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KINEMATIC ANALYSIS OF THE EXECHON ROBOT ACCOUNTING OFFSETS IN THE JOINT AXES

P.C. López-Custodio*, J.S. Dai

Kinematics and Mechanisms Group,
Centre for Robotics Research,
King's College London,
University of London,
Strand, London WC2R 2LS, UK

Email: pablo.lopez-custodio@kcl.ac.uk, jian.dai@kcl.ac.uk

R. Fu, Y. Jin

Centre for Intelligent and
Autonomous Manufacturing Systems,
School of Mechanical and Aerospace Engineering,
Queen's University Belfast,
123 Stranmillis Road BT9 5AH. Belfast, UK
Email: r.fu@qub.ac.uk, y.jin@qub.ac.uk

ABSTRACT

An Exechon robot with offsets between the axes of the joints that connect the legs to the fixed platform is analyzed for the first time. Ideally these axes intersect constituting two universal and one spherical joint. The insertion of imperfections in these universal and spherical joints leads to more complex forward and inverse kinematics which are solved in this paper. It is proved that the equations used for the kinematics of the ideal Exechon robot are no longer applicable when these offsets are added. The constraint system is also obtained and it is found to be different to the one of the ideal case. Finally, the combination of offsets that lead to the largest deviation in the position of the parallel platform is determined.

NOMENCLATURE

$a_{i,j}, \alpha_{i,j}$ DH parameters of link length and twist angle, respectively, between axes i and j .
 d_i, θ_i DH parameters of axial displacement and joint angle, respectively, for joint i .
 $d(P, Q)$ Euclidean distance between points P and Q .
 S_i Instantaneous screw axis associated to kinematic pair i .
 $\mathbb{T}^n$ n -dimensional torus.

$\text{Rot}(\hat{\mathbf{u}}, \theta)$ Rotation matrix representing a rotation of $\theta \in \mathbb{T}$ radians about an axis that contains the origin and is parallel to $\hat{\mathbf{u}} \in \mathbb{R}^3$.
 $\exp(a\mathbf{S})$ Exponential mapping that generates a finite displacement of magnitude a , about/along axis $\mathbf{S}$.
 im Image of a map.
 $\text{Adj}(\psi)$ Adjoint representation of $\psi \in \text{SE}(3)$.
 ${}^A\mathbf{r}_P$ Position vector of point P with respect to the origin of the coordinate system A .
 ${}^A\mathbf{R}_B$ Rotation matrix that gives the orientation of frame B with respect to frame A .
 $\pi_{\hat{\mathbf{u}}}(\mathbf{v})$ Projection of vector $\mathbf{v}$ onto a plane that is perpendicular to $\hat{\mathbf{u}}$ and contains the origin.

INTRODUCTION

Since the Stewart platform was developed in the sixties [1,2], applications for parallel kinematic machines (PKM) of parallel robots [3,4] have been expanding in industry. Nevertheless, the use of PKMs in manufacturing is relatively recent. It was until 1996 that Boeing incorporated the Tricept technology [5,6] in their production lines. Since then, the use of PKMs in manufacturing became a topic of interest of many researchers.

The big advantage of the Tricept robot is its topology, which is *hybrid*, this means that the robot consists of a parallel module

*Address all correspondence to this author.

with a serial module attached to the moving platform of the parallel one. This combination of characteristics allows to get the benefits from both worlds: a hybrid robot has the high stiffness which can be obtained from any parallel robot, but at the same time, the hybrid robot overcomes the important drawback of a parallel robot which is workspace limitations.

However, the topology of the Tricept robot has an important problem: it has many passive joints, leading to control problems and costly manufacturing. The Tricept has a completely unactuated leg which is in addition prone to high torsion and tension stresses. To overcome this, the same team which developed the Tricept and which is led by Neumann, designed in 2004 a new hybrid robot with a reduced number of passive joints but with all the advantages brought by the Tricept. The result is the Exechon robot [7, 8] (see figure 1) which again a 5-DOF robot with a serial module attached to the moving platform of a fully parallel mechanism. The Exechon has been already used in several applications, particularly, the Exechon is been used in aircraft manufacturing [9].

FIGURE 1. Exechon Xmini in operation in Queen's University Belfast.

In addition to the applications of the Exechon robot, its striking topology has called the attention of many researchers who have worked in studying its kinematics [10, 11, 12], stiffness [13, 14, 15, 16], constraint space and singularities [17, 18], dimensional synthesis [11], equivalent PKMs [19].

In the material provided by the developer of the Exechon, the diagram introducing the dimensions of the Exechon robot presents a more general topology [7]. The diagram is shown in figure 2, it can be seen that the revolute joints that connect the legs to the fixed platform are not intersecting as in the ideal model, where the nominal lengths of the common perpendiculars between these axes is zero.

The error propagation in the structure of the Exechon robot is currently being studied in a research carried out in conjunction

FIGURE 2. Diagram of a more general Exechon robot, taken from [7].

between King's College London and Queen's University Belfast. The study is made using the Exechon XMini, which is the model shown in figure 1. As part of this research this paper aims to kinematically analyze for the first time the Exechon robot considering the geometric imperfection [20] of the joint axes connecting the legs to the fixed platform. The equations applied to solve both the inverse and the forward kinematics of the ideal Exechon robot will no longer apply when considering these offsets. The study only considers the five offsets of the ideally intersecting axes in the base of the robot, but considers that the joint axes in the spherical wrist of the serial module are perfect, i.e., they intersect. The analysis considering offsets in the wrist will be carried out in future research.

The study will reveal the forward and inverse kinematics of the robot, its system of constraints which will also turn out to be different to that of the ideal Exechon robot and finally it will shortly study the effects of these offsets in the position of the moving platform, revealing also which combination of errors produces the largest deviation in position.

This paper is organized as follows: Section 1 describes the geometry of the ideal Exechon robot, this one is then compared in section 2 to the geometry of the Exechon robot with offsets to be studied in the paper, in section 3 the inverse kinematics of the Exechon robot with offsets is solved, followed by its forward kinematics in section 4 and the computation of its constraint system 5. The results of the forward kinematics problem are used in section 6 in order to determine the deviation of the position of the parallel platform of the robot due to the introduction of offsets. Finally, some conclusions are drawn in section 7.

