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ALGEBRAIC JUMP LOCI FOR RANK AND BETTI NUMBERS OVER LAURENT POLYNOMIAL RINGS

THOMAS HÜTTEMANN AND ZUHONG ZHANG

ABSTRACT. Let C be a chain complex of finitely generated free modules over a commutative LAURENT polynomial ring L_s in s indeterminates. Given a group homomorphism $p: \mathbb{Z}^s \longrightarrow \mathbb{Z}^t$ we let $p_!(C) = C \otimes_{L_s} L_t$ denote the resulting induced complex over the LAURENT polynomial ring L_t in t indeterminates. We prove that the BETTI number jump loci, that is, the sets of those homomorphisms p such that $b_k(p_!(C)) > b_k(C)$, have a surprisingly simple structure. We allow non-unital commutative rings of coefficients, and work with a notion of BETTI numbers that generalises both the usual one for integral domains, and the analogous concept involving MCCOY ranks in case of unital commutative rings.

INTRODUCTION

In this paper we examine the behaviour of ranks of matrices and BETTI numbers of chain complexes over LAURENT polynomial rings in several indeterminates under a linear change of variables. Here “rank” and “BETTI number” are taken relative to a prescribed family of ideals in the ground ring; in spirit, this is close to (and generalises) considering the MCCOY rank of a matrix over a commutative ring rather than the usual rank in linear algebra. The resulting purely algebraic “jump loci” have a particularly simple structure, which is a surprising feature of the theory; we start with a digression to demonstrate that, in general, jump loci can be almost arbitrarily complicated. We then sketch the motivating results of KOHNO and PAJITNOV, before we finally start discussing the specific set-up under consideration.

General jump loci are complicated. Jump loci are subsets of a moduli space M encoding precisely which of the objects parametrised by M have a certain desirable property, and which do not. For example, the space of rank one local systems on a finite CW-complex X may be identified with the algebraic variety

$$M = \operatorname{hom}(\pi_1(X), \mathbb{C}^\times) = (\mathbb{C}^\times)^{b_1(X)} \times F$$

with F a discrete finite abelian group. For $k > 0$ let $\Sigma^k(X)$ be the subset of M corresponding to those rank one systems ρ satisfying $\dim H^k(X, \rho) \geq 1$;

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this is an example of (cohomological) jump loci. No general classification of such jump loci can be given; in fact, for any algebraic subvariety Z of the algebraic torus $(\mathbb{C}^\times)^n$ and any $k \geq 1$ there is a finite CW complex X with $M = (\mathbb{C}^\times)^n$ and $\Sigma^k(X) = Z \cup \{\mathbf{1}\}$, see SUCIU, YANG and ZHAO [SYZ15, Lemma 10.3], SIMPSON [Sim97] and WANG [Wan13, Theorem 1.1].

KOHNO and PAJITNOV considered, in a similar spirit, the twisted NOVIKOV homology of a finite CW complex. In contrast to the case of Σ^k described above, the ensuing (homological) jump loci have a surprisingly simple structure: they are finite unions of linear subspaces of $\mathbb{C}^n$. This striking result rests on an analysis of jump loci of a purely algebraic nature, which we hasten to describe in some detail now.

The Kohno-Pajitnov algebraic jump loci. Let R be a commutative integral domain with unit. A group homomorphism $p: \mathbb{Z}^s \longrightarrow \mathbb{Z}^t$ determines a homomorphism $p_*: R[\mathbb{Z}^s] \longrightarrow R[\mathbb{Z}^t]$ of group rings. Given a bounded chain complex C of finitely generated free $R[\mathbb{Z}^s]$ -modules we obtain the induced chain complex $p_!(C) = C \otimes_{R[\mathbb{Z}^s]} R[\mathbb{Z}^t]$ of finitely generated free $R[\mathbb{Z}^t]$ -modules. Over a group ring $R[\mathbb{Z}^n]$ we have a meaningful notion of “rank” for matrices and free modules, and can thus define the BETTI numbers $b_k(C)$ and $b_k(p_!(C))$. KOHNO and PAJITNOV proved the following result characterising the jump loci of the BETTI numbers with respect to varying the group homomorphism p :

Theorem 0.1 (KOHNO and PAJITNOV [KP14, Theorem 7.3]). *Let $k \in \mathbb{Z}$ and $q \geq 0$ be given. There exists a finite family of proper direct summands $G_i \subset \text{hom}(\mathbb{Z}^s, \mathbb{Z})$ such that the inequality $b_k(C) + q < b_k(p_!(C))$ holds if and only if there is an index i with $p \in G_i^t \subseteq \text{hom}(\mathbb{Z}^s, \mathbb{Z})^t = \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$.*

In the language used before, the “moduli space” in question is $M = \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. — The first step of the proof is to characterise those p which satisfy $p_*(\Delta) = 0$, for a fixed non-zero $\Delta \in R[\mathbb{Z}^s]$. Next one establishes a variant of the theorem for jump loci of the rank of matrices, characterising the condition $\text{rank}(p_*(A)) < \text{rank}(A)$ for a fixed matrix A with entries in $R[\mathbb{Z}^s]$. Finally the actual theorem can be verified by considering the ranks of the differentials in the chain complex.

Jump loci from families of ideals. The KOHNO-PAJITNOV jump loci described above result from considering vanishing conditions of the type $p_*(\Delta) = 0$. In the present paper, we take the jump to more general conditions, showing that a much larger class of jump loci has structure as described in Theorem 0.1. To this end, let $\mathcal{K}$ be a non-empty set of ideals of R such that¹ if $I \in \mathcal{K}$ and J is an ideal contained in I , then $J \in \mathcal{K}$. With respect to $\mathcal{K}$ and Δ , we formulate the following condition on p : *The ideal of R generated by the coefficients of the element $p_*(\Delta) \in R[\mathbb{Z}^t]$ lies in $\mathcal{K}$.* This should be thought of as saying that $p_*(\Delta)$ satisfies a certain property encoded by $\mathcal{K}$. If $\mathcal{K}$ contains a unique inclusion-maximal ideal M , the property in question is that of being zero in R/M .

¹Such a set $\mathcal{K}$ is usually called an order ideal in the partially ordered set of ideals of R , but this terminology seems less than ideal in the present context.

