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Stochastic Model Predictive Control for Constrained Discrete-time Markovian Switching Systems [★]

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Abstract

In this paper we study constrained stochastic optimal control problems for Markovian switching systems, an extension of Markovian jump linear systems (MJLS), where the subsystems are allowed to be nonlinear. We develop appropriate notions of invariance and stability for such systems and provide terminal conditions for stochastic MPC that guarantee mean-square stability and robust constraint fulfillment of the Markovian switching system in closed-loop with the SMPC law under very weak assumptions. In the special but important case of constrained MJLS we present an algorithm for computing explicitly the stochastic MPC control law off-line, that combines dynamic programming with parametric piecewise quadratic optimization.

1 Introduction

Markovian switching systems consist of a family of nonlinear subsystems (usually called *modes*) and a Markov chain that orchestrates the switching among them. Since their introduction [21], they have found numerous applications due to their ability to model dynamical systems with random abrupt dynamic changes (failures and repairs) and random time-delays. Some of the applications include manufacturing systems [2], bioreactors [14], macroeconomics [35], and networked control systems [25], to name a few.

Due to these reasons, a large amount of research has been conducted concerning various notions of stability such as mean square stability [17], stochastic stability [9], almost sure stability [8] and uniform stability [22]. Furthermore, finite and infinite horizon optimal control both in discrete [1, 7] and continuous time [31, 34] have been studied extensively. Notably, all the aforementioned works deal with a special instance of Markovian switching systems, where individual mode dynamics are linear, namely Markov jump linear systems (MJLS) [12]. Regarding the infinite horizon linear quadratic optimal control problem for unconstrained MJLS, it can be solved efficiently via a Coupled Algebraic Riccati equations (CARE) approach [1, 7], or a linear matrix inequalities (LMI) approach [27].

However, almost all physical systems are subject to constraints dictated by physical limits and performance, safety, or economical considerations. Nonetheless, only a few works exist in the literature concerning optimal control of *constrained* Markovian switching systems. Specifically, in [11], the framework of [20] for robust model predictive control (MPC) of uncertain linear systems is extended to MJLS subject to hard symmetric state and control constraints, while the transition matrix of the Markov chain is known to lie in a convex set. This suboptimal approach calculates, on-line, a mode-dependent, linear, state-feedback control law that minimizes an upper bound on the worst-case expected infinite horizon

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cost, by solving an LMI problem. In [33], the MPC problem for MJLS with constraints on the first and second moments for the input and state vector and unobservable modes is studied. More recently [4, 5], a Stochastic Model Predictive Control (SMPC) framework for stochastic constrained linear systems was proposed. The authors impose a stochastic Lyapunov decrease condition for the first step of the SMPC algorithm that is robust with respect to constraint enforcement, and allows to guarantee mean-square stability and robust invariance so that scenario trees are only used for performance optimization.

This paper studies the constrained finite horizon stochastic optimal control problem for discrete-time Markovian switching systems. Here, the constraints must be satisfied uniformly, over all admissible switching paths. Properties of the value function and the mode-dependent optimal policy are derived under a variety of assumptions. Furthermore, an appropriate notion of control invariance, namely uniform control invariance, is defined for Markovian switching systems. In addition, we employ dynamic programming coupled with the parametric piecewise quadratic optimization solver [24] to solve explicitly the constrained finite-horizon constrained stochastic optimal control problem arising in SMPC for MJLS, without gridding the state-space. For general nonlinear Markovian switching systems we show how the finite-horizon stochastic optimal control problem can be formulated as a finite-dimensional optimization problem. Conditions that guarantee mean-square (exponential) stability for the system in closed-loop with the SMPC law are established.

This work was motivated by the increased interest of the research community on Markovian Switching Systems and the need for a Stochastic Model Predictive Control methodology that provides robust uniform satisfaction of the state and input constraints (a very important requirement for engineering applications) and mean-square stability.

2 Mathematical Preliminaries

Let \mathbb{R} , \mathbb{R}_+ , \mathbb{N} and \mathbb{N}_+ denote the sets of real numbers, nonnegative real numbers, nonnegative integers and positive integers, respectively. The notation we use in this paper comes from [29]. For $k_1, k_2 \in \mathbb{N}$, $\mathbb{N}_{[k_1, k_2]} \triangleq \{k \in \mathbb{N} | k_1 \leq k \leq k_2\}$. $\bar{\mathbb{R}} \triangleq [-\infty, \infty]$ denotes the extended-real line. For an extended-real-valued function $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$, its *epigraph* is $\text{epi } f \triangleq \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} | \alpha \geq f(x)\}$, its *effective domain* is $\text{dom } f \triangleq \{x \in \mathbb{R}^n | f(x) < \infty\}$ and for any $\alpha \in \mathbb{R}$, the corresponding *level-set* of f is $\text{lev}_{\leq \alpha} f \triangleq \{x \in \mathbb{R}^n | f(x) \leq \alpha\}$. We call f *proper* if $f(x) < \infty$ for at least one $x \in \mathbb{R}^n$, and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$. A function $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ is *lower-semicontinuous (lsc)* at \bar{x} if $\liminf_{x \rightarrow \bar{x}} f(x) = f(\bar{x})$. A function $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$

with values $f(x, u)$ is *level-bounded in u locally uniformly in x* if for each $\bar{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ there exists a neighborhood $\mathcal{N}(\bar{x})$ of \bar{x} , along with a bounded set $B \subset \mathbb{R}^m$ such that $\{u | f(x, u) \leq \alpha\} \subset B$ for all $x \in \mathcal{N}(\bar{x})$. A function $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ is called *piecewise quadratic (PWQ)* if $\text{dom } f$ can be represented as the union of a finite number of polyhedral sets, relative to each of which f is quadratic.

Let $\mathcal{S} \subset \mathbb{N}_+$. For ease of notation we define the class of functions

$$\text{fcns}(\mathbb{R}^n, \mathcal{S}) \triangleq \{f : \mathbb{R}^n \times \mathcal{S} \rightarrow \bar{\mathbb{R}} | f \geq 0, f(0, i) = 0, i \in \mathcal{S}\}.$$

We use the notation $\text{lsc}(\mathbb{R}^n, \mathcal{S})$, $\text{conv}(\mathbb{R}^n, \mathcal{S})$ and $\text{pwq}(\mathbb{R}^n, \mathcal{S})$ for the subclasses of $\text{fcns}(\mathbb{R}^n, \mathcal{S})$ whose members $f(\cdot, i)$ are lsc, convex and PWQ respectively for all $i \in \mathcal{S}$. We define the class of sets

$$\text{sets}(\mathbb{R}^n, \mathcal{S}) \triangleq \{C = \{C_i\}_{i \in \mathcal{S}} | 0 \in C_i \subseteq \mathbb{R}^n, i \in \mathcal{S}\},$$

and we use the notation $\text{cl-sets}(\mathbb{R}^n, \mathcal{S})$, $\text{conv-sets}(\mathbb{R}^n, \mathcal{S})$ and $\text{poly-sets}(\mathbb{R}^n, \mathcal{S})$ for the subclasses of $\text{sets}(\mathbb{R}^n, \mathcal{S})$ whose member C_i are closed, convex and polyhedral respectively for all $i \in \mathcal{S}$. With a slight abuse of notation, for $f \in \text{fcns}(\mathbb{R}^n, \mathcal{S})$ we write $\text{dom } f = C$, meaning that $C \in \text{sets}(\mathbb{R}^n, \mathcal{S})$ and $\text{dom } f(\cdot, i) = C_i, i \in \mathcal{S}$. $f_1 \leq f_2$ for $f_1, f_2 \in \text{fcns}(\mathbb{R}^n, \mathcal{S})$ means $f_1(x, i) \leq f_2(x, i)$ for every $(x, i) \in \mathbb{R}^n \times \mathcal{S}$. Likewise, $C^1 = C^2$ ($C^1 \subseteq C^2$) for $C^1, C^2 \in \text{sets}(\mathbb{R}^n, \mathcal{S})$ means $C_i^1 = C_i^2$ ($C_i^1 \subseteq C_i^2$) for every $i \in \mathcal{S}$.

The indicator function δ_C of a set $C \subseteq \mathbb{R}^n$ is defined by $\delta_C(x) = 0$, if $x \in C$ and $\delta_C(x) = \infty$, otherwise. For $C \in \text{sets}(\mathbb{R}^n, \mathcal{S})$, let $\delta_C : \mathbb{R}^n \times \mathcal{S} \rightarrow \bar{\mathbb{R}}$ with $\delta_C(\cdot, i) = \delta_{C_i}, i \in \mathcal{S}$. The domain of a set-valued mapping $\mathcal{S} : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$, is the set $\text{dom } \mathcal{S} = \{p | \mathcal{S}(p) \neq \emptyset\}$. If C is a finite set, then $|C|$ denotes the cardinality of C .

3 Constrained Markovian Switching Systems

Consider the following discrete-time Markovian switching system (MSS):

$$x_{k+1} = f_{r_k}(x_k, u_k) \quad (1)$$

Here, $\{r_k\}_{k \in \mathbb{N}}$ is a discrete-time, time-homogeneous Markov chain taking values in a finite set $\mathcal{S} \triangleq \{1, \dots, S\}$ with transition matrix $P \triangleq (p_{ij}) \in \mathbb{R}^{S \times S}$ and initial distribution $v = (v_1, \dots, v_S)$. We assume that $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$. The standing assumption valid throughout the paper is:

Assumption 1 *The mappings $f_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ are continuous and satisfy $f_i(0, 0) = 0, i \in \mathcal{S}$.*

When needed, we will impose the following assumption:

Assumption 2 $f_i(x, u) = A_i x + B_i u, \forall i \in \mathcal{S}$.