1 GEOMETRY OF THE IDEAL EXECHON ROBOT

Figure 3 shows a representation of the ideal Exechon hybrid robot. The hybrid robot is conformed by a 3-DOF parallel module and a 3-DOF serial module. The parallel module consists of a moving platform and a fixed platform connected by three legs. Legs 1 and 3 are UPR serial chains, while leg 2 is an SPR kinematic chain. In the ideal model joint axes S_{l1} and S_{l2} intersect in point A_l , $l = 1, 3$ and S_{21} , S_{22} and S_{23} intersect in A_2 . In addition, $S_{l3} \perp S_{l2}$, $l = 1, 3$, $S_{24} \perp S_{22}$ and $S_{24} \parallel S_{23}$.

Refer to figure 4, points A_1 , A_2 and A_3 constitute the vertices of an isosceles triangle with base $2d_{A1} := d(A_1, A_3)$ and height $d_{A2} := d(A_2, S_{11})$. Points B_1 , B_2 and B_3 also form an isosceles triangle with base $2d_{B1} := d(B_1, B_3)$ and height $d_{B2} := d(B_2, B_1B_3)$. Points A_1 , A_3 , B_1 and B_3 are coplanar, we call the plane that these point lie on Π , while plane Λ is the one containing B_1 , B_2 and B_3 . The joint variables of the actuated joints of each leg are measured as follows: $q_{l3} := d(A_l, B_l)$, $l = 1, 3$, and $q_{24} := d(A_2, B_2)$.

FIGURE 3. The ideal Exechon hybrid robot

The serial module is mounted on the moving platform. This module is a serial 3R chain with its three axes S_{S1} , S_{S2} and S_{S3} intersecting at point S . Revolute joint with axis S_{S3} represents the rotation of the tool, the tip of the tool is the point T . Point S is located a distance $h_z := d(S, \Lambda)$ from plane Λ and a distance

$h_x := d(S, \Pi)$ from plane Π . Axis S_{S1} is perpendicular to Λ and the three axes constituting the serial module are perpendicular to each other. The distance between T and S is $d_T := d(T, S)$.

FIGURE 4. Structural dimensions of the ideal Exechon hybrid robot and planes Π and Λ

2 THE EXECHON ROBOT WITH OFFSET ERROR IN THE BASE REVOLUTE JOINTS

2.1 Geometry of the inserted offsets

Figure 5 shows an Exechon hybrid robot in which offsets were introduced between the axes of the revolute joints connecting the fixed platform. The condition of having axes intersecting at points A_i , $i = 1, 2, 3$, no longer holds. In the model shown in figure 5, former points A_i , $i = 1, 2, 3$, have been renamed as A_{ia} , $i = 1, 2, 3$. Error E_4 is the length of the common perpendicular between now skew axes S_{11} and S_{12} , this common perpendicular intersects S_{11} at A_{1a} and S_{12} at A_{1b} . Similarly, the common perpendicular between S_{31} and S_{32} intersects these axes at A_{3a} and A_{3b} , respectively, and has a length of E_5 . The joint variable of the actuated prismatic joint in legs $i = 1, 3$ is defined as

$q_{i3} := d(A_{ia}, B_i)$. In summary, if $E_{l1} := E_4$ and $E_{l3} := E_5$, the DH parameters for legs $i = 1, 3$ are given by:

$$\begin{aligned} a_{i1,i2} &= E_{li}, & \alpha_{i1,i2} &= \frac{\pi}{2}, & d_{i1} &= 0, & \theta_{i1} &= q_{i1}, \\ a_{i2,i3} &= 0, & \alpha_{i2,i3} &= \frac{\pi}{2}, & d_{i2} &= 0, & \theta_{i2} &= q_{i2}, \\ a_{i3,i4} &= 0, & \alpha_{i3,i4} &= -\frac{\pi}{2}, & d_{i3} &= q_{i3}, & \theta_{i3} &= 0, \\ & & & & d_{i5} &= 0, & \theta_{i5} &= q_{i4}. \end{aligned}$$

E_4 and E_5 are considered in the kinematic analysis carried out in [12], however the rest of joints are considered ideal. It is seen in [12], that the introduction of E_4 and E_5 does not really affect the analysis which is basically the same as that for the wholly ideal Exechon.

FIGURE 5. Exechon robot with offset errors in the base R joints

E_1 and E_3 are the lengths of the common perpendiculars between adjacent axes S_{21} and S_{22} , and S_{22} and S_{23} , respectively, and the distance between these two common perpendiculars is E_2 . The common perpendicular between S_{21} and S_{22} intersects S_{21} at A_{2a} .

For leg 2, the common perpendicular between adjacent axes S_{21} and S_{22} has a length of E_1 and intersects S_{21} at A_{2a} . Similarly, the common perpendicular between adjacent axes S_{22} and S_{23} has a length of E_3 and intersects S_{23} at A_{2b} . The distance

between these two common perpendiculars is E_2 . Joint variable of the actuated prismatic joint is defined as $q_{24} := d(A_{2b}, B_2)$. In summary, the DH parameters of leg 2 in the distal convention are:

$$\begin{aligned} a_{21,22} &= E_1, & \alpha_{21,22} &= -\frac{\pi}{2}, & d_{21} &= 0, & \theta_{21} &= q_{21}, \\ a_{22,23} &= E_3, & \alpha_{22,23} &= \frac{\pi}{2}, & d_{22} &= E_2, & \theta_{22} &= q_{22}, \\ a_{23,24} &= 0, & \alpha_{23,24} &= 0, & d_{23} &= 0, & \theta_{23} &= q_{23}, \\ a_{24,25} &= 0, & \alpha_{24,25} &= \frac{\pi}{2}, & d_{24} &= q_{24}, & \theta_{24} &= -\frac{\pi}{2}, \\ & & & & d_{25} &= 0, & \theta_{25} &= q_{25}. \end{aligned}$$

