3.10. Notice that the even more drastic local deformation, namely the operator $-\mathrm{d}^2/\mathrm{d}x^2 + i \operatorname{sgn}(x)$ subject to an *additional* Dirichlet boundary condition at 0, exhibits *no decay* for $\lambda \rightarrow +\infty$ in (1.2), since such an operator becomes normal.

Laptev-Safronov eigenvalue bounds. Our results for decaying potentials from Section 3.5 show that the bound on individual eigenvalues of one-dimensional Schrödinger operators due to Laptev and Safronov (see [13, Thm. 5] and [1, Open Problem 7.1]) cannot be improved using the Birman-Schwinger technique (since the norm estimate on the Birman-Schwinger operator provides a resolvent estimate). To justify the latter, we find simple L^p -potentials with $p > 1$ for which (1.2) holds, with the decay rate faster than any power of $1/|\lambda|$, in a region Ω determined by (3.24), which essentially coincides with the set appearing in [13, Thm. 5]. Thus the very natural reason for the appearance of such Ω is provided.

The existence of this region Ω , where the spectrum of H_V is extremely unstable with respect to further, even tiny, perturbations, is a crucial difference with respect to the L^1 -potentials. In the latter case, the resolvent estimate preventing that the resolvent of H_V explodes for large λ 's again follows from the Birman-Schwinger estimate, see [2].

Higher dimensions. The results and methods of this paper are essentially one-dimensional. Nonetheless, the results have consequences for multi-dimensional Schrödinger operators with (at least local) symmetries and their not too strong perturbations. The pseudomodes in $L^2(\mathbb{R})$ from Section 3 are obviously applicable for problems allowing for the separation of variables in Cartesian coordinates, while the pseudomodes in $L^2(\mathbb{R}_+)$ from Section 5 are applicable for radially symmetric problems. Finally, the pseudomodes from Example 5.5 arising due to the strongly singular potential at 0, namely

$$V(r) := \frac{c}{r^2} + \frac{i}{r^\alpha}, \quad c \in \mathbb{R}, \quad \alpha > 2, \quad r > 0, \quad (1.6)$$

localise in a vicinity of 0 and so are applicable for multi-dimensional potentials with a local radial singularity of the type (1.6). Unlike in one dimension, these pseudomodes do not show the optimality of region Ω in Laptev-Safronov multi-dimensional eigenvalue bounds since the condition $\alpha > 2$ cannot be satisfied for $V \in L^p(\mathbb{R}^d)$ with $p \geq d/2$ (or $p > 1$ for $d = 2$).

Organisation of the paper. In Section 2 we outline our strategy to construct the pseudomodes and settle a number of important prerequisites for the subsequent applications. Section 3 is devoted to large positive pseudoeigenvalues, while the case of general complex regions is treated only in Section 5; these two sections are concerned with sufficiently regular potentials (at least continuous). Large positive pseudoeigenvalues for discontinuous and singular potentials are dealt with in the intermediate Section 4.

Notations. Let us fix some notations employed throughout the paper. We use the following conventions for number sets, $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{R}_- := (-\infty, 0)$. Given an interval $I \subset \mathbb{R}$, the norm of $L^p(I)$ is denoted by $\|\cdot\|_{L^p(I)}$. If $I = \mathbb{R}$, we abbreviate $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R})}$ and $\|\cdot\| := \|\cdot\|_2$. The L^p spaces with a weight are denoted by

$$L_\alpha^p(I) := \{f \text{ measurable} : \langle x \rangle^\alpha f(x) \in L^p(I)\}, \quad \alpha \in \mathbb{R},$$

where $\langle x \rangle := (1 + x^2)^{\frac{1}{2}}$. For an “integer interval” we use the double brackets, $[[m, n]] := [m, n] \cap \mathbb{Z}$. To avoid using many irrelevant constants, we employ the convention that $a \lesssim b$ if there is a constant $C > 0$, independent of λ and x (or any other relevant parameter), such that $a \leq Cb$; the convention for $\gtrsim$ is analogous. By $a \approx b$ it is meant that $a \lesssim b$ and $a \gtrsim b$.

2. PRELIMINARIES

A standing hypothesis of this paper is that the complex-valued potential V satisfies the local square-integrability condition $V \in L^2_{\text{loc}}(\mathbb{R})$. We understand the Schrödinger operator (2.1) as the maximal operator generated by the differential expression, *i.e.*,

$$\begin{aligned} H_V f &:= -f'' + Vf, \\ \mathcal{D}(H_V) &:= \{f \in L^2(\mathbb{R}) : -f'' + Vf \in L^2(\mathbb{R})\}. \end{aligned} \quad (2.1)$$

If $\text{Re } V$ is bounded from below, Kato's theorem (*cf.* [8, Sec. VII.2.2]) yields that H_V is quasi-m-accretive and, moreover, $C_0^\infty(\mathbb{R})$ is a core of H_V . The quasi-m-accretivity ensures that (2.1) is well defined as a closed operator with non-empty resolvent set containing some open left half-plane. The latter properties of H_V are valid also in the non-accretive case under alternative assumptions on V , see [10]. For the pseudomode constructions performed in the present paper, however, not even the closedness of H_V is necessary.

2.1. The JWKB ansatz. Our construction of pseudomodes is based on the Liouville-Green approximation (also known as the JWKB method), see *e.g.* [5, 17].

If V were constant, *i.e.* $V(x) = V_0$ for all $x \in \mathbb{R}$, exact solutions of the differential equation $-g'' + V_0 g = \lambda g$ would be given by

$$e^{\pm i \int_0^x \sqrt{\lambda - V_0} dt}. \quad (2.2)$$

We shall be particularly interested in the limit $\lambda \rightarrow +\infty$ and consistently consider the *principal branch* of the square root. More generally, we always restrict to

$$\lambda \in \mathbb{C} \setminus (-\infty, 0). \quad (2.3)$$

For a variable potential V , we still take (2.2) with V_0 replaced by V and with the minus sign (due to assumptions on the signs of $\text{Im } V$, see (3.1)) as a basic ansatz to get the approximate solutions (1.2). Nonetheless, usually more terms will be needed for unbounded potentials or when V is sufficiently regular and more information on the decay rates in (1.2) are sought. In general, we therefore take

$$g(x) := \exp \left(- \sum_{k=1}^{n-1} \lambda^{-\frac{k}{2}} \psi_k(x) \right), \quad (2.4)$$

where functions ψ_k are to be determined. Not surprisingly, ψ_{-1} will turn out to be given by $\psi_{-1}(x) := i\lambda^{-1/2} \int_0^x \sqrt{\lambda - V(t)} dt$. As we will show in examples in Section 3.4, most of interesting potentials can be treated already with the expansion (2.4) up to $n = 2$.

2.2. The cut-off. To obtain admissible pseudomodes, it is important to employ a λ -dependent cut-off of the JWKB ansatz (2.4). To this aim, we consider a function $\xi : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following properties:

$$\begin{aligned} \xi &\in C_0^\infty(\mathbb{R}), \quad 0 \leq \xi \leq 1, \\ \forall x \in (-\delta_- + \Delta_-, \delta_+ - \Delta_+), \quad \xi(x) &= 1, \\ \forall x \notin (-\delta_-, \delta_+), \quad \xi(x) &= 0; \end{aligned} \quad (2.5)$$

the λ -dependent positive numbers $\delta_\pm = \delta_\pm(\lambda)$ and $\Delta_\pm = \Delta_\pm(\lambda) < \delta_\pm$ will be determined later. Notice that ξ can be selected in such a way that

$$\|\xi^{(j)}\|_{L^\infty(\mathbb{R}_\pm)} \lesssim \Delta_\pm^{-j}, \quad j = 1, 2. \quad (2.6)$$

To simplify notations, we also define intervals

$$\begin{aligned}\mathcal{J} &:= (-\delta_-, \delta_+), & \mathcal{J}_\pm &:= \{x \in \mathbb{R}_\pm : |x| < \delta_\pm\}, \\ \mathcal{J}' &:= (-\delta_- + \Delta_-, \delta_+ - \Delta_+), & \mathcal{J}'_\pm &:= \{x \in \mathbb{R}_\pm : |x| < \delta_\pm - \Delta_\pm\}.\end{aligned}$$

Our ansatz for a general potential V then reads

$$f := \xi g, \quad (2.7)$$

where g is defined in (2.4) and the index $n \in \mathbb{N}_0$ will be chosen according to the smoothness of V .

2.3. The strategy. Let us informally describe the strategy. Recalling (2.4), we have

$$\begin{aligned}-f'' + (V - \lambda)f &= -(\xi g)'' + (V - \lambda)\xi g \\ &= -\xi''g - 2\xi'g' + \xi[-g'' + (V - \lambda)g].\end{aligned} \quad (2.8)$$

When $n = 0$, the appearing terms read

$$g' = -i\sqrt{\lambda - V}g, \quad -g'' + (V - \lambda)g = \frac{-iV'}{\sqrt{\lambda - V}}g,$$

which already suggests what needs to be done. First, V must be sufficiently regular so that $f \in \mathcal{D}(H_V)$; in fact, the more terms in (2.4) are taken, the more regular V is needed. Second, the functions ψ_k in (2.4) and the cut-off ξ must be selected in such a way that the L^2 -norm of the third term on the second line of (2.8) is as small as possible when λ is large. Third, the assumption on the sign of $\text{Im } V$, see (3.1), implies that $|g|$ decays exponentially, see Lemma 3.5, and so the terms with ξ' and ξ'' are expected to be small; nevertheless, an appropriate restriction of $\delta_\pm, \Delta_\pm$ must be given.

Since our goal is to deal with potentials of low regularity, the construction consists of more steps. First we deal with sufficiently regular potentials V , later we add a singular term W and follow various possible strategies how to treat it, see Section 4.

2.4. The expansion. For g given in (2.4), we have

$$\begin{aligned}-g'' + (V - \lambda)g &= \left(\sum_{k=-1}^{n-1} \lambda^{-\frac{k}{2}} \psi_k'' \right) g - \left(\sum_{k=-1}^{n-1} \lambda^{-\frac{k}{2}} \psi_k' \right)^2 g + (V - \lambda)g \\ &=: \left(\sum_{k=-2}^{2(n-1)} \lambda^{-\frac{k}{2}} \phi_{k+1} \right) g, \quad n \in \mathbb{N}.\end{aligned} \quad (2.9)$$

Here the functions ϕ_k with $k \in [[-1, 2n-1]]$ are naturally defined after grouping together the terms with the same power of λ on the right hand side of the first line in (2.9), with the exception of V which we include in the leading order term:

$$\begin{aligned}(k = -2) \quad \lambda^1 &: & -(\psi_{-1}')^2 + \frac{V - \lambda}{\lambda} &=: \phi_{-1}, \\ (k = -1) \quad \lambda^{\frac{1}{2}} &: & \psi_{-1}'' - 2\psi_{-1}'\psi_0' &=: \phi_0, \\ (k = 0) \quad \lambda^0 &: & \psi_0'' - 2\psi_{-1}'\psi_1' - (\psi_0')^2 &=: \phi_1, \\ (k = 1) \quad \lambda^{-\frac{1}{2}} &: & \psi_1'' - 2\psi_{-1}'\psi_2' - 2\psi_0'\psi_1' &=: \phi_2, \\ &\dots & &\end{aligned} \quad (2.10)$$

For $-1 \leq k \leq 2(n-1)$, the formulae can be written concisely as

$$\psi_k'' - \sum_{\alpha+\beta=k} \psi_\alpha' \psi_\beta' = \phi_{k+1}, \quad (2.11)$$

with the convention that $\psi_\alpha = 0$ whenever $\alpha \geq n$ or $\alpha \leq -2$.

For the given $n \in \mathbb{N}$, we have $n+1$ functions $\psi_{-1}, \dots, \psi_{n-1}$ and $2n+1$ functions $\phi_{-1}, \dots, \phi_{2n-1}$. The strategy is to require that the first $n+1$ functions $\phi_{-1}, \dots, \phi_{n-1}$ are equal to zero, which determines all available ψ_k . Using (2.11), this leads to a system of $n+1$ first-order differential equations that the functions $\psi_{-1}, \dots, \psi_{n-1}$ must satisfy:

$$\begin{aligned} \psi'_{-1} &= i\lambda^{-\frac{1}{2}}(\lambda - V)^{\frac{1}{2}}, \\ \psi'_{k+1} &= \frac{1}{2\psi'_{-1}} \left(\psi''_k - \sum_{\substack{\alpha+\beta=k \\ \alpha, \beta \neq -1}} \psi'_\alpha \psi'_\beta \right), \quad k \in [[-1, n-2]], \end{aligned} \quad (2.12)$$

with the convention as above that $\psi_\alpha = 0$ whenever $\alpha \geq n$ or $\alpha \leq -2$. Here and in the sequel λ is, in addition to (2.3), assumed to be such that $\lambda - V(x) \in \mathbb{C} \setminus (-\infty, 0)$ for all $x \in \mathbb{R}$. Recall that the principal branch of the square root is considered in this paper.

Notice that we were free to choose the sign in the definition of ψ'_{-1} to make $\phi_{-1} = 0$, see (2.10). Our choice made in (2.12) will be consistently followed in this paper.

Finally, with this choice of functions ψ_k we get

$$-g'' + (V - \lambda)g = \left(\sum_{k=n-1}^{2(n-1)} \lambda^{-\frac{k}{2}} \phi_{k+1} \right) g =: r_n g, \quad n \in \mathbb{N}. \quad (2.13)$$

The essential point for estimating the resulting term is the understanding of the structure of functions ψ'_k and remainders r_n , which is the content of the following lemmata. The proof is based on a straightforward but rather lengthy induction argument, see Appendix.

Lemma 2.1. *Let $n \in \mathbb{N}_0$, $V \in W_{\text{loc}}^{n+1,2}(\mathbb{R})$ and functions $\{\psi'_k\}_{k \in [[-1, n-1]]}$ be determined by (2.12). Then*

$$\psi_k^{(m)} = \frac{\lambda^{\frac{k}{2}}}{(\lambda - V)^{\frac{k}{2}}} \sum_{j=0}^{k+m} \frac{T_j^{k+m, k+m+1-j}}{(\lambda - V)^j}, \quad m \in [[1, n+1-k]], \quad (2.14)$$

where (with some $c_\alpha \in \mathbb{C}$)

$$\begin{aligned} T_j^{r,s} &:= \sum_{\alpha \in \mathcal{I}_j^{r,s}} c_\alpha (V^{(1)})^{\alpha_1} (V^{(2)})^{\alpha_2} \dots (V^{(s)})^{\alpha_s}, \\ \mathcal{I}_j^{r,s} &:= \left\{ \alpha \in \mathbb{N}_0^s : \sum_{i=1}^s i\alpha_i = r \text{ \& } \sum_{i=1}^s \alpha_i = j \right\}. \end{aligned} \quad (2.15)$$

Moreover, if $r \geq 1$, then $\mathcal{I}_0^{r,r+1} = \emptyset$.

Lemma 2.2. *Let $n \in \mathbb{N}_0$, $V \in W_{\text{loc}}^{n+1,2}(\mathbb{R})$ and functions $\{\psi'_k\}_{k \in [[-1, n-1]]}$ be determined by (2.12), $\{\phi_k\}_{k \in [[-1, 2n-1]]}$ be as in (2.11) and r_n as in (2.13). Then*

$$|r_n| \lesssim \frac{|V^{(n+1)}|}{|\lambda - V|^{\frac{n+1}{2}}} + \sum_{k=0}^{n-1} \frac{1}{|\lambda - V|^{\frac{n-1+k}{2}}} \sum_{l=2}^{n+1+k} \frac{|T_l^{n+1+k,n}|}{|\lambda - V|^l}, \quad (2.16)$$

where $T_j^{r,s}$ are as in (2.15).

As an illustration for the expansions above with $n = 0, 1, 2$ we list functions ψ'_k :

$$\begin{aligned} (n=0) \quad \psi'_{-1} &= i \frac{(\lambda - V)^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}, \\ (n=1) \quad \psi'_0 &= -\frac{1}{4} \frac{V'}{\lambda - V}, \\ (n=2) \quad \psi'_1 &= \frac{i}{2} \frac{\lambda^{\frac{1}{2}}}{(\lambda - V)^{\frac{1}{2}}} \left(\frac{1}{4} \frac{V''}{\lambda - V} + \frac{5}{16} \frac{V'^2}{(\lambda - V)^2} \right), \end{aligned} \quad (2.17)$$

together with the remainders r_n on the right of (2.13):

$$\begin{aligned} r_0 &= -\frac{i}{2} \frac{V'}{(\lambda - V)^{\frac{1}{2}}}, \\ r_1 &= -\frac{1}{4} \frac{V''}{\lambda - V} - \frac{5}{16} \frac{V'^2}{(\lambda - V)^2}, \\ r_2 &= \frac{i}{8} \frac{1}{(\lambda - V)^{\frac{1}{2}}} \left(\frac{V'''}{(\lambda - V)} + \frac{9}{2} \frac{V'V''}{(\lambda - V)^2} + \frac{15}{4} \frac{V'^3}{(\lambda - V)^3} \right) \\ &\quad + \frac{1}{64} \frac{1}{(\lambda - V)} \left(\frac{V''^2}{(\lambda - V)^2} + \frac{5}{2} \frac{V'^2 V''}{(\lambda - V)^3} + \frac{25}{16} \frac{V'^4}{(\lambda - V)^4} \right). \end{aligned} \quad (2.18)$$

3. PSEUDOMODES FOR $\lambda \rightarrow +\infty$

In this section, unless otherwise stated, we always assume that λ is *positive* and typically very large.