Let \mathcal{S} consist of all subsets of \mathcal{S} , and $\Omega \triangleq \prod_{k \in \mathbb{N}} (\mathbb{R}^n \times \mathbb{R}^m \times \mathcal{S})$. Let \mathfrak{F}_k be the minimal σ -field over the Borel-measurable rectangles of Ω with k -dimensional base and \mathfrak{F} be the minimal σ -field over all Borel-measurable rectangles, *i.e.*, the product σ -field measurable space. Define the filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_k\}_{k \in \mathbb{N}}, \mathbb{P})$ where \mathbb{P} is the unique product probability measure according to the infinite dimensional product measure theorem [3, Th. 2.7.2], with $\mathbb{P}(r_0 = i_0, r_1 = i_1, \dots, r_k = i_k) = v_{i_0} p_{i_0 i_1} \dots p_{i_{k-1} i_k}$ for any $i_0, i_1, \dots, i_k \in \mathcal{S}$ and $k \in \mathbb{N}$, where r_k is a random variable from Ω to \mathcal{S} . Let $\mathbb{E}[\cdot]$ denote the expectation of a random variable with respect to \mathbb{P} and $\mathbb{E}[\cdot | \mathfrak{F}_k]$ the conditional expectation. It can be shown [32] that the augmented state (x_k, r_k) contains all the probabilistic information relevant to the evolution of the Markovian switching system for times $t > k$. We call realizations of the Markov chain, *switching paths*.

Definition 1 The cover \mathcal{S}_i of a mode $i \in \mathcal{S}$ is the set of all modes $j \in \mathcal{S}$ accessible from i in one time step, *i.e.* $\mathcal{S}_i \triangleq \{j \in \mathcal{S} | p_{ij} > 0\}$.

Definition 2 An admissible switching path of length $N \in \mathbb{N} \cup \{+\infty\}$, $\mathbf{r} \triangleq (r_0, \dots, r_N)$ for (1) is a switching path for which $r_{k+1} \in \mathcal{S}_{r_k}$, for any $k \in \mathbb{N}_{[0, N-1]}$. We denote by \mathfrak{G} the set of all admissible switching paths (of infinite length), and by \mathfrak{G}_N the set of all admissible switching paths of length N . For any $i \in \mathcal{S}$, $\mathfrak{G}(i) \triangleq \{\mathbf{r} \in \mathfrak{G} | r_0 = i\}$ and $\mathfrak{G}_N(i) \triangleq \{\mathbf{r} \in \mathfrak{G}_N | r_0 = i\}$ denote the set of all admissible switching paths emanating from i , of infinite length and length N , respectively.

It is assumed that (1) must satisfy the following hard joint state and input constraints, uniformly, over all admissible switching paths:

$$(x_k, u_k) \in Y_{r_k}, k \in \mathbb{N}, \mathbf{r} \in \mathfrak{G} \quad (2)$$

where $Y_i \subseteq \mathbb{R}^n \times \mathbb{R}^m$, $i \in \mathcal{S}$. For each $i \in \mathcal{S}$ let $\mathcal{U}_i(x) \triangleq \{u \in \mathbb{R}^m | (x, u) \in Y_i\}$ and $X_i \triangleq \text{dom} \mathcal{U}_i$. Let $\mathcal{Y} \triangleq \{Y_i\}_{i \in \mathcal{S}}$ and $\mathcal{X} \triangleq \{X_i\}_{i \in \mathcal{S}}$. A Borel measurable mapping $\mu : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^m$, such that $\mu(x, i) \in \mathcal{U}_i(x)$ for each $x \in X_i$ and $i \in \mathcal{S}$, is called a (*mode-dependent*) *control law*. A sequence of control laws $\pi \triangleq \{\mu_0, \mu_1, \dots\}$ is called a (*mode-dependent*) *policy*. Since we are only dealing with mode-dependent control laws and policies, the adjective mode-dependent will be omitted for brevity since now on. We denote by $\Pi \triangleq \{\pi = \{\mu_0, \mu_1, \dots\} | \mu_k(x, i) \in \mathcal{U}_i(x), i \in \mathcal{S}, k \in \mathbb{N}\}$ the set of all policies, and by Π_N the set of all policies of length N . If the policy is of the form $\{\mu, \mu, \dots\}$ then it is called *stationary* and is simply denoted by μ . The solution of (1) at time k , given a policy π and a switching path \mathbf{r} with $r_0 = i$ and $x_0 = x$, is denoted by $\phi(k; x, i, \pi, \mathbf{r})$.

4 Finite-Horizon Stochastic Optimal Control for MSS

In this section, the finite-horizon stochastic optimal control problem for constrained MSS is formulated. The stage cost ℓ is assumed to be (possibly) mode-dependent. To improve clarity of exposition and express the results of the paper in a more general setting, we will work with extended-real-valued stage costs where for each mode $i \in \mathcal{S}$, their effective domain is equal to Y_i , *i.e.*, $\ell \in \text{fncs}(\mathbb{R}^{n+m}, \mathcal{S})$ with $\text{dom} \ell = Y$. Furthermore, the terminal cost function can be mode-dependent, *i.e.*, $V_f \in \text{fncs}(\mathbb{R}^n, \mathcal{S})$. Let $X^f \triangleq \text{dom} V_f \subseteq X$. The finite-horizon cost of the policy $\pi \in \Pi_N$ for (1), starting from $x_0 = x$, $r_0 = i$ is:

$$V_{N, \pi}(x, i) \triangleq \mathbb{E} \left[\sum_{k=0}^{N-1} \ell(x_k, u_k, r_k) + V_f(x_N, r_N) \right] \quad (3)$$

where $x_k \triangleq \phi(k; x, i, \pi, \mathbf{r})$, $u_k \triangleq \mu_k(\phi(k; x, i, \pi, \mathbf{r}), r_k)$ and N is the finite horizon length. It is apparent that given a pair $(x, i) \in \mathbb{R}^n \times \mathcal{S}$ and a policy $\pi \in \Pi_N$, the finite-horizon cost (3) is finite if and only if $(x_k, u_k) \in Y_{r_k}$ and $x_N \in X_{r_N}^f$ for all $\mathbf{r} \in \mathfrak{G}_N(i)$. The constrained finite-horizon stochastic optimal control problem is:

$$\mathbb{P}_N^*(x, i) : V_N^*(x, i) \triangleq \inf_{\pi \in \Pi_N} V_{N, \pi}(x, i) \quad (4a)$$

$$\Pi_N^*(x, i) \triangleq \underset{\pi \in \Pi_N}{\text{argmin}} V_{N, \pi}(x, i) \quad (4b)$$

We call $V_N^* : \mathbb{R}^n \times \mathcal{S} \rightarrow \overline{\mathbb{R}}$, $\Pi_N^* \subset \Pi_N$ the *value function* and *optimal policy mapping*, respectively.

4.1 Dynamic Programming Solution

In this subsection, we study properties of (4) using dynamic programming. We also define an appropriate notion of controlled invariance for MSS, namely *uniform control invariance* and establish a connection with dynamic programming. In order to study properties of (4) we introduce some notation due to [6].

Definition 3 For any $V \in \text{fncs}(\mathbb{R}^n, \mathcal{S})$ and any control law $\mu : \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^m$ define the operator \mathbf{T}_μ as

$$\mathbf{T}_\mu V(x, i) \triangleq \ell(x, \mu(x, i), i) + \sum_{j \in \mathcal{S}} p_{ij} V(f_i(x, \mu(x, i)), j)$$

Definition 4 For any $V \in \text{fncs}(\mathbb{R}^n, \mathcal{S})$ define the operators \mathbf{T} and \mathbf{S} , respectively as

$$\mathbf{T}V(x, i) \triangleq \inf_u \{\ell(x, u, i) + \sum_{j \in \mathcal{S}} p_{ij} V(f_i(x, u), j)\}$$

$$\mathbf{S}V(x, i) \triangleq \underset{u}{\text{argmin}} \{\ell(x, u, i) + \sum_{j \in \mathcal{S}} p_{ij} V(f_i(x, u), j)\}$$

We call \mathbf{T} and \mathbf{S} , the DP operator and the optimal control operator, respectively. For any $k \in \mathbb{N}$, denote by \mathbf{T}^k the composition of \mathbf{T} with itself k times. Similarly, for any feedback policy π , and any $k \in \mathbb{N}$, $\mathbf{T}_{\mu_0} \mathbf{T}_{\mu_1} \cdots \mathbf{T}_{\mu_k}$ denotes the composition of operators $\mathbf{T}_{\mu_0}, \mathbf{T}_{\mu_1}, \dots, \mathbf{T}_{\mu_k}$. Then the finite-horizon cost (cf. (3)) of the feedback policy π for (1), starting from $x_0 = x, r_0 = i$ can be expressed as

$$V_{N,\pi}(x, i) = (\mathbf{T}_{\mu_0} \mathbf{T}_{\mu_1} \cdots \mathbf{T}_{\mu_{N-1}})(V_f)(x, i),$$

while the value function can be expressed as

$$V_N^*(x, i) = \mathbf{T}^N V_f(x, i),$$

The standard DP algorithm to compute the value function (4a) and the optimal policy mapping (4b) is expressed as

$$V_0^* = V_f, \quad (6a)$$

$$V_{k+1}^* = \mathbf{T}V_k^*, \mathcal{U}_{k+1}^* = \mathbf{S}V_k^*, k \in \mathbb{N}_{[0, N-1]}. \quad (6b)$$

Upon termination of the DP algorithm, the value function is V_N^* and the optimal policy mapping is $\Pi_N^* = \mathcal{U}_N^* \times \cdots \times \mathcal{U}_1^*$ ($\mathcal{U}_k^* : \mathbb{R}^n \times \mathcal{S} \rightrightarrows \mathbb{R}^m$).

In parallel with the DP operator, the so-called predecessor operator is introduced below.