Plane Λ is defined the same as for the ideal Exechon robot, whilst plane Π is now defined as the plane containing points A_{1b} , A_{3b} , B_1 and B_3 . Three coordinate systems that will be useful throughout the analysis are defined as shown in figure 5: Coordinate system O , is fixed to the fixed platform with origin at point O , the middle point of segment $A_{1a}A_{3a}$, so that $d(O, A_{3a}) = d(O, A_{1a}) = d_{A1}$ and $d(O, A_{2a}) = d_{A2}$, axes x_O and y_O are parallel to $A_{1a}A_{3a}$ and OA_{2a} , respectively. Frame E is fixed to the moving platform, its origin, point E , is the middle point of segment B_1B_3 so that $d(E, B_3) = d(E, B_1) = d_{B1}$ and $d(E, B_2) = d_{B2}$, axes x_E and y_E are parallel to B_1A_3 and EB_2 , respectively. Finally, a frame S is fixed to the end-effector tool, with origin at S , the center of the spherical wrist of the serial module, axis z_S is coincident with S_{S3} , so that ${}^S\mathbf{r}_{T/S} = (0, 0, d_T)$.

2.2 Mobility of the parallel module of the Exechon robot with offsets

The mobility of the modified Exechon robot is the same as that of the ideal Exechon robot. A very simple way to prove this mobility is described. Considering legs $i = 1, 3$ and locking joints $i1$, the result is a planar mechanism, since S_{i2} , S_{i3} and S_{i4} constitute a generator of the group of general planar displacements $G(\hat{\mathbf{u}}_\Pi)$, where $\hat{\mathbf{u}}_\Pi$ is the normal vector to Π , the moving platform undergoes general planar motions in this situation. Therefore, both legs can be replaced by any kinematic chain that can generate $G(\hat{\mathbf{u}}_\Pi)$ connected to the fixed platform by means of an R joint with axis S_{i1} . In figure 6a, legs 1 and 3 have been replaced by a single 4R leg, with axes S_{11} , S_{G1} , S_{G2} and S_{G3} , where S_{Gj} , $j = 1 \dots 3$ are a generator of $G(\hat{\mathbf{u}}_\Pi)$. Since the only conditions for these three axes is them being non-coincident to each other and being parallel to $\hat{\mathbf{u}}_\Pi$, we can choose S_{G3} to intersect B_2 .

It can be seen that in leg 2 $S_{23} \parallel S_{24}$ and since these two joints are adjacent, they represent a generator of the group of cylindrical displacements $C(\mathbf{r}_{B_2/A_{2b}}/|\mathbf{r}_{B_2/A_{2b}}|, B_2)$, and as such, permutation of these joints is allowed. In figure 6a joints 23 and 24 have been permuted.

The moving platform of the equivalent mechanism shown in figure 6a undergoes exactly the same motion pattern than the moving platform of the Exechon robot with offsets in figure 5. It can be seen that in such a mechanism axes S_{G3} , S_{23} and S_{25}

FIGURE 7. Definition of coordinate system C and variables a) α and b) θ .

Unfortunately, the first condition does not hold for the Exechon with offsets in the base joints. It can be seen in figure 7 that ${}^O\hat{\mathbf{i}}_E \not\perp {}^O\mathbf{r}_{S/A_{2a}}$. Instead, our version of the condition is ${}^O\hat{\mathbf{i}}_E \perp {}^O\mathbf{r}_{S/A_{2b}}$. However, unlike $A_{2a} = A_2$, A_{2b} is unknown as it is not fixed to the ground. As a result of this, the analysis becomes much more complicated.

Point A_{2b} which in the ideal Exechon robot is fixed and coincident with A_{2a} is now moving in a toroid $\mathcal{S}$ [22,23,24] as shown in figure 8. The common perpendicular between axes $\mathbf{S}_{21}$ and $\mathbf{S}_{22}$ intersects the later at point A_{2c} . This point draws a circle $\mathcal{C}_1$ lying on plane $y_O z_O$ with radius E_1 . The common perpendicular between axes $\mathbf{S}_{22}$ and $\mathbf{S}_{23}$ intersects $\mathbf{S}_{22}$ at A_{2d} . Point A_{2b} draws a circle $\mathcal{C}_2$ of radius E_3 , with center at A_{2d} and normal parallel to $\mathbf{S}_{22}$. when actuating joint 21 $\mathcal{C}_2(q_{21})$ is swept along $\mathcal{C}_1$ generating a toroid. The axial displacement E_2 gives a banana shape to the cross section of the toroid, see [22].

Point A_{2b} can be located by means of any parametrization of $\mathcal{S}$, here the following is used:

$$\begin{aligned} {}^O\sigma(q_{21}, q_{22}) &:= {}^O\mathbf{r}_{A_{2a}} + \text{Rot}\left(\frac{\pi}{2}, {}^O\hat{\mathbf{j}}_O\right) \text{Rot}(q_{21}, {}^O\hat{\mathbf{k}}_O) (E_1 {}^O\hat{\mathbf{i}}_O \\ &\quad - E_2 {}^O\hat{\mathbf{j}}_O + E_3 \text{Rot}(q_{22}, -{}^O\hat{\mathbf{j}}_O) {}^O\hat{\mathbf{i}}_O) \\ &= (E_3 \sin q_{22}, d_{A2} + (E_1 + E_3 \cos q_{22}) \sin q_{21} \\ &\quad - E_2 \cos q_{21}, -(E_1 + E_3 \cos q_{22}) \cos q_{21} \\ &\quad - E_2 \sin q_{21}) \end{aligned} \quad (1)$$

so that ${}^O\hat{\mathbf{r}}_{A_{2b}} = {}^O\sigma(q_{21}, q_{22})$ and $\mathcal{S} = \text{im}(\sigma(\mathbb{T}^2))$. This parametrization uses the joint variables (q_{21}, q_{22}) as parameters,

FIGURE 8. Point A_{2b} lying on a toroid.

q_{21} being measured from z_O axis to the common perpendicular of length E_1 , about $\mathbf{S}_{21}$ and q_{22} measured from the common perpendicular of length E_1 to the common perpendicular of length E_3 about $-\mathbf{S}_{22}$.

