**QUEEN'S
UNIVERSITY
BELFAST**

A "quite superfluous" characterization of the Jacobson radical of an associative ring

Hüttemann, T. (2019). A "quite superfluous" characterization of the Jacobson radical of an associative ring. *Communications in Algebra*, 47(11), 4740-4742. <https://doi.org/10.1080/00927872.2019.1593429>

Published in:
Communications in Algebra

Document Version:
Peer reviewed version

Queen's University Belfast - Research Portal:
[Link to publication record in Queen's University Belfast Research Portal](#)

Publisher rights

© 2019 Taylor & Francis Group, LLC. This work is made available online in accordance with the publisher's policies. Please refer to any applicable terms of use of the publisher.

General rights

Copyright for the publications made accessible via the Queen's University Belfast Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The Research Portal is Queen's institutional repository that provides access to Queen's research output. Every effort has been made to ensure that content in the Research Portal does not infringe any person's rights, or applicable UK laws. If you discover content in the Research Portal that you believe breaches copyright or violates any law, please contact openaccess@qub.ac.uk.

A “QUITE SUPERFLUOUS” CHARACTERISATION OF THE JACOBSON RADICAL OF AN ASSOCIATIVE RING

THOMAS HÜTTEMANN

ABSTRACT. It is well-known that the JACOBSON radical of a unital ring R is the largest superfluous right ideal of R . While a simple example shows this to be false for non-unital rings, it is shown that the result remains valid in the absence of a unit provided the notion of “superfluous” is taken relative to all *regular* right ideals. The purpose of this note is to provide a proof for this little-known fact.

A standard result in ring theory asserts that the JACOBSON radical of a unital ring is the largest superfluous right ideal. Apparently the analogous situation for non-unital rings has not been documented. We show by a simple modification of a well-known example that the result fails for non-unital rings in general, but remains valid provided the notion of “superfluous ideal” is adapted suitably.

1. THE JACOBSON RADICAL OF A RING

Let R be an associative ring, possibly non-unital. An element $a \in R$ is said to be *right quasi-regular* if there exists $x \in R$ with $ax = a + x$. (In case R is a unital ring, this is equivalent to $1 - a$ having right inverse $1 - x$.) The JACOBSON *radical* $J(R)$ of R is the two-sided ideal

$$J(R) = \{a \in R \mid \forall b \in R : ab \text{ is right quasi-regular}\},$$

see, for example, [McC64, Definition 6.6].

Example 1.1. Write Q for the (commutative) unital ring of rational numbers of the form s/t for $s, t \in \mathbb{Z}$ with $2 \nmid t$ and $3 \nmid t$. The units in Q are precisely the elements represented by fractions s/t with $2, 3 \nmid t$ and $2, 3 \nmid s$, in which case the inverse of s/t is t/s .

Let $R_0 = (2)$, the (right) ideal consisting of fractions $2s/t$ with $2, 3 \nmid t$, considered as a non-unital ring. The JACOBSON *radical* of R_0 is the ideal $J = (6)$ of fractions of the form $6s/t$ with $2, 3 \nmid t$. Indeed, as $R_0 \triangleleft Q$ we have $J(R_0) = R_0 \cap J(Q)$, by [McC64, Theorem 6.14]. Hence it is enough to verify that $J(Q) = J$, that is, $J = \{a \in Q \mid \forall b \in Q : 1 - ab \text{ is a unit in } Q\}$. This is straightforward, and the details are omitted. $\square$

2. SUPERFLUOUS IDEALS AND THE JACOBSON RADICAL

A right ideal $J \triangleleft R$ is called *superfluous* if for every right ideal $I \triangleleft R$,

$$J + I = R \implies I = R.$$

The following well-known result can be found in ANDERSON-FULLER [AF92, Theorem 15.3].

Theorem 2.1. *Let R be a unital ring. Its JACOBSON radical $J(R)$ is a superfluous right ideal of R which contains every superfluous right ideal of R (i.e., $J(R)$ is the unique largest superfluous right ideal of R).* $\square$

Date: February 21, 2019.

2000 Mathematics Subject Classification. Primary 16N20; Secondary 16D25, 16D80.

In the next section we will generalise this result to include the case of non-unital rings. We note first that $J(R)$ need not be superfluous in general:

Example 2.2. Consider the ring $R_0 = (2) \triangleleft Q$ from Example 1.1, and let $I = (4) \triangleleft R_0$ be the proper (right) ideal of R_0 consisting of fractions $4s/t$ with $2, 3 \nmid t$. Then $J(R_0) + I = R_0$; for given $a = 2s/t \in R_0$ with $2, 3 \nmid t$ we have trivially

$$a = \frac{2s}{t} = \frac{6s}{t} - \frac{4s}{t} \in (6) + (4) = J(R_0) + I.$$

As $I \subsetneq R_0$ this shows that $J(R_0)$ is not a superfluous right ideal. $\square$

Such considerations are not without merit; for example, it is known that a left ARTINIAN ring possesses a left unit if and only if its JACOBSON radical is a superfluous right ideal, cf. KERTÉSZ [Ker68, Satz 10.16].