Definition 5 Given a family of sets $C \in \text{sets}(\mathbb{R}^n, \mathcal{S})$, let $\mathfrak{R}(C) \triangleq \{\mathfrak{R}(C, i)\}_{i \in \mathcal{S}}$ where:

$$\mathfrak{R}(C, i) \triangleq \left\{ x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ s.t. } \begin{array}{l} (x, u) \in Y_i \\ f_i(x, u) \in C_{r_1}, \forall r \in \mathfrak{G}_1(i) \end{array} \right\} \quad (7)$$

Using Definition 2, Eq. (7) becomes:

$$\mathfrak{R}(C, i) = \text{Proj}_x(\mathcal{Z}(C, i)) \quad (8a)$$

$$\mathcal{Z}(C, i) \triangleq \{(x, u) \in Y_i \mid f_i(x, u) \in \bigcap_{j \in \mathcal{S}_i} C_j\} \quad (8b)$$

For any $i \in \mathcal{S}$, $\mathfrak{R}(C, i)$ denotes the set of states x , for which there exists an admissible input such that, for all admissible switching paths of length 1 emanating from i , the next state is in C_{r_1} .

For any $k \in \mathbb{N}$, denote by \mathcal{R}^k the composition of \mathcal{R} , k times with itself, i.e., $\mathcal{R}^k(C) \triangleq \mathcal{R}(\mathcal{R}^{k-1}(C)) = \{\mathfrak{R}(\mathcal{R}^{k-1}(C), i)\}_{i \in \mathcal{S}}$. Let $\mathfrak{R}^k(C, i) \triangleq \mathfrak{R}(\mathcal{R}^{k-1}(C), i)$. Obviously, $\mathcal{R}^k(C) = \{\mathfrak{R}^k(C, i)\}_{i \in \mathcal{S}}$. Here we make the convention that $\mathcal{R}^0(C) = C$.

Theorem 1 presents properties of V_k^*, \mathcal{U}_k^* , $k \in \mathbb{N}_{[1, N]}$, inherited by properties of ℓ and V_f . These properties will be studied under the following assumptions on the stage cost, ℓ :

Assumption 3 $\ell \in \text{lsc}(\mathbb{R}^{n+m}, \mathcal{S})$, $\text{dom } \ell = Y$ and $\ell(\cdot, \cdot, i)$ is level-bounded in u locally uniformly in x , for every $i \in \mathcal{S}$.

Assumption 4 In addition to Assumption 3, $\ell \in \text{conv}(\mathbb{R}^{n+m}, \mathcal{S})$.

Assumption 5 In addition to Assumption 4, $\ell \in \text{pwq}(\mathbb{R}^{n+m}, \mathcal{S})$.

Assumption 3 is the minimal assumption (along with Assumption 1) for which we will guarantee existence of an optimal policy. The stronger Assumptions 4 and 5 lead to more favorable properties of V_k^* and \mathcal{U}_k^* .

Theorem 1 Consider a $V_f \in \text{fncs}(\mathbb{R}^n, \mathcal{S})$ with $\text{dom } V_f = X^f$. Then $V_k^* \in \text{fncs}(\mathbb{R}^n, \mathcal{S})$, $k \in \mathbb{N}_{[1, N]}$. Furthermore:

- If Assumptions 1 and 3 hold and $V_f \in \text{lsc}(\mathbb{R}^n, \mathcal{S})$, then $V_k^* \in \text{lsc}(\mathbb{R}^n, \mathcal{S})$, $k \in \mathbb{N}_{[1, N]}$. In addition, $\text{dom } V_k^* = \text{dom } \mathcal{U}_k^* = \mathcal{R}^k(X^f)$, and for each $x \in \text{dom } \mathcal{U}_k^*(\cdot, i)$ the set $\text{dom } \mathcal{U}_k^*(x, i)$ is compact, for any $i \in \mathcal{S}$, $k \in \mathbb{N}_{[1, N]}$.
- If Assumptions 2 and 4 hold and $V_f \in \text{conv}(\mathbb{R}^n, \mathcal{S})$, then $V_k^* \in \text{conv}(\mathbb{R}^n, \mathcal{S})$ and $\mathcal{U}_k^*(\cdot, i)$ is convex-valued and outer-semicontinuous relative to $\text{int}(\text{dom } \mathcal{U}_k^*(\cdot, i))$ for any $i \in \mathcal{S}$, $k \in \mathbb{N}_{[1, N]}$. Furthermore, if $\ell(\cdot, \cdot, i)$ is strictly convex for some $i \in \mathcal{S}$, then $V_k^*(\cdot, i)$ is strictly convex and $\mathcal{U}_k^*(\cdot, i)$ is single-valued on $\text{dom } \mathcal{U}_k^*(\cdot, i)$ and continuous relative to $\text{int}(\text{dom } \mathcal{U}_k^*(\cdot, i))$, $k \in \mathbb{N}_{[1, N]}$.
- If Assumptions 2 and 5 hold and $V_f \in \text{pwq}(\mathbb{R}^n, \mathcal{S})$, then $V_k^* \in \text{pwq}(\mathbb{R}^n, \mathcal{S})$ and $\mathcal{U}_k^*(\cdot, i)$ is a polyhedral multifunction, thus outer-semicontinuous relative to $\text{dom } \mathcal{U}_k^*(\cdot, i)$ for any $i \in \mathcal{S}$, $k \in \mathbb{N}_{[1, N]}$. Furthermore, if $\ell(\cdot, \cdot, i)$ is strictly convex for some $i \in \mathcal{S}$, then $\mathcal{U}_k^*(\cdot, i)$ is a single-valued, piecewise-affine mapping, thus Lipschitz continuous relative to $\mathcal{U}_k^*(\cdot, i)$, for any $k \in \mathbb{N}_{[1, N]}$.

PROOF. It suffices to prove the claims for $k = 0$. Then using a simple induction argument, the corresponding properties for V_k^*, \mathcal{U}_k^* will hold for all $k \in \mathbb{N}_{[1, N]}$. Let $h_i^V(x, u) \triangleq \ell(x, u, i) + \sum_{j \in \mathcal{S}} p_{ij} V(f_i(x, u), j)$, $i \in \mathcal{S}$. Then (6b) becomes

$$V_{k+1}^*(x, i) = \inf_u h_i^{V_k^*}(x, u) \quad (9a)$$

$$\mathcal{U}_{k+1}^*(x, i) = \underset{u}{\text{argmin}} h_i^{V_k^*}(x, u) \quad (9b)$$

Therefore, properties of the dynamic programming operator can be inferred by properties of the parametric optimization problem (9). Obviously, from (8b) $\text{dom } h_i^{V_f} = \mathcal{Z}(X^f, i)$. Since $h_i^{V_f} \geq 0$ and $h_i^{V_f}(0, 0) = 0$ it follows that $V_1^* \geq 0$ and $V_1^*(0, i) = 0$, $i \in \mathcal{S}$, hence $V_1^* \in \text{fncs}(\mathbb{R}^n, \mathcal{S})$.

(a) Because of [29, Props. 1.39, 1.40], $h_i^{V_f}$ is lsc for every $i \in \mathcal{S}$. Since V_f is bounded below by zero and $p_{ij} \geq 0$, it follows that $\sum_{j \in \mathcal{S}} p_{ij} V_f(f_i(x, u), j) \geq 0$. From the uniform level-boundedness of $\ell(\cdot, \cdot, i)$ we have that for any $\bar{x} \in \mathbb{R}^n$ and any $\alpha \in \mathbb{R}$ there exists a neighborhood $\mathcal{N}(\bar{x})$ along with a bounded set $B \subset \mathbb{R}^m$ such that $\{u | \ell(x, u, i) \leq \alpha\} \subset B$ for all $x \in \mathcal{N}(\bar{x})$. Therefore, $\{u | h_i^{V_f}(x, u) \leq \alpha\} \subset \{u | \ell(x, u, i) \leq \alpha\} \subset B$. Hence, $h_i^{V_f}$ is proper, lsc and level-bounded in u locally uniformly in x , for every $i \in \mathcal{S}$. By [29, Th. 1.17], it follows that $V_1^*(\cdot, i)$ is proper, lsc, $\text{dom } V_1^*(\cdot, i) = \text{dom } \mathcal{U}_1^*(\cdot, i)$, and for each $x \in \text{dom } \mathcal{U}_1^*(\cdot, i)$, the set $\mathcal{U}_1^*(x, i)$ is compact, for every $i \in \mathcal{S}$. Furthermore, $V_1^*(\cdot, i) = \{x | \exists \alpha \in \mathbb{R} \text{ s.t. } (x, \alpha) \in \text{epi } V_1^*(\cdot, i)\} = \{x | \exists \alpha \in \mathbb{R} \exists u \text{ s.t. } (x, \alpha) \in \text{epi } h_i^{V_f}\} = \{x | \exists u \text{ s.t. } (x, u) \in \text{dom } h_i^{V_f}\} = \mathfrak{R}(X^f, i)$. The first and the third equality follows from the relationship between epigraphs and effective domains, the second from the fact that for any $x \in \text{dom } V_1^*(\cdot, i)$, the minimum is attained and [29, Prop. 1.18], and the last by (8a).