3.1. Admissible class of potentials. We proceed under the following hypothesis about the (possibly unbounded) potential V .

Assumption I. Let $N \in \mathbb{N}$ and let $V \in W_{\text{loc}}^{N,\infty}(\mathbb{R})$ satisfy the following conditions:

a) $\text{Im } V$ has a different asymptotic behaviour at $\pm\infty$:

$$\limsup_{x \rightarrow -\infty} \text{Im } V(x) < 0, \quad \liminf_{x \rightarrow +\infty} \text{Im } V(x) > 0; \quad (3.1)$$

b) derivatives of V are controlled by V : $\exists \nu_{\pm} \in \mathbb{R}, \forall m \in [[1, N]]$,

$$\begin{aligned} |\text{Im } V^{(m)}(x)| &= \mathcal{O}(|\text{Im } V(x)| \langle x \rangle^{m\nu_{\pm}}), \quad x \rightarrow \pm\infty, \\ |V^{(m)}(x)| &= \mathcal{O}(|V(x)| \langle x \rangle^{m\nu_{\pm}}), \quad x \rightarrow \pm\infty; \end{aligned} \quad (3.2)$$

c) $\text{Im } V$ is sufficiently large:

i) if V is unbounded at $\pm\infty$, then suppose that: $\exists \varepsilon_1 > 0$,

$$\langle x \rangle^{4(\nu_{\pm} + \varepsilon_1) + 2} (\langle x \rangle^{4\nu_{\pm} + 2} + |\text{Re } V(x)|) = \mathcal{O}(|\text{Im } V(x)|^2), \quad x \rightarrow \pm\infty; \quad (3.3)$$

ii) if V is bounded at $\pm\infty$, then suppose that $\nu_{\pm} < 1$, where $\nu_{\pm}$ are the numbers from (3.2).

Example 3.1. A model case of V satisfying Assumption I is a sufficiently regular polynomial-like potential satisfying

$$V(x) = |x|^{\beta} + i \text{sgn } x |x|^{\gamma}, \quad |x| \geq 1,$$

for which $\nu_{\pm} = -1$, cf. (3.2). The conditions (3.1) and (3.3) hold if $\gamma \geq 0$ and $\gamma > (\beta - 2)/2$, respectively. More examples can be found in Section 3.4 and the optimality of the conditions in Assumption I is discussed around (1.3).

Several comments on the assumption are in place. First of all, notice that V is necessarily continuous due to $V \in W_{\text{loc}}^{1,\infty}(\mathbb{R})$.

The condition (3.1) ensures that the operator (1.1) is “significantly non-self-adjoint”. More precisely, H_V is not normal as a consequence of (3.1), the normality is equivalent to the condition that $\text{Im } V$ is constant. Furthermore, hypothesis (3.1) ensures that the pseudomode g from (2.4) is exponentially decaying. The correct sign for the decay can be seen (if the shape of g is determined mainly by ψ_{-1}) by employing (3.1) and the complex square root formula

$$\begin{aligned} \text{Re} \left(\lambda^{\frac{1}{2}} \psi'_{-1} \right) &= -\text{Im}(\lambda - V)^{\frac{1}{2}} \\ &= \frac{1}{2^{\frac{1}{2}}} \frac{\text{Im } V}{\left([(\lambda - \text{Re } V)^2 + (\text{Im } V)^2]^{\frac{1}{2}} + (\lambda - \text{Re } V) \right)^{\frac{1}{2}}}, \end{aligned} \quad (3.4)$$

valid for all positive λ satisfying in addition the requirement $\lambda - V(x) \in \mathbb{C} \setminus (-\infty, 0)$ for all $x \in \mathbb{R}$. The remaining two intertwined conditions guarantee that the exponential decay of g is not spoiled by too large $\text{Re } V$ or too wild behaviour of the derivatives of V .

The condition (3.2) restricts the growth and oscillations of V , nonetheless, it is still quite flexible as *e.g.* $V(x) := \pm i e^{x^2}$ for $x \rightarrow \pm\infty$ is covered. Notice that Gronwall’s inequality implies that (with some $M > 0$)

$$\forall x \geq 0, \quad |V(x)| \lesssim \begin{cases} e^{Mx^{1+\nu_+}}, & \nu_+ > -1, \\ \langle x \rangle^M, & \nu_+ = -1, \\ 1, & \nu_+ < -1; \end{cases} \quad (3.5)$$

an analogous estimate holds also for $x \leq 0$.

If V is bounded, we do not require that the derivatives of V are bounded in general, thus *e.g.* rapidly oscillating potentials are allowed. In such cases, the estimate from Gronwall’s inequality becomes very crude.

The condition (3.2) also implies that for $\nu_+ \geq -1$ and all sufficiently large $x > 0$ and every $|\Delta| \leq x^{-\nu_+}/4$, we have

$$\frac{|\text{Im } V(x + 2\Delta)|}{|\text{Im } V(x)|} \approx 1. \quad (3.6)$$

Indeed, for $\nu_+ > -1$,

$$\begin{aligned} \left| \log \frac{|\text{Im } V(x + 2\Delta)|}{|\text{Im } V(x)|} \right| &= \left| \int_x^{x+2\Delta} \frac{\text{Im } V'(t)}{\text{Im } V(t)} dt \right| \lesssim ||x + 2\Delta|^{\nu_++1} - |x|^{\nu_++1}| \\ &\lesssim x^{\nu_+} |\Delta| + \mathcal{O}(|\Delta|^2 x^{\nu_+-1}), \end{aligned}$$

and similarly for $\nu_+ = -1$. The conclusion (3.6) is clearly valid also for a bounded V as we require (3.1).

3.2. Localisation of the pseudomode and cut-off. To estimate $|g|$ we first show that under Assumption I the function $\int_0^x [\lambda^{\frac{1}{2}} \psi'_{-1}(t) + \psi'_0(t)] dt$ in the expansion (2.4) dominates over the other terms with $k \geq 1$. Thus estimates simplify significantly even for many terms in (2.4). Already at this step, it is important to employ a

suitable cut-off. Namely, for every $\lambda > 0$ we define:

$$\delta_{\pm} := \begin{cases} \inf \left\{ \delta \geq 0 : \frac{|\operatorname{Im} V(\pm\delta)|^2}{\langle \delta \rangle^{4\nu_{\pm} + 2\varepsilon_1 + 2}} = \lambda \right\} & \text{if } V \text{ is unbounded at } \pm\infty, \\ \lambda^{\frac{1+\varepsilon_2}{2}} & \text{with } 0 < \varepsilon_2 < 1 - \nu_{\pm} \text{ if } V \text{ is bounded at } \pm\infty, \end{cases} \quad (3.7)$$

$$\Delta_{\pm} := \frac{1}{4} \begin{cases} \delta_{\pm}^{-\nu_{\pm}} & \text{if } V \text{ is unbounded at } \pm\infty, \\ \delta_{\pm} & \text{if } V \text{ is bounded at } \pm\infty. \end{cases}$$

Remark 3.2 (Properties of $\delta_{\pm}$ and $\Delta_{\pm}$). The following hold.

- i) The infimum can be infinite ($\inf \emptyset = +\infty$), however, for all sufficiently large $\lambda > 0$, the numbers $\delta_{\pm}$ are finite and

$$\lim_{\lambda \rightarrow +\infty} \delta_{\pm} = +\infty; \quad (3.8)$$

- ii) $\Delta_{\pm}$ are so small that the values of $\operatorname{Im} V(x)$ are comparable if $|x - \delta_{\pm}| \leq 2\Delta_{\pm}$;
- iii) for all sufficiently large $\lambda > 0$,

$$\forall x \in \mathcal{J}, \quad |\lambda - V(x)| \approx \lambda. \quad (3.9)$$

Proof. All the three properties are obvious for bounded V . We further assume that V is unbounded at $+\infty$ and prove the claims; the case of V unbounded at $-\infty$ is analogous.

- i) It follows from the assumption (3.3) that for all sufficiently large $\delta > 0$

$$\frac{|\operatorname{Im} V(\delta)|^2}{\langle \delta \rangle^{4\nu_+ + 2\varepsilon_1 + 2}} \gtrsim \langle \delta \rangle^{2\varepsilon_1},$$

thus for all

$$\lambda > \min_{\delta \geq 0} \frac{|\operatorname{Im} V(\delta)|^2}{\langle \delta \rangle^{4\nu_+ + 2\varepsilon_1 + 2}}$$

the number δ_+ is finite. Since $\operatorname{Im} V$ is continuous, (3.8) follows.

- ii) See (3.6).

- iii) From (3.3), we obtain that for all $x > x_0$ with x_0 sufficiently large,

$$|\operatorname{Im} V(x)| \lesssim \frac{|\operatorname{Im} V(x)|^2}{\langle x \rangle^{4\nu_+ + 2\varepsilon_1 + 2}}, \quad |\operatorname{Re} V(x)| \lesssim \frac{|\operatorname{Im} V(x)|^2}{\langle x \rangle^{4\nu_+ + 4\varepsilon_1 + 2}};$$

thus we may assume that x_0 is chosen so large that for all $x > x_0$, we have

$$|\operatorname{Re} V(x)| \leq \frac{1}{2} \frac{|\operatorname{Im} V(x)|^2}{\langle x \rangle^{4\nu_+ + 2\varepsilon_1 + 2}}.$$

Thus, using (3.7), already proved i) and the continuity of V , we can select sufficiently large $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ and all $x \in [0, \delta_+]$, we have

$$|\operatorname{Im} V(x)| \lesssim \lambda, \quad |\operatorname{Re} V(x)| \leq \frac{\lambda}{2}. \quad (3.10)$$

Hence (3.9) follows. $\square$

Remark 3.3 (More on the assumption on $\operatorname{Re} V$). The restriction on $\operatorname{Re} V$ made in (3.3) arises in a very natural way. As an illustration, let us consider the potential $V(x) := |x|^{\beta} + i \operatorname{sgn}(x)|x|^{\gamma}$ with positive powers β, γ . In this case we can take $\nu_{\pm} := -1$ to satisfy hypothesis (3.2) and assumption (3.3) imposes the relationship $\beta - 2 + 4\varepsilon_1 \leq 2\gamma$. Choosing on the contrary $\beta > 2\gamma + 2$ so that (3.3) is violated, the dominant part in the expansion (2.4), which is given by $\operatorname{Re} \left(\lambda^{\frac{1}{2}} \psi'_{-1} \right)$ as we show

later, becomes (uniformly in λ) *integrable* in $x \in \mathbb{R}$. Indeed, with the substitution $t = \lambda^{1/\beta} s$ and by straightforward estimates,

$$\int_0^{\delta_+} \left| \operatorname{Re} \left(\lambda^{\frac{1}{2}} \psi'_{-1}(t) \right) \right| dt \lesssim \lambda^{\frac{2+2\gamma-\beta}{2\beta}} \int_0^\infty \frac{s^\gamma}{|1-s^\beta|^{\frac{1}{2}}} ds = o(1), \quad \lambda \rightarrow +\infty.$$

Moreover, notice that by taking a larger δ , so that it is possible that $\operatorname{Re} V(x) \geq \lambda$, problems appear in the control of the decay of $|r_n|$ for which the estimate $|\operatorname{Re} V - \lambda| \gtrsim \lambda$ is essential.

Using the properties of $\delta_\pm$ and $\Delta_\pm$, we obtain the following estimates.

Lemma 3.4. *Let Assumption I hold, let $0 \leq n \leq N$ and $\{\psi'_k\}_{k \in [[-1, n-1]]}$ be determined by (2.12) and let $\delta_\pm$ be as in (3.7). Then for all sufficiently large $\lambda > 0$*

$$\forall x \in \mathcal{J}, \quad \lambda^{\frac{1}{2}} \operatorname{Re} \psi'_{-1}(x) \approx \lambda^{-\frac{1}{2}} \operatorname{Im} V(x), \quad (3.11)$$

and

$$\forall k \in [[0, n-1]], \quad \forall x \in \mathcal{J}_\pm, \quad \lambda^{-\frac{k}{2}} |\psi'_k(x)| \lesssim \frac{|V(x)| \langle x \rangle^{(k+1)\nu_\pm}}{\lambda^{\frac{k}{2}+1}}. \quad (3.12)$$

Proof. The estimate (3.11) follows immediately from (3.4) using (3.9) and (3.10). The rest is based on Lemma 2.1 and assumptions (3.2), (3.3).

For $k \geq 0$, Lemma 2.1 yields

$$\begin{aligned} \lambda^{-\frac{k}{2}} |\psi'_k| &\leq \frac{1}{|\lambda - V|^{\frac{k}{2}}} \sum_{j=1}^{k+1} \frac{|T_j^{k+1, k+2-j}|}{|\lambda - V|^j} \\ &\lesssim \frac{1}{|\lambda - V|^{\frac{k}{2}}} \sum_{j=1}^{k+1} \frac{\sum_{\alpha \in \mathcal{I}_j^{k+1, k+2-j}} |V'|^{\alpha_1} |V''|^{\alpha_2} \dots |V^{(k+2-j)}|^{\alpha_{k+2-j}}}{|\lambda - V|^j}. \end{aligned} \quad (3.13)$$

Notice that the highest appearing derivative of V is $V^{(n)}$ and that the product of $|V^{(i)}|$ always consists of j factors (counting with powers) since $\sum_{i=1}^{k+2-j} \alpha_i = j$; see (2.15). Thus all appearing derivatives of V are continuous and the assumption (3.2) with $\sum_{i=1}^{k+2-j} i\alpha_i = k+1$ from (2.15) yields that for all sufficiently large $x > 0$

$$\lambda^{-\frac{k}{2}} |\psi'_k(x)| \lesssim \frac{\langle x \rangle^{(k+1)\nu_+}}{|\lambda - V(x)|^{\frac{k}{2}}} \sum_{j=1}^{k+1} \frac{|V(x)|^j}{|\lambda - V(x)|^j} \lesssim \frac{|V(x)| \langle x \rangle^{(k+1)\nu_+}}{\lambda^{\frac{k}{2}+1}}; \quad (3.14)$$

in the last step we have used (3.9). For small $x > 0$, the estimate (3.12) follows immediately from (3.13) and the continuity of the derivatives of V . For $x < 0$, the estimate is analogous. $\square$

Localising the ansatz (2.4) on the interval $\mathcal{J}$, the preceding lemma shows that the shape of g is determined mainly by ψ_{-1} and ψ_0 . More specifically, given the derivatives ψ'_k from (2.12), henceforth we choose the primitive functions ψ_k by fixing the integration constant by the requirement

$$\psi_k(0) := 0, \quad k \in [[-1, n-1]].$$

With this standing convention, we have the following two-sided bounds.

Lemma 3.5. *Let Assumption I hold, g be as in (2.4) with $\{\psi'_k\}_{k \in [[-1, n-1]]}$, $0 \leq n \leq N$, determined by (2.12) and let $\delta_\pm$ be as in (3.7). Then there exist $c_1, c_2 > 0$ such that for all sufficiently large $\lambda > 0$ and all $x \in \mathcal{J}$ we have*

$$\exp \left(-\frac{c_1}{\lambda^{\frac{1}{2}}} \int_0^{|x|} |\operatorname{Im} V(t)| dt \right) \lesssim |g(x)| \lesssim \exp \left(-\frac{c_2}{\lambda^{\frac{1}{2}}} \int_0^{|x|} |\operatorname{Im} V(t)| dt \right). \quad (3.15)$$

Proof. Notice that the formula (2.17) for ψ'_0 is exceptional since it can be easily integrated, hence

$$g(x) = \frac{[\lambda - V(0)]^{\frac{1}{4}}}{[\lambda - V(x)]^{\frac{1}{4}}} \exp \left(- \sum_{\substack{k=-1 \\ k \neq 0}}^{n-1} \lambda^{-\frac{k}{2}} \int_0^x \psi'_k(t) dt \right). \quad (3.16)$$

From (3.9) we get

$$\forall x \in \mathcal{J}, \quad \left| \frac{\lambda - V(0)}{\lambda - V(x)} \right| \approx 1.$$

We continue with estimates for $x > 0$, the other case is analogous. For any $x_0 > 0$ fixed, we have from Lemma 3.4 that

$$\left| \operatorname{Re} \sum_{k=-1}^{n-1} \lambda^{-\frac{k}{2}} \int_0^{x_0} \psi'_k(t) dt \right| \lesssim \lambda^{-\frac{1}{2}}.$$

The remaining estimate for $x > x_0$ follows from (3.12), (3.7) and assumption (3.3), namely

$$\operatorname{Re} \sum_{\substack{k=-1 \\ k \neq 0}}^{n-1} \lambda^{-\frac{k}{2}} \int_{x_0}^x \psi'_k(t) dt = \int_{x_0}^x \lambda^{\frac{1}{2}} \operatorname{Re} \psi'_{-1}(t) [1 + S(t)] dt,$$

where

$$|S(t)| \lesssim \begin{cases} \lambda^{-\frac{1}{2}} & \text{if } V \text{ is unbounded,} \\ \lambda^{-\frac{1}{2}} & \text{if } V \text{ is bounded and } \nu_+ < 0, \\ \lambda^{-\frac{1-(1+\varepsilon_2)\nu_+}{2}} & \text{if } V \text{ is bounded and } \nu_+ \geq 0. \end{cases}$$

Indeed, in the first case, Lemma 3.4, assumption (3.3), (3.9) and (3.11) give

$$\frac{\lambda^{\frac{k}{2}} |\psi'_k(x)|}{\lambda^{\frac{1}{2}} |\operatorname{Re} \psi'_{-1}(x)|} \lesssim \frac{\langle x \rangle^{(k+1)\nu_{\pm}}}{\lambda^{\frac{k-1}{2}} |\operatorname{Im} V(x)|} \lesssim \frac{1}{\langle x \rangle^{\frac{k+1}{2}(\varepsilon_1+1)} \lambda^{\frac{k+1}{4}}};$$

the other cases can be verified similarly.