With respect to the “property” $\mathcal{K}$ we establish a notion of $\mathcal{K}$ -rank of matrices and $\mathcal{K}$ -BETTI numbers for chain complexes of finitely generated free modules, and characterise their jump loci. We allow R to be an arbitrary commutative ring which may even be non-unital. It is surprising that the aforementioned structure theory can be established in this generality. The price to pay is that one cannot expect to have *proper* direct summands G_i any more.

Among the many possible sets $\mathcal{K}$ two deserve special mention. We assume a unital commutative ring R for now. If we take $\mathcal{K}$ to be the set of all ideals having non-trivial annihilator then the $\mathcal{K}$ -rank of a matrix (Definition 4.1) is precisely the MCCOY-rank of a matrix as introduced in [McC42, §2]. If R is an integral domain and $\mathcal{K}$ consists of the zero ideal only, we recover the original result of KOHNO and PAJITNOV [KP14].

Notation and conventions. Throughout the paper, R denotes a fixed commutative ring, possibly non-unital. The notation $I \leq R$ is used to indicate that I is an ideal of R , possibly $\{0\}$ or R itself. We will concern ourselves with the group rings $R[\mathbb{Z}^s]$ and $R[\mathbb{Z}^t]$; their elements are written in the form $\sum_{a \in \mathbb{Z}^s} r_a x^a$ and $\sum_{b \in \mathbb{Z}^t} \rho_b y^b$, respectively, with almost all of the coefficients r_a and ρ_b being zero. We let $\mathbb{Z}^{s*} = \text{hom}(\mathbb{Z}^s, \mathbb{Z})$ stand for the $\mathbb{Z}$ -dual of $\mathbb{Z}^s$. The group $\text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ is often identified with $(\mathbb{Z}^{s*})^t$. Any homomorphism $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ induces an R -algebra homomorphism $p_*: R[\mathbb{Z}^s] \longrightarrow R[\mathbb{Z}^t]$, which maps $\Delta = \sum_{a \in \mathbb{Z}^s} r_a x^a \in R[\mathbb{Z}^s]$ to

$$p_*(\Delta) = \sum_{b \in \mathbb{Z}^t} \left(\sum_{a \in p^{-1}(b)} r_a \right) \cdot y^b \in R[\mathbb{Z}^t]. \quad (0.2)$$

The map $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ also induces a map of sets of matrices

$$p_*: \mathcal{M}_{m,n}(R[\mathbb{Z}^s]) \longrightarrow \mathcal{M}_{m,n}(R[\mathbb{Z}^t])$$

by applying the ring homomorphism p_* to each matrix element.

1. PROPERTIES OF IDEALS AND MODULES

Suppose that $\mathcal{K}$ is a set of ideals of R . This set encodes a “property” that elements of the group ring $R[\mathbb{Z}^n]$ may or may not possess. The simplest case is that $\mathcal{K}$ consists of the zero ideal only, in which case the property in question is “being 0”.

Definition 1.1. (1) An ideal I of R is called a $\mathcal{K}$ -ideal if $I \in \mathcal{K}$.
 (2) A subset $X \subseteq R$ is called a $\mathcal{K}$ -set if the ideal $\langle X \rangle$ generated by X is a $\mathcal{K}$ -ideal.

Definition 1.2. Let X be subset of the group ring $R[\mathbb{Z}^n]$.

- (1) We let iX denote the ideal generated by the set of coefficients of the elements of X .
- (2) We say that X is a $\mathcal{K}$ -set provided that iX is a $\mathcal{K}$ -ideal, *i.e.*, provided that the set of coefficients of elements of X is a $\mathcal{K}$ -set.
- (3) An R -submodule M of $R[\mathbb{Z}^n]$ is called a $\mathcal{K}$ -module provided it is a $\mathcal{K}$ -set.

In general, we will only be interested in sets $\mathcal{K}$ which are closed under taking smaller ideals:

Definition 1.3. We call $\mathcal{K}$ *hereditary* if $\mathcal{K}$ is non-empty, and if $I \in \mathcal{K}$ and $J \subseteq I$ together imply $J \in \mathcal{K}$, for any $J \trianglelefteq R$.

As mentioned before, two relevant hereditary sets are

$$\mathcal{K}_0 = \{\{0\}\} \quad \text{and} \quad \mathcal{K}_1 = \{I \trianglelefteq R \mid \text{ann}_R(I) \neq \{0\}\} ;$$

the former corresponds to vanishing conditions, the latter relates to the MCCOY-rank of matrices as explained in §4 below. If V is a fixed injective R -module we have the hereditary set

$$\mathcal{K}_{E,V} = \{I \trianglelefteq R \mid E(I) \text{ embeds into } V\} ,$$

where $E(I)$ denotes an injective hull of the R -module I ; more generally, if $\mathcal{V}$ is a family of injective modules, we obtain a hereditary set

$$\mathcal{K}_{E,\mathcal{V}} = \{I \trianglelefteq R \mid E(I) \text{ embeds into } V \text{ for some } V \in \mathcal{V}\} .$$

Any union of hereditary sets is hereditary, and in fact $\mathcal{K}_{E,\mathcal{V}} = \bigcup_{V \in \mathcal{V}} \mathcal{K}_{E,V}$. Any subset $X \subseteq R$ containing 0 gives rise to the hereditary set $\mathcal{K}_{\subseteq X} = \{I \trianglelefteq R \mid I \subseteq X\}$, and, if X contains at least one element other than 0, the hereditary set $\mathcal{K}_{\subsetneq X} = \{I \trianglelefteq R \mid I \subsetneq X\}$. In particular, we can consider the hereditary set $\mathcal{K}_{\subsetneq J(R)}$ in case the JACOBSON radical $J(R)$ of R is non-trivial. For a unital ring R , this is the hereditary set of all superfluous ideals different from $J(R)$, where I is *superfluous* if $I + J = R$ implies $J = R$, for all $J \trianglelefteq R$.

Before we construct more examples of hereditary sets, we prove a basic result relating the property of “being a $\mathcal{K}$ -set” for a set $X \subseteq R[\mathbb{Z}^s]$ and its image $p_*(X)$, for $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. Those p where $p_*(X)$ is a $\mathcal{K}$ -set form the “jump loci” from the title of the paper.