(b) Convexity is preserved under composition with affine mappings and nonnegative sums, hence $h_i^{V_f}$ is proper and convex. The convexity of $V_1^*(\cdot, i)$ and the convex-valuedness of $\mathcal{U}_1^*(\cdot, i)$ follow by [29, Prop. 2.22]. The outer-semicontinuity of $\mathcal{U}_1^*(\cdot, i)$ relative to $\text{int}(\text{dom } \mathcal{U}_1^*(\cdot, i))$ follows by [29, Th. 7.41]. If $\ell(\cdot, \cdot, i)$ is strictly convex, one can easily show by definition of strict convexity that $h_i^{V_f}$ is strictly convex as well. The single-valuedness of $\mathcal{U}_1^*(\cdot, i)$ on $\text{dom } \mathcal{U}_1^*(\cdot, i)$ and its continuity on $\text{int}(\mathcal{U}_1^*(\cdot, i))$ follow by [29, Th. 7.43]. In order to prove strict convexity of $V_1^*(\cdot, i)$, for any $x_1, x_2 \in \text{dom } V_1^*(\cdot, i)$, $x_1 \neq x_2$, let $u_1 \in \mathcal{U}_1^*(x_1, i)$ and $u_2 \in \mathcal{U}_1^*(x_2, i)$. Then $V_1^*(\tau x_1 + (1 - \tau)x_2, i) = \inf_u h_i^{V_f}(\tau x_1 + (1 - \tau)x_2, u) \leq h_i^{V_f}(\tau x_1 + (1 - \tau)x_2, \tau u_1 + (1 - \tau)u_2) < \tau V_1^*(x_1, i) + (1 - \tau)V_1^*(x_2, i)$, for any $\tau \in (0, 1)$.

(c) The PWQ property is preserved under composition with affine mappings and nonnegative sums [29, Ex. 10.22], hence $h_i^{V_f}$ is proper, convex, PWQ. That $V_1^* \in \text{pwq}(\mathbb{R}^n, \mathcal{S})$ follows by [29, Cor. 11.32(c)]. That $\mathcal{U}_1^*(\cdot, i)$ is a polyhedral multifunction is proved in [24, Prop. 5]. Outer-semicontinuity of $\mathcal{U}_1^*(\cdot, i)$ on $V_1^*(\cdot, i)$ follows from [16, Th. 3D.1], and Lipschitz continuity of $\mathcal{U}_1^*(\cdot, i)$ on $V_1^*(\cdot, i)$ in case of strict convexity follows from part (b), convexity of $V_1^*(\cdot, i)$ and [16, Cor. 3D.5]. \square

Remark 1 *In the case of constrained MJLS, the value function and an optimal policy $\pi^* \in \Pi_N^*$ can be calculated explicitly, using the DP recursion (6) and the convex parametric piecewise quadratic optimization solver of [24]. The solver uses a computable formula for calculating the graphical derivative [23] of the solution mapping under a graph traversal framework, to enumerate all critical regions, i.e., all full-dimensional polyhedral sets on which the solution mapping is polyhedral. For each $k \in \mathbb{N}_{[0, N-1]}$, the proposed algorithm calculates a*

$\mu_k^* \in \mathcal{U}_k^*$, where μ_k^* is a piecewise affine (PWA) mapping for each mode $i \in \mathcal{S}$, i.e., $\text{dom } V_k^*(\cdot, i)$ is decomposed in a finite number of polyhedral sets $\{\mathcal{P}_{k,i}^j\}_{j \in \mathcal{J}_{k,i}}$ where $\mu_k^*(\cdot, i)$ is affine, i.e., $\mu_k^*(x, i) = K_{k,i}^j x + \kappa_{k,i}^j$ if $x \in \mathcal{P}_{k,i}^j$.

Tracing a parallel with invariant set theory for discrete-time nonlinear systems [19, 26] we introduce the following notion of invariance for MSS.

Definition 6 *A family of sets $C \in \text{sets}(\mathbb{R}^n, \mathcal{S})$ with $C_i \subseteq X$ is said to be uniformly control invariant for the constrained MSS (1), if there exists a policy π such that $x_0 \in C_{r_0} \Rightarrow \phi(k; x, r_0, \pi, \mathbf{r}) \in C_{r_k}$, $k \in \mathbb{N}, \forall \mathbf{r} \in \mathfrak{G}(r_0)$*

Remark 2 *Uniform control invariance is a less conservative notion than classical robust control invariance. By taking into consideration the mode of the MSS as an additional discrete-valued state, a uniform control invariant set is allowed to be dependent of the current mode while ensuring satisfaction of constraints for every possible transition of the underlying Markov chain.*

Lemma 1 below presents the monotonicity property of the DP operator. Its proof can be easily inferred by e.g., [6, Ch. 3] and is omitted for brevity.

Lemma 1 *If $V, V' \in \text{fcns}(\mathbb{R}^n, \mathcal{S})$ with $V \leq V'$ then $\mathbf{T}^k V \leq \mathbf{T}^k V'$ for any $k \in \mathbb{N}$.*

The following lemma gives a geometric characterization of uniform control invariance.

Lemma 2 *A family of sets $C \in \text{sets}(\mathbb{R}^n, \mathcal{S})$ with $C \subseteq X$ is uniformly control invariant for the Markovian switching system (1), (2) if and only if $C \subseteq \mathcal{R}(C)$.*

PROOF. For the reverse implication suppose that $C_i \not\subseteq \mathfrak{R}(C, i)$ for some $i \in \mathcal{S}$. Then there exists a $x \in C_i$ such that $f_i(x, u) \notin C_j$ for some $j \in \mathcal{S}_i$ and for any $u \in \mathcal{U}_i(x)$. Pick a switching path $\mathbf{r} \in \mathcal{G}$ with $r_k = i$ and $r_{k+1} = j$. It then follows that for some $x_k \in C_i$, there does not exist a $u_k \in \mathcal{U}_{r_k}(x_k)$ such that $x_{k+1} \in C_{r_{k+1}}$ contradicting the definition of uniform control invariance. The opposite direction follows by an analogous argument. \square

Lemma 3 below presents a link between uniform control invariance and the DP operator.

Lemma 3 *Suppose that Assumptions 1 and 3 hold. If $V_f \in \text{lsc}(\mathbb{R}^n, \mathcal{S})$ and $V_f \geq \mathbf{TV}_f$ then*

- (a) $V_k^* \geq V_{k+1}^*$,
- (b) $\text{dom } V_k^*$ is uniformly control invariant for any $k \in \mathbb{N}_{[0, N]}$.

PROOF. (a) Since $V_f \geq \mathbf{T}V_f$, using Lemma 1, $V_k^* = \mathbf{T}^k V_f \geq \mathbf{T}^{k+1} V_f = V_{k+1}^*$.

(b) From the assumptions of the lemma, Theorem 1(a) is valid, hence $\text{dom } V_k^* = \mathcal{R}^k(X^f)$ where $X^f = \text{dom } V_f$. Notice that part (a) implies that $\text{dom } V_k^* \subseteq \text{dom } V_{k+1}^*$, or $\mathcal{R}^k(X^f) \subseteq \mathcal{R}^{k+1}(X^f)$. Equivalently, this can be expressed as $\mathcal{R}^k(X^f) \subseteq \mathcal{R}(\mathcal{R}^k(X^f))$. Invoking Lemma 2, the claim is proved. \square

4.2 Conversion to a Finite-dimensional Optimization Problem

This section shows how the constrained finite-horizon stochastic optimal control problem (4) can be converted to a finite-dimensional optimization problem. For each $i \in \mathcal{S}$ let $\mathcal{Q}_N(i) \triangleq \mathbb{N}_{[1, |\mathfrak{G}_N(i)|]}$ and associate with each $q \in \mathcal{Q}_N(i)$ the corresponding switching path emanating from i , i.e., $\mathbf{r}^q \in \mathfrak{G}_N(i)$. Also let $\mathbf{u}^q \triangleq \{u_0^q, \dots, u_{N-1}^q\}$ denote a control sequence associated with the q -th switching path and let $\mathbf{x}^q \triangleq \{x_0^q, \dots, x_N^q\}$ represent the sequence of solutions of:

$$x_{k+1}^q = f_{r_k}^q(x_k^q, u_k^q). \quad (10)$$

Let $\mathbf{x} \triangleq \{\mathbf{x}^q\}_{q \in \mathcal{Q}_N(i)}$, $\mathbf{u} \triangleq \{\mathbf{u}^q\}_{q \in \mathcal{Q}_N(i)}$ and

$$\mathbf{p}_0^q = 1, \text{ and } \mathbf{p}_{k+1}^q = \mathbf{p}_k^q p_{r_k r_{k+1}}^q, \quad k \in \mathbb{N}_{[0, N-1]}. \quad (11)$$

Then (4) is equivalent to [30]

$$V_N^*(x, i) = \inf_{\mathbf{x}, \mathbf{u}} \sum_{q \in \mathcal{Q}_N(i)} \sum_{k=0}^{N-1} \mathbf{p}_k^q \ell(x_k^q, u_k^q, r_k^q) + \mathbf{p}_N^q V_f(x_N^q, r_N^q) \quad (12a)$$

$$\text{s.t. } x_0^q = x, \quad \forall q \in \mathcal{Q}_N(i) \quad (12b)$$

$$x_{k+1}^q = f_{r_k}^q(x_k^q, u_k^q), \quad \forall q \in \mathcal{Q}_N(i), \quad k \in \mathbb{N}_{[0, N-1]}, \quad (12c)$$

$$x_k^{q_1} = x_k^{q_2}, \quad \forall q_1, q_2 \in \mathcal{Q}_N(i), \text{ with } \mathbf{r}_{[k]}^{q_1} = \mathbf{r}_{[k]}^{q_2}, \quad k \in \mathbb{N}_{[0, N-1]}, \quad (12d)$$

$$u_k^{q_1} = u_k^{q_2}, \quad \forall q_1, q_2 \in \mathcal{Q}_N(i), \text{ with } \mathbf{r}_{[k]}^{q_1} = \mathbf{r}_{[k]}^{q_2}, \quad k \in \mathbb{N}_{[0, N-1]}, \quad (12e)$$

$$(x_k^q, u_k^q) \in Y_{r_k}^q, \quad \forall q \in \mathcal{Q}_N(i), \quad k \in \mathbb{N}_{[0, N-1]}, \quad (12f)$$

$$x_N^q \in X_{r_N}^q, \quad \forall q \in \mathcal{Q}_N(i) \quad (12g)$$

The above problem possesses a favourable structure which can be used its efficient numerical solution based on techniques on dual decomposition [30]. It is easy to notice that if Assumptions 2 and 4 hold then (12) is a convex optimization problem, for which efficient solution algorithms exist. However, it can be highly complex for large number of modes and large prediction horizons. This complexity can be mitigated at the expense of introducing some conservatism based on the reduction of the scenario tree [5].