Since $\overline{A_{2d}A_{2b}} \perp \mathbf{S}_{23}$, it follows that $\mathbf{S}_{23}(q_{22}, q_{23})$ is tangent to $\mathcal{C}_2(q_{22})$, therefore, the following condition holds:

$$\mathbf{S}_{23}(q_{21}, q_{22}) \parallel \left. \frac{\partial \sigma}{\partial q_{22}} \right|_{(q_{21}, q_{22})} \quad (2)$$

With the information in hand so far we also know that B_2 lies somewhere in a circle $\mathcal{C}$ of radius h_z , center at C and contained in the $x_C z_C$ plane (see figure 7b). We define the angle θ as the angle from z_C to z_E measured about y_C (or y_E). With this angle, $\mathcal{C}$ is drawn by:

$${}^O\mathbf{r}_{B_2/C}(\theta) = h_z \text{Rot}(\alpha, {}^O\hat{\mathbf{i}}_O) \text{Rot}(\theta, {}^O\hat{\mathbf{j}}_O) (-{}^O\hat{\mathbf{k}}_O) \quad (3)$$

and the position of B_2 is given by: ${}^O\hat{\mathbf{r}}_{B_2}(\theta) = {}^O\mathbf{r}_C + {}^O\mathbf{r}_{B_2/C}(\theta)$.

A first condition to consider is that $\hat{\mathbf{r}}_{B_2/C}$, $\hat{\mathbf{j}}_C$ and $\partial \sigma / \partial q_{22}$ must be coplanar. Therefore:

$$\begin{aligned} \det \left(\text{aug} \left(\frac{{}^O\mathbf{r}_{B_2/C}}{|{}^O\mathbf{r}_{B_2/C}|}, \frac{\sigma_{q_{22}}}{|\sigma_{q_{22}}|}, {}^O\hat{\mathbf{i}}_O \right) \right) &= 0 \\ \sin q_{q_{22}} \sin \theta (\cos \alpha \cos q_{21} + \sin \alpha \sin q_{21}) - \cos q_{22} \cos \theta &= 0 \end{aligned} \quad (4)$$

where $\sigma_{q_{22}} := \partial \sigma / \partial q_{22}$. Eq. (4) is in terms of unknowns θ , q_{21} and q_{22} . The computation of these unknowns fully solves the IKP of the parallel module, hence it is required to find more conditions that involve these variables.

Note in figure 7 that $\mathbf{S}_{23}$ must intersect axis y_C , let ${}^O\mathbf{S}_{23} := (\hat{\mathbf{s}}_{23}; \mathbf{v}_{23})$, with $\hat{\mathbf{s}}_{23} := \sigma_{q_{22}}/|\sigma_{q_{22}}|$ and $\mathbf{v}_{23} := {}^O\sigma_{q_{22}} \times {}^O\mathbf{r}_{A_{2b}}$. Similarly, let the Plücker coordinates of the y_C axis with respect to frame O be $(\hat{\mathbf{s}}_{y_C}; \mathbf{v}_{y_C})$, with $\hat{\mathbf{s}}_{y_C} := {}^O\hat{\mathbf{j}}_C$ and $\mathbf{v}_{y_C} := {}^O\mathbf{r}_S$. Then, if both lines intersect the following holds:

$$\begin{aligned} & \hat{\mathbf{s}}_{23} \cdot \mathbf{v}_{y_C} - \hat{\mathbf{s}}_{y_C} \cdot \mathbf{v}_{23} = 0 \\ & ((-E_2 \cos q_{21} + E_1 \sin q_{21} + d_{A_2} - S_y) \sin \alpha + (E_1 \cos q_{21} \\ & \quad + E_2 \sin q_{21} + S_z) \cos \alpha) \cos q_{22} \\ & - E_3 (S_x \sin q_{22} - E_3) (\cos \alpha \cos q_{21} + \sin \alpha \sin q_{21}) = 0 \end{aligned} \quad (5)$$

A third equation to complete the system can be obtained from condition in Eq. (2) by considering $\mathbf{r}_{B_2/A_{2b}} \parallel \mathbf{S}_{23}$ then $\mathbf{r}_{B_2/A_{2b}} \parallel \sigma_{q_{22}}$. Since ${}^O\mathbf{r}_{B_2}$ and ${}^O\mathbf{r}_{A_{2b}}$ are known in terms of the three unknowns, the condition can be used as follows:

$$\begin{aligned} & {}^O\mathbf{r}_{B_2/A_{2b}} \cdot (0, 1, 0) {}^O\sigma_{q_{22}} \cdot (0, 0, 1) \\ & - {}^O\mathbf{r}_{B_2/A_{2b}} \cdot (0, 0, 1) {}^O\sigma_{q_{22}} \cdot (0, 1, 0) = 0 \\ & \Leftrightarrow \\ & ((d_{B_2} - h_y) \cos \alpha + h_z \sin \alpha \cos \theta - d_{A_2} + S_y) \cos q_{21} + \\ & (S_z - h_z \cos \alpha \cos \theta + (d_{B_2} - h_y) \sin \alpha) \sin q_{21} + E_2 = 0 \end{aligned} \quad (6)$$

Eqs. (4-6) represent a system of 3 non-linear equations in 3 unknowns. However, note that Eq. (5) is only in terms of unknowns q_{21} and q_{22} , and Eq. (6) is only in terms θ and q_{21} . It is possible to solve these two equations so that two variables are eliminated.