Lemma 1.4. *Let $\mathcal{K}$ be a hereditary set of ideals of R , and let $X \subseteq R[\mathbb{Z}^s]$ be a subset. For any $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$, if X is a $\mathcal{K}$ -set then so is $p_*(X)$.*

Proof. Let I be the ideal generated by the coefficients of elements of X ; we have $I \in \mathcal{K}$ by hypothesis on X . Let J be the ideal generated by the coefficients of elements of $p_*(X)$. The elements of $p_*(X)$ are of the form $p_*(\Delta)$, for $\Delta \in M$, and by formula (0.2) the coefficients of $p_*(\Delta)$ are sums of coefficients of Δ . But the latter coefficients are elements of I , hence so are the former. That is, the generators of J are elements of I whence $J \subseteq I$. As $\mathcal{K}$ is hereditary we conclude $J \in \mathcal{K}$ so that $p_*(X)$ is a $\mathcal{K}$ -set as claimed. $\square$

As our set $\mathcal{K}$ always contains the zero ideal, we also have the following:

Lemma 1.5. *If X is a subset of the augmentation ideal $I_s = \ker 0_*$ of $\mathbb{Z}^s$, then $0_*(X)$ is a $\mathcal{K}$ -set.* $\square$

Remark 1.6. If the hereditary set $\mathcal{K}$ contains a unique maximal element J , then the conditions “being a $\mathcal{K}$ -module” can be transformed into the annihilation condition “being trivial” by replacing the ground ring R with R/J .

Hereditary sets and filters of ideals. A *filter* of ideals is a non-empty set $\mathcal{F}$ of ideals of R such that $J \in \mathcal{F}$ and $I \supseteq J$ together imply $I \in \mathcal{F}$. There is an intimate connection between hereditary sets of ideals and filters of ideals:

Proposition 1.7. (1) *Every filter $\mathcal{F}$ of ideals determines a hereditary set of ideals given by $\mathcal{F}' = \{I \trianglelefteq R \mid \text{ann}_R(I) \in \mathcal{F}\}$, and the assignment $\mathcal{F} \mapsto \mathcal{F}'$ is inclusion-reversing.*
 (2) *Every hereditary set $\mathcal{K}$ of ideals determines a filter of ideals given by $\mathcal{K}' = \{J \trianglelefteq R \mid \text{ann}_R(J) \in \mathcal{K}\}$, and the assignment $\mathcal{K} \mapsto \mathcal{K}'$ is inclusion-reversing.* $\square$

Hereditary sets of ideals can be constructed with the aid of Proposition 1.7, for example from the filter $\mathcal{F}_{\text{ess}} = \{J \trianglelefteq R \mid J \text{ is an essential ideal}\}$, where J is *essential* if $J \cap I = \{0\}$ implies $I = \{0\}$, for all $I \trianglelefteq R$. Using Proposition 1.7 twice, any hereditary set $\mathcal{K}$ gives rise to another hereditary set $\mathcal{K}'' \supseteq \mathcal{K}$; specifically, $\mathcal{K}'_0 \subseteq \mathcal{F}_{\text{ess}}$ and hence $\mathcal{K}''_0 \supseteq \mathcal{F}'_{\text{ess}}$.

Any subset X of R defines a filter $\mathcal{F}_{\supseteq X} = \{J \trianglelefteq R \mid J \supseteq X\}$, and, if $X \neq R$, also the filter $\mathcal{F}_{\not\supseteq X} = \{J \trianglelefteq R \mid J \not\supseteq X\}$. Finally, we observe that any intersection of filters of ideals is again a filter.

2. PARTITION SUBGROUPS OF $\text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$

Definition 2.1. Suppose $\pi = (\pi_1, \pi_2, \dots, \pi_k)$ is a partition of a subset of $\mathbb{Z}^s$, so that $\pi = \coprod_{j=1}^k \pi_j \subseteq \mathbb{Z}^s$. (We allow the case $\pi = \emptyset$ and $k = 0$ here.) To π we associate the abelian group

$$H(\pi) = \{p \in \mathbb{Z}^{s*} \mid \forall j = 1, 2, \dots, k : p|_{\pi_j} \text{ is constant}\}$$

called the *partition subgroup associated to π* . For a set P of partitions of (possibly distinct) subsets of $\mathbb{Z}^s$, we define

$$H(P) = \bigcap_{\pi \in P} H(\pi),$$

and call $H(P)$ the *partition subgroup associated to P* .

Lemma 2.2. *Suppose P is a set of partitions of subsets of $\mathbb{Z}^s$.*

- (1) *If at least one part π_j of at least one partition $\pi \in P$ has at least two elements, then $H(P)$ is a proper subgroup of $\mathbb{Z}^{s*}$.*
- (2) *The subgroup $H(P)$ is contained in $\mathbb{Z}^{s*}$ as a direct summand.*

Proof. Part (1) is trivial. Let us prove (2). In view of the canonical short exact sequence

$$0 \longrightarrow H(P) \xrightarrow{\subseteq} \mathbb{Z}^{s*} \longrightarrow \mathbb{Z}^{s*}/H(P) \longrightarrow 0$$

it is enough to show that $\mathbb{Z}^{s*}/H(P)$ is torsion-free. For then $\mathbb{Z}^{s*}/H(P)$ is free abelian, whence the short exact sequence splits.