5 Stability of Autonomous Markovian Switching Systems

In this section, we proceed with the establishment of sufficient conditions for mean-square stability and exponential mean-square stability of constrained autonomous MSS. Consider the autonomous MSS:

$$x_{k+1} = f_{r_k}(x_k) \quad (13)$$

with $f_i(0) = 0$, $i \in \mathcal{S}$. Since the system has no input, ‘‘uniformly control invariant’’ is replaced with ‘‘uniformly positive invariant’’ in Definition 6, the predecessor operator (7) becomes $\mathfrak{R}(C, i) = \{x \in X_i \mid f_i(x) \in \cap_{j \in \mathcal{S}_i} C_j\}$ and Lemma 2 remains valid with the appropriate modifications. The solution of (13) at time $k \in \mathbb{N}$ given a switching path \mathbf{r} with $r_0 = i$ and $x_0 = x$ is denoted by $\phi(k; x, i, \mathbf{r})$.

Definition 7 Let $X \in \text{sets}(\mathbb{R}^n, \mathcal{S})$ be a uniformly positive invariant set for (13). We say that the origin is:

(a) Mean square (MS) stable in X if

$$\lim_{k \rightarrow \infty} \mathbb{E}[\|\phi(k; x, i, \mathbf{r})\|^2] = 0, \quad \forall x \in X_i, \quad i \in \mathcal{S}$$

(b) Exponentially mean square (EMS) stable in X if there exist $\theta > 1$, $0 < \zeta \leq 1$ such that

$$\mathbb{E}[\|\phi(k; x, i, \mathbf{r})\|^2] \leq \theta \zeta^k \|x\|^2, \quad \forall x \in X_i, \quad i \in \mathcal{S}$$

The assumption that X is uniformly positive invariant for (13) ensures that $\phi(k; x, i, \mathbf{r}) \in X_{r_k}$ for all $x \in X_i$, $\mathbf{r} \in \mathfrak{G}(i)$ and $i \in \mathcal{S}$. For any $V : X \times \mathcal{S} \rightarrow \mathbb{R}$ let:

$$\mathcal{L}V(x_k, r_k) \triangleq \mathbb{E}[V(x_{k+1}, r_{k+1}) - V(x_k, r_k) \mid \mathfrak{F}_k]$$

Due to the Markov property one has:

$$\mathcal{L}V(x_k, r_k) = \sum_{r_{k+1} \in \mathcal{S}} p_{r_k r_{k+1}} V(f_{r_k}(x_k), r_{k+1}) - V(x_k, r_k)$$

Lemma 4 For any $0 \leq k_1 \leq k_2$

$$\mathbb{E}[V(x_{k_2}, r_{k_2}) - V(x_{k_1}, r_{k_1}) \mid \mathfrak{F}_{k_1}] = \mathbb{E} \left[\sum_{k=k_1}^{k_2-1} \mathcal{L}V(x_k, r_k) \mid \mathfrak{F}_{k_1} \right]$$

PROOF. Notice that $V(x_{k_2}, r_{k_2}) - V(x_{k_1}, r_{k_1}) = \sum_{k=k_1}^{k_2-1} [V(x_{k+1}, r_{k+1}) - V(x_k, r_k)]$. Taking the conditional expectation: $\mathbb{E}[V(x_{k_2}, r_{k_2}) - V(x_{k_1}, r_{k_1}) \mid \mathfrak{F}_{k_1}] = \mathbb{E}[\sum_{k=k_1}^{k_2-1} [V(x_{k+1}, r_{k+1}) - V(x_k, r_k)] \mid \mathfrak{F}_{k_1}]$. Using properties of the conditional expectation, the right-hand side of the above becomes: $\mathbb{E}[\sum_{k=k_1}^{k_2-1} \mathbb{E}[V(x_{k+1}, r_{k+1}) - V(x_k, r_k) \mid \mathfrak{F}_k] \mid \mathfrak{F}_{k_1}] = \mathbb{E}[\sum_{k=k_1}^{k_2-1} \mathcal{L}V(x_k, r_k) \mid \mathfrak{F}_{k_1}]$ and the statement is valid. \square

In the next theorem, sufficient stochastic Lyapunov-like conditions for MS and EMS stability of (13) are presented.

Theorem 2 Consider the autonomous MSS (13). Let X be a uniformly positive invariant set for (13).

- (a) Suppose that there exists a $V \in \text{fens}(\mathbb{R}^n, \mathcal{S})$ and $\gamma > 0$ satisfying $\mathcal{L}V(x, i) \leq -\gamma\|x\|^2, \forall x \in X_i, i \in \mathcal{S}$. Then the origin is MS stable in X for (13).
- (b) Assume that there exists a $V \in \text{fens}(\mathbb{R}^n, \mathcal{S})$ and positive scalars α, β and γ satisfying the following properties.

$$\alpha\|x\|^2 \leq V(x, i) \leq \beta\|x\|^2, \forall x \in X_i, i \in \mathcal{S}, \quad (14a)$$

$$\mathcal{L}V(x, i) \leq -\gamma\|x\|^2, \forall x \in X_i, i \in \mathcal{S} \quad (14b)$$

Then the origin is EMS stable in X for (13).

For better readability of the paper the proof of Theorem 2 can be found in the appendix. For the rest of this section the focus is on autonomous MJLS:

$$x_{k+1} = A_{r_k} x_k \quad (15)$$

where the state vector must satisfy the constraint $x_k \in X_{r_k}, k \in \mathbb{N}$ for all $\mathbf{r} \in \mathfrak{G}$ where $X \in \text{cl-sets}(\mathbb{R}^n, \mathcal{S})$. Let $\Phi(k; \mathbf{r}) = A_{r_0} A_{r_1} \cdots A_{r_k}$ for $k > 0$ and $\Phi(0; \mathbf{r}) = I$. The maximal uniformly positive invariant set is $X^* = \{X_i^*\}_{i \in \mathcal{S}}$ with $X_i^* = \{x \in \mathbb{R}^n | \Phi(k; \mathbf{r})x \in X_{r_k}, \forall k \in \mathbb{N}, \mathbf{r} \in \mathfrak{G}(i)\}$ and can be calculated via the recursion $X^{k+1} = \mathcal{R}(X^k)$, with $X^0 = X$. It is not difficult to see that for any $i \in \mathcal{S}$:

$$X_i^k = \{x \in \mathbb{R}^n | \Phi(t; \mathbf{r})x \in X_{r_t}, \forall t \in \mathbb{N}_{[0, k]}, \mathbf{r} \in \mathfrak{G}_k(i)\} \quad (16)$$

For autonomous LTI systems ($|\mathcal{S}|=1$) it is known that asymptotic stability of (15) implies that the maximal positive invariant set is finitely determined and the origin belongs to its interior [18]. However, when $\mathcal{S} > 1$, MS stability of (15) is not sufficient, neither for finite determinedness of X^* , nor for its full-dimensionality. For that matter, a stronger notion of stability is required, *i.e.*, uniform asymptotic stability. The MJLS (15) can be viewed as a discrete-time linear switched system [13, 22], where the switching path is constrained by the matrix $Q = (q_{ij}) \in \{0, 1\}^{\mathcal{S} \times \mathcal{S}}$ where $S=|\mathcal{S}|$, $q_{ij} = 1$ if $p_{ij} > 0$, and $q_{ij} = 0$ otherwise. The MJLS (15) is said to be *uniformly asymptotically stable* if for every $x \in \mathbb{R}^n$, $\Phi(k; \mathbf{r})x$ converges to zero uniformly, over all $\mathbf{r} \in \mathfrak{G}$, as k approaches infinity. A necessary and sufficient condition for uniform asymptotic stability of (15) is the existence of $P_i \in \mathbb{R}^{n \times n}$, such that $P_i > 0$ and $P_i - A_i^* P_j A_i < 0$ for all $j \in \mathcal{S}_i, i \in \mathcal{S}$ [13]. Notice that uniform asymptotic stability implies mean-square (exponential) stability. Next, we will establish a sufficient condition for finite determinedness of X^* .

Lemma 5 Suppose that (15) is uniformly asymptotically stable, X_i is bounded and $0 \in \text{int } X_i, i \in \mathcal{S}$. Then X^* is finitely determined and $0 \in \text{int } X_i^*$.

PROOF. By monotonicity, the sequence $\{X^k\}$ is non-increasing. X^* is finitely determined if and only if there exists a k^* such that $X^k = X^{k+1}$, for all $k \geq k^*$. Since X_i is bounded there exists an $\epsilon > 0$ such that $X_i \subseteq B(\epsilon)$, for every $i \in \mathcal{S}$. This fact, and the monotonicity of the sequence lead to $X_i^k \subseteq B(\epsilon)$, for every $k \in \mathbb{N}, i \in \mathcal{S}$. Since $0 \in \text{int } X_i$ and $\lim_{k \rightarrow \infty} \|\Phi(k; \mathbf{r})\| = 0$ for every $\mathbf{r} \in \mathfrak{G}$, it follows that there exists a $k \in \mathbb{N}$ such that $\Phi(k+1; \mathbf{r})B(\epsilon) \subseteq X_{r_{k+1}}$, for every $\mathbf{r} \in \mathfrak{G}_{k+1}(i)$ and since $X_i^k \subseteq B(\epsilon)$, we get $\Phi(k+1; \mathbf{r})X_i^k \subseteq X_{r_{k+1}}$, for every $\mathbf{r} \in \mathfrak{G}_{k+1}(i), i \in \mathcal{S}$. This shows that $x \in X_i^k$ implies $\Phi(k+1; \mathbf{r})x \in X_{r_{k+1}}$. Using (16) this is translated to $X^k \subseteq X^{k+1}$, therefore $X^k = X^{k+1}$, and X^* is finitely determined.