Hence using (3.11) we get

$$\operatorname{Re} \sum_{\substack{k=-1 \\ k \neq 0}}^{n-1} \lambda^{-\frac{k}{2}} \int_{x_0}^x \psi'_k(t) dt \approx \lambda^{-\frac{1}{2}} \int_{x_0}^x \operatorname{Im} V(t) dt.$$

Putting all estimates from above together, we obtain (3.15). $\square$

The following proposition ensures that the terms in (2.8) containing derivatives of the cut-off function ξ are negligible in a sense.

Proposition 3.6. *Let Assumption I hold, g be as in (2.4) with $\{\psi'_k\}_{k \in [-1, n-1]}$, $0 \leq n \leq N$, determined by (2.12) and ξ be as in (2.5) with $\delta_{\pm}$, $\Delta_{\pm}$ as in (3.7). Then*

$$\kappa(\lambda) := \frac{\|\xi''g\| + \|\xi'g'\|}{\|\xi g\|} = o(1), \quad \lambda \rightarrow +\infty. \quad (3.17)$$

More precisely, $\kappa(\lambda) = \kappa_-(\lambda) + \kappa_+(\lambda)$ where (with some $c > 0$)

$$\kappa_{\pm}(\lambda) = \begin{cases} \mathcal{O} \left(\exp(-c \delta_{\pm}^{\nu_{\pm}+1+\varepsilon_1}) \right) & \text{if } V \text{ is unbounded at } \pm \infty, \\ \mathcal{O} \left(\exp(-c \lambda^{\frac{\varepsilon_2}{2}}) \right) & \text{if } V \text{ is bounded at } \pm \infty. \end{cases} \quad (3.18)$$

Proof. First notice that we have $\|\xi g\| \gtrsim 1$ from (3.15). The main step is to estimate $|g(x)|^2$ for $x \in \overline{\mathcal{J} \setminus \mathcal{J}'}$ where ξ' and ξ'' are supported. We give details only for $x > 0$; the other case is analogous.

We start with the case when V is unbounded at $+\infty$. Let $x_0 > 0$ be so large that $\operatorname{Im} V(x) > 0$ for all $x > x_0$. From the property (3.6) and the selected size of Δ_+ , see (3.7), we obtain for $x \in \mathcal{J}_+ \setminus \mathcal{J}'_+$ that

$$\int_{x_0}^x \operatorname{Im} V(t) dt \geq \int_{\delta_+ - 2\Delta_+}^x \operatorname{Im} V(t) dt \gtrsim \Delta_+ \operatorname{Im} V(\delta_+) \gtrsim \frac{\operatorname{Im} V(\delta_+)}{\delta_+^{\nu_+}}.$$

Thus using (3.7), we get

$$\begin{aligned} \lambda^{-\frac{1}{2}} \int_0^x \operatorname{Im} V(t) dt &= \lambda^{-\frac{1}{2}} \int_0^{x_0} \operatorname{Im} V(t) dt + \lambda^{-\frac{1}{2}} \int_{x_0}^x |\operatorname{Im} V(t)| dt \\ &\gtrsim -\lambda^{-\frac{1}{2}} + \frac{\delta_+^{2\nu_+ + \varepsilon_1 + 1}}{\operatorname{Im} V(\delta_+)} \frac{\operatorname{Im} V(\delta_+)}{\delta_+^{\nu_+}} \gtrsim \delta_+^{\nu_+ + \varepsilon_1 + 1}. \end{aligned}$$

Hence it follows from (3.15) that (with some $c_3 > 0$)

$$\forall x \in \mathcal{J}_+ \setminus \mathcal{J}'_+, \quad |g(x)| \lesssim \exp(-c_3 \delta_+^{\nu_+ + \varepsilon_1 + 1}).$$

Additional terms appearing in $\|\xi' g'\|_{L^2(\mathbb{R}_+)}$ can be estimated using (2.6), (3.7), (3.12), (3.9) and (3.5). In detail, for all $x \in \mathcal{J}_+ \setminus \mathcal{J}'_+$ we have (with some $c_4 > 0$)

$$\begin{aligned} |\xi'(x)g'(x)| &\lesssim \delta_+^{\nu_+} \sum_{k=-1}^n \lambda^{\frac{k}{2}} |\psi'_k(x)| \exp(-c_3 \delta_+^{\nu_+ + \varepsilon_1 + 1}) \\ &\lesssim \delta_+^{\nu_+} \left(\lambda^{\frac{1}{2}} + \sum_{k=0}^n \frac{\langle x \rangle^{(k+1)\nu_+}}{\lambda^{\frac{k}{2}}} \right) \exp(-c_3 \delta_+^{\nu_+ + \varepsilon_1 + 1}) \\ &\lesssim \delta_+^{\nu_+} \left(\frac{|V(\delta_+)|}{\delta_+^{2\nu_+ + \varepsilon_1 + 1}} + \delta_+^{(n+1)\nu_+} \right) \exp(-c_3 \delta_+^{\nu_+ + \varepsilon_1 + 1}) \\ &\lesssim \exp(-c_4 \delta_+^{\nu_+ + \varepsilon_1 + 1}). \end{aligned}$$

The term $\|\xi'' g\|_{L^2(\mathbb{R}_+)}$ is estimated similarly (and in fact more easily).

Putting everything together, we obtain (with some $c_5 > 0$)

$$\frac{\|\xi'' g\|_{L^2(\mathbb{R}_+)} + \|\xi' g'\|_{L^2(\mathbb{R}_+)}}{\|\xi g\|_{L^2(\mathbb{R}_+)}} \lesssim \exp(-c_5 \delta_+^{\nu_+ + \varepsilon_1 + 1}).$$

If V is bounded at $+\infty$, the appropriate rate in (3.18) follows immediately from (3.15) and the selected size of $\delta_{\pm}$ and $\Delta_{\pm}$, see (3.7). $\square$

3.3. Remainder estimate.

Theorem 3.7. *Let Assumption I hold and set $n := N - 1$. Let g be as in (2.4) with $\{\psi'_k\}_{k \in [-1, n-2]}$ determined by (2.12), ξ be as in (2.5) with $\delta_{\pm}$, $\Delta_{\pm}$ as in (3.7) and f be as in (2.7). Then*

$$\frac{\|(H_V - \lambda)f\|}{\|f\|} = \kappa(\lambda) + \sigma^{(n)}(\lambda),$$

where κ is as in (3.18) and $\sigma^{(n)} = \sigma_-^{(n)} + \sigma_+^{(n)}$ with, as $\lambda \rightarrow +\infty$,

i) if V is unbounded at $\pm\infty$

$$\sigma_{\pm}^{(n)}(\lambda) = \begin{cases} \mathcal{O}(\lambda^{-\frac{n+1}{2}} \sup_{x \in \mathcal{J}_{\pm}} |V(x)| \langle x \rangle^{(n+1)\nu_{\pm}}), & \nu_{\pm} < 0, \\ \mathcal{O}(\delta_{\pm}^{(n+1)\nu_{\pm}} \lambda^{\frac{1-n}{2}}), & \nu_{\pm} \geq 0, \end{cases}$$

ii) if V is bounded at $\pm\infty$

$$\sigma_{\pm}^{(n)}(\lambda) = \begin{cases} \mathcal{O}\left(\lambda^{-\frac{n+1}{2}}\right), & \nu_{\pm} < 0, \\ \mathcal{O}\left(\lambda^{-\frac{n+1}{2}(1-(1+\varepsilon_2)\nu_{\pm})}\right), & \nu_{\pm} \geq 0. \end{cases} \quad (3.19)$$

Proof. We employ the pseudomode construction for $n = N - 1$. The estimate of the remainder r_n , see (2.16), and the assumption (3.2) together with (3.9) and (3.10) yield that for $x > 0$ and V unbounded at $+\infty$ we have

$$|r_n(x)| \lesssim \begin{cases} |V(x)| \langle x \rangle^{(n+1)\nu_+} \lambda^{-\frac{n+1}{2}}, & \nu_+ < 0, \\ \delta_+^{(n+1)\nu_+} \lambda^{\frac{1-n}{2}}, & \nu_+ \geq 0, \end{cases}$$

and similarly for $x < 0$. Here the case $\nu_+ \geq 0$ also employs $\lambda \gtrsim \langle \delta_+ \rangle^{4\nu_+ + 2\varepsilon_1 + 2}$, which is a consequence of (3.3) and (3.7). If V is bounded at $\pm\infty$, the estimate of r_n follows straightforwardly from (2.16), assumptions (3.2), (3.3) and the choice of $\delta_{\pm}$ in (3.7). $\square$

3.4. Examples.

Example 3.8 (Polynomial-like potentials). Consider V satisfying Assumption I with $\nu_- = \nu_+ = -1$ and having the form

$$V := P_{\beta} + iQ_{\gamma},$$

where P_{β} and Q_{γ} are real-valued functions satisfying

$$\forall |x| \gtrsim 1, \quad |P_{\beta}(x)| \lesssim \langle x \rangle^{\beta}, \quad |Q_{\gamma}(x)| \approx \langle x \rangle^{\gamma},$$

with some numbers $\beta \in \mathbb{R}$ and $\gamma \geq 0$. Typical examples of P_{β} and Q_{γ} are polynomials of degree β and γ , respectively. Notice that a necessary condition to satisfy (3.1) is $\gamma \geq 0$, while the sufficient one, which additionally guarantees (3.3), requires $\gamma > (\beta - 2)/2$. In particular for $\beta < 2$ (i.e. $|\operatorname{Re} V(x)|$ grows slower than x^2) even a bounded $\operatorname{Im} V$ fits.

We define the quantity

$$\omega := \max\{\beta, \gamma\} \geq 0$$

and observe that $\omega = 0$ if, and only if, V is bounded. If ω is positive, then (3.7) yields

$$\delta = \delta_- = \delta_+ \approx \lambda^{\frac{1}{2(\gamma+1)} + \epsilon}, \quad (3.20)$$

where $\epsilon > 0$ can be made arbitrarily small by an appropriate choice of (small) $\varepsilon_1 > 0$. Hence the application of Theorem 3.7 yields (with $n := N - 1$)

$$\frac{\|(H_V - \lambda)f\|}{\|f\|} = \begin{cases} \mathcal{O}\left(\lambda^{-\frac{n+1}{2}}\right), & \omega \leq n + 1, \\ \mathcal{O}\left(\lambda^{-\frac{n+1}{2} + \frac{\omega - n - 1}{2(\gamma+1)} + \epsilon(\omega - n - 1)}\right), & \omega > n + 1, \end{cases} \quad (3.21)$$

as $\lambda \rightarrow +\infty$. Notice that the first case particularly involves bounded potentials (because $N \geq 1$) and that the decay rate in the second case improves by diminishing ϵ . It is also worth noticing that the restrictions on β and γ made above imply the uniform bounds

$$\frac{\omega - n - 1}{2(\gamma + 1)} < \begin{cases} 1/2 & \text{if } \gamma \geq \beta, \\ 1 & \text{if } \gamma < \beta, \end{cases} \quad (3.22)$$

which provides a rough estimate on the decay rate in the second case of (3.21).

Observe that the pseudomode with $n = 1$ (i.e. we require $N \geq 2$) is sufficient to treat all polynomial-like potentials. The pseudomode with $n = 0$ (i.e. $N \geq 1$)

suffices for potentials growing not faster than linearly. Notice also that for smooth potentials ($N = \infty$) the obtained rate is faster than any power of λ^{-1} .

Example 3.9 (Exponential potentials). Consider V satisfying Assumption I with $\nu_- = \nu_+ = 0$ and $N \geq 3$; a simple smooth choice is $V(x) := \cosh x + i \sinh x$. Since $|V(x)| \lesssim e^{|x|}$, see (3.5), we have for sufficiently large $\lambda > 0$ that

$$\delta = \delta_- = \delta_+ \approx \log \lambda.$$

Theorem 3.7 then gives

$$\frac{\|(H_V - \lambda)f\|}{\|f\|} = \mathcal{O}\left(\lambda^{\frac{2-N}{2}}\right),$$

thus exponential-type potentials can be treated using the pseudomodes with $n = 2$.

Example 3.10 (Bounded oscillating potentials). Consider two smooth potentials

$$V_1(x) := i \arctan x, \quad V_2(x) := 2i \arctan x + i \sin(\langle x \rangle^{1+\mu}), \quad 0 < \mu < 1.$$

Clearly, $\nu_{\pm} = -2$ for V_1 , however $\nu_{\pm} = \mu$ for V_2 . Since both potentials are smooth, we can achieve an arbitrarily fast decay in (3.19) in both cases by taking N large, nevertheless, substantially more terms in the pseudomode construction must be taken in the second case if μ is close to 1.

3.5. Decaying potentials. Finally, we discuss a class of potentials that do not satisfy the basic assumption (3.1), but the method of the present section still enables one to construct the desired pseudomodes. Indeed, the inequalities (3.15) suggest that the assumption (3.1) can be relaxed basically to $\operatorname{Im} V \notin L^1(\mathbb{R})$ if $\operatorname{Im} V$ has an appropriate sign for $x \gtrsim 1$ and $x \lesssim 1$. Here we analyse the simplest examples, namely a class of smooth potentials with the asymptotic behaviour

$$V(x) := i \frac{\operatorname{sgn}(x)}{\langle x \rangle^{\gamma}}, \quad |x| \gtrsim 1, \quad 0 < \gamma < 1.$$

Since the essential spectrum of H_V with this potential covers $[0, +\infty)$ and the numerical range of H_V is a shrinking neighbourhood of this set, we will consider $\lambda = a + ib$ with $a \rightarrow +\infty$ and $b \rightarrow 0+$.

The selection of a suitable $\delta_{\pm}$ for the cut-off is inspired by the estimate for $x \gtrsim 1$ (the case $x \lesssim -1$ and upper bounds are simpler)

$$\int_0^x \operatorname{Re}(\lambda^{\frac{1}{2}} \psi'_{-1}(t)) dt \gtrsim a^{-\frac{1}{2}} \int_0^x (\langle t \rangle^{-\gamma} - b) dt \gtrsim \frac{x^{1-\gamma}[1 - (1-\gamma)bx^{\gamma}] - C}{a^{\frac{1}{2}}}$$

with some $C \geq 0$. Thus, requiring that the first term in the expansion (2.4) leads to an integrable exponential, sought restrictions on δ_+ read

$$a^{\frac{1}{2}} \delta_+^{\gamma-1} + b \delta_+^{\gamma} = o(1), \quad \lambda \rightarrow \infty; \quad (3.23)$$

δ_- can be selected similarly and we can take $\Delta_{\pm} := \delta_{\pm}/4$. It can be also checked that the other terms in the expansion are negligible. Since V is bounded, it is clear that the cut-off works and we indeed have a decay like in (3.17). Regarding the remainders r_n , by taking sufficiently many terms in the expansion, we obtain a decay in (1.2) that is faster than any power of $1/a$.

The set Ω where (1.2) holds can be obtained from (3.23); in detail, we need

$$ba^{\frac{\gamma}{2(1-\gamma)}} = o(1), \quad \lambda \rightarrow \infty.$$

Observing that $V \in L^p(\mathbb{R})$ if $p\gamma > 1$, we can further describe Ω by a condition essentially appearing in [13, Thm. 5]:

$$b^{p-1} = o(a^{-\frac{1}{2}}), \quad \lambda \rightarrow \infty. \quad (3.24)$$

4. LOWER REGULARITY

Our goal in this section is to treat potentials of lower regularity. The first possibility is a perturbative approach, *i.e.* we search for conditions on a possibly singular perturbation W guaranteeing that the pseudomodes constructed for a regular part V , thus ignoring W , still exhibit a decay in (1.2). The second option is to introduce a λ -dependent mollification W^ε of W with $\varepsilon = \varepsilon(\lambda)$ and perform the construction for $V + W^\varepsilon$; naturally the crucial point is to determine suitable dependence of the mollification on λ .

In both approaches, eventually, we need more precise information on the L^p -norms of pseudomodes. We make here additional assumptions on the growth of V ; in fact we analyse in detail potentials with a polynomial growth, nonetheless, other cases may be treated similarly.