So let $q \in \mathbb{Z}^{s*}$ be such that $[q] \in \mathbb{Z}^{s*}/H(P)$ is torsion. Then there exists a natural number $n \geq 1$ with

$$[nq] = n \cdot [q] = 0 \in \mathbb{Z}^{s*}/H(P),$$

that is, $nq \in H(P)$. By definition of $H(P)$ this means that the homomorphism nq is constant on each part π_j of each partition $\pi \in P$, which implies that q has the same property. Consequently, $q \in H(P)$ and thus $[q] = 0$. $\square$

Let $\Delta = \{\Delta^{(1)}, \Delta^{(2)}, \dots, \Delta^{(d)}\}$, for $d \geq 0$, be a finite subset of $R[\mathbb{Z}^s]$. To fix notation, we write

$$\Delta^{(i)} = \sum_{a \in \mathbb{Z}^s} r_a^{(i)} x^a \quad (\text{for } 1 \leq i \leq d).$$

Let $\pi = (\pi_1, \pi_2, \dots, \pi_k)$ be a partition of $\text{supp}(\Delta) = \bigcup_{i=1}^d \text{supp}(\Delta^{(i)})$. We define ring elements

$$\Delta_j^{(i)} = \sum_{a \in \pi_j} r_a^{(i)} \in R \quad (\text{for } 1 \leq j \leq k \text{ and } 1 \leq i \leq d), \quad (2.3)$$

and denote the ideal generated by these elements by

$$\Delta_\pi = \langle \Delta_j^{(i)} \mid 1 \leq j \leq k, 1 \leq i \leq d \rangle \trianglelefteq R.$$

Definition 2.4. Let $\mathcal{K}$ be a hereditary set of ideals of R , cf. Definition 1.3. A $\mathcal{K}$ -partition is a partition π of $\text{supp}(\Delta)$ such that Δ_π is a $\mathcal{K}$ -ideal.

Lemma 2.5. Let $\mathcal{K}$ be a hereditary set of ideals of R . Suppose that $i\Delta \notin \mathcal{K}$, that is, $i\Delta$ is not a $\mathcal{K}$ -ideal. Suppose P is a set of partitions of subsets of $\mathbb{Z}^s$ which contains at least one $\mathcal{K}$ -partition π of $\text{supp}(\Delta)$. Then $H(P)$ is a proper subgroup of $\mathbb{Z}^{s*}$.

Proof. Since $i\Delta \notin \mathcal{K}$ we have $i\Delta \neq \{0\}$, so the set Δ contains a non-zero element whence $\text{supp}(\Delta) \neq \emptyset$. Let $\tau = (\tau_1, \tau_2, \dots, \tau_k) \in P$ be any partition of $\text{supp}(\Delta)$. If all parts τ_j of τ are singletons then the generators of Δ_τ , as specified in (2.3), are precisely the generators of $i\Delta$ so that $\Delta_\tau = i\Delta \notin \mathcal{K}$. That is, such a τ is not a $\mathcal{K}$ -partition. But the stipulated partition π is a $\mathcal{K}$ -partition; it follows that at least one part of π must have at least two elements. Now Lemma 2.2 applies, assuring us that $H(P)$ is a proper direct summand of $\mathbb{Z}^{s*}$. $\square$

3. JUMP LOCI FOR MODULES

In this section, $\mathcal{K}$ denotes a fixed hereditary set of ideals of R in the sense of Definition 1.3.

Proposition 3.1. Let M be a non-trivial, finitely generated R -submodule of the group ring $R[\mathbb{Z}^s]$. There exist partition subgroups $G_1, G_2, \dots, G_\ell \subseteq \mathbb{Z}^{s*}$, where $\ell \geq 0$, such that for every $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$,

$$p_*(M) \text{ is a } \mathcal{K}\text{-module} \quad \Longleftrightarrow \quad p \in \bigcup_{j=1}^{\ell} G_j^t.$$

Here $G_j^t = \bigoplus_1^t G_j \subseteq (\mathbb{Z}^{s*})^t = \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. — More precisely, writing

$$\Delta = \{\Delta^{(1)}, \Delta^{(2)}, \dots, \Delta^{(d)}\} \subseteq M$$

for a finite generating set of the R -module M , the number ℓ is the number of $\mathcal{K}$ -partitions of $\text{supp}(\Delta)$, and the groups G_j are the partition subgroups $H(\pi)$ associated to these partitions.

If M is not a $\mathcal{K}$ -module then the groups G_j are proper subgroups of $\mathbb{Z}^{s*}$ so that $\bigcup_{j=1}^{\ell} G_j^t \neq \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. We have $\ell > 0$ if and only if $0_*(M)$ is a $\mathcal{K}$ -module, which certainly is the case if M is contained in the augmentation ideal of $R[\mathbb{Z}^s]$.

For ease of reading we delegate the main step of the proof to the following Lemma, which is based on the argument from [KP14, Theorem 7.3]. Write $p = (p_1, p_2, \dots, p_t)$, with each component $p_i: \mathbb{Z}^s \longrightarrow \mathbb{Z}$ being an element of $\mathbb{Z}^{s*}$.

Lemma 3.2. *Let M and Δ be as in Proposition 3.1. The R -submodule $p_*(M)$ of $R[\mathbb{Z}^t]$ is a $\mathcal{K}$ -module if and only if there exists a $\mathcal{K}$ -partition π of $\text{supp}(\Delta)$ such that p is constant on each part π_j of π , that is, such that $p \in H(\pi)^t$.*

Proof. Let us prove the “if” implication first. Suppose that there is a $\mathcal{K}$ -partition π of $\text{supp}(\Delta)$ with $p \in H(\pi)^t$; this last condition is equivalent to p being constant on each part π_j of π . As $p_*(M)$ is generated (as an R -module) by the set $p_*(\Delta)$, the ideal $ip_*(M)$ of R equals $ip_*(\Delta)$. We write

$$p_*(\Delta^{(i)}) = \sum_{b \in \mathbb{Z}^t} \rho_b^{(i)} y^b = \sum_{b \in \mathbb{Z}^t} \left(\sum_{a \in p^{-1}(b)} r_a^{(i)} \right) \cdot y^b$$

where $\rho_b^{(i)} = \sum_{a \in p^{-1}(b)} r_a^{(i)}$. As π is a partition of $\text{supp}(\Delta)$, and as p is constant on each part of π , we have the equality

$$\rho_b^{(i)} = \sum_j \sum_{\substack{a \in \pi_j \\ a \in p^{-1}(b)}} r_a^{(i)} = \sum_j \sum_{p|_{\pi_j} \equiv b} r_a^{(i)} = \sum_j \Delta_j^{(i)}. \quad (3.3)$$

As π is a $\mathcal{K}$ -partition, the ideal $\Delta_\pi = \langle \Delta_j^{(i)} \mid 1 \leq j \leq k, 1 \leq i \leq d \rangle$ is an element of $\mathcal{K}$. It contains all the $\Delta_j^{(i)}$ and hence all the $\rho_b^{(i)}$, by (3.3). As $\mathcal{K}$ is hereditary the ideal generated by the $\rho_b^{(i)}$ thus also lies in $\mathcal{K}$. But this ideal is precisely $ip_*(M)$, as observed above, so $p_*(M)$ is a $\mathcal{K}$ -module as desired.