To prove that $0 \in \text{int } X_i^*, i \in \mathcal{S}$, from the uniform asymptotic stability of (15) we have that there exists a constant $\gamma_1 > 0$ such that $\|\Phi(k; \mathbf{r})x\| \leq \gamma_1\|x\|$. Since $0 \in \text{int } X_i$, there exists a $\gamma_2 > 0$ such that $B(\gamma_2) \subseteq X_i, i \in \mathcal{S}$. Then $\gamma_1\|x\| \leq \gamma_2$ implies $\Phi(k; \mathbf{r})x \in X_i$ for all $i \in \mathcal{S}$ and all $\mathbf{r} \in \mathfrak{G}$. Hence $B(\gamma_2/\gamma_1) \subseteq X_i^*$ and consequently $0 \in \text{int } X_i^*$ for every $i \in \mathcal{S}$. \square

6 Stochastic MPC for MSS

In stochastic MPC the stationary policy $\mu_N^* \in \text{SV}_{N-1}^*$, *i.e.*, $\mathbf{T}_{\mu_N^*} V_{N-1}^* = \mathbf{T}V_{N-1}^* = V_N^*$ is implemented to system (1). For future reference, the following notation for the MSS in closed-loop with the receding horizon controller is introduced:

$$x_{k+1} = f_{r_k}^{\mu_N^*}(x_k), \quad (17)$$

where $f_i^{\mu_N^*}(x) \triangleq f_i(x, \mu_N^*(x, i))$. If Assumptions 2 and 5 hold, then the procedure described in Remark 1 can be employed to calculate off-line the mode-dependent, PWA receding horizon controller μ_N^* . The implementation of the receding-horizon controller is trivial, since only a minimal number of computations is performed on line. Specifically, at time k , after the state $(x(k), r(k))$ of (1) is measured, one needs to find a $j \in \mathcal{J}_{N, r(k)}$ such that $x(k) \in \mathcal{P}_{N, r(k)}^j$, and apply $u(k) = K_{N, r(k)}^j x(k) + \kappa_{N, r(k)}^j$ to the system.

In any other case, if merely Assumptions 1 and 3 hold, one can calculate on-line the receding horizon control action, by solving at each time instant k , the optimization problem (12). The following standard assumption is imposed for the stage cost.

Assumption 6 The stage cost satisfies $\ell(x, u, i) \geq \alpha\|x\|^2$ for every $(x, u) \in Y_i, i \in \mathcal{S}$.

Mean-square stability can be guaranteed under the following assumption for the terminal cost function.

Assumption 7 $V_f \in \text{lsc}(\mathbb{R}^n \times \mathcal{S})$, with $V_f \geq \mathbf{TV}_f$.

Assumption 7 is trivially satisfied when $V_f = \delta_{\{0\}}$.

Theorem 3 Suppose that Assumptions 1, 3, 6 and 7 hold. Then the origin is mean-square stable in $X_N^* \triangleq \text{dom } V_N^*$ for (17).

PROOF. By virtue of the fact that $V_N^* = \mathbf{T}_{\mu_N^*} V_{N-1}$:

$$\mathcal{L}V_N^*(x, i) = \sum_{j \in \mathcal{S}} p_{ij} V_N^*(f_i^{\mu_N^*}(x), j) - V_N^*(x, i) \quad (18a)$$

$$= \sum_{j \in \mathcal{S}} p_{ij} V_N^*(f_i^{\mu_N^*}(x), j) - \ell(x, \mu_N^*(x, i), i), \quad (18b)$$

and from Assumption 7 and Lemma 3(a) it is $\mathcal{L}V_N^*(x, i) \leq -\ell(x, \mu_N^*(x, i), i)$, and then because of Assumption 6 it is $\mathcal{L}V_N^*(x, i) \leq -\alpha\|x\|^2$. The claim is proved by invoking Theorem 2(a). \square

Assumption 8 Stage cost $\ell(x, u, i) = x'Q_i x + u'R_i u + \delta_{Y_i}$ with $Q_i > 0$, $R_i > 0$, $i \in \mathcal{S}$. Furthermore $Y \in \text{poly-sets}(\mathbb{R}^{n+m}, \mathcal{S})$ with Y_i bounded.

For constrained MJLS (Assumption 2), if the stage cost satisfies Assumption 8, one can choose

$$V_f(x, i) = x'P_i^f x + \delta_{X_i^f}, \quad (19)$$

where P_i^f , $i \in \mathcal{S}$ solve the CARE, [12] (ch. 4)

$$P_i^f = A_i' \mathcal{E}_i(P^f) A_i + Q_i \quad (20)$$

$$- A_i' \mathcal{E}_i(P^f) B_i (R_i + B_i' \mathcal{E}_i(P^f) B_i)^{-1} B_i' \mathcal{E}_i(P^f) A_i, i \in \mathcal{S}$$

where $\mathcal{E}_i(P^f) = \sum_{j \in \mathcal{S}} p_{ij} P_j^f$, and $X^f = \{X_i^f\}_{i \in \mathcal{S}}$ is the maximal uniformly positive invariant set for the MJLS in closed loop with the unconstrained optimal policy:

$$\mu(x, i) = -(R_i + B_i' \mathcal{E}_i(P^f) B_i)^{-1} B_i' \mathcal{E}_i(P^f) A_i \quad (21)$$

In order to assure mean square exponential stability the following stronger assumption on the terminal cost is required:

Assumption 9 $V_f \in \text{lsc}(\mathbb{R}^n \times \mathcal{S})$, with $V_f \geq \mathbf{TV}_f$, $V_f(x, i) \leq \delta\|x\|^2$ and $0 \in \text{int}(\text{dom } V_f(\cdot, i))$, $i \in \mathcal{S}$.

Theorem 4 Suppose that Assumptions 1, 3, 6 and 9 hold and $0 \in \text{int}(\text{dom } V_f)$, $V_N^*(\cdot, i)$ is continuous on $X_i^N \triangleq \text{dom } V_N^*(\cdot, i)$ and X_i^N is compact for every $i \in \mathcal{S}$. Then the origin is mean square exponentially stable in X^N for (17).

PROOF. Because of Assumption 6 and $V_N^*(x, i) = \ell(x, \mu_N^*(x, i)) + \sum_{j \in \mathcal{S}} p_{ij} V_{N-1}(f_i^{\mu_N^*}(x), j)$ it follows that $\alpha\|x\|^2 \leq V_N^*(x, i)$, $x \in X_i^N$, $i \in \mathcal{S}$. Since $V_f \geq \mathbf{TV}_f$ (Assumption 9), using the monotonicity of the DP operator (Lemma 3(a)), we arrive at $V_f \geq V_N^*$. Therefore, through Assumption 9, $V_N^*(x, i) \leq \delta\|x\|^2$. This fact along with the extra assumption $0 \in \text{int}(\text{dom } V_f)$, in conjunction with the continuity and compactness assumption provide an upper bound for V_N^* relative to X^N , [28, Prop. 2.18], i.e., there exists a $\beta > 0$ such that $V_N^*(x, i) \leq \beta\|x\|^2$ for any $x \in X_i^N$, $i \in \mathcal{S}$. As it was shown in Theorem 3, X^N is uniformly positive invariant for system (17) and $\mathcal{L}V^*(x, i) \leq -\alpha\|x\|^2$, for any $x \in X_i^N$, $i \in \mathcal{S}$. In virtue of Theorem 2(b), the origin is exponentially mean-square stable in X^N for (17). \square

An important case where Theorem 4 is valid is SMPC of constrained MJLS.

Corollary 1 Let Assumptions 2 and 6 hold. Consider the LMI

$$\begin{bmatrix} Z_i & (A_i Z_i + B_i Y_i)' F_i & Z_i Q_i^{1/2} & Y_i' R_i^{1/2} \\ * & Z & 0 & 0 \\ * & * & I & 0 \\ * & * & * & I \end{bmatrix} \geq 0, i \in \mathcal{S} \quad (22a)$$

$$\begin{bmatrix} Z_i & (A_i Z_i + B_i Y_i)' \\ * & Z_j \end{bmatrix} \geq 0, j \in \mathcal{S}_i, i \in \mathcal{S} \quad (22b)$$

where $F_i = [\sqrt{p_{i1}} I \dots \sqrt{p_{iS}} I]$, $i \in \mathcal{S}$ and $Z = \bigoplus_{i \in \mathcal{S}} Z_i$. If (22) is feasible, consider the terminal cost

$$V_f(x, i) = x'P_i^f x + \delta_{X_i^f}, \quad (23)$$

where $P_i^f = Z_i^{-1}$, and $X^f = \{X_i^f\}_{i \in \mathcal{S}}$ is the maximal uniformly positive invariant set for the MSS in closed-loop with $\mu(x, i) = K_i x_i$ ($K_i = Y_i Z_i^{-1}$), $i \in \mathcal{S}$. Then the origin is mean-square exponentially stable in $X^N = \text{dom } V_N^*$ for (17).

PROOF. Consider the closed-loop system $x_{k+1} = (A_{r_k} + B_{r_k} K_{r_k}) x_k$. Using the Schur complement formula, Eq. (22a) is equivalent to $P_i^f \geq (A_i + B_i K_i)' (\sum_{j \in \mathcal{S}} p_{ij} P_j^f) (A_i + B_i K_i) + (Q_i + K_i' R_i K_i)$, for $i \in \mathcal{S}$. Therefore $V \geq \mathbf{T}_\mu V \geq \mathbf{TV}$. Using the Schur complement formula, equation (22b) becomes $P_i^f \geq (A_i + B_i K_i)' P_j^f (A_i + B_i K_i)$, $j \in \mathcal{S}_i$, $i \in \mathcal{S}$, implying that the origin is uniformly asymptotically stable for the close-loop system.