3. QUITE SUPERFLUOUS IDEALS AND THE JACOBSON RADICAL

Recall that a right ideal I of a ring R is called *regular* if there exists an element $e \in R$ such that $er - r \in I$ for all $r \in R$.

Definition 3.1. A right ideal J of R is called *quite superfluous* if $J + I = R$ implies $I = R$ for every regular right ideal I of R .

There are of course analogous definitions of “regular” and “quite superfluous” for left ideals. Every superfluous (right or left) ideal is quite superfluous. Every right ideal contained in a quite superfluous right ideal is quite superfluous. — The following well-known fact is central to the theory:

Lemma 3.2 (MCCOY [McC64, Lemma 6.18]). *If $I \neq R$ is a regular right ideal of R , there exists a regular maximal right ideal M containing I .* $\square$

The JACOBSON radical $J(R)$ is the intersection of all regular maximal right ideals of R [McC64, Theorem 6.20]. Hence $J(R) = R$ if and only if R has no regular right ideals distinct from R , in which case every right ideal of R is quite superfluous.

Theorem 3.3. *Let R be an associative ring. Its JACOBSON radical $J(R)$ is a quite superfluous right ideal containing every quite superfluous right ideal of R (i.e., $J(R)$ is the unique largest quite superfluous right ideal of R).*

If R possesses a left unit e_ℓ , as is the case when R is unital, then every right ideal is regular (choose $e = e_\ell$), and “quite superfluous” is the same as “superfluous”. Thus Theorem 3.3 indeed generalises Theorem 2.1.

Proof of Theorem 3.3. In view of the remarks preceding the Theorem, we may assume that $J(R) \neq R$. Suppose that $I \subsetneq R$ is a regular right ideal of R . By Lemma 3.2 there is a regular maximal right ideal M containing I . As the JACOBSON radical is contained in every regular maximal ideal, we conclude $J(R) + I \subseteq J(R) + M = M \neq R$. Hence $J(R)$ is quite superfluous.

Next, let K be a quite superfluous right ideal of R . If M is a regular maximal right ideal then $K + M \neq R$ since K is quite superfluous; this implies that $K \subseteq M$. As this is true for every M we have $K \subseteq J(R)$ as claimed. $\square$

Corollary 3.4. *A right ideal I of R is quite superfluous if and only if it is contained in $J(R)$. $\square$*

Since the JACOBSON radical is left-right symmetric, we can also say that *a left ideal I of R is quite superfluous if and only if it is contained in $J(R)$.*

By the Corollary above the JACOBSON radical of R can be described as

$$J(R) = \{a \in R \mid (a) \text{ is quite superfluous}\} , \quad (3.5)$$

where (a) denotes the smallest right ideal of R containing a . As the JACOBSON radical of the ring R_0 of Example 1.1 contains the element $a = 6$, but $(a) = J(R_0)$ is not a superfluous (right) ideal by Example 2.2, we cannot replace “quite superfluous” by “superfluous” in (3.5).

A characterisation of the JACOBSON radical closely related to (3.5) has been given by KERTÉSZ [Ker63]:

$$J(R) = \{a \in R \mid \forall s \in R: (sa) \text{ is superfluous}\} . \quad (3.6)$$

By comparing (3.5) with (3.6) we see that the right ideal (a) is quite superfluous if and only if (sa) is superfluous for all $s \in R$.

REFERENCES

- [AF92] Frank W. Anderson and Kent R. Fuller. *Rings and categories of modules*, volume 13 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992.
- [Ker63] Andor Kertész. A characterization of the Jacobson radical. *Proc. Amer. Math. Soc.*, 14:595–597, 1963.
- [Ker68] Andor Kertész. *Vorlesungen über artinsche Ringe*. Akadémiai Kiadó, Budapest, 1968.
- [McC64] Neal H. McCoy. *The theory of rings*. The Macmillan Co., New York; Collier-Macmillan Ltd., London, 1964.

THOMAS HÜTTEMANN, QUEEN’S UNIVERSITY BELFAST, SCHOOL OF MATHEMATICS AND PHYSICS, MATHEMATICAL SCIENCES RESEARCH CENTRE, BELFAST BT7 1NN, NORTHERN IRELAND, UK

E-mail address: t.huettemann@qub.ac.uk