4.1. Weighted L^p -norms of pseudomodes.

Lemma 4.1. *Let Assumption I hold, let f be as in (2.7) with $0 \leq n \leq N$. Then for all sufficiently large $\lambda > 0$ the following holds.*

i) *If there is $\gamma \geq 0$ such that*

$$\forall x \gtrsim 1, \quad |\operatorname{Im} V(x)| \lesssim |x|^\gamma, \quad \text{or} \quad \forall x \lesssim -1, \quad |\operatorname{Im} V(x)| \lesssim |x|^\gamma, \quad (4.1)$$

then

$$\|f\|_p \gtrsim \lambda^{\frac{1}{2p(\gamma+1)}}, \quad 2 \leq p \leq \infty. \quad (4.2)$$

ii) *If there are $\gamma_\pm \geq 0$ such that*

$$|\operatorname{Im} V(x)| \gtrsim \begin{cases} |x|^{\gamma_+}, & x \gtrsim 1, \\ |x|^{\gamma_-}, & x \lesssim -1, \end{cases} \quad (4.3)$$

then

$$\|\langle x \rangle^\alpha f(x)\|_{L^p(\mathbb{R}_\pm)} \lesssim \lambda^{\frac{1+p\alpha}{2p(\gamma_\pm+1)}}, \quad 2 \leq p \leq \infty, \quad \alpha \geq 0. \quad (4.4)$$

Proof. i) Suppose that the first inequality in (4.1) holds. From (3.15) we have (with some $C \geq 0$, $c > 0$)

$$\|f\|_p^p \gtrsim \int_C^{\delta_+ - \Delta_+} e^{-pc\lambda^{-\frac{1}{2}}x^{\gamma+1}} dx = \lambda^{\frac{1}{2(\gamma+1)}} \int_{C\lambda^{-\frac{1}{2(\gamma+1)}}}^{(\delta_+ - \Delta_+)\lambda^{-\frac{1}{2(\gamma+1)}}} e^{-pcy^{\gamma+1}} dy.$$

Thus it remains to verify that $(\delta_+ - \Delta_+)\lambda^{-\frac{1}{2(\gamma+1)}} \gtrsim 1$. The latter follows from (3.7). The case of a bounded V is simple and in the unbounded case (necessarily with $\nu_+ \geq -1$, see (3.5)) we get from (4.1) and (3.7) that

$$\frac{\delta_+^{2(\gamma+1)}}{\lambda} \approx \frac{\delta_+^{2(\gamma+1)+4\nu_++2\varepsilon_1+2}}{|\operatorname{Im} V(\delta_+)|^2} \gtrsim \delta_+^{4\nu_++4+2\varepsilon_1} \gtrsim 1.$$

This proves (4.2) for $p \in [2, \infty)$ under the first of the assumptions in (4.1), the second alternative is treated similarly. The case $p = \infty$ is even simpler to show.

ii) For $x \geq 1$ we have $\langle x \rangle \approx x$, thus (3.15) and (4.3) yield (with some $C \geq 1$, $c > 0$)

$$\left(\int_0^C + \int_C^{\delta_+} \right) \langle x \rangle^{p\alpha} e^{-pc\lambda^{-\frac{1}{2}}x^{\gamma+1}} dx \lesssim 1 + \lambda^{\frac{1+p\alpha}{2(\gamma+1)}} \int_0^\infty y^{p\alpha} e^{-pcy^{\gamma+1}} dy.$$

The case $p = \infty$ can be checked by calculating the maximum of $|f|$ and the second case for $x \lesssim -1$ is analogous. $\square$

The immediate consequence is a possibility to employ pseudomodes constructed for V even for $V + W$, where W is an L^r -perturbation.

Theorem 4.2. *Let Assumption I hold and set $n := N - 1$. Let $\text{Im } V$ satisfy (4.1) and (4.3) and let $W \in L^{r-}(\mathbb{R}_-) + L^{r+}(\mathbb{R}_+)$ with some $2 \leq r_{\pm} < \infty$. Then*

$$\frac{\|(H_{V+W} - \lambda)f\|}{\|f\|} = \kappa(\lambda) + \sigma^{(n)}(\lambda) + \rho(\lambda),$$

where f , κ and $\sigma^{(n)}$ are as in Theorem 3.7 and $\rho = \rho_- + \rho_+$ with

$$\rho_{\pm}(\lambda) = \mathcal{O}\left(\lambda^{\frac{\gamma - \gamma_{\pm} - \frac{2}{r_{\pm}}(\gamma+1)}{4(\gamma_{\pm}+1)(\gamma+1)}}\right), \quad \lambda \rightarrow +\infty,$$

where γ and $\gamma_{\pm}$ are as in Lemma 4.1.

Proof. The estimate follows from (4.2), (4.4) with $\alpha = 0$ and Hölder inequality. In detail, with $2/r_{\pm} + 2/s_{\pm} = 1$, we have

$$\frac{\|Wf\|_{L^2(\mathbb{R}_{\pm})}}{\|f\|} \leq \frac{\|W\|_{L^{r_{\pm}}(\mathbb{R}_{\pm})}\|f\|_{L^{s_{\pm}}(\mathbb{R}_{\pm})}}{\|f\|} \lesssim \lambda^{\frac{2(\gamma+1) - s_{\pm}(\gamma_{\pm}+1)}{4s_{\pm}(\gamma+1)(\gamma_{\pm}+1)}}$$

and the claim follows when $s_{\pm}$ is expressed in terms of $r_{\pm}$. $\square$

The weighted L^p -estimates of f can be used also to employ the pseudomode with $n = N$, instead of $n = N - 1$ in Theorem 3.7, and thereby lower assumptions on the regularity of V .

Theorem 4.3. *Let Assumption I hold and set $n := N$. Let $\text{Im } V$ satisfy (4.1) and (4.3) and let $V^{(N+1)} \in L^2(\mathbb{R}) + L_{-\alpha_-}^{\infty}(\mathbb{R}_-) + L_{-\alpha_+}^{\infty}(\mathbb{R}_+)$ with some $\alpha_{\pm} \geq 0$. Then*

$$\frac{\|(H_V - \lambda)f\|}{\|f\|} = \kappa(\lambda) + \sigma^{(n)}(\lambda) + \tau(\lambda), \quad (4.5)$$

where f is the pseudomode with $n = N$, κ and $\sigma^{(n)}$ are as in Theorem 3.7 and $\tau = \tau_- + \tau_+$ with

$$\tau_{\pm}(\lambda) = \mathcal{O}\left(\lambda^{-\frac{N+1}{2} - \frac{1}{4(\gamma+1)}} + \lambda^{-\frac{N+1}{2} + \frac{\gamma - \gamma_{\pm} + 2\alpha_{\pm}(\gamma+1)}{4(\gamma_{\pm}+1)(\gamma+1)}}\right), \quad \lambda \rightarrow +\infty,$$

where γ and $\gamma_{\pm}$ are as in Lemma 4.1.

Proof. If f is taken as the pseudomode with $n = N$, the terms κ and $\sigma^{(n)}$ in (4.5) are estimated in the same way as in Theorem 3.7. The difference arises in the first term of r_n , see (2.16), since it contains $V^{(N+1)}$, more precisely, we need to estimate

$$\lambda^{-\frac{N+1}{2}} \|V^{(N+1)}f\|.$$

The claim follows straightforwardly from the assumption on $V^{(N+1)}$, Hölder inequality, (4.2) and (4.4). $\square$

4.2. Examples.

Example 4.4 (Singularly perturbed polynomial-like potentials). Let V be as in Example 3.8 and $W \in L^{r-}(\mathbb{R}_-) + L^{r+}(\mathbb{R}_+)$ with $2 \leq r_{\pm} < \infty$. If Assumption I holds with $N \geq 2$, Theorem 4.2 and the already obtained rates $\sigma^{(n)}$, see Example 3.8 and in particular (3.22), yield

$$\begin{aligned} \frac{\|(H_{V+W} - \lambda)f\|}{\|f\|} &= \mathcal{O}\left(\lambda^{-\frac{1}{2r_{\pm}(\gamma+1)}}\right) + \begin{cases} \mathcal{O}\left(\lambda^{-\frac{N}{2}}\right), & \omega \leq N, \\ \mathcal{O}\left(\lambda^{-\frac{N}{2} + \frac{\omega - N}{2(\gamma+1) + \epsilon(\omega - N)}}\right) & \omega > N, \end{cases} \\ &= \mathcal{O}\left(\lambda^{-\frac{1}{2r_{\pm}(\gamma+1)}}\right), \end{aligned}$$

as $\lambda \rightarrow +\infty$. Here the second equality follows by the restrictions made on β and γ in Example 3.8 (cf. particularly (3.22)). In other words, adding the singularity W deteriorates the decay rate (3.21) (at least when the result of Theorem 4.2 is used).

Example 4.5 (Imaginary step-like potential). Now we would like to treat the discontinuous example from (1.5). First, to apply Theorem 4.2, we specify a suitable splitting (to have a sufficiently regular V)

$$V(x) := i(1 - \eta(x)) \operatorname{sgn}(x), \quad W(x) := i\eta(x) \operatorname{sgn}(x), \quad (4.6)$$

with some $\eta \in C_0^\infty((-1, 1))$ and $\eta = 1$ on a neighbourhood of 0. Then Theorem 4.2 (with $N \geq 1$, $r_\pm := 2$ and $\gamma_\pm := 0 =: \gamma$) yields

$$\frac{\|(H_{i \operatorname{sgn}} - \lambda)f\|}{\|f\|} = \mathcal{O}\left(\lambda^{-\frac{1}{4}}\right), \quad \lambda \rightarrow +\infty.$$

Example 4.6 (Polynomial growth with a local singularity). As an application of Theorem 4.3, let us consider the following class of potentials

$$V(x) := i \operatorname{sgn}(x) |x|^\gamma (2 + \sin |x|^{-\mu}), \quad \mu \in (0, 1), \quad \gamma \in \mathbb{N}.$$

If $\gamma \geq 2$ and $N := \gamma - 1$, it is easy to verify that V also satisfies the other items of Assumption I (with $\nu_\pm := -1$), namely the basic regularity requirement $V \in W^{N, \infty}(\mathbb{R})$. Since the derivative $V^{(\gamma)}$ has a singularity at zero, however, the best decay rate we can obtain by directly applying Theorem 3.7 is

$$\frac{\|(H_V - \lambda)f\|}{\|f\|} = \mathcal{O}\left(\lambda^{-\frac{\gamma-1}{2} + \frac{1}{2(\gamma+1)} + \epsilon}\right), \quad \lambda \rightarrow +\infty,$$

where $\epsilon > 0$ can be made arbitrarily small (cf. (3.20)). On the other hand, observing that $V^{(\gamma)} \in L^2(\mathbb{R}) + L^\infty(\mathbb{R})$ and applying Theorem 4.3 (with $\alpha_\pm := 0$ and $\gamma_\pm := \gamma$), where the pseudomode with one more term in the expansion is employed, we obtain a better result, namely

$$\frac{\|(H_V - \lambda)f\|}{\|f\|} = \mathcal{O}\left(\lambda^{-\frac{\gamma}{2}}\right), \quad \lambda \rightarrow +\infty.$$

4.3. Mollification strategy. Now we turn to the alternative approach to deal with irregular potentials.

Let $w \in C_0^\infty(\mathbb{R})$ with $0 \leq w \leq 1$, $\operatorname{supp} w = [-1, 1]$ and $\|w\|_1 = 1$ and define

$$w_\varepsilon(x) := \frac{1}{\varepsilon} w\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}, \quad \varepsilon > 0. \quad (4.7)$$

For $\phi \in L_{\operatorname{loc}}^p(\mathbb{R})$, we introduce the L^p modulus of continuity on an interval $\mathcal{J} \subset \mathbb{R}$ by

$$\omega_p(\varepsilon; \phi, \mathcal{J}) := \sup_{0 < |t| < \varepsilon} \|\phi(\cdot + t) - \phi\|_{L^p(\mathcal{J})}, \quad 1 \leq p < \infty.$$

Finally, we introduce an ε -neighbourhood of $\mathcal{J}$, $\mathcal{J}_\varepsilon := \{x \in \mathbb{R} : \operatorname{dist}(x, \mathcal{J}) < \varepsilon\}$.

The main idea in what follows is the mollification of a singular part of the potential. For $\phi \in L_{\operatorname{loc}}^p(\mathbb{R})$ and w_ε as in (4.7), we denote

$$\phi^\varepsilon := w_\varepsilon * \phi. \quad (4.8)$$

To be able to estimate newly constructed pseudomodes, we need several basic properties of mollifications and their relation to the L^p modulus of continuity summarised in the following lemma; the proof relies on Minkowski's integral inequality and properties of the convolution and of w .

Lemma 4.7. *Let $\phi \in L_{\operatorname{loc}}^p(\mathbb{R})$ with $1 \leq p < \infty$, ϕ^ε be as in (4.8), $\mathcal{J}$ be an interval and $\mathcal{J}_\varepsilon$ its ε -neighbourhood. Then for every $1 \leq p < \infty$, $j \in \mathbb{N}$ and $\varepsilon > 0$ we have*

$$\begin{aligned} \|\phi^\varepsilon\|_{L^p(\mathcal{J})} &\leq \|\phi\|_{L^p(\mathcal{J}_\varepsilon)}, & \|\phi^\varepsilon\|_{L^\infty(\mathcal{J})} &\leq \varepsilon^{-\frac{1}{p}} \|\phi\|_{L^p(\mathcal{J}_\varepsilon)}, \\ \|\phi - \phi^\varepsilon\|_{L^p(\mathcal{J})} &\leq \omega_p(\varepsilon; \phi, \mathcal{J}), & \|(\phi^\varepsilon)^{(j)}\|_{L^p(\mathcal{J})} &\leq \varepsilon^{-j} \omega_p(\varepsilon; \phi, \mathcal{J}) \|w^{(j)}\|_{L^1}. \end{aligned}$$

We proceed with the construction of pseudomodes for a potential $V+W$ where W is possibly discontinuous and singular. In fact the pseudomodes are constructed for $V + W^\varepsilon$ with a certain λ -dependent mollification. Thus besides usual remainders (2.13) we need to estimate also $\|(W - W^\varepsilon)f\|$.

While the construction can be in principle performed with an arbitrary number of terms, we restrict ourselves to the case $n \in [[0, 1]]$ since the assumptions on the singular part W would become more complicated and implicit for $n > 1$. In spite of this restriction, we can still treat potentials with $\nu_\pm < 0$, *i.e.* even with some super-polynomial growth or oscillations. More precisely, new pseudomodes are constructed under the following assumptions.

Assumption II. Let V satisfy Assumption I with $N \in [[1, 2]]$ and $\nu_\pm < 0$ and suppose that $W = W_1 + W_2$ satisfy

- a) $|\operatorname{Im} W_1| \leq (1 - \varepsilon)|\operatorname{Im} V|$ with some $0 < \varepsilon < 1$ and with $\varepsilon_1 > 0$ from Assumption I

$$\forall x \in \mathbb{R}_\pm, \quad |\operatorname{Re} W_1(x)| \lesssim |\operatorname{Im} V(x)|^2 \langle x \rangle^{-4(\nu_\pm + \varepsilon_1) - 2}. \quad (4.9)$$

- b) $W_2 \in L^2(\mathbb{R})$ and $\operatorname{supp} W_2$ is compact.

The mollification (4.8) is done separately for three parts of W , namely

$$\tilde{W} := (\chi_- W_1)^{\varepsilon_-} + (\chi_+ W_1)^{\varepsilon_+} + W_2^{\varepsilon_0}$$

with $\chi_\pm$ being the characteristic function of $\mathbb{R}_\pm$ and

$$\varepsilon_\iota := \lambda^{-\alpha_\iota}, \quad \alpha_\iota \in (0, 1), \quad \iota \in \{-, +, 0\}. \quad (4.10)$$

The expansion (see (2.4))

$$\tilde{g} := \exp \left(- \sum_{k=-1}^{n-1} \lambda^{-\frac{k}{2}} \psi_k(x) \right), \quad n \leq 1, \quad (4.11)$$

is determined by functions ψ'_k satisfying (2.17) with V replaced by

$$\tilde{V} := V + \tilde{W}. \quad (4.12)$$

On the other hand, we keep the size of the cut-off the same as for V only, *i.e.* the new pseudomodes read

$$\tilde{f} := \xi \tilde{g}, \quad (4.13)$$

where ξ is as in (2.5) with $\delta_\pm$, $\Delta_\pm$ as in (3.7) with V .

Lemma 4.8. *Let Assumption II hold and $\tilde{g}$ be as in (4.11) with (4.12). Then*

$$\kappa(\lambda) := \frac{\|\xi'' \tilde{g}\| + \|\xi' \tilde{g}'\|}{\|\tilde{g}\|} = o(1), \quad \lambda \rightarrow +\infty, \quad (4.14)$$

with κ as in (3.18) (with possibly a smaller positive constant $c > 0$).