Eq. (5) has the form $K_1 \cos q_{22} + K_2 \sin q_{22} + K_3 = 0$, which be solved for q_{22} with two solutions:

$$q_{22}(q_{21}) := \arctan \left(\frac{K_2(K_1(K_1 K_3 \mp K_4) - K_3(k_1^2 + K_2^2))}{-K_1 K_3 \pm K_4} \right) \quad (7)$$

where

$$\begin{aligned} K_1 &:= (-E_2 \cos q_{21} + E_1 \sin q_{21} + d_{A_2} - S_y) \sin \alpha + (E_1 \cos q_{21} \\ & \quad + E_2 \sin q_{21} + S_z) \cos \alpha, \\ K_2 &:= S_x (\cos \alpha \cos q_{21} + \sin \alpha \sin q_{21}), \\ K_3 &:= E_3 (\cos \alpha \cos q_{21} + \sin \alpha \sin q_{21}), \\ K_4 &:= \sqrt{K_1^2 K_2^2 + K_2^4 - K_2^2 K_3^2}. \end{aligned}$$

Eq. (6) can be easily solved for θ to obtain:

$$\begin{aligned} \theta(q_{21}) &:= \pi \pm \arccos \left((d_{B_2} \sin \alpha \sin q_{21} - h_y \sin \alpha \sin q_{21} + \right. \\ & \quad + d_{B_2} \cos \alpha \cos q_{21} - h_y \cos \alpha \cos q_{21} + S_z \sin q_{21} + (S_y \\ & \quad - d_{A_2}) \cos q_{21} + E_2) (h_z (\sin \alpha \cos q_{21} \\ & \quad - \cos \alpha \sin q_{21})^{-1}) \left. \right) \end{aligned} \quad (8)$$

Replacing the expressions for q_{22} and θ from Eqs. (7) and (8) in Eq. (4) the system is reduced to a single equation in a single variable, q_{21} . However, this equation is too complex to obtain a closed form solution and a numerical method has to be applied. There are two solutions in each expression in Eq. (7) and (8), therefore, at least four solutions can be obtained for the system of equations, however, it is not possible to determine how many solutions can be obtained from Eq. (4) once each expression for q_{22} and θ are replaced in it and it is solved numerically.

When the values of q_{21} , q_{22} and θ are defined frame E is determined by ${}^O\mathbf{R} = {}^O\mathbf{R} \text{Rot}(\theta, {}^C\hat{\mathbf{j}}_C) = \text{Rot}(\alpha, {}^O\hat{\mathbf{i}}_O) \text{Rot}(\theta, {}^O\hat{\mathbf{j}}_C)$ and ${}^O\mathbf{r}_E = {}^O\mathbf{r}_S + \text{Rot}(\theta, {}^O\hat{\mathbf{j}}_C)(0, -h_y, -h_z)$. ${}^O\mathbf{r}_{B_i}$, $i = 1, 2, 3$, are also easily obtained now that frame E is known. ${}^O\mathbf{r}_{A_{2b}}$ is also known by evaluating parametrization in Eq. (1).

The screw coordinates of all joint axes in the parallel module are given by:

Leg $i = 1, 3$:

$$\begin{aligned} {}^O\mathbf{S}_{i1} &= ({}^O\hat{\mathbf{i}}_O; \mathbf{0}), \\ {}^O\mathbf{S}_{i2} &= ({}^O\hat{\mathbf{j}}_E; {}^O\hat{\mathbf{j}}_E \times ({}^O\mathbf{r}_{A_{ia}} + E_l {}^O\hat{\mathbf{k}}_C)), \\ {}^O\mathbf{S}_{i3} &= \left(\mathbf{0}; \frac{{}^O\mathbf{r}_{B_i} - {}^O\mathbf{r}_{A_{ia}} - E_l {}^O\hat{\mathbf{k}}_C}{|{}^O\mathbf{r}_{B_i} - {}^O\mathbf{r}_{A_{ia}} - E_l {}^O\hat{\mathbf{k}}_C|} \right) \\ {}^O\mathbf{S}_{i4} &= ({}^O\hat{\mathbf{j}}_E; {}^O\hat{\mathbf{j}}_E \times \mathbf{r}_{B_i}) \end{aligned}$$

Leg 2:

$$\begin{aligned} {}^O\mathbf{S}_{21} &= ({}^O\hat{\mathbf{i}}_O; d_{A_{2a}} {}^O\hat{\mathbf{i}}_O \times {}^O\hat{\mathbf{j}}_O), \\ {}^O\mathbf{S}_{22} &= \text{Adj}(\exp(q_{21}, {}^O\mathbf{S}_{21})) (-{}^O\hat{\mathbf{j}}_O; {}^O\hat{\mathbf{j}}_O \times (-E_1 {}^O\hat{\mathbf{k}}_O)), \\ {}^O\mathbf{S}_{23} &= \left(\frac{{}^O\sigma_{q_{22}}}{|{}^O\sigma_{q_{22}}|}; \frac{{}^O\sigma_{q_{22}}}{|{}^O\sigma_{q_{22}}|} \times {}^O\sigma \right), \\ {}^O\mathbf{S}_{24} &= \left(\mathbf{0}; \frac{{}^O\sigma_{q_{22}}}{|{}^O\sigma_{q_{22}}|} \right) \\ {}^O\mathbf{S}_{25} &= ({}^O\hat{\mathbf{i}}_E; {}^O\hat{\mathbf{i}}_E \times \mathbf{r}_{B_2}) \end{aligned}$$

Joint variables of the actuated joints are given by: $q_{i3} = |{}^O\mathbf{r}_{B_i} - {}^O\mathbf{r}_{A_{ia}} - E_l {}^O\hat{\mathbf{k}}_C|$ and $q_{24} := |{}^O\mathbf{r}_{B_2} - {}^O\sigma|$.

3.2 Analysis of the serial module for the IKP.

After the analysis of the parallel module the orientation of frame E, ${}^O_E\mathbf{R}$, is known. From the input information of the IKP ${}^O_S\mathbf{R}$ is also known.

It can be seen that ${}^E_S\mathbf{R} = ({}^O_E\mathbf{R})^t({}^O_S\mathbf{R})$. The same transformation must be achieved by actuating the motors of the serial module from an initial configuration $\mathbf{q}_0$ in which frames E and S coincide. Such a configuration is shown in figure 9.

FIGURE 9. Initial configuration of the serial module.