By Lemma 5, $0 \in \text{int } X_i^f$, $i \in \mathcal{S}$. Therefore, the terminal cost (23) satisfies Assumption 9. Furthermore, Assumption 8 obviously implies Assumption 5. Therefore, Theorem 1(c) is valid, hence $V^* \in \text{pwq}(\mathbb{R}^n, \mathcal{S})$, implying that $V^*(\cdot, i)$ is continuous relative to its effective domain for

$i \in \mathcal{S}$. Furthermore, $\text{dom } V^*(\cdot, i)$ is compact, hence Theorem 4 is valid, proving EMS of the origin in $\text{dom } V^*$ for (17). \square

Note that the LMI (22) is feasible if and only if the set of pair $\{(A_i, B_i)\}_{i \in \mathcal{S}}$ is mean-square stabilizable, *i.e.*, if there exist feedback gains $\{K_i\}_{i \in \mathcal{S}}$ so that the closed-loop system is mean-square stable. The following corollary allows us to perform MPC for MSS using local linearization. This result is reminiscent of the standard nonlinear MPC approach that can be found in [28].

Corollary 2 *Let Assumptions 1 and 6 hold and for $i \in \mathcal{S}$ define $A_i = \frac{\partial f_i}{\partial x}(0, 0)$, and $B_i = \frac{\partial f_i}{\partial u}(0, 0)$. Let P_i be given by the LMI (22) and $X_i^f = \text{lev}_{\leq \alpha_i}(x' P_i x)$ where $\alpha_i > 0$. Then, the α_i can be chosen in such a way so that Assumption 9 is satisfied and the origin becomes mean-square exponentially stable in $X^N = \text{dom } V_N^*$ for the nonlinear MSS (17).*

7 Illustrative Examples

7.1 Samuelson's macroeconomic model

In this example we compare the SMPC scheme for constrained MJLS against the algorithm of [11]. The algorithm of [11] is an extension of the robust MPC algorithm of [20] to stochastic MPC of MJLS with symmetric input and state constraints. Essentially, it is an MPC scheme with prediction horizon 1, where in real-time an LMI problem is solved, to compute a mode-dependent, linear control law which minimizes an upper bound of the infinite-horizon cost. The two techniques will be compared on Samuelson's multiplier-accelerator macroeconomic model. The system has three operating modes and satisfies Assumption 2 with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2.5 & 3.2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -4.3 & 4.5 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ 5.3 & -5.2 \end{bmatrix},$$

$B_1 = B_2 = B_3 = [0 \ 1]'$. The mode-dependent polyhedra constraint sets are $Y_1 = \mathbb{R}_{[-10, 10]}^2$, $Y_2 = \mathbb{R}_{[-8, 8]} \times \mathbb{R}_{[-10, 10]}$, $Y_3 = \mathbb{R}_{[-12, 12]} \times \mathbb{R}_{[-10, 10]}$. The stage-cost satisfies Assumption 8 with

$$Q_1 = \begin{bmatrix} 3.6 & -3.8 \\ -3.8 & 4.87 \end{bmatrix}, Q_2 = \begin{bmatrix} 10 & -3 \\ -3 & 8 \end{bmatrix}, Q_3 = \begin{bmatrix} 5 & -4.5 \\ -4.5 & 5 \end{bmatrix},$$

and $R_1 = 2.6$, $R_2 = 1.165$, $R_3 = 1.111$. The transition matrix of the Markov chain is

$$P = \begin{bmatrix} 0.67 & 0.17 & 0.16 \\ 0.3 & 0.47 & 0.23 \\ 0.26 & 0.1 & 0.64 \end{bmatrix}.$$

The terminal cost is chosen so as to satisfy Eqs. (19), (20). The maximal uniformly positive invariant set for the system in closed-loop with (21) is chosen as a terminal set. The prediction horizon is $N = 6$. The SMPC problem was solved explicitly off-line, using

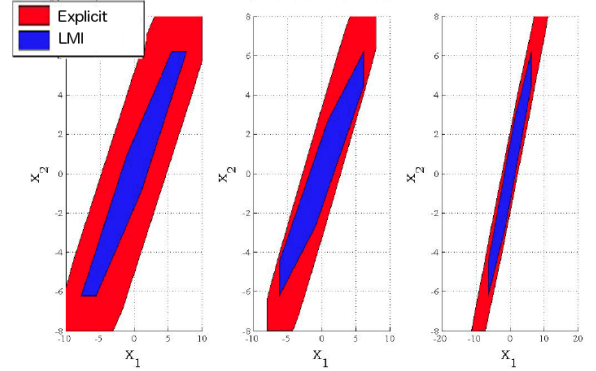


Fig. 1. Region of Attraction for SMPC (red) and LMI-based MPC of [11] (blue).

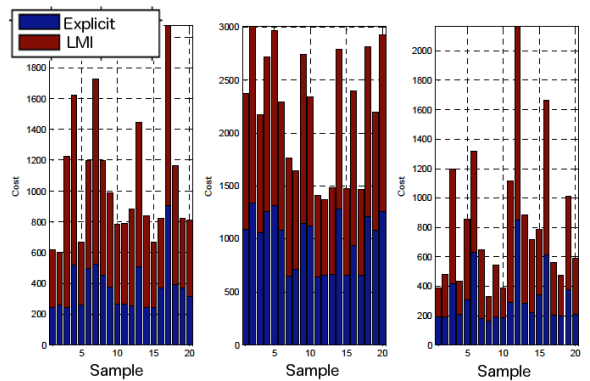


Fig. 2. Simulation cost comparison for 20 random switching paths for the system in closed-loop with the proposed SMPC law and the LMI-based MPC law of [11].

the technique outlined in Remark 1. The effective domain $\text{dom } V_6^*$ (the region of attraction of the system in closed-loop with the SMPC controller) consists of 393, 409 and 465 polyhedral sets, for each one of the three modes, respectively. The region of attraction of the LMI algorithm [11] is computed approximately by gridding the polyhedral set $\text{Proj}_x Y$. As expected, the region of attraction of the proposed SMPC algorithm is a superset of the one corresponding to the LMI-based MPC algorithm, for every mode of the Markov chain (Figure 1).

Next, we simulate the MJLS in closed-loop with the SMPC and the LMI-based controller for 30 time steps starting from a vertex of the region of attraction of the LMI-based approach, by selecting randomly 20 admissible switching paths, for each mode. The goal of this task is to compare the two design methodologies in terms of closed-loop simulated cost. As it can be seen from Figure 2, the proposed SMPC algorithm always results in a smaller simulation cost.

Figure 3 depicts statistical results for simulations of the

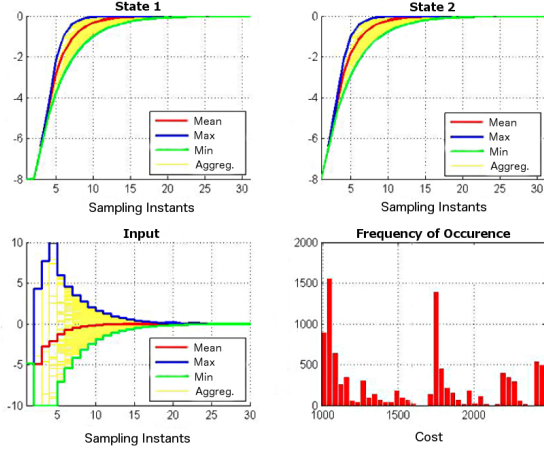


Fig. 3. Simulation of MJLS on closed-loop with SMPC for 10000 switching paths from $\mathcal{G}_{30}(2)$ starting from $x_0 = -[8 \ 8]'$.

MJLS system in closed-loop with the SMPC controller for 10000 randomly generated admissible switching paths of length 30 emanating from mode $i = 2$ and initial state $x_0 = -[8 \ 8]'$.

7.2 Constrained networked control with random time delay

We apply the proposed SMPC design on a networked control system (NCS) and manifest its advantages over alternative approaches found in literature. We consider the NCS model consisting of a printer described via a linear time-invariant, continuous-time plant that is controlled using a discrete-time controller that is connected to the system through a communication network with induced sensor-to-controller (SC), τ^{sc} , and controller-to-actuator (CA), τ^{ca} , delays [10]. The controller delay (the time needed by the controller to perform computations) is assumed to be incorporated into the CA delay. The full state of the system is sampled by a time-driven sensor with a constant sampling interval $h > 0$. The discrete-time controller is event-driven and able to monitor the SC delay, via timestamping. The CA delay is considered to be constant by using the buffering technique. The discrete-time control signal u_k is transformed to a continuous-time control input $u(t)$ by a zero-order hold device (ZOH). Based on these assumptions, the NCS model is:

$$\dot{x}(t) = A_c x(t) + B_c u(t) \quad (24a)$$

$$u(t) = u_k, \quad t \in [kh + \tau_k^{\text{sc}} + \tau_k^{\text{ca}}, (k+1)h + \tau_{k+1}^{\text{sc}} + \tau_{k+1}^{\text{ca}}) \quad (24b)$$

where $A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B_c = \begin{bmatrix} 0 \\ 126.70 \end{bmatrix}$. The system is subject to continuous-time state, $x(t) \in X \triangleq [-10, 10]^2$, $t \in \mathbb{R}_+$, input constraints $u_k \in U = [-2, 2]$ and $Q = 10I_2$ and $R = 1$ are the state and input weight matrices for the continuous-time optimal control problem. The sampling

interval is $h = 20$ ms while the SC delay can take the values $\tau^{\text{sc},1} = 3$ ms and $\tau^{\text{sc},2} = 15$ ms with transition matrix $P = \begin{bmatrix} 0.67 & 0.33 \\ 0.30 & 0.70 \end{bmatrix}$. The CA delay is considered constant with $\tau^{\text{ca}} = 1$ ms. Using the technique described in [25], (24) is transformed into a discrete-time MJLS in the extended state space $\xi_k \triangleq [x'_k \ u'_{k-1}]' \in \mathbb{R}^{n_x + n_u}$ ($x_k = x(kh)$), whereas the continuous time constraints on the state vector X , have been replaced with polyhedral constraint set $Y \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_u}$ that guarantees *continuous-time* constraint satisfaction for the NCS.