Proof. We start with showing

$$\|\tilde{f}\|^2 \gtrsim \int_{\mathcal{J}} \exp \left(-c_3 \lambda^{-\frac{1}{2}} \int_0^{|x|} |\operatorname{Im} V(t)| dt \right) dx \quad (4.15)$$

with some $c_3 > 0$, where $\tilde{f}$ is defined in (4.13). We give details on estimates on $\mathbb{R}_+$, the other case is analogous. First notice that $\tilde{W}$ is locally bounded, see Lemma 4.7.

Moreover, since $\varepsilon_\pm = o(\Delta_\pm)$, we obtain from (3.2), (3.6) and assumptions on W that

$$\begin{aligned} |\operatorname{Im}(\chi_\pm W_1)^{\varepsilon_\pm}(x)| &\leq \int_{\mathbb{R}} w_{\varepsilon_\pm}(y) |\operatorname{Im} W_1(x-y)| dy \leq \sup_{|y| < \varepsilon_\pm} |\operatorname{Im} W_1(x-y)| \\ &\leq (1-\varepsilon) \sup_{|y| < \varepsilon_\pm} |\operatorname{Im} V(x-y)| \\ &\leq (1-\varepsilon) \left(|\operatorname{Im} V(x)| + \sup_{|y| < \varepsilon_\pm} \left| \int_x^{x-y} |\operatorname{Im} V'(t)| dt \right| \right) \\ &\leq (1-\varepsilon) |\operatorname{Im} V(x)| (1 + \mathcal{O}(\varepsilon_\pm)) \end{aligned} \quad (4.16)$$

and similarly, using (4.9) and (3.6),

$$\forall x \in \mathbb{R}_\pm, \quad |\operatorname{Re}(\chi_\pm W_1)^{\varepsilon_\pm}(x)| \lesssim |\operatorname{Im} V(x)|^2 \langle x \rangle^{-4(\nu_\pm + \varepsilon_1) - 2}. \quad (4.17)$$

For W_2 , Lemma 4.7 yields immediately

$$|W_2^{\varepsilon_0}(x)| \leq \varepsilon_0^{-\frac{1}{2}} \|W_2\| = o\left(\lambda^{\frac{1}{2}}\right), \quad \lambda \rightarrow +\infty. \quad (4.18)$$

The estimates above imply that $\tilde{W}$ can be absorbed by V or λ in all relevant estimates in Lemmata 3.4, 3.5 and Proposition 3.6; in particular notice that W_2 affects the estimates only on a compact set due to the assumed boundedness of $\operatorname{supp} W_2$, and that the size of $\operatorname{Re} \tilde{W}$ is the largest possible complying with (3.3) and (3.7). Straightforward estimates of (3.16) with $n \in [[0, 1]]$ and with V replaced by $\tilde{V}$ lead to (with some $c_1, c_2 > 0$)

$$e^{-c_1 \lambda^{-\frac{1}{2}} \int_0^{|x|} |\operatorname{Im} V(t)| dt} \lesssim |\tilde{g}(x)| \lesssim e^{-c_2 \lambda^{-\frac{1}{2}} \int_0^{|x|} |\operatorname{Im} V(t)| dt}$$

for $n \in [[0, 1]]$, all sufficiently large λ and all $x \in \mathcal{J}$; here (4.16), (4.17), (4.18) and the boundedness of $\operatorname{supp} W_2$ were used. Hence (4.15) follows.

To verify (4.14), we need in addition that

$$\forall x \in \mathcal{J}_\pm, \quad |((\chi_\pm W_1)^{\varepsilon_\pm})'(x)| \lesssim \frac{|\operatorname{Im} V(x)| + |\operatorname{Im} V(x)|^2 \langle x \rangle^{-4(\nu_\pm + \varepsilon_1) - 2}}{\varepsilon_\pm}; \quad (4.19)$$

the proof is similar to (4.16) and (4.17). Hence, using (2.17), (4.19) and (4.10), we obtain

$$\forall x \in \mathcal{J}_\pm, \quad \lambda^{\frac{1}{2}} |\psi'_{-1}(x)| + |\psi'_0(x)| \lesssim \lambda^{\frac{1}{2}} + |V(x)| + |\operatorname{Im} V(x)|^2 \langle x \rangle^{-4(\nu_\pm + \varepsilon_1) - 2}.$$

The rest of the proof is a simple modification of the one of Proposition 3.6. $\square$

Now we are in a position to state the main result of this section.

Theorem 4.9. *Let Assumption II hold and $\tilde{f}$ be as in (4.13) with $n \in [[0, 1]]$. Then*

$$\frac{\|(H_{V+W} - \lambda)\tilde{f}\|}{\|\tilde{f}\|} = \kappa(\lambda) + \sigma^{(n)}(\lambda) + \frac{\zeta^{(n)}(\lambda)}{\|\tilde{f}\|}, \quad (4.20)$$

where κ and $\sigma^{(n)}$ are as in Theorem 3.7 and $\zeta^{(n)} = \zeta_-^{(n)} + \zeta_+^{(n)} + \zeta_0^{(n)}$ with, as $\lambda \rightarrow +\infty$,

$$\begin{aligned} \zeta_\pm^{(0)}(\lambda) &= \mathcal{O}\left(\omega_2(\varepsilon_\pm; W_1, \mathcal{J}_\pm) \left(1 + \varepsilon_\pm^{-1} \lambda^{-\frac{1}{2}}\right)\right), \\ \zeta_0^{(0)}(\lambda) &= \mathcal{O}\left(\omega_2(\varepsilon_0; W_2, \mathbb{R}) \left(1 + \varepsilon_0^{-1} \lambda^{-\frac{1}{2}}\right)\right), \\ \zeta_\pm^{(1)}(\lambda) &= \mathcal{O}\left(\omega_2(\varepsilon_\pm; W_1, \mathcal{J}_\pm) \left(1 + \varepsilon_\pm^{-2} \lambda^{-1}\right) + \omega_4(\varepsilon_\pm; W_1, \mathcal{J}_\pm) \varepsilon_\pm^{-2} \lambda^{-2}\right), \\ \zeta_0^{(1)}(\lambda) &= \mathcal{O}\left(\omega_2(\varepsilon_0; W_2, \mathbb{R}) \left(1 + \varepsilon_0^{-2} \lambda^{-1}\right) + \omega_4(\varepsilon_0; W_2, \mathbb{R}) \varepsilon_0^{-2} \lambda^{-2}\right), \end{aligned}$$

where ε_i are as in (4.10).

Proof. Inserting the pseudomode $\tilde{f}$, we obtain

$$\|(H_{V+W} - \lambda)\tilde{f}\| \leq \|(H_{\tilde{V}} - \lambda)\tilde{f}\| + \|(\tilde{W} - W)\tilde{f}\|. \quad (4.21)$$

We need to estimate remainders (2.13) with $\tilde{V}$ and the second term in (4.21). The claim follows straightforwardly from (2.18) and the properties of the mollification, see Lemma 4.7. $\square$

4.4. Examples. First we prove a lemma on the L^p modulus of continuity of piecewise C^1 potentials with a controlled growth.

Lemma 4.10. *Let W be a piece-wise C^1 function, more precisely $W \in C^1(\mathbb{R} \setminus \mathcal{M})$ with $\mathcal{M} := \{a_k\}_{k \in \mathbb{Z}}$ such that for all $k \in \mathbb{Z}$, $a_{k+1} - a_k \gtrsim 1$ and for all $k \in \mathbb{Z}$ the one-sided limits $\lim_{x \rightarrow a_k \pm} W(x)$ exist and are finite. Moreover, let W satisfy*

$$\exists \beta_{\pm} \in \mathbb{R}, \quad \forall x \in \mathbb{R}_{\pm}, \quad |W(x)| \lesssim \langle x \rangle^{\beta_{\pm}},$$

and

$$\exists \gamma_{\pm} \in \mathbb{R}, \quad \forall x \in \mathbb{R}_{\pm} \setminus \mathcal{M}, \quad |W'(x)| \lesssim \langle x \rangle^{\gamma_{\pm}}.$$

Then, for all ε small and $\delta_{\pm}$ large,

$$\omega_p(\varepsilon; W, \mathcal{J}_{\pm}) \lesssim \varepsilon \delta_{\pm}^{\gamma_{\pm} + \frac{1}{p}} + \varepsilon^{\frac{1}{p}} \delta_{\pm}^{\beta_{\pm} + \frac{1}{p}}, \quad 2 \leq p < \infty. \quad (4.22)$$

If in addition $\text{supp } W$ is bounded, then

$$\omega_p(\varepsilon; W, \mathbb{R}) \lesssim \varepsilon^{\frac{1}{p}}, \quad 2 \leq p < \infty. \quad (4.23)$$

Proof. We analyse only the case with $\mathcal{J}_+$, the other situation is analogous. We can assume that $a_0 = 0$ and $a_{L+1} = \delta_+$ with some $L \in \mathbb{N}$. Splitting the intervals (a_k, a_{k+1}) to ε -neighbourhoods of the discontinuities and the rest and employing the assumptions on W and W' , we have, for every $|t| < \varepsilon$,

$$\begin{aligned} \int_{\mathcal{J}_+} |W(x+t) - W(x)|^p dx &= \sum_{k=0}^L \int_{a_k}^{a_{k+1}} |W(x+t) - W(x)|^p dx \\ &= \sum_{k=0}^L \int_{a_k}^{a_{k+1}-\varepsilon} \left| \int_x^{x+t} W'(\xi) d\xi \right|^p dx + \sum_{k=0}^L \int_{a_{k+1}-\varepsilon}^{a_{k+1}} |W(x+t) - W(x)|^p dx \\ &\lesssim \sum_{k=0}^L \int_{a_k}^{a_{k+1}-\varepsilon} dx \left(\text{esssup}_{(a_0-\varepsilon, a_{L+1})} |W'| \right)^p \varepsilon^p + \sum_{k=0}^L \int_{a_{k+1}-\varepsilon}^{a_{k+1}} dx \sup_{(a_{k+1}-\varepsilon, a_{k+1}+\varepsilon)} |W|^p \\ &\lesssim \delta_+^{1+p\gamma_+} \varepsilon^p + \varepsilon \sum_{k=0}^L a_{k+1}^{p\beta_+}. \end{aligned}$$

Consequently, (4.22) follows since $a_{k+1} - a_k \gtrsim 1$ and the last sum can be estimated by an integral (details are omitted).

If $\text{supp } W$ is bounded, then the estimates are performed on a bounded interval independent of $\delta_{\pm}$ and so (4.23) follows as well. $\square$

Example 4.11 (Imaginary step-like potential continued). Following Example 4.5, we keep the splitting of the imaginary sign potential $i \text{sgn}$ to the sum of the smooth potential V and the discontinuous W with a compact support, see (4.6). The latter obeys Assumption II with $W_1 := 0$. Applying Theorem 4.9 (with $n := 1$ and $\alpha_0 := 1/2$ in (4.10)) with help of Lemma 4.1 (with $\gamma := 0$ and $p := 2$) to estimate $\|\tilde{f}\|$ and Lemma 4.10 to estimate the moduli of continuity in $\zeta^{(1)}(\lambda)$, we arrive at

$$\frac{\|(H_{i \text{sgn}} - \lambda)\tilde{f}\|}{\|\tilde{f}\|} = \mathcal{O}\left(\lambda^{-\frac{1}{2}}\right), \quad \lambda \rightarrow +\infty.$$

This is an improvement with respect to the rate $\lambda^{-\frac{1}{4}}$ provided by Theorem 4.2, see Example 4.5. Nevertheless, even this better rate is not optimal, as it is known from [9] that there exists a pseudomode with the decay rate $O(\lambda^{-1})$ and that it is actually the best possible.

Example 4.12 (Infinite steps). Let us consider the step-like (odd) potential

$$U(x) := i \lfloor |x| \rfloor^\gamma \operatorname{sgn}(x), \quad \gamma > 0,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Hence U represents a piece-wise approximation of $x \mapsto i |x|^\gamma \operatorname{sgn}(x)$ (cf. Example 3.8 with $P_\beta := 0$). The basic hypothesis (3.1) is clearly satisfied, so it is expected that H_U admits pseudomodes. However, Theorem 3.7 cannot be used because of the lack of regularity required by Assumption I.

We show how Theorem 4.9 can be used instead. To this end, we split U as

$$U = V + W, \quad W = W_1 + W_2,$$

where

$$\begin{aligned} V(x) &:= i(1 - \eta(x)) |x|^\gamma \operatorname{sgn}(x), \\ W_1(x) &:= i(1 - \eta(x)) (\lfloor |x| \rfloor^\gamma - |x|^\gamma) \operatorname{sgn}(x), \\ W_2(x) &:= i\eta(x) \lfloor |x| \rfloor^\gamma \operatorname{sgn}(x), \end{aligned}$$

and $\eta \in C_0^\infty(\mathbb{R})$ is such that $0 \leq |\eta| \leq 1$ and $\eta = 1$ on the interval $[-\gamma - 1, \gamma + 1]$. Using the mean value theorem and properties of the floor function, we have

$$|W_1(x)| \leq (1 - \eta(x)) \gamma |x|^{\gamma-1} |\lfloor |x| \rfloor - |x|| \leq (1 - \eta(x)) \gamma |x|^{\gamma-1}$$

for every $x \in \mathbb{R}$. Since $W_1(x)$ equals zero if $|x| \leq \gamma + 1$, we see that Assumption II holds with $\varepsilon := 1/(\gamma + 1)$.

Now we are in a position to apply Theorem 4.9 with $n := 1$. For $\sigma^{(1)}(\lambda)$, we always have a decay, see Example 3.8. Lemma 4.1 with $p := 2$ yields

$$\|\tilde{f}\| \gtrsim \lambda^{\frac{1}{4(\gamma+1)}}$$

for all sufficiently large λ and Lemma 4.10 immediately implies (we take $\alpha_0 := 1/2$)

$$\zeta_0^{(1)}(\lambda) = \mathcal{O}\left(\lambda^{-\frac{1}{4}}\right), \quad \lambda \rightarrow +\infty.$$

Again from Lemma 4.10 (with $\beta_\pm := \gamma - 1$ and $\gamma_\pm$ arbitrarily large negative), we obtain for W_1 (with $\alpha_\pm := \alpha \in (0, 1)$)

$$\lambda^{-\frac{1}{4(\gamma+1)}} \omega_p(\lambda^{-\alpha}; W_1, \mathcal{J}_\pm) = \mathcal{O}\left(\lambda^{-\frac{\alpha}{p} + \frac{1}{2p(\gamma+1)} + \frac{2\gamma-3}{4(\gamma+1)} + \epsilon}\right), \quad \lambda \rightarrow +\infty,$$

where $\epsilon = \epsilon(\gamma, p) > 0$ can be made arbitrarily small. Calculating the individual terms in $\zeta_\pm^{(1)}$, we obtain the following conditions on α to have a decay in (4.20):

$$\frac{\gamma - 1}{\gamma + 1} < \alpha < \frac{1}{3} \frac{\gamma + 3}{\gamma + 1}.$$

These can be satisfied only if $\gamma < 3$ and the corresponding decay rate in (4.20) can be calculated in a straightforward way.

5. PSEUDOMODES FOR GENERAL CURVES

In this section, we focus on potentials V with unbounded $\operatorname{Im} V$ and investigate pseudomodes for other curves in the complex plane than lines parallel to the real axis. The construction is basically the same as in Section 3, however, instead of having the pseudomode localised around 0, we work around a λ -dependent point.

As the support of the pseudomode will be contained in $\mathbb{R}_+$, this construction is suitable also for operators in $L^2(\mathbb{R}_+)$. In fact we shall rather proceed reversely

and formulate the strategy for such a situation, the subsequent applicability of the results for problems in $L^2(\mathbb{R})$ is obvious.

5.1. Admissible class of potentials and curves. To keep the previous strategy working without more complicated and implicit conditions on V , we add an additional condition on $\text{Im } V$, namely a control of $\text{Im } V'(x)$. In detail, we assume the following.

Assumption III. Let $N \in \mathbb{N}$, $N > 1$, let $V \in W_{\text{loc}}^{N,\infty}(\overline{\mathbb{R}_+})$ satisfy

$$\lim_{x \rightarrow +\infty} \text{Im } V(x) = +\infty \quad (5.1)$$

together with all the conditions of Assumption I for $x > 0$. In addition suppose that

$$\forall x \gtrsim 1, \quad \text{Im } V'(x) \gtrsim \text{Im } V(x) \langle x \rangle^\nu, \quad (5.2)$$

where $\nu := \nu_+$.