After actuation of the three motors of the serial module, the orientation of frame S with respect to E is given by ${}^E_S\mathbf{R} = \text{Rot}(q_{S1}, {}^E\hat{\mathbf{k}}_E)\text{Rot}(q_{S2}, {}^E\hat{\mathbf{j}}_E)\text{Rot}(q_{S3}, {}^E\hat{\mathbf{k}}_E)$. Matching this rotation matrix with ${}^E_S\mathbf{R} = ({}^O_E\mathbf{R})^t({}^O_S\mathbf{R})$, the values of q_{Si} , $i = 1, 2, 3$, are obtained as:

$$\begin{aligned} q_{S2} &= \text{atan2}(\pm R_{2,3}, \pm R_{1,3}), \\ q_{S2} &= \text{atan2}(\pm \sqrt{1 - R_{3,3}}, R_{3,3}), \\ q_{S3} &= \text{atan2}(\pm R_{3,2}, \mp R_{3,1}) \end{aligned}$$

where $R_{i,j}$ is the (i, j) entry of known matrix ${}^E_S\mathbf{R}$. Note that two possible solutions are obtained for a desired orientation of the tool.

The screw coordinates of the joint axes in the serial module

are given by ${}^O\mathbf{S}_{Si} = (\hat{\mathbf{s}}_{Si}; \hat{\mathbf{s}}_{Si} \times {}^O\mathbf{r}_S)$, where:

$$\hat{\mathbf{s}}_{S1} = {}^O\hat{\mathbf{k}}_E, \hat{\mathbf{s}}_{S2} = \text{Rot}(q_{S1}, {}^O\hat{\mathbf{k}}_E){}^O\hat{\mathbf{i}}_E, \hat{\mathbf{s}}_{S3} = {}^O\hat{\mathbf{k}}_S.$$

4 FORWARD KINEMATICS OF THE EXECHON ROBOT WITH OFFSETS IN THE BASE R JOINTS

In the forward kinematics problem (FKP) of the Exechon robot with offsets the input information are the values of joint variables q_{13} , q_{24} , q_{33} , q_{S1} , q_{S2} and q_{S3} and the outputs are ${}^O_S\mathbf{R}$ and ${}^O\mathbf{r}_T$.

In a similar way to the IKP, it is possible to analyze the parallel and the serial modules separately. However, since the forward kinematics of the 3-DOF spherical wrist of the serial module is straightforward it is skipped in this paper, so the FKP is reduced to find the position and orientation of the moving platform, i.e. ${}^O_E\mathbf{R}$ and ${}^O\mathbf{r}_E$, respectively, when q_{13} , q_{24} and q_{33} are given. Again, as done with the IKP, the procedure in [10, 12] is first attempted to follow.

Refer to figure 7, a new variable k is introduced and it is defined as $k := |\pi_{i_0}({}^O\mathbf{r}_E)| = \sqrt{e_y^2 + e_z^2}$, where ${}^O\mathbf{r}_E := (e_x, e_y, e_z)$. We consider $k > 0$ due to the limits of the actuators of the actual robot. The following two conditions hold analyzing the projection on plane Π shown in figure 7b:

$$\begin{aligned} (e_x + d_{B1} \cos \theta - d_{A1})^2 + (k - d_{B1} \sin \theta - E_4)^2 - q_{13}^2 &= 0 \quad (9) \\ (e_x + d_{B1} \cos \theta - d_{A1})^2 + (k - d_{B1} \sin \theta - E_4)^2 - q_{13}^2 &= Q(10) \end{aligned}$$

Eqs. (9) and (10) are in terms of unknowns k , e_x and θ . Two sets of closed-form solutions for k and e_x in terms of θ can be obtained, however the expressions are too long to be presented in this paper.

Note that ${}^O\mathbf{r}_E = (e_x, -k \sin \alpha, k \cos \alpha)$, therefore, finding e_x , k and α solves the FKP since α also defines ${}^O\hat{\mathbf{j}}_E$ and using Eqs. (9) and (10), θ would be obtained which allows the computation of ${}^O\hat{\mathbf{i}}_E$ and ${}^O\hat{\mathbf{k}}_E$.

To obtain equations involving e_x , k and α two expressions for point B_2 are written. The first one describes the vectors from point O to E to B_2 :

$$\begin{aligned} {}^O\mathbf{r}_{B_2} &= {}^O\mathbf{r}_E + d_{B2} \text{Rot}(\alpha, {}^O\hat{\mathbf{i}}_O){}^O\hat{\mathbf{j}}_O \\ &= (e_x(\theta), -k(\theta) \sin \alpha, k(\theta) \cos \alpha) + d_{B2} \text{Rot}(\alpha, {}^O\hat{\mathbf{i}}_O){}^O\hat{\mathbf{j}}_O \end{aligned} \quad (11)$$

and the second equation takes the vectors from point O to A_{2a} to B_2 :

$${}^O\mathbf{r}_{B_2} = {}^O\mathbf{r}_{A_{2a}} + {}^O\sigma(q_{21}, q_{22}) + q_{24} \frac{{}^O\sigma_{q_{22}}(q_{21}, q_{22})}{{}^O\sigma_{q_{22}}(q_{21}, q_{22})} \quad (12)$$

Plugging Eqs. (11) and (12), three scalar equations in terms of 4 unknowns, θ , α , q_{21} and q_{22} , are obtained. In order to get rid of one of the variables, the condition of axis x_E being always perpendicular to joint axes S_{23} and S_{24} is considered:

$$\begin{aligned} \sigma_{q_{22}}(q_{21}, q_{22}) \cdot \hat{\mathbf{i}}_E &= 0 \\ \Leftrightarrow \\ {}^O\sigma_{q_{22}}(q_{21}, q_{22}) \cdot \text{Rot}(\alpha, {}^O\hat{\mathbf{i}}_O) \text{Rot}(\theta, {}^O\hat{\mathbf{j}}_O) {}^O\hat{\mathbf{i}}_O &= 0 \\ \Leftrightarrow \\ -\cos q_{21} \sin q_{22} \cos \alpha \sin \theta - \sin q_{21} \sin q_{22} \sin \alpha \sin \theta + \\ \cos q_{22} \cos \theta &= 0 \end{aligned} \quad (13)$$

Eq. (13) can be solved for q_{22} to obtain two solutions:

$$q_{22}(\theta, \alpha, q_{21}) \in \left\{ \arctan\left(\frac{\text{ctg}\theta}{\cos(\alpha - q_{21})}\right), \arctan\left(\frac{\text{ctg}\theta}{\cos(\alpha - q_{21})}\right) - \pi \right\}$$

Replacing this solution in Eqs. (11) and (12) a system of three equations in unknowns three unknowns, θ , α and q_{21} , is obtained. The expressions are large and have to be solved numerically.