To show the reverse implication suppose that $p_*(M)$ is a $\mathcal{K}$ -module. Written more explicitly (using the notation from the previous paragraph) this means that the ideal $ip_*(M) = ip_*(\Delta)$ generated by the elements $\rho_b^{(i)} = \sum_{a \in p^{-1}(b)} r_a^{(i)}$, for $b \in \mathbb{Z}^t$ and $1 \leq i \leq d$, lies in $\mathcal{K}$. The requisite partition π of $\text{supp}(\Delta)$ is defined by declaring those intersections $\text{supp}(\Delta) \cap p^{-1}(b)$ which are non-empty to be the parts of π , where b varies over all of $\mathbb{Z}^t$. By construction, each component p_i of p is constant on each part of π ; on the part corresponding to $b = (b_1, b_2, \dots, b_t) \in \mathbb{Z}^t$, the component p_i takes the constant value b_i . The corresponding elements $\Delta_j^{(i)}$ are exactly the elements of the form $\rho_b^{(i)}$, so $\Delta_\pi = ip_*(\Delta) = ip_*(M)$ is a $\mathcal{K}$ -ideal. We have thus shown that π is a $\mathcal{K}$ -partition and $p \in H(\pi)^t$, as required. $\square$

Proof of Proposition 3.1. We re-state the conclusion of Lemma 3.2:

$$p_*(M) \text{ is a } \mathcal{K}\text{-module} \quad \Longleftrightarrow \quad p \in \bigcup_{\pi} H(\pi)^t,$$

the union extending over the finite set of all $\mathcal{K}$ -partitions π of $\text{supp}(\Delta)$. Up to renaming the groups occurring on the right-hand side, this is the condition stated in the Proposition. We have verified that each $H(\pi)$ is a direct summand of $\mathbb{Z}^{s*}$ in Lemma 2.2 above.

In case M is not a $\mathcal{K}$ -module we know from Lemma 2.5 that the partition subgroups $H(\pi)$ are proper subgroups of $\mathbb{Z}^{s*}$. We have $\ell > 0$ if and only if $0 \in \bigcup_{j=1}^{\ell} G_j^t$ if and only if $0_*(M)$ is a $\mathcal{K}$ -module. If M is contained in the augmentation ideal $I_s = \ker(0_*)$ of $R[\mathbb{Z}^s]$ then $0_*(M) = \{0\}$ is a $\mathcal{K}$ -module so that the union $\bigcup_{j=1}^{\ell} G_j^t$ must contain 0; this forces $\ell > 0$. This finishes the proof of Proposition 3.1. $\square$

Corollary 3.4. *Let $M_1, M_2, \dots, M_k$ be finitely generated R -submodules of $R[\mathbb{Z}^s]$. There are partition subgroups $G_1, G_2, \dots, G_{\ell} \subseteq \mathbb{Z}^{s*}$, where $\ell \geq 0$, such that for all $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$,*

$$\forall i = 1, 2, \dots, k : p_*(M_i) \text{ is a } \mathcal{K}\text{-module} \quad \Longleftrightarrow \quad p \in \bigcup_{j=1}^{\ell} G_j^t.$$

If at least one of the modules M_j is not a $\mathcal{K}$ -module the groups G_j are proper subgroups of $\mathbb{Z}^{s}$ so that $\bigcup_{j=1}^{\ell} G_j^t \neq \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. Moreover, $\ell > 0$ if and only if all the modules $0_*(M_j)$, for $1 \leq j \leq k$, are $\mathcal{K}$ -modules, which is certainly the case if all of the modules M_j are contained in the augmentation ideal I_s of $R[\mathbb{Z}^s]$.*

Proof. Let $\Delta_{(i)}$ be a finite generating set for the R -module M_i . By Proposition 3.1, we know that the statement

$$\forall i = 1, 2, \dots, k : p_*(M_i) \text{ is a } \mathcal{K}\text{-module}$$

holds if and only if there exist $\mathcal{K}$ -partitions $\pi_{(i)}$ of $\text{supp}(\Delta_{(i)})$, for $1 \leq i \leq k$, such that all components of p are constant on all parts of $\pi_{(i)}$. The latter condition is equivalent to saying $p \in \bigcap_{i=1}^k H(\pi_{(i)})^t$. So the requisite finite family of subgroups of $\mathbb{Z}^{s*}$ is given by the family of intersections $\bigcap_{i=1}^k H(\pi_{(i)}) = H(\{\pi_{(i)} \mid 1 \leq i \leq k\})$, with the $\pi_{(i)}$ ranging independently over all $\mathcal{K}$ -partitions of $\text{supp}(\Delta_{(i)})$. — The additional properties follow as in Proposition 3.1. $\square$

4. JUMP LOCI FOR THE RANK OF MATRICES

Let A be a matrix with entries in $R[\mathbb{Z}^s]$. We apply the results of the previous section to analyse the dependence of the rank of the matrix $p_*(A)$ from the homomorphism $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. First, we need to clarify what we mean by “rank”.

The $\mathcal{K}$ -rank of a matrix. We denote by $\mathcal{K}$ a hereditary set of ideals of R in the sense of Definition 1.3, with R an arbitrary commutative ring. Given any matrix A we write $|A|$ for the set of its entries. Now let A be specifically an $m \times n$ -matrix with entries in $R[\mathbb{Z}^s]$, and let z be a k -minor of A , that is, the determinant of a square submatrix of A of size k . (We remark here that determinants are defined in the usual fashion, *via* a sum indexed by the symmetric group or, equivalently, using LAPLACE expansion and induction

on k , and that determinants have all the usual properties. See the discussion in §2 of [McC39].) Let z' be the minor of $p_*(A)$ corresponding to z , for some fixed $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. Then we have $p_*(z) = z'$, as $p_*: R[\mathbb{Z}^s] \longrightarrow R[\mathbb{Z}^t]$ is a ring homomorphism. We also have $p_*(|A|) = |p_*(A)|$.