We set the horizon length to $N = 10$ steps. In the following illustrations we present a visualization of the polyhedral decomposition of the feasible state space on which the control law is defined as a PWA function over these regions. The mode-dependent PWA control law consists of 61 and 73 critical regions (cf. Figure ??) for each of the two modes.

In order to elucidate the benefits of SMPC we compare our results with alternative control approaches. The first approach (*Delay-free MPC*) is a deterministic MPC scheme for the exact discretization of the continuous-time system without taking into consideration the time-varying delay, *i.e.*, for the system $x_{k+1} = e^{A_c h} x_k + \Gamma_0(h) u_k$, where $\Gamma_0(t) \triangleq \int_0^t e^{A_c s} ds B_c$. Constraints are imposed only on discrete sampling times while the cost function is considered to be quadratic, $\ell(x, u) = \frac{1}{2}(x' Q_h x + u' R_h u)$ where $Q_h = hQ$ and $R_h = hR$. The second alternative scheme (*Non-switched MPC*) is a deterministic MPC controller for the exact discretization of the continuous-time system where the delay is considered constant and equal to its greatest value (worst case scenario, $\tau_{\text{max}} = 16$ ms), *i.e.*, for the discrete-time system $\xi_{k+1} = [e^{A_c h} \ \Gamma_0(h) - \Gamma_0(h - \tau_{\text{max}})] \xi_k + \Gamma_0(h - \tau_{\text{max}}) u_k$ and the constraints are imposed only for the sampling times. In order to compare SMPC against the alternative schemes, 20 simulations (corresponding to 20 switching paths according to the transition matrix) for every extreme point of the effective domain of $V_N^*(\cdot, i)$, $i \in \mathcal{S}$ are performed. For every single one of them, SMPC achieved mean-square stability for the continuous time closed loop system while respecting the constraints in the continuous time. Non-switched MPC achieved this goal only in 66.77% of the cases while for delay-free MPC the percentage drops to 8.47%. An illustrative simulation of the NCS in closed-loop with the SMPC controller is depicted in Figure 4.

7.3 Control of a nonlinear Lotka-Volterra model

Consider a discrete-time two-state nonlinear Lotka-Volterra model whose dynamics is described by:

$$x_{k+1} = \frac{a_{\tau_k} x_k - b x_k y_k}{1 + c x_k} + u_k, \quad y_{k+1} = \frac{d y_k - h x_k y_k}{1 + g y_k}, \quad (25)$$

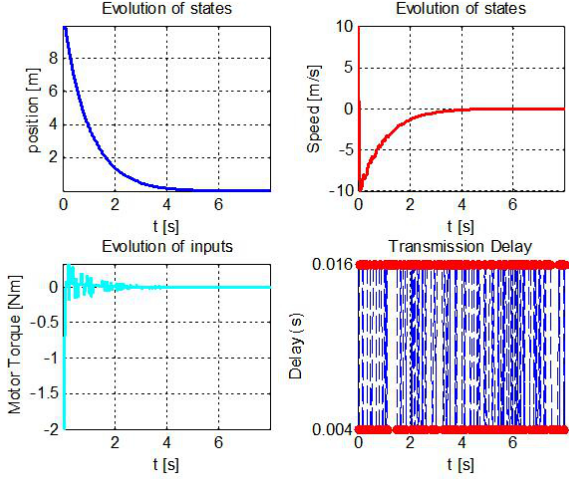


Fig. 4. Simulation of the closed-loop system using the SMPC controller, in continuous time, starting from $x(0) = [9.72 \ 8.98]'$ and $r_0 = 1$.

where the parameter a_{r_k} is governed by a time-homogeneous Markov chain with states $\mathcal{S} = \{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} 0.85 & 0.1 & 0.05 \\ 0.2 & 0 & 0.8 \\ 0.1 & 0.2 & 0.7 \end{bmatrix},$$

so that $a_{r_k} = a_i$ whenever $r_k = i$ and $a_1 = 0.8$, $a_2 = 1.1$, $a_3 = 1$. The linearization matrices A_i and B_i about the origin which are given by Corollary 2 are

$$A_i = \begin{bmatrix} a_i & 0 \\ 0 & d \end{bmatrix}, \text{ and } B_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We introduce the state and input constraints $x_k \in \mathcal{X} = \{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$ and $u_k \in \mathcal{U} = \{u \in \mathbb{R} \mid -0.1 \leq u \leq 0.1\}$. The other parameters of the system were chosen to be $b = 0.2$, $c = 0.1$, $d = 0.95$, $h = 0.1$ and $g = 0.5$. We formulated the Nonlinear MPC problem described in Corollary 2 using $\alpha_i = 0.04$ for all $i \in \mathcal{S}$ and the prediction horizon $N = 8$. The closed-loop trajectories of the Lotka-Volterra system are presented in Figure 5.

8 Conclusions

The present paper has proposed a new SMPC algorithm for constrained MSS. This class of stochastic switching systems is an extension of MJLS, a type of systems that have been studied thoroughly in the literature. In this work, the general case of nonlinear mode dynamics and state-input constraints is investigated in detail. Specifically, a new type of positive invariance is introduced, namely uniform positive invariance, that is less conservative than robust positive invariance and stochastic Lyapunov-type conditions for mean-square stability are stated and proved. Furthermore, conditions that the terminal cost and terminal set must satisfy are given, that

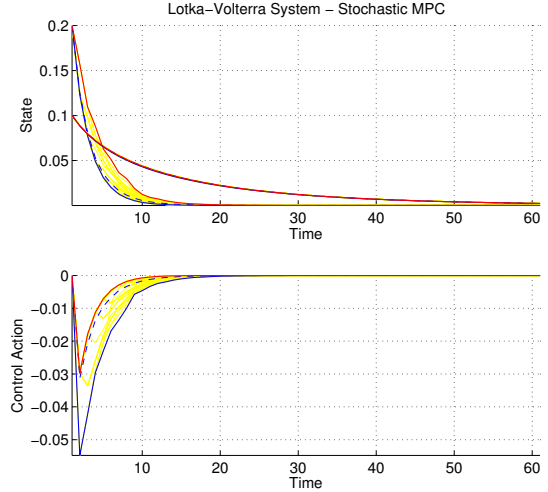


Fig. 5. Closed-loop trajectories of the Lotka-Volterra system from the initial point $(x_0, y_0) = (0.2, 0.1)$. in closed-loop with the nonlinear SMPC controller. (Blue) Lower bound, (Red) Upper bound, (Dashed) Average value, (Yellow) Individual trajectories.

guarantee mean-square stability of the system in closed loop with the proposed SMPC controller. The new approach is shown to be significantly less conservative than the ones proposed in the literature, through simulations. For the special case of MJLS with quadratic costs and polyhedral constraint sets, we show how one can compute the explicit SMPC law by combining DP and parametric optimization.

Future work will focus on the extension of the principles introduced in this paper to address the problem of state estimation in a moving-horizon fashion [28] especially for nonlinear cases that involve sensor saturation [15].

Appendix - Proof of Theorem 2

(a) Using Lemma 4 for $k_1 = 0$ and $k_2 = k$ $\mathbb{E}[V(x_k, r_k) - V(x_0, r_0)] = \mathbb{E}[\sum_{j=0}^{k-1} \mathcal{L}V(x_j, r_j)] \leq -\gamma \sum_{j=0}^{k-1} \mathbb{E}[||x_j||^2]$ implying in turn $\gamma \sum_{j=0}^{k-1} \mathbb{E}[||x_j||^2] \leq V(x_0, r_0) - \mathbb{E}[V(x_k, r_k)] \leq V(x_0, r_0)$. This yields $\sum_{j=0}^{k-1} \mathbb{E}[||x_j||^2] \leq V(x_0, r_0)/\gamma$, *i.e.*, the partial sums of $\sum_{j=0}^{\infty} \mathbb{E}[||x_j||^2]$ form a bounded sequence, therefore the series converges, implying that one must have $\lim_{k \rightarrow \infty} \mathbb{E}[||x_k||^2] = 0$.

(b) We have $\mathbb{E}[V(x_{k+1}, r_{k+1}) - V(x_k, r_k)] \leq -\gamma \mathbb{E}[||x_k||^2] \leq -(\gamma/\beta) \mathbb{E}[V(x_k, r_k)]$ where the first inequality follows from (14b) and the second from (14a). Therefore:

$$\mathbb{E}[V(x_{k+1}, r_{k+1})] \leq \zeta \mathbb{E}[V(x_k, r_k)] \quad (26)$$

where $\zeta \triangleq 1 - (\gamma/\beta)$. Using (14b) and (14a) it is $0 \leq \mathbb{E}[V(x_{k+1}, r_{k+1})] \leq \mathbb{E}[V(x_k, r_k)] - \gamma ||x_k||^2 \leq$

$(\beta - \gamma)\|x_k\|^2$ and it can be inferred that $0 \leq \zeta \leq 1$. Applying recursively (26), we arrive at $\mathbb{E}[V(x_k, r_k)] \leq \zeta^k V(x_0, r_0)$. Using (14a) we have $\alpha \mathbb{E}[\|x_k\|^2] \leq \mathbb{E}[V(x_k, r_k)] \leq \zeta^k V(x_0, r_0) \leq \zeta^k \beta \|x_0\|^2$. Finally we arrive at $\mathbb{E}[\|x_k\|^2] \leq \theta \zeta^k \|x_0\|^2$ where $\theta \triangleq \beta/\alpha > 1$. \square

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