When θ , α and q_{21} are known the FKP is solved as: ${}^O\mathbf{R} = \text{Rot}(\alpha, {}^O\hat{\mathbf{i}}_O) \text{Rot}(\theta, {}^O\hat{\mathbf{j}}_O)$ and ${}^O\mathbf{r}_E = \text{Rot}(\alpha, {}^O\hat{\mathbf{i}}_O)(e_x(\theta), 0, k(\theta))$. In addition, the center of the spherical wrist, S , can be easily located by means of: ${}^O\mathbf{r}_S = {}^O\mathbf{r}_E + \text{Rot}(\alpha, {}^O\hat{\mathbf{i}}_O) \text{Rot}(\theta, {}^O\hat{\mathbf{j}}_O)(0, h_y, h_z)$.

It is important to mention that the solutions $k(\theta)$ and $e_x(\theta)$, from Eqs. (9) and (10) do not exist should both $\theta = 0$ and $E_4 = E_5$ happen. In such a situation, Eqs. (9) and (10) are reduced to two equations in $k(\theta)$ and $e_x(\theta)$ with the following solutions:

$$\begin{aligned} e_x &= -\frac{q_{13}^2 - q_{33}^2}{4(d_{A1} - d_{B1})}, \\ k &= \frac{1}{4(d_{A1} - d_{B1})} \left(\pm \left((8d_{A1}^2 - 16d_{A1}d_{B1} + 8d_{B1}^2 + 2q_{13}^2)q_{33}^2 \right. \right. \\ &\quad \left. \left. - q_{33}^4 - 16 \left(d_{A1} - d_{B1} - \frac{q_{13}}{2} \right)^2 \left(d_{A1} - d_{B1} + \frac{q_{13}}{2} \right)^2 \right)^{\frac{1}{2}} + \right. \\ &\quad \left. 4E_4(d_{A1} - d_{B1}) \right) \end{aligned} \quad (14)$$

Eq. (13) now leads to $q_{22} \in \{\pi/2, 3\pi/2\}$, replacing the known values of e_x , k and q_{22} in Eqs. (11) and (12) three

equations in unknowns q_{21} and α is obtained. The extra equations gives a necessary condition for $\theta = 0$ to happen: $q_{33} = \sqrt{4E_3(d_{A1} - d_{B1}) + q_{13}^2}$. It can be seen that in the ideal Exechon robot, i.e. $E_3 = 0$, the parallel module adopts a symmetric configuration with respect to the $y_O z_O$ plane.

5 CONSTRAINT SYSTEM

It is desired to know the system of constraints for a given position and orientation of the end-effector. After solving the IKP for such a pose of the tool, it is possible to compute the wrench system of constraints. In this section, only the computation of the wrench system of the parallel module is addressed since the process is straightforward for the serial module.

The constraint system of each leg of the parallel module of the Exechon robot with offsets is different from that of the ideal Exechon robot. However, for legs 1 and 3 the wrench system is the same and still can be found by means of geometry as shown in figure 10 for leg 1: Similarly to legs $i = 1, 3$ of the ideal Exechon robot, the wrench system of constraints consists of two wrenches, the first one, $\mathbf{W}_{ci1}$ being a pure moment that is perpendicular to axes S_{i1} and S_{i2} and, in consequence, also perpendicular to S_{i4} . Since it is $\mathbf{W}_{ci1}$ is a pure moment, then $\text{Kl}(\mathbf{W}_{ci1}, S_{i3}) = 0$ is also ensured. The second wrench, $\mathbf{W}_{ci2}$, is a pure force intersecting S_{i1} and parallel to S_{i2} and S_{i4} . Since $\mathbf{W}_{ci2} \perp S_{i3}$ their reciprocity is ensured.

FIGURE 10. Constraint system of leg 1.

Since the IKP is considered to be known, the two wrenches for legs $i = 1, 3$ can be computed by means of:

$$\begin{aligned} {}^0\mathbf{W}_{ci1} &= (\mathbf{0}; \text{Rot}(\alpha, {}^0\hat{\mathbf{i}}_O) {}^0\hat{\mathbf{k}}_O), \\ {}^0\mathbf{W}_{ci2} &= (\text{Rot}(\alpha, {}^0\hat{\mathbf{i}}_O) {}^0\hat{\mathbf{j}}_O; \mathbf{0}). \end{aligned}$$

Unfortunately, for leg 2, unlike in the ideal Exechon robot, the sole wrench the spans the system of constraints of the leg is neither 0- nor infinite-pitched, and therefore it cannot be located by means of geometry, at least not easily. $\mathbf{W}_{c21}$, the only wrench in this system, can be found by computation of a basis of $\text{Null}((\mathbf{J}_2(\mathbf{q}))^t)$, where $\mathbf{J}_2 := \text{aug}({}^0\mathbf{S}_{21}, \dots, {}^0\mathbf{S}_{25})$.

Although IKP is solved numerically, it is possible to obtain a closed form solution for ${}^0\mathbf{W}_{c21}$ if $\mathbf{J}_2$ is expressed in terms of α , θ , q_{21} and q_{22} . When these values are obtained in the IKP it is easier to evaluate the closed form solution of ${}^0\mathbf{W}_{c21}$ than computing $\text{Null}((\mathbf{J}_2(\mathbf{q}))^t)$ for the given configuration $\mathbf{q}$.