In this section, we write

$$\lambda = a + ib, \quad a \in \mathbb{R}, \quad b \in \mathbb{R}_+.$$

For sufficiently large b we define the *turning point* x_b of $\text{Im } V$ by the equation

$$\text{Im } V(x_b) = b,$$

which is well-defined due to (5.2). The cut-off is taken around the turning point x_b , namely:

$$\begin{aligned} \xi &\in C_0^\infty(\mathbb{R}_+), \quad 0 \leq \xi \leq 1, \\ \forall x \in (x_b - \delta + \Delta, x_b + \delta - \Delta), \quad \xi(x) &= 1, \\ \forall x \notin (x_b - \delta, x_b + \delta), \quad \xi(x) &= 0. \end{aligned} \quad (5.3)$$

Here we take

$$\delta := \frac{x_b^{-\nu}}{2}, \quad \Delta := \frac{\delta}{4},$$

and denote

$$\mathcal{J}_b := (x_b - \delta, x_b + \delta), \quad \mathcal{J}'_b := (x_b - \delta + \Delta, x_b + \delta - \Delta). \quad (5.4)$$

Finally, we restrict the real part of λ by

$$\forall x \in \mathcal{J}_b, \quad b^{\frac{2}{3}} x_b^{\frac{2\nu}{3}} \lesssim |a| \lesssim a - \text{Re } V(x) \lesssim b^2 x_b^{-4\nu - 4\varepsilon_1 - 2}. \quad (5.5)$$

The set of admissible a 's is non-empty since $\sup_{x \in \mathcal{J}_b} |\text{Re } V(x)| \lesssim b^2 x_b^{-4\nu - 4\varepsilon_1 - 2}$ by assumption (3.3) and the choice of $\mathcal{J}_b$ in (5.4); moreover it follows from (3.3) that $b^{\frac{2}{3}} x_b^{\frac{2\nu}{3}} \lesssim b^2 x_b^{-4\nu - 4\varepsilon_1 - 2}$ for every sufficiently small $\varepsilon_1 > 0$.

5.2. Pseudomode construction. The pseudomode will have the form

$$f(x) := \xi(x)g(x) \quad \text{with} \quad g(x) := \exp \left(- \sum_{k=-1}^{n-1} \lambda^{-\frac{k}{2}} \int_{x_b}^x \psi'_k(t) dt \right), \quad (5.6)$$

where $\{\psi'_k\}_{k \in [-1, n-1]}$ are determined by (2.12).

Proposition 5.1. *Let Assumption III hold, $0 \leq n \leq N$, $\{\psi'_k\}_{k \in [-1, n-1]}$ be determined by (2.12), $\mathcal{J}_b, \mathcal{J}'_b$ be as in (5.4), ξ, g be as in (5.3), (5.6), respectively, and a satisfy (5.5). Then there exists $c > 0$ such that*

$$\frac{\|\xi'' g\|_{L^2(\mathbb{R}_+)} + \|\xi' g'\|_{L^2(\mathbb{R}_+)}}{\|\xi g\|_{L^2(\mathbb{R}_+)}} = \mathcal{O}(\exp(-cx_b^{\nu+1+2\varepsilon_1})), \quad b \rightarrow +\infty. \quad (5.7)$$

Proof. Let us first estimate $\operatorname{sgn}(x - x_b) \int_{x_b}^x \operatorname{Re}(\lambda^{\frac{1}{2}} \psi'_{-1}(t)) dt$. For $x_b < x < x_b + \delta$ (the other case is analogous), an analogue of the complex square root formula (3.4), the choice of a in (5.5) and the mean value theorem lead to

$$\begin{aligned} \operatorname{Re}(\lambda^{\frac{1}{2}} \psi'_{-1}(x)) &\gtrsim \frac{\operatorname{Im} V(x) - b}{(a - \operatorname{Re} V(x))^{\frac{1}{2}} + (\operatorname{Im} V(x) - b)^{\frac{1}{2}}} \\ &\gtrsim \frac{\operatorname{Im} V'(x_b)(x - x_b)}{|a|^{\frac{1}{2}} + (\operatorname{Im} V'(x_b)(x - x_b))^{\frac{1}{2}}}. \end{aligned} \quad (5.8)$$

In the second inequality we have used also that the values of $\operatorname{Im} V'$ at $\mathcal{J}_b$ are comparable, see (3.2) and (3.6). Hence, for every $x \in \mathcal{J}_b \setminus \mathcal{J}'_b$, we have

$$\int_{x_b}^x \operatorname{Re}(\lambda^{\frac{1}{2}} \psi'_{-1}(t)) dt \gtrsim \frac{b x_b^{-\nu}}{|a|^{\frac{1}{2}} + b^{\frac{1}{2}}} \gtrsim \begin{cases} x_b^{\nu+1+2\varepsilon_1}, & |a| > b, \\ b^{\frac{1}{2}} x_b^{-\nu}, & |a| \leq b. \end{cases} \quad (5.9)$$

Here the first inequality employs (5.2) in the numerator and (3.2) in the denominator, while the second inequality follows from (5.5). Notice that by (3.3) we have $b^{\frac{1}{2}} x_b^{-\nu} \gtrsim x_b^{\nu+1+2\varepsilon_1}$, so the left hand side tends to infinity as $b \rightarrow +\infty$ too.

Next we investigate $\int_{x_b}^x |\lambda^{-\frac{k}{2}} \psi'_k|$ for $k \in [[0, n-1]]$ and $x \in \mathcal{J}_b$. The estimates analogous to (3.13), (3.14) and the choice of a in (5.5) yield

$$\int_{x_b}^x |\lambda^{-\frac{k}{2}} \psi'_k(t)| dt \lesssim \sum_{j=1}^{k+1} \int_{x_b}^x \frac{|V(t)|^j x_b^{(k+1)\nu}}{|a - \operatorname{Re} V(t)|^{j+\frac{k}{2}}} dt \lesssim \sum_{j=1}^{k+1} \frac{(|a|^j + b^j) x_b^{k\nu}}{|a|^{j+\frac{k}{2}}}. \quad (5.10)$$

Further from (5.5) and (3.3)

$$\sum_{j=1}^{k+1} \frac{|a|^j x_b^{k\nu}}{|a|^{j+\frac{k}{2}}} \lesssim \left(\frac{x_b^\nu}{|a|^{\frac{1}{2}}} \right)^k \lesssim \left(\frac{x_b^{2\nu}}{b} \right)^{\frac{k}{3}} \lesssim x_b^{-\frac{2}{3}k(\nu+1+\varepsilon_1)}$$

and

$$\sum_{j=1}^{k+1} \frac{b^j x_b^{k\nu}}{|a|^{j+\frac{k}{2}}} \lesssim \begin{cases} x_b^{-k(\nu+1+\varepsilon_1)}, & |a| > b, \\ \sum_{j=1}^{k+1} (b x_b^{-2\nu})^{\frac{j-k}{3}}, & |a| \leq b. \end{cases}$$

Thus, using again (3.3), for every $x \in \mathcal{J}_b \setminus \mathcal{J}'_b$ we get

$$\frac{\int_{x_b}^x |\lambda^{-\frac{k}{2}} \psi'_k(t)| dt}{\int_{x_b}^x \operatorname{Re}(\lambda^{\frac{1}{2}} \psi'_{-1}(t)) dt} \lesssim x_b^{-(\frac{2}{3}k+1)(\nu+1+\varepsilon_1)} + \begin{cases} x_b^{-(k+1)(\nu+1+\varepsilon_1)}, & |a| > b, \\ x_b^{-(\nu+1+\varepsilon_1)}, & |a| \leq b. \end{cases} \quad (5.11)$$

Using (5.9) with help of (5.11) and (2.6), we obtain (with some $C_1 > 0$)

$$\|\xi'' g\|_{L^2(\mathbb{R}_+)} + \|\xi' g'\|_{L^2(\mathbb{R}_+)} \lesssim \exp\left(-C_1 \frac{b x_b^{-\nu}}{|a|^{\frac{1}{2}} + b^{\frac{1}{2}}}\right).$$

The estimate is clear for the first norm on the left hand side. To control the extra terms obtained by differentiating g , we employ the bounds coming from Gronwall's inequality (3.5) for the term $\lambda^{\frac{1}{2}} \psi'_{-1}$ and the other terms $\lambda^{-\frac{k}{2}} \psi'_k$ can be estimated similarly as in (5.10).

Finally, to show (5.7), we need to verify that $\|\xi g\|_{L^2(\mathbb{R}_+)}$ is not too small. To this end notice that for $a < b$

$$\begin{aligned} \int_{x_b}^{x_b + x_b^{-2|\nu|}} |\operatorname{Re}(\lambda^{\frac{1}{2}} \psi'_{-1}(t))| dt &\lesssim \int_{x_b}^{x_b + x_b^{-2|\nu|}} |\operatorname{Im} V(t) - b|^{\frac{1}{2}} dt \\ &\lesssim b^{\frac{1}{2}} x_b^{-3|\nu| + \frac{1}{2}\nu} \end{aligned} \quad (5.12)$$

and for $a \geq b$

$$\begin{aligned} \int_{x_b}^{x_b + x_b^{-2|\nu|}} |\operatorname{Re}(\lambda^{\frac{1}{2}} \psi'_{-1}(t))| dt &\lesssim \int_{x_b}^{x_b + x_b^{-2|\nu|}} \frac{\operatorname{Im} V'(x_b)(t - x_b)}{|a|^{\frac{1}{2}}} dt \\ &\lesssim |a|^{-\frac{1}{2}} b x_b^{\nu-4|\nu|}. \end{aligned} \quad (5.13)$$

Since

$$\begin{cases} b^{\frac{1}{2}} x_b^{-3|\nu| + \frac{1}{2}\nu} = o\left(\frac{b x_b^{-\nu}}{|a|^{\frac{1}{2}} + b^{\frac{1}{2}}}\right), & a < b, \\ |a|^{-\frac{1}{2}} b x_b^{\nu-4|\nu|} = o\left(\frac{b x_b^{-\nu}}{|a|^{\frac{1}{2}} + b^{\frac{1}{2}}}\right), & a \geq b, \end{cases}$$

we obtain in both cases (with some $C_2 > 0$)

$$\frac{\|\xi'' g\|_{L^2(\mathbb{R}_+)} + \|\xi' g'\|_{L^2(\mathbb{R}_+)}}{\|\xi g\|_{L^2(\mathbb{R}_+)}} \lesssim \exp\left(-C_2 \frac{b x_b^{-\nu}}{|a|^{\frac{1}{2}} + b^{\frac{1}{2}}}\right). \quad (5.14)$$

The claim (5.7) follows from the last inequality in (5.9). $\square$

Theorem 5.2. *Let Assumption III hold, f be as in (5.6) with $n = N - 1$ and a satisfy (5.5). Then, as $b \rightarrow +\infty$,*

$$\begin{aligned} \frac{\|(H_V - \lambda)f\|_{L^2(\mathbb{R}_+)}}{\|f\|_{L^2(\mathbb{R}_+)}} &= \mathcal{O}\left(\exp(-c x_b^{\nu+1+2\varepsilon_1}) + x_b^{N\nu} \sup_{x \in \mathcal{J}_b} \frac{b + |\operatorname{Re} V(x)|}{(a - \operatorname{Re} V(x))^{\frac{N}{2}}}\right. \\ &\quad \left. + \sum_{k=0}^{N-2} \sum_{l=2}^{N+k} x_b^{(N+k)\nu} \sup_{x \in \mathcal{J}_b} \frac{(b + |\operatorname{Re} V(x)|)^l}{(a - \operatorname{Re} V(x))^{\frac{N-2+k}{2} + l}}\right). \end{aligned} \quad (5.15)$$

Proof. The claim follows straightforwardly from Proposition 5.1, the estimate of the remainder $|r_n|$, see (2.16), and the choice of a , see (5.5). $\square$

5.3. Examples.

Example 5.3 (Example 3.8 continued). We illustrate applicability of Theorem 5.2 on the imaginary monomial potentials, namely $V(x) = ix^\gamma$ for $x > 0$ and $\gamma \geq 1$. With this choice, we have $\nu = -1$, $x_b = b^{\frac{1}{\gamma}}$ and we may take a as (with $\varepsilon > 0$)

$$b^{\frac{2}{3} \frac{\gamma-1}{\gamma} + \varepsilon} \lesssim a \lesssim b^{2 \frac{\gamma+1}{\gamma} - \varepsilon}, \quad (5.16)$$

see (5.5). Straightforward calculations yield that for a sufficiently large N we get a decay in (5.15). In other words we show that there are pseudomodes (with a decay in (5.15)) in a region bounded by curves $\Gamma_{1,2}$ in $\mathbb{C}$ given by

$$\Gamma_1(t) := t^{\frac{2}{3} \frac{\gamma-1}{\gamma} + \varepsilon} + it, \quad \Gamma_2(t) := t^{2 \frac{\gamma+1}{\gamma} - \varepsilon} + it.$$

Notice that for $\gamma = 2$, we obtain (with an obvious re-parametrisation) the curves $\eta + i\eta^p$ with $1/3 < p < 3$ of the Boulton's conjecture, cf. [4], which are known to be optimal, cf. [19].

Example 5.4 (Semiclassical operators). Let us briefly explain how the semiclassical setting, see e.g. [5], can be treated using our approach and how previously used assumptions can be relaxed. For a sufficiently regular potential U , we search for pseudomodes of the semiclassical operator

$$-h^2 \frac{d^2}{dx^2} + U(x) - z, \quad h > 0,$$

corresponding to a pseudoeigenvalue $z \in \mathbb{C}$, in the limit $h \rightarrow 0$.

First we factor the parameter h^2 out and obtain (1.1) with the scaled potential $V(x) := h^{-2}U(x)$ and pseudoeigenvalue $\lambda := h^{-2}z$ in our notations, see (1.2). The pseudomode is constructed around the point x_0 satisfying the equation $\operatorname{Im} V(x_0) =$

$\operatorname{Im} \lambda$, *i.e.* $\operatorname{Im} U(x_0) = \operatorname{Im} z$. Notice that x_0 is determined only by $\operatorname{Im} z$, which is fixed here.

The cut-off is successful if there exist $\delta_{\pm}$ such that for all $x \in (x_0 + \delta_+/2, x_0 + \delta_+)$

$$\int_{x_0}^x \frac{\operatorname{Im} U(t) - \operatorname{Im} U(x_0)}{(\operatorname{Re} z - \operatorname{Re} U(t))^{\frac{1}{2}} + |\operatorname{Im} U(t) - \operatorname{Im} U(x_0)|^{\frac{1}{2}}} dt \gtrsim h^{1-\varepsilon} \quad (5.17)$$

with some $\varepsilon > 0$ and similarly for δ_- . Indeed, an appropriately modified first inequality in (5.8) yields

$$\operatorname{Re}(\lambda^{\frac{1}{2}} \psi'_{-1}(t)) \gtrsim h^{-1} \frac{\operatorname{Im} U(t) - \operatorname{Im} U(x_0)}{(\operatorname{Re} z - \operatorname{Re} U(t))^{\frac{1}{2}} + |\operatorname{Im} U(t) - \operatorname{Im} U(x_0)|^{\frac{1}{2}}}.$$

However, (5.17) can be satisfied *e.g.* when the Davies' condition [5]

$$\operatorname{Im} U'(x_0) > 0 \quad \text{and} \quad z = \eta^2 + U(x_0) \quad \text{with} \quad \eta^2 > 0 \quad (5.18)$$

is imposed; indeed, Taylor's theorem yields

$$\begin{aligned} \operatorname{Im} U(t) - \operatorname{Im} U(x_0) &= \operatorname{Im} U'(x_0)(t - x_0) + \mathcal{O}((t - x_0)^2), \\ \operatorname{Re} z - \operatorname{Re} U(t) &= \eta^2 + \mathcal{O}(|t - x_0|), \end{aligned} \quad t \rightarrow x_0,$$

and so the choice $\delta_+ := h^{\frac{1-\varepsilon}{2}}$ works. It can be easily checked that the other terms in the expansion are harmless. Finally, the decay of the remainders r_n follows easily if $|\operatorname{Re} z - \operatorname{Re} U(x)|$ is not too small on $(x_0 - \delta_-, x_0 + \delta_+)$, which is satisfied when the Davies' condition (5.18) holds; as an illustration, we have

$$h^2 |r_0| \lesssim \frac{h}{|\operatorname{Re} z - \operatorname{Re} U(x)|^{\frac{1}{2}}}$$

for all $x \in (x_0 - \delta_-, x_0 + \delta_+)$.

In summary, the semiclassical setting allows for many simplifications and a suitable behaviour of U around a fixed point x_0 only is needed to obtain pseudomodes (localising around x_0) as $h \rightarrow 0$. It is also clear that the previously used conditions of the type $\operatorname{Im} U'(x_0) \neq 0$ are not needed as we may use a larger neighbourhood of x_0 and take a sufficiently large η to satisfy (5.17) and obtain a decay of r_n .

Example 5.5 (Strong local singularities). In all previous pseudomode constructions, we used the behaviour of the potential V at infinity. If V is sufficiently singular at a finite point, the construction of the present Section 5 can be adapted accordingly. We illustrate this on an example in $L^2(\mathbb{R}_-)$ with

$$V(x) := \frac{i}{|x|^\alpha} \quad \text{for} \quad x \in (-1, 0), \quad \alpha > 2, \quad (5.19)$$

and an arbitrary behaviour outside $(-1, 0)$. We consider $\mathbb{R}_-$ for convenience only so that (5.1) holds for $x \rightarrow 0-$ and the shape of already derived formulas is preserved. Considering $\mathbb{R}_+$ instead of $\mathbb{R}_-$ and further generalisations in the sense of Section 5.1 (like $\operatorname{Re} V \neq 0$ or $\nu > -1$) are straightforward. We emphasise in particular the potentials with $\operatorname{Re} V(x) = c/|x|^2$, $c \in \mathbb{R}$, appearing in the radial part of higher-dimensional Schrödinger operators.