Definition 4.1. We say that A has $\mathcal{K}$ -rank 0 and write $\text{rank}_{\mathcal{K}}(A) = 0$ if $|A|$ is a $\mathcal{K}$ -set (that is, if the ideal generated by the coefficients of the entries of A is a $\mathcal{K}$ -ideal). Otherwise, the $\mathcal{K}$ -rank of A is the maximal integer $k = \text{rank}_{\mathcal{K}}(A) > 0$ such that the set of k -minors of A is *not* a $\mathcal{K}$ -set (that is, the ideal generated by the coefficients of the k -minors of A is *not* a $\mathcal{K}$ -ideal).

The following Lemma sheds some light on this definition.

Lemma 4.2. *If the set of k -minors of A is a $\mathcal{K}$ -set, then so is the set of $(k+1)$ -minors.*

Proof. By the familiar expansion formula of determinants, each minor of size $k+1$ is a linear combination of minors of size k , and the claim follows as $\mathcal{K}$ is a hereditary property. $\square$

If R is a (commutative) *unital* ring, the $\mathcal{K}_1$ -rank coincides with the rank of matrices over $R[\mathbb{Z}^s]$ as considered by MCCOY [McC42, §2]; we will show this in Proposition 4.6 below. If R is a field, the $\mathcal{K}_0$ -rank coincides with the usual rank from linear algebra.

Proposition 4.3. *For each $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ we have the inequality*

$$\text{rank}_{\mathcal{K}}(p_*(A)) \leq \text{rank}_{\mathcal{K}}(A) .$$

Proof. Let $k = 1 + \text{rank}_{\mathcal{K}}(A)$. The set of $k \times k$ -minors of A is a $\mathcal{K}$ -set, by definition of rank, hence so is its image under p_* by Lemma 1.4. But this image is precisely the set of $k \times k$ -minors of $p_*(A)$, whence $p_*(A)$ must have $\mathcal{K}$ -rank strictly less than k . $\square$

The McCoy-rank of a matrix. Let S denote a commutative *unital* ring, and let A be an $m \times n$ -matrix with entries in S .

Definition 4.4. We say that the matrix A has McCoy-rank 0, written $\text{rank}_{\text{McCoy}}(A) = 0$, if there exists a non-zero element of S annihilating every entry of A . Otherwise, the McCoy-rank of A is the maximal integer $k = \text{rank}_{\text{McCoy}}(A)$ such that the set of k -minors of A is *not* annihilated by a non-zero element of S .

In complete analogy to Lemma 4.2 one can show that *if the set of k -minors of A is annihilated by a non-zero element of S , then so is the set of $(k+1)$ -minors*. The relevance of the McCoy-rank is revealed by the following result:

Theorem 4.5 (McCoy [McC42, Theorem 1]). *The homogeneous system of m linear equations in n variables represented by the matrix A has a non-trivial solution over S if and only if $\text{rank}_{\text{McCoy}}(A) < n$.* $\square$

We are of course mainly interested in the special case $S = R[\mathbb{Z}^s]$ with R a commutative unital ring, in which case we also have the notion of $\mathcal{K}_1$ -rank at our disposal. Note that we have the inequality

$$\text{rank}_{\mathcal{K}_1}(A) \geq \text{rank}_{\text{McCoy}}(A) ;$$

indeed, if the set of coefficients of the k -minors of A is annihilated by a non-zero element y of R so that $\text{rank}_{\mathcal{K}_1}(A) < k$, then y also annihilates the k -minors themselves so that $\text{rank}_{\text{McCoY}}(A) < k$ as well.

Conversely, assume that the (finite) set $X \subseteq R[\mathbb{Z}^s]$ of k -minors of A is annihilated by a non-zero element $x \in R[\mathbb{Z}^s] \cong R[X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_s^{\pm 1}]$. For every $\ell \in \mathbb{Z}$ the element $u = X_1^\ell X_2^\ell \cdots X_s^\ell$ is a unit; we choose $\ell \gg 0$ such that $X' = uX$ and $x' = ux$ lie in the monoid ring

$$P = R[\mathbb{N}^s] \cong R[X_1, X_2, \dots, X_s] .$$

The ideal $\langle X' \rangle \trianglelefteq P$ generated by X' is annihilated by $x' \neq 0$. By a result of McCoY [McC57, Theorem] we can thus find a non-zero element $y \in R$ annihilating $X' = uX$. But then y also annihilates $u^{-1}X' = X$, and as y is an element of R this implies that y annihilates the coefficients of the elements of X individually. This yields the inequality $\text{rank}_{\mathcal{K}_1}(A) \leq \text{rank}_{\text{McCoY}}(A)$. — We have shown:

Proposition 4.6. *If R is a commutative unital ring, the two quantities $\text{rank}_{\mathcal{K}_1}(A)$ and $\text{rank}_{\text{McCoY}}(A)$ coincide.* $\square$

Jump loci. As before, we denote by $\mathcal{K}$ a hereditary set of ideals of R in the sense of Definition 1.3. Let $q \geq 0$, and let $A_1, A_2, \dots, A_k$ be matrices (of various sizes) over $R[\mathbb{Z}^s]$, with $\mathcal{K}$ -ranks $r_i = \text{rank}_{\mathcal{K}}(A_i)$. We want to characterise those $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ such that

$$\forall i = 1, 2, \dots, k : \text{rank}_{\mathcal{K}}(p_*(A_i)) < \text{rank}_{\mathcal{K}}(A_i) - q . \quad (4.7)$$

Of course such p cannot exist if $r_i \leq q$ for some i . Otherwise, we see from Lemma 4.2 that (4.7) is true if and only if for all indices i , the image of the set of $(r_i - q)$ -minors of A_i under p_* is a $\mathcal{K}$ -set. Writing M_i for the R -submodule of $R[\mathbb{Z}^s]$ generated by the $(r_i - q)$ -minors of A_i , this is equivalent to saying that the modules $p_*(M_i)$ are $\mathcal{K}$ -modules. Now Corollary 3.4 applies. Note that, by definition of rank, the modules M_i are *not* $\mathcal{K}$ -modules. We have shown:

Theorem 4.8. *Let $A_1, A_2, \dots, A_k$ be matrices (of various sizes) over the ring $R[\mathbb{Z}^s]$, and let $q \geq 0$. There are a number $\ell \geq 0$ and direct summands $G_1, G_2, \dots, G_\ell$ of $\mathbb{Z}^{s*}$ such that for $p \in H$,*

$$\begin{aligned} \forall i = 1, 2, \dots, k : \text{rank}_{\mathcal{K}}(p_*(A_i)) < \text{rank}_{\mathcal{K}}(A_i) - q \\ \iff p \in \bigcup_{j=1}^{\ell} G_j^t . \end{aligned}$$

If there is an index i such that $r_i \leq q$ then $\ell = 0$. Otherwise, the groups G_j are proper subgroups of $\mathbb{Z}^{s}$ so that $\bigcup_{j=1}^{\ell} G_j^t \neq \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$, and moreover $\ell > 0$ if the set of $(r_i - q)$ -minors of A_i is contained in the augmentation ideal I_s of $R[\mathbb{Z}^s]$ for all i .* $\square$

5. $\mathcal{K}$ -BETTI NUMBERS AND THEIR JUMP LOCI

Let C be a chain complex (possibly unbounded) consisting of finitely generated free based $R[\mathbb{Z}^s]$ -modules; more precisely, we suppose that $C_k = (R[\mathbb{Z}^s])^{r_k}$ for certain integers $r_k \geq 0$. We call r_k the rank of C_k ; if R is unital,

the (commutative) ring $R[\mathbb{Z}^s]$ has IBN and thus the isomorphism type of C_k determines r_k uniquely.

Our differentials lower the degree, $d_k: C_k \longrightarrow C_{k-1}$, and we assume that each map d_k is given by multiplication by a matrix D_k with entries in $R[\mathbb{Z}^s]$. (For non-unital R this potentially restricts the set of allowed differentials.) We define the $\mathcal{K}$ -rank of the homomorphism $d_k: C_k \longrightarrow C_{k-1}$, denoted $\text{rank}_{\mathcal{K}}(d_k)$, to be the $\mathcal{K}$ -rank $\text{rank}_{\mathcal{K}}(D_k)$ of the matrix D_k , as defined previously.

Definition 5.1. The k th $\mathcal{K}$ -BETTI number $b_k^{\mathcal{K}} = b_k^{\mathcal{K}}(C)$ of C is

$$b_k^{\mathcal{K}}(C) = r_k - \text{rank}_{\mathcal{K}}(d_k) - \text{rank}_{\mathcal{K}}(d_{k+1}) .$$

Given a homomorphism $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$, we define a new chain complex $p_!(C)$ by setting $p_!(C)_k = (R[\mathbb{Z}^t])^{r_k}$, equipped with differentials denoted $p_!(d_k)$ given by the matrices $p_*(D_k)$. In case the multiplication map $R[\mathbb{Z}^s] \otimes_{R[\mathbb{Z}^s]} R[\mathbb{Z}^t] \longrightarrow R[\mathbb{Z}^t]$, $x \otimes y \mapsto p_*(x) \cdot y$ is an isomorphism (this happens, for example, if R is unital), we have $p_!(C) = C \otimes_{R[\mathbb{Z}^s]} R[\mathbb{Z}^t]$. — In general we have $b_k^{\mathcal{K}}(p_!(C)) \geq b_k^{\mathcal{K}}(C)$ by Proposition 4.3, with strict inequality if and only if at least one of the strict inequalities $\text{rank}_{\mathcal{K}}(p_!(d_k)) < \text{rank}_{\mathcal{K}}(d_k)$ and $\text{rank}_{\mathcal{K}}(p_!(d_{k+1})) < \text{rank}_{\mathcal{K}}(d_{k+1})$ is satisfied.

More generally, given $q \geq 0$ we want to characterise those $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ such that

$$b_k^{\mathcal{K}}(p_!(C)) > b_k^{\mathcal{K}}(C) + q . \quad (5.2)$$

This happens if and only if applying $p_!$ lowers the $\mathcal{K}$ -rank of d_k by at least $q + 1 - j$, and lowers the $\mathcal{K}$ -rank of d_{k+1} by at least j , for some j in the range $0 \leq j \leq q + 1$. Stated more formally:

Lemma 5.3. *The inequality (5.2) holds if and only if there exists a number j with $0 \leq j \leq q + 1$ such that*

$$\text{rank}_{\mathcal{K}}(p_!(d_k)) \leq \text{rank}_{\mathcal{K}}(d_k) - (q + 1 - j) \quad (5.4a)$$

and

$$\text{rank}_{\mathcal{K}}(p_!(d_{k+1})) \leq \text{rank}_{\mathcal{K}}(d_{k+1}) - j . \quad (5.4b)$$

Proof. If such j exists, then we have indeed

$$\begin{aligned} b_k^{\mathcal{K}}(p_!(C)) &= r_k - \text{rank}_{\mathcal{K}}(p_!(d_k)) - \text{rank}_{\mathcal{K}}(p_!(d_{k+1})) \\ &\geq r_k - (\text{rank}_{\mathcal{K}}(d_k) - (q + 1 - j)) - (\text{rank}_{\mathcal{K}}(d_{k+1}) - j) \\ &= r_k - \text{rank}_{\mathcal{K}}(d_k) - \text{rank}_{\mathcal{K}}(d_{k+1}) + q + 1 \\ &> b_k^{\mathcal{K}}(C) + q . \end{aligned}$$

For the converse, suppose that for each j at least one of the inequalities (5.4a) and (5.4b) is violated. Inequality (5.4b) holds for $j = 0$, by Proposition 4.3; let $m \leq q + 1$ be maximal such that (5.4b) is true for $0 \leq j \leq m$. We must have $m \leq q$; otherwise (5.4a) must be violated for $j = m = q + 1$, resulting in the inequality $\text{rank}_{\mathcal{K}}(p_!(d_k)) > \text{rank}_{\mathcal{K}}(d_k)$ which is known to be nonsense, by Proposition 4.3 again.