We follow the notations of Section 5.1 and construct a pseudomode of the type (5.6) around the turning point x_b of $\operatorname{Im} V$ that tends to $0-$ as $b \rightarrow +\infty$. In more detail, we take here

$$\begin{aligned} \lambda &= a + ib, \quad a, b \in \mathbb{R}_+, \\ \operatorname{Im} V(x_b) &= b, \quad \delta := \frac{|x_b|}{2}, \quad \Delta := \frac{\delta}{4}, \end{aligned}$$

with δ going to zero as $b \rightarrow \infty$, and the cut-off ξ as well as intervals $\mathcal{J}_b$ and $\mathcal{J}'_b$ are as in (5.3), (5.4), respectively. The new condition on admissible a 's (corresponding to the simple case (5.19)) reads

$$b^{\frac{2}{3}(1+\frac{1}{\alpha})} \lesssim a \lesssim b^{2(1-\frac{1}{\alpha})-\epsilon} \quad (5.20)$$

with some $\epsilon > 0$.

Following and slightly adapting the estimates in the proof of Proposition 5.1, we get for every $x \in \mathcal{J}_b \setminus \mathcal{J}'_b$ with $x > x_b$ that

$$\int_{x_b}^x \operatorname{Re}(\lambda^{\frac{1}{2}} \psi'_{-1}(t)) dt \gtrsim \frac{b^{1-\frac{1}{\alpha}}}{a^{\frac{1}{2}} + b^{\frac{1}{2}}}.$$

Here the importance of the assumed condition $\alpha > 2$, as well as of (5.20), is clearly visible in order to ensure that the right-hand side tends to infinity as $b \rightarrow +\infty$. Further, it can be straightforwardly checked that the cut-off is indeed successful and an analogue of (5.14) holds; we remark that in the estimates like (5.12) and (5.13) we integrate *e.g.* over $(x_b, x_b + x_b^2)$.

The remainder estimate is also straightforward, using (2.16), we obtain altogether that with V as in (5.19) there exists a positive constant c such that

$$\frac{\|(H_V - \lambda)f\|_{L^2(\mathbb{R}_-)}}{\|f\|_{L^2(\mathbb{R}_-)}} = \mathcal{O} \left(\exp(-cb^{\frac{\epsilon}{2}}) + \frac{b^{1+\frac{n+1}{\alpha}}}{a^{\frac{n+1}{2}}} + \sum_{k=0}^{n-1} \sum_{l=2}^{n+1+k} \frac{b^{l+\frac{n+1+k}{\alpha}}}{a^{l+\frac{n-1+k}{2}}} \right) \quad (5.21)$$

as $b \rightarrow +\infty$ (then necessarily also $a \rightarrow +\infty$ due to (5.20)). Similarly as in Example 5.3, we can check that if we strengthen (5.20) to

$$b^{\frac{2}{3}(1+\frac{1}{\alpha})+\epsilon} \lesssim a \lesssim b^{2(1-\frac{1}{\alpha})-\epsilon}$$

with some $\epsilon > 0$, then for a sufficiently large n we indeed have a decay in (5.21).

APPENDIX A. PROOFS OF LEMMATA 2.1 AND 2.2

In the following, notations of Lemmata 2.1 and 2.2 are used and V is assumed to be sufficiently regular so that all appearing derivatives of it exist.

In the first step, we investigate certain operations on $T_j^{r,s}$, defined in (2.15). To simplify notations, we view $T_j^{r,s}$ as a set of functions of the prescribed form with all possible choices of constants c_α . We start with the following simple observations.

- (a) If $r \geq 1$, then $\mathcal{I}_0^{r,r+1} = \emptyset$ and so $T_0^{r,r+1} = \{0\}$.
- (b) $T_0^{0,1} = \mathbb{C}$ since $\mathcal{I}_0^{0,1} = \{0\}$.
- (c) $T_j^{r,s} + T_j^{r,s} = T_j^{r,s}$ and so $\sum_{j=0}^s T_j^{r,s-j} + \sum_{j=0}^s T_j^{r,s-j} = \sum_{j=0}^s T_j^{r,s-j}$.
- (d) For $s_1 \leq s_2$, $T_j^{r,s_1} \subset T_j^{r,s_2}$ since taking $\alpha_i = 0$ is allowed.
- (e) $cT_j^{r,s} = T_j^{r,s}$ for any constant $c \in \mathbb{C}$.

Lemma A.1. *Let $T_j^{r,s}$ be as in (2.15). Then*

$$(T_j^{r,s})' \subset T_j^{r+1,s+1}, \quad (A.1)$$

$$V'T_j^{r,s} \subset T_{j+1}^{r+1,s}, \quad (A.2)$$

$$T_{j_1}^{r_1,s_1} T_{j_2}^{r_2,s_2} \subset T_{j_1+j_2}^{r_1+r_2,\max\{s_1,s_2\}} \subset T_{j_1+j_2}^{r_1+r_2,s_1+s_2-1}. \quad (A.3)$$

Proof. To prove (A.1), the product rule yields

$$\begin{aligned} & \left((V^{(1)})^{\alpha_1} (V^{(2)})^{\alpha_2+1} \dots (V^{(s)})^{\alpha_s} \right)' \\ &= \alpha_1 (V^{(1)})^{\alpha_1-1} (V^{(2)})^{\alpha_2+1} \dots (V^{(s)})^{\alpha_s} \\ & \quad + \alpha_2 (V^{(1)})^{\alpha_1} (V^{(2)})^{\alpha_2-1} (V^{(3)})^{\alpha_3+1} \dots (V^{(s)})^{\alpha_s} \\ & \quad + \dots + \alpha_s (V^{(1)})^{\alpha_1} \dots (V^{(s)})^{\alpha_s-1} V^{(s+1)}. \end{aligned}$$

Since $\sum_{i=1}^s \alpha_i = j$ and $\sum_{i=1}^s i\alpha_i = r$, we obtain that the sum of powers in every term is again j , *e.g.*

$$\alpha_1 - 1 + \alpha_2 + 1 + \dots + \alpha_s = \alpha_1 + \alpha_2 + \dots + \alpha_s = j,$$

and the sum of powers multiplied by the order of the derivative is $r + 1$, *e.g.*

$$1(\alpha_1 - 1) + 2(\alpha_2 + 1) + \dots + s\alpha_s = 1 + 1\alpha_1 + 2\alpha_2 + \dots + s\alpha_s = r + 1.$$

So (A.1) follows by the rule (c).

To verify (A.2), observe that

$$V'(V^{(1)})^{\alpha_1} (V^{(2)})^{\alpha_2} \dots (V^{(s)})^{\alpha_s} = (V^{(1)})^{\alpha_1+1} (V^{(2)})^{\alpha_2} \dots (V^{(s)})^{\alpha_s},$$

and

$$(\alpha_1 + 1) + \alpha_2 + \dots + \alpha_s = j + 1,$$

$$1(\alpha_1 + 1) + 2\alpha_2 + \dots + s\alpha_s = 1\alpha_1 + 2\alpha_2 + \dots + s\alpha_s = s + 1.$$

To show (A.3), we assume without the loss of generality that $s_1 \leq s_2$. Hence

$$\begin{aligned} & \sum_{\alpha \in \mathcal{I}_{j_1}^{r_1, s_1}} c_\alpha (V^{(1)})^{\alpha_1} (V^{(2)})^{\alpha_2} \dots (V^{(s_1)})^{\alpha_{s_1}} \sum_{\beta \in \mathcal{I}_{j_2}^{r_2, s_2}} c_\beta (V^{(1)})^{\beta_1} (V^{(2)})^{\beta_2} \dots (V^{(s_2)})^{\beta_{s_2}} \\ &= \sum_{\substack{\alpha \in \mathcal{I}_{j_1}^{r_1, s_1}, \\ \beta \in \mathcal{I}_{j_2}^{r_2, s_2}}} c_{\alpha, \beta} (V^{(1)})^{\alpha_1 + \beta_1} (V^{(2)})^{\alpha_2 + \beta_2} \dots (V^{(s_1)})^{\alpha_{s_1} + \beta_{s_1}} (V^{(s_1+1)})^{\beta_{s_1+1}} \dots (V^{(s_2)})^{\beta_{s_2}}. \end{aligned}$$

For the powers in the resulting sum, we have

$$\begin{aligned} & 1(\alpha_1 + \beta_1) + \dots + s_1(\alpha_{s_1} + \beta_{s_1}) + (s_1 + 1)\beta_{s_1+1} + \dots + s_2\beta_{s_2} \\ &= 1\alpha_1 + \dots + s_1\alpha_{s_1} + 1\beta_1 + \dots + s_2\beta_{s_2} = r_1 + r_2, \\ & (\alpha_1 + \beta_1) + \dots + (\alpha_{s_1} + \beta_{s_1}) + \beta_{s_1+1} + \dots + \beta_{s_2} \\ &= \alpha_1 + \dots + \alpha_{s_1} + \beta_1 + \dots + \beta_{s_2} = j_1 + j_2, \end{aligned}$$

hence,

$$T_{j_1}^{r_1, s_1} T_{j_2}^{r_2, s_2} \subset T_{j_1 + j_2}^{r_1 + r_2, \max\{s_1, s_2\}}.$$

Since $s_1 \geq 1$ and $s_2 \geq 1$, we have $\max\{s_1, s_2\} \leq s_1 + s_2 - 1$ and thus the rule (d) yields the second inclusion in (A.3). $\square$

In the next step, we find the form of the derivatives of ψ'_{-1} and $1/\psi'_{-1}$.

Lemma A.2. *Let ψ'_{-1} be as in (2.12) and $T_j^{r, s}$ be defined as in (2.15). Then, for every $m \in \mathbb{N}_0$, we have*

$$\psi_{-1}^{(m+1)} \in \frac{\lambda^{-\frac{1}{2}}}{(\lambda - V)^{-\frac{1}{2}}} \sum_{j=0}^m \frac{T_j^{m, m+1-j}}{(\lambda - V)^j}, \quad (\text{A.4})$$

$$\left(\frac{1}{\psi'_{-1}} \right)^{(m)} \in \frac{\lambda^{\frac{1}{2}}}{(\lambda - V)^{\frac{1}{2}}} \sum_{j=0}^m \frac{T_j^{m, m+1-j}}{(\lambda - V)^j}. \quad (\text{A.5})$$

Proof. We give the detailed induction proof with respect to $m \in \mathbb{N}_0$ for (A.4) only, the proof of (A.5) is fully analogous. Recall that $\psi'_{-1} = i\lambda^{-\frac{1}{2}}(\lambda - V)^{\frac{1}{2}}$ and so the base step can be easily verified in both cases using the rule (b).

We make the induction step $m \rightarrow m+1$, *i.e.* we assume that the formula for $\psi_{-1}^{(m)}$ holds. Using the rule (c) and always incorporating the constants arising by taking the derivatives in the constants in the sets T , we arrive at

$$\begin{aligned} \psi_{-1}^{(m+1)} &= \left(\psi_{-1}^{(m)} \right)' \in \left(\frac{\lambda^{-\frac{1}{2}}}{(\lambda - V)^{-\frac{1}{2}}} \sum_{j=0}^{m-1} \frac{T_j^{m-1, m-j}}{(\lambda - V)^j} \right)' \\ &= \left(\frac{\lambda^{-\frac{1}{2}}}{(\lambda - V)^{-\frac{1}{2}}} \right)' \sum_{j=0}^{m-1} \frac{T_j^{m-1, m-j}}{(\lambda - V)^j} + \frac{\lambda^{-\frac{1}{2}}}{(\lambda - V)^{-\frac{1}{2}}} \sum_{j=0}^{m-1} \left(\frac{T_j^{m-1, m-j}}{(\lambda - V)^j} \right)' \\ &= \frac{\lambda^{-\frac{1}{2}} V'}{(\lambda - V)^{\frac{1}{2}}} \sum_{j=0}^{m-1} \frac{T_j^{m-1, m-j}}{(\lambda - V)^j} + \frac{\lambda^{-\frac{1}{2}}}{(\lambda - V)^{-\frac{1}{2}}} \sum_{j=0}^{m-1} \left(\frac{(T_j^{m-1, m-j})'}{(\lambda - V)^j} + \frac{V' T_j^{m-1, m-j}}{(\lambda - V)^{j+1}} \right) \\ &= \frac{\lambda^{-\frac{1}{2}}}{(\lambda - V)^{-\frac{1}{2}}} \sum_{j=0}^{m-1} \frac{V' T_j^{m-1, m-j}}{(\lambda - V)^{j+1}} + \frac{\lambda^{-\frac{1}{2}}}{(\lambda - V)^{-\frac{1}{2}}} \sum_{j=0}^{m-1} \frac{(T_j^{m-1, m-j})'}{(\lambda - V)^j}. \end{aligned}$$

Using the rule (a), we can let the first sum start with $j = -1$ and Lemma A.1 yields

$$\begin{aligned} \psi_{-1}^{(m+1)} &\in \frac{\lambda^{-\frac{1}{2}}}{(\lambda - V)^{-\frac{1}{2}}} \sum_{j=-1}^{m-1} \frac{T_{j+1}^{m, m-j}}{(\lambda - V)^{j+1}} + \frac{\lambda^{-\frac{1}{2}}}{(\lambda - V)^{-\frac{1}{2}}} \sum_{j=0}^{m-1} \frac{T_j^{m, m-j+1}}{(\lambda - V)^j} \\ &\subset \frac{\lambda^{-\frac{1}{2}}}{(\lambda - V)^{-\frac{1}{2}}} \left(\sum_{j=0}^m \frac{T_j^{m, m-j+1}}{(\lambda - V)^j} + \sum_{j=0}^{m-1} \frac{T_j^{m, m-j+1}}{(\lambda - V)^j} \right) \\ &= \frac{\lambda^{-\frac{1}{2}}}{(\lambda - V)^{-\frac{1}{2}}} \sum_{j=0}^m \frac{T_j^{m, m-j+1}}{(\lambda - V)^j}, \end{aligned}$$

where we use the rule (c) in the last step. Thus (A.4) is proved. $\square$

Now we are ready to prove Lemmata 2.1 and 2.2.

Proof of Lemma 2.1. We give an induction argument with respect to the index $k \in [[-1, n-1]]$. To verify the base case $k = -1$ and $m \in [[1, n+2]]$, we apply (A.4) from Lemma A.2 above. As the induction step we take $k \rightarrow k+1$, *i.e.* we start with assuming that for a fixed k (2.14) holds for all $m \in [[1, n+1-k]]$ and we have to show that

$$\psi_{k+1}^{(m)} \in \frac{\lambda^{\frac{k+1}{2}}}{(\lambda - V)^{\frac{k+1}{2}}} \sum_{j=0}^{k+1+m} \frac{T_j^{k+1+m, k+m+2-j}}{(\lambda - V)^j}, \quad m \in [[1, n-k]].$$

By formula (2.12), we have

$$\begin{aligned} \psi_{k+1}^{(m)} &= (\psi'_{k+1})^{(m-1)} = \left[\frac{1}{2\psi'_{-1}} \left(\psi''_k - \sum_{\substack{\omega+\chi=k \\ \omega, \chi \neq -1}} \psi'_\omega \psi'_\chi \right) \right]^{(m-1)} \\ &= \left(\frac{1}{2\psi'_{-1}} \psi''_k \right)^{(m-1)} - \left(\frac{1}{2\psi'_{-1}} \sum_{\substack{\omega+\chi=k \\ \omega, \chi \neq -1}} \psi'_\omega \psi'_\chi \right)^{(m-1)}. \end{aligned}$$

Using the Leibniz product rule once for the first term and twice for the second one, we arrive at

$$\begin{aligned} \psi_{k+1}^{(m)} &= \sum_{i=0}^{m-1} \binom{m-1}{i} \left(\frac{1}{2\psi'_{-1}} \right)^{(i)} \psi_k^{(m+1-i)} \\ &\quad - \sum_{i=0}^{m-1} \left[\binom{m-1}{i} \left(\frac{1}{2\psi'_{-1}} \right)^{(i)} \sum_{\substack{\omega+\chi=k \\ \omega, \chi \neq -1}} \sum_{p=0}^{m-1-i} \binom{m-1-i}{p} \psi_\omega^{(p+1)} \psi_\chi^{(m-i-p)} \right]. \end{aligned} \quad (\text{A.6})$$

Let us now have a more detailed look at the term $(\psi_\omega)^{(p+1)}(\psi_\chi)^{(m-i-p)}$. We notice first that $\omega + \chi = k$, $\omega, \chi \neq -1$ and $(p+1), (m-i-p) \in [[1, n-k]]$, which enables us to use the induction assumption to rewrite both $\psi_\omega^{(p+1)}$ and $\psi_\chi^{(m-i-p)}$. Namely, the formula for the product of series yields

$$\begin{aligned} &\psi_\omega^{(p+1)} \psi_\chi^{(m-i-p)} \\ &\in \frac{\lambda^{\frac{\omega}{2}}}{(\lambda-V)^{\frac{\omega}{2}}} \sum_{j_1=0}^{\omega+p+1} \frac{T_{j_1}^{\omega+p+1, \omega+p+2-j_1}}{(\lambda-V)^{j_1}} \frac{\lambda^{\frac{\chi}{2}}}{(\lambda-V)^{\frac{\chi}{2}}} \sum_{j_2=0}^{\chi+m-i-p} \frac{T_{j_2}^{\chi+m-i-p, \chi+m-i-p+1-j_2}}{(\lambda-V)^{j_2}} \\ &\subset \frac{\lambda^{\frac{\omega+\chi}{2}}}{(\lambda-V)^{\frac{\omega+\chi}{2}}} \sum_{j=0}^{\omega+\chi+m+1-i} \sum_{q=0}^j \frac{T_q^{\omega+p+1, \omega+p+2-q}}{(\lambda-V)^q} \frac{T_{j-q}^{\chi+m-i-p, \chi+m-i-p+1-j+q}}{(\lambda-V)^{j-q}}, \end{aligned}$$

where we set

$$T_l^{m,n} = \emptyset, \quad l \geq m. \quad (\text{A.7})$$

Using Lemma A.1 and the fact that $\omega + \chi = k$, we obtain

$$\psi_\omega^{(p+1)} \psi_\chi^{(m-i-p)} \in \frac{\lambda^{\frac{k}{2}}}{(\lambda-V)^{\frac{k}{2}}} \sum_{j=0}^{k+m+1-i} \sum_{q=0}^j \frac{T_j^{k+m+1-i, k+m+2-i-j}}{(\lambda-V)^j},$$

where the resulting terms no longer depend on p, ω or χ . Coming back to (A.6), by Lemma A.2 and the induction assumption, we get further that

$$\begin{aligned} \psi_{k+1}^{(m)} &\in \sum_{i=0}^{m-1} \frac{\lambda^{\frac{1}{2}}}{(\lambda-V)^{\frac{1}{2}}} \sum_{j_3=0}^i \frac{T_{j_3}^{i, i+1-j_3}}{(\lambda-V)^{j_3}} \left(\frac{\lambda^{\frac{k}{2}}}{(\lambda-V)^{\frac{k}{2}}} \sum_{j_4=0}^{k+m+1-i} \frac{T_{j_4}^{k+m+1-i, k+m+2-i-j_4}}{(\lambda-V)^{j_4}} \right. \\ &\quad \left. + \sum_{\substack{\omega+\chi=k \\ \omega, \chi \neq -1}} \sum_{p=0}^{m-1-i} \frac{\lambda^{\frac{k}{2}}}{(\lambda-V)^{\frac{k}{2}}} \sum_{j_4=0}^{k+m+1-i} \frac{T_{j_4}^{k+m+1-i, k+m+2-i-j_4}}{(\lambda-V)^{j_4}} \right). \end{aligned}$$

By the rule (c), the formula for the product of series (with the convention (A.7)) and the rule (e) in the last step, we obtain

$$\begin{aligned} \psi_{k+1}^{(m)} &\in \sum_{i=0}^{m-1} \frac{\lambda^{\frac{k+1}{2}}}{(\lambda-V)^{\frac{k+1}{2}}} \sum_{j_3=0}^i \frac{T_{j_3}^{i, i+1-j_3}}{(\lambda-V)^{j_3}} \sum_{j_4=0}^{k+m+1-i} \frac{T_{j_4}^{k+m+1-i, k+m+2-i-j_4}}{(\lambda-V)^{j_4}} \\ &= \sum_{i=0}^{m-1} \frac{\lambda^{\frac{k+1}{2}}}{(\lambda-V)^{\frac{k+1}{2}}} \sum_{j=0}^{k+m+1} \sum_{q=0}^j \frac{T_q^{i, i+1-q}}{(\lambda-V)^q} \frac{T_{j-q}^{k+m+1-i, k+m+2-i-j+q}}{(\lambda-V)^{j-q}} \\ &\subset \sum_{i=0}^{m-1} \frac{\lambda^{\frac{k+1}{2}}}{(\lambda-V)^{\frac{k+1}{2}}} \sum_{j=0}^{k+m+1} \sum_{q=0}^j \frac{T_j^{k+m+1, k+m+2-j}}{(\lambda-V)^j} \\ &= \frac{\lambda^{\frac{k+1}{2}}}{(\lambda-V)^{\frac{k+1}{2}}} \sum_{j=0}^{k+m+1} \frac{T_j^{k+m+1, k+m+2-j}}{(\lambda-V)^j}. \end{aligned}$$

This concludes the proof of Lemma 2.1. $\square$

Proof of Lemma 2.2. Recalling (2.11), we can write the remainder r_n in the following way

$$\begin{aligned} r_n &= \sum_{k=n-1}^{2(n-1)} \lambda^{-\frac{k}{2}} \phi_{k+1} = \sum_{k=n-1}^{2(n-1)} \lambda^{-\frac{k}{2}} \left(\psi_k'' - \sum_{\substack{\omega+\chi=k \\ -1 \leq \omega, \chi \leq n-1}} \psi_\omega' \psi_\chi' \right) \\ &= \underbrace{\sum_{k=n-1}^{2(n-1)} \lambda^{-\frac{k}{2}} \psi_k''}_{S_1} - \underbrace{\sum_{k=n-1}^{2(n-1)} \lambda^{-\frac{k}{2}} \sum_{\substack{\omega+\chi=k \\ -1 \leq \omega, \chi \leq n-1}} \psi_\omega' \psi_\chi'}_{S_2}. \end{aligned} \quad (\text{A.8})$$

By the rule (b), the fact that $\psi_k = 0$ for $k > n-1$ and Lemma 2.1, we have

$$\begin{aligned} S_1 &\in \lambda^{-\frac{n-1}{2}} \frac{\lambda^{\frac{n-1}{2}}}{(\lambda-V)^{\frac{n-1}{2}}} \sum_{j=0}^{n+1} \frac{T_j^{n+1, n+2-j}}{(\lambda-V)^j} = \frac{1}{(\lambda-V)^{\frac{n-1}{2}}} \sum_{j=1}^{n+1} \frac{T_j^{n+1, n+2-j}}{(\lambda-V)^j}, \\ S_2 &\in \sum_{k=n-1}^{2(n-1)} \lambda^{-\frac{k}{2}} \sum_{\substack{\omega+\chi=k \\ -1 \leq \omega, \chi \leq n-1}} \frac{\lambda^{\frac{\omega}{2}}}{(\lambda-V)^{\frac{\omega}{2}}} \sum_{j_1=0}^{\omega+1} \frac{T_{j_1}^{\omega+1, \omega+2-j_1}}{(\lambda-V)^{j_1}} \frac{\lambda^{\frac{\chi}{2}}}{(\lambda-V)^{\frac{\chi}{2}}} \sum_{j_2=0}^{\chi+1} \frac{T_{j_2}^{\chi+1, \chi+2-j_2}}{(\lambda-V)^{j_2}} \\ &= \sum_{k=n-1}^{2(n-1)} \frac{1}{(\lambda-V)^{\frac{k}{2}}} \sum_{\substack{\omega+\chi=k \\ -1 \leq \omega, \chi \leq n-1}} \underbrace{\left(\sum_{j_1=1}^{\omega+1} \frac{T_{j_1}^{\omega+1, \omega+2-j_1}}{(\lambda-V)^{j_1}} \sum_{j_2=1}^{\chi+1} \frac{T_{j_2}^{\chi+1, \chi+2-j_2}}{(\lambda-V)^{j_2}} \right)}_{S_3}. \end{aligned}$$

Since $\max\{\omega+2-j_1\}, \max\{\chi+2-j_2\} \leq n$ and $\omega+\chi=k$, by the formula for the product of series, the rule (d) and Lemma A.1 with the maximum in (A.3), we get

$$\begin{aligned} S_3 &\subset \sum_{j_1=1}^{\omega+1} \frac{T_{j_1}^{\omega+1, n}}{(\lambda-V)^{j_1}} \sum_{j_2=1}^{\chi+1} \frac{T_{j_2}^{\chi+1, n}}{(\lambda-V)^{j_2}} = \sum_{j_1=0}^{\omega} \frac{T_{j_1+1}^{\omega+1, n}}{(\lambda-V)^{j_1+1}} \sum_{j_2=0}^{\chi} \frac{T_{j_2+1}^{\chi+1, n}}{(\lambda-V)^{j_2+1}} \\ &= \sum_{j=0}^{\omega+\chi} \sum_{q=0}^j \frac{T_{q+1}^{\omega+1, n}}{(\lambda-V)^{q+1}} \frac{T_{j-q+1}^{\chi+1, n}}{(\lambda-V)^{j-q+1}} \subset \sum_{j=0}^{\omega+\chi} \frac{T_{j+2}^{\omega+\chi+2, n}}{(\lambda-V)^{j+2}} \\ &= \sum_{j=2}^{\omega+\chi+2} \frac{T_j^{\omega+\chi+2, n}}{(\lambda-V)^j} = \sum_{j=2}^{k+2} \frac{T_j^{k+2, n}}{(\lambda-V)^j}. \end{aligned}$$

Observe that S_3 does not depend on ω and χ wherefore the sum over ω and χ in S_2 disappears (for it can be included in the constants appearing in T).

Inserting formulas for S_1 , S_2 and S_3 to (A.8), we have

$$\begin{aligned} r_n &\in \frac{1}{(\lambda-V)^{\frac{n-1}{2}}} \sum_{j=1}^{n+1} \frac{T_j^{n+1, n+2-j}}{(\lambda-V)^j} + \sum_{k=n-1}^{2(n-1)} \frac{1}{(\lambda-V)^{\frac{k}{2}}} \sum_{j=2}^{k+2} \frac{T_j^{k+2, n}}{(\lambda-V)^j} \\ &= \frac{1}{(\lambda-V)^{\frac{n-1}{2}}} \underbrace{\sum_{j=1}^{n+1} \frac{T_j^{n+1, n+2-j}}{(\lambda-V)^j}}_{S_4} + \underbrace{\sum_{k=0}^{n-1} \frac{1}{(\lambda-V)^{\frac{n-1+k}{2}}} \sum_{j=2}^{n+1+k} \frac{T_j^{n+1+k, n}}{(\lambda-V)^j}}_{S_5}. \end{aligned}$$

Writing the first summand of S_4 separately, we get

$$r_n \in \frac{1}{(\lambda - V)^{\frac{n-1}{2}}} \frac{T_1^{n+1, n+1}}{\lambda - V} + \frac{1}{(\lambda - V)^{\frac{n-1}{2}}} \sum_{j=2}^{n+1} \frac{T_j^{n+1, n+2-j}}{(\lambda - V)^j} \\ + \sum_{k=0}^{n-1} \frac{1}{(\lambda - V)^{\frac{n-1+k}{2}}} \sum_{j=2}^{n+1+k} \frac{T_j^{n+1+k, n}}{(\lambda - V)^j}.$$

Using the fact that $T_1^{n+1, n+1} = \mathbb{C}V^{(n+1)}$ and writing the first summand of S_5 separately, we obtain

$$r_n \in \frac{V^{(n+1)}}{(\lambda - V)^{\frac{n+1}{2}}} + \frac{1}{(\lambda - V)^{\frac{n-1}{2}}} \sum_{j=2}^{n+1} \frac{T_j^{n+1, n+2-j}}{(\lambda - V)^j} \\ + \frac{1}{(\lambda - V)^{\frac{n-1}{2}}} \sum_{j=2}^{n+1} \frac{T_j^{n+1, n}}{(\lambda - V)^j} + \sum_{k=1}^{n-1} \frac{1}{(\lambda - V)^{\frac{n-1+k}{2}}} \sum_{j=2}^{n+1+k} \frac{T_j^{n+1+k, n}}{(\lambda - V)^j}.$$

Observe that $\max\{n+2-j\} \leq n$. Thus, by the rules (d) and (c), we get

$$r_n \in \frac{V^{(n+1)}}{(\lambda - V)^{\frac{n+1}{2}}} + \underbrace{\frac{1}{(\lambda - V)^{\frac{n-1}{2}}} \sum_{j=2}^{n+1} \frac{T_j^{n+1, n}}{(\lambda - V)^j}}_{\text{the summand for } k=0} + \sum_{k=1}^{n-1} \frac{1}{(\lambda - V)^{\frac{n-1+k}{2}}} \sum_{j=2}^{n+1+k} \frac{T_j^{n+1+k, n}}{(\lambda - V)^j} \\ = \frac{V^{(n+1)}}{(\lambda - V)^{\frac{n+1}{2}}} + \sum_{k=0}^{n-1} \frac{1}{(\lambda - V)^{\frac{n-1+k}{2}}} \sum_{j=2}^{n+1+k} \frac{T_j^{n+1+k, n}}{(\lambda - V)^j}.$$

Hence, the estimate (2.16) follows. $\square$

Acknowledgements. The authors would like to express their gratitude to the *American Institute of Mathematics* (AIM) for a support to organise the workshop [1], which partially stimulated the present research. D.K. was partially supported by the GACR grant No. 18-08835S. Until December 2017, the research of P.S. was supported by the *Swiss National Science Foundation*, SNF Ambizione grant No. PZ00P2_154786.

REFERENCES

- [1] AIM workshop: Mathematical aspects of physics with non-self-adjoint operators, <http://www.aimath.org/pastworkshops/nonselfadjoint.html>, June 2015, San Jose, CA, USA.
- [2] A. A. Abramov, A. Aslanyan, and E. B. Davies, *Bounds on complex eigenvalues and resonances*, J. Phys. A: Math. Gen. **34** (2001), 57–72.
- [3] W. Bordeaux Montrieux, *Estimation de résolvante et construction de quasimode près du bord du pseudospectre*, arXiv:1301.3102 [math.SP] (2013).
- [4] L. Boulton, *The non-self-adjoint harmonic oscillator, compact semigroups and pseudospectra*, J. Operator Theory **47** (2002), 413–429.
- [5] E. B. Davies, *Semi-classical states for non-self-adjoint Schrödinger operators*, Comm. Math. Phys. **200** (1999), 35–41.
- [6] ———, *Linear operators and their spectra*, Cambridge University Press, 2007.
- [7] N. Dencker, J. Sjöstrand, and M. Zworski, *Pseudospectra of semiclassical (pseudo-) differential operators*, Comm. Pure Appl. Math. **57** (2004), 384–415.
- [8] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*, Oxford University Press, Oxford, 1987.
- [9] R. Henry and D. Krejčířík, *Pseudospectra of the Schrödinger operator with a discontinuous complex potential*, J. Spectr. Theory **7** (2017), 659–697.
- [10] D. Krejčířík, N. Raymond, J. Royer, and P. Siegl, *Non-accretive Schrödinger operators and exponential decay of their eigenfunctions*, Israel J. Math. **221** (2017), 779–802.
- [11] D. Krejčířík and P. Siegl, *Elements of spectral theory without the spectral theorem*, In *Non-selfadjoint operators in quantum physics: Mathematical aspects* (432 pages), F. Bagarello, J.-P. Gazeau, F. H. Szafraniec, and M. Znojil, Eds., Wiley-Interscience, 2015.

- [12] D. Krejčířík, P. Siegl, M. Tater, and J. Viola, *Pseudospectra in non-Hermitian quantum mechanics*, J. Math. Phys. **56** (2015), 103513.
- [13] A. Laptev and O. Safronov, *Eigenvalue estimates for Schrödinger operators with complex potentials*, Comm. Math. Phys. **292** (2009), 29–54.
- [14] B. Mityagin and P. Siegl, *Local form-subordination condition and Riesz basisness of root systems*, J. Anal. Math. (to appear, arxiv:1608.00224).
- [15] B. Mityagin, P. Siegl, and J. Viola, *Differential operators admitting various rates of spectral projection growth*, J. Funct. Anal. **272** (2017), 3129–3175.
- [16] R. Novák, *On the pseudospectrum of the harmonic oscillator with imaginary cubic potential*, Int. J. Theor. Phys. **54** (2015), 4142–4153.
- [17] F. W. J. Olver, *Asymptotics and special functions*, A K Peters Ltd., Wellesley, MA, 1997.
- [18] K. Pravda-Starov, *A general result about the pseudo-spectrum of Schrödinger operators*, Proc. R. Soc. A **460** (2004), 471–477.
- [19] ———, *A complete study of the pseudo-spectrum for the rotated harmonic oscillator*, J. London Math. Soc. **73** (2006), 745–761.
- [20] P. Siegl and D. Krejčířík, *On the metric operator for the imaginary cubic oscillator*, Phys. Rev. D **86** (2012), 121702(R).
- [21] L. N. Trefethen and M. Embree, *Spectra and pseudospectra*, Princeton University Press, 2005.
- [22] M. Zworski, *A remark on a paper of E. B. Davies*, Proc. Amer. Math. Soc. **129** (2001), 2955–2957.

(David Krejčířík) DEPARTMENT OF MATHEMATICS, FACULTY OF NUCLEAR SCIENCES AND PHYSICAL ENGINEERING, CZECH TECHNICAL UNIVERSITY IN PRAGUE, TROJANOVA 13, 12000 PRAGUE 2, CZECH REPUBLIC

Email address: david.krejcirik@fjfi.cvut.cz

(Petr Siegl) SCHOOL OF MATHEMATICS AND PHYSICS, QUEEN’S UNIVERSITY BELFAST, UNIVERSITY ROAD, BELFAST BT7 1NN, NORTHERN IRELAND, UK

Email address: p.siegl@qub.ac.uk