We have established that (5.4b) holds for $j = m \leq q$ but is violated for $j = m + 1$, that is, we know

$$\text{rank}_{\mathcal{K}}(p_!(d_{k+1})) \leq \text{rank}_{\mathcal{K}}(d_{k+1}) - m$$

and

$$\text{rank}_{\mathcal{K}}(p_!(d_{k+1})) > \text{rank}_{\mathcal{K}}(d_{k+1}) - (m + 1)$$

which yields the equality

$$\text{rank}_{\mathcal{K}}(p_!(d_{k+1})) = \text{rank}_{\mathcal{K}}(d_{k+1}) - m .$$

As (5.4b) holds for $j = m$ we know that (5.4a) must be false for $j = m$. This, together with the previous equality, provides the estimate

$$\begin{aligned} b_k^{\mathcal{K}}(p_!(C)) &= r_k - \text{rank}_{\mathcal{K}}(p_!(d_k)) - \text{rank}_{\mathcal{K}}(p_!(d_{k+1})) \\ &< r_k - (\text{rank}_{\mathcal{K}}(d_k) - (q + 1 - m)) - (\text{rank}_{\mathcal{K}}(d_{k+1}) - m) \\ &= r_k - \text{rank}_{\mathcal{K}}(d_k) - \text{rank}_{\mathcal{K}}(d_{k+1}) + q + 1 \\ &= b_k^{\mathcal{K}}(C) + q + 1 , \end{aligned}$$

whence $b_k^{\mathcal{K}}(p_!(C)) \leq b_k^{\mathcal{K}}(C) + q$ so that (5.2) does not hold, as required. $\square$

We can now apply Theorem 4.8 for each *fixed* j in the range $0 \leq j \leq q + 1$ to the two matrices D_k and D_{k+1} , yielding a family of direct summands of $\mathbb{Z}^{s*}$ characterising for which $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ the inequalities (5.4a) and (5.4b) hold for our given choice of j . To characterise for which p inequality (5.2) holds, we allow j to vary and take the union of all the groups G_j occurring. Collecting the information then results in the following:

Theorem 5.5. *Let C be a (not necessarily bounded) chain complex of finitely generated based free $R[\mathbb{Z}^s]$ -modules, with differentials given by matrices over $R[\mathbb{Z}^s]$. Let $q \geq 0$ and $k \in \mathbb{Z}$. There are a number $\ell \geq 0$ and direct summands $G_1, G_2, \dots, G_\ell$ of $\mathbb{Z}^{s*}$ such that for $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$,*

$$b_k^{\mathcal{K}}(p_!(C)) > b_k^{\mathcal{K}}(C) + q \quad \Longleftrightarrow \quad p \in \bigcup_{j=1}^{\ell} G_j^t . \quad \square$$

Empty jump locus. We finish the paper with a curious observation on the minors of differentials in certain chain complexes of free modules. We keep the notation from above: R is a commutative ring, $\mathcal{K}$ a hereditary set of ideals, and C a (possibly unbounded) chain complex consisting of finitely generated free based $R[\mathbb{Z}^s]$ -modules. As before we denote the k th differential by d_k , and insist that d_k is given by multiplication by a matrix D_k . In Theorem 5.5 we consider $q = 0$ and a fixed $k \in \mathbb{Z}$, and assume that $\ell = 0$ so the jump locus is the empty set. This means that for all $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ the induced complex $p_!(C)$ has the same k th $\mathcal{K}$ -BETTI number as C , that is, $b_k^{\mathcal{K}}(p_!(C)) = b_k^{\mathcal{K}}(C)$.

By definition of BETTI numbers the equality $b_k^{\mathcal{K}}(p_!(C)) = b_k^{\mathcal{K}}(C)$ necessitates that $\text{rank}_{\mathcal{K}}(d_i) = \text{rank}_{\mathcal{K}}(p_!(d_i))$ for $i = k, k + 1$. Write $r = \text{rank}_{\mathcal{K}}(d_i)$. In case $r > 0$ we let $j \leq r$ be a positive integer. The set of j -minors of D_i is not a $\mathcal{K}$ -set, by definition of the $\mathcal{K}$ -rank and Lemma 4.2. It follows that the set of j -minors of $p_*(D_i)$ is not a $\mathcal{K}$ -set either. (Indeed, if it was a $\mathcal{K}$ -set then $\text{rank}_{\mathcal{K}}(p_!(C)) < j \leq r = \text{rank}_{\mathcal{K}}(d_i)$ so that $b_i^{\mathcal{K}}(p_!(C)) > b_i^{\mathcal{K}}(C)$ contrary to

our hypothesis.) In particular, not all j -minors of D_i can be contained in the augmentation ideal of $R[\mathbb{Z}^s]$ as otherwise the j -minors of $0_*(D_i)$ would all be trivial, and would thus form a $\mathcal{K}$ -set. We have shown:

Proposition 5.6. *Suppose that $b_k^{\mathcal{K}}(p_!(C)) = b_k^{\mathcal{K}}(C)$ for all $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. Let $i = k, k+1$. For every positive integer $j \leq \text{rank}_{\mathcal{K}}(d_i)$ at least one j -minor of D_i is not contained in the augmentation ideal of $R[\mathbb{Z}^s]$. $\square$*

The Proposition applies to the following special case: R a unital integral domain, $\mathcal{K} = \mathcal{K}_0$ and C a contractible complex. For then $p_!(C)$ is contractible as well since tensor products preserve homotopies. It follows that C and $p_!(C)$ are acyclic, for any p ; as R is an integral domain, the (usual) BETTI numbers, corresponding to the specific choice of $\mathcal{K}_0$ as hereditary set of ideals, can be computed as the rank (in the usual sense) of the homology modules, which are all trivial. That is, both C and $p_!(C)$ have vanishing k th BETTI number for all $k \in \mathbb{Z}$. As $\text{rank}_{\mathcal{K}_0}(d_k) > 0$ is equivalent to $d_k \neq 0$, we conclude:

Corollary 5.7. *Suppose that R is a unital integral domain, and that C is a contractible chain complex of finitely generated free $R[\mathbb{Z}^s]$ -modules. For every non-zero differential d_k of C , and every positive $j \leq \text{rank}(d_k)$, at least one j -minor of its representing matrix D_k is not contained in the augmentation ideal of $R[\mathbb{Z}^s]$. $\square$*

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