Let $\lambda {}^0\tilde{\mathbf{W}}_{c21}$ be the only element of the basis of the computed null space [25] (it is multiplied by λ since it is not necessary a unit screw), then $\text{span}({}^0\tilde{\mathbf{W}}_{c21}) = \text{Null}((\mathbf{J}_2^t(\mathbf{q}))^t)$, where, if $\mathbf{W}_{c21} := (\hat{\mathbf{s}}; \mathbf{v})$, then $\tilde{\mathbf{W}}_{c21} := (\mathbf{v}; \hat{\mathbf{s}})$. Clearly, $\text{Kl}({}^0\mathbf{S}_{2j}, {}^0\mathbf{W}_{c21}) = 0$, $j = 1, \dots, 5$, if and only if $({}^0\mathbf{S}_{2j})^t \tilde{\mathbf{W}}_{c21} = 0$.

Using symbolic algebra software Maple $\text{Null}((\mathbf{J}_2(\mathbf{q}))^t)$ is computed and a closed-form solution is found in terms of the previously mentioned intermediate variables. The solution is long and is presented in Appendix A.

6 EFFECTS OF OFFSETS IN THE POSITION OF THE MOVING PLATFORM

It is possible that not all five offsets are present in the robot, therefore a total of $2^5 = 32$ situations are possible, depending on which offsets present error and which are ideal. For example, the link between joints 11 and 12 can be ideal and $E_4 = 0$, while the other offsets are still present. It is possible to check which of these 32 cases produces the biggest error in the position of the parallel platform. The 32 cases will be tested in different points of the reachable workspace of the robot, and in each case the deviation of point E will be compared to its ideal position that would be obtained in the ideal Exechon robot using the same values of the actuated joint variables.

The comparison is made using the dimensions of the Exechon XMini, which are shown in table 1. The stroke of each actuator is split in 11 positions, generating a total of $11^3 = 1331$ configurations to be tested. Since in each configuration 32 different cases of combinations of offsets are to be analyzed, a total of $1331 \times 32 = 42592$ iterations are required.

In each of the 1331 configurations the case of offsets combinations that produces the biggest deviation of point E is saved. When all configuration have been analyzed the number of times that every case of offsets combinations is counted. According to

Parameter	Value (mm)
h_y	83
h_z	408.1
d_{A1}	250
d_{A2}	400
d_{B1}	133
d_{B2}	166
q_{13}, q_{24} and q_{33} , actuator fully contracted	563
Stroke of actuators	300

TABLE 1. Dimensions and joint limits of the Exechon Xmini

these results, the case that affected the position of E the worst is when all offsets are present except E_5 , the length of the common perpendicular between joint axes $\mathbf{S}_{31}$ and $\mathbf{S}_{33}$, which was the worst case in 428 configurations.

Figure 11 shows the points reached by the ideal Exechon robot in the 1331 configurations analyzed. In such plots, the clearer the dots are, the larger the deviation of point E in the parallel platform will be when all offsets, except the one in leg 3 are introduced. It can be seen that the safest area is when x_O -coordinate of E is negative, while its y_O -coordinate is positive. These results were obtained using an offset of 1 mm, for which the largest deviation of the position of E with respect to the ideal case was 2.8428193 mm, while the mean was a deviation of 2.216036073 mm.

7 CONCLUSION

In this paper the Exechon robot was analyzed considering imperfect joint axes in the joints connecting the legs to the fixed platform. The forward and inverse kinematics of the robot were solved and its constraint system was obtained. It can be conclude that the insertion of these offsets becomes the analysis of the robot a much more complex problem which had to be solved numerically, although the systems of equations were reduced from their original form. The constraint system is the same as in the ideal case only for legs 1 and 3, however for leg 2 the single wrench that spans the system of constraints of such leg cannot be obtained by geometrical means. The use of the FKP showed that the combination of all offsets except that of leg 1 leads to the largest deviation in the position of the parallel platform.

The results obtained in this paper allow a more precise prediction of the position and orientation of the tool once the dimensions of each offset are known. These results will also allow the determination of a stiffness model of this robot with imperfections for a more detailed analysis of the errors produced during

FIGURE 11. a) Workspace of the ideal Exechon robot. The clearer the dots, the larger the deviation of the moving platform if all offsets are introduced except that of leg 3. b) Projection of this workspace on plane x_{Oy_O} and c) on plane y_{Oz_O} .

machining operations.

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Appendix A: Wrench of constraints of leg 2.

The following wrench of constraints of leg 2 is computed:

$$\lambda^O \mathbf{W}_{c21} = (K_5, K_6, K_7; d_{A_2}, K_8, K_9)$$

where:

$$\begin{aligned} K_5 := & ((-S_x(\cos \alpha \cos q_{21} + \sin \alpha \sin q_{21}) \sin \theta + (-\cos \theta(b_{B_2} \\ & - h_y) \sin \alpha + \cos \alpha h_z - \cos \theta S_z) \cos q_{21} - \sin q_{21}(((h_y \\ & - d_{B_2}) \cos \alpha + d_{A_2} - S_y) \cos \theta - \sin \alpha h_z)) \sin q_{22} + \\ & \sin \theta(-\sin \alpha \cos q_{22}(d_{A_1} - d_{A_2}) \cos^2 q_{21} + (\cos q_{22}(d_{A_1} \\ & - d_{A_2}) \sin q_{21} + \cos q_{22} E_1 + E_3) \cos \alpha \cos q_{21} + \\ & ((\cos q_{22} E_1 + E_3) \sin q_{21} + \cos q_{22}(d_{A_1} - d_{A_2})) \sin \alpha)) \sin q_{22} \end{aligned}$$

$$\begin{aligned} K_6 := & (-\cos \alpha(\cos q_{22} - 1)(\cos q_{22} + 1)(d_{A_1} - d_{A_2}) \cos^3 q_{21} + \\ & (-\sin \alpha \sin q_{21}(d_{A_1} - d_{A_2}) \cos^2 q_{22} - \sin \alpha E_3 \cos q_{22} + \\ & ((d_{A_1} - d_{A_2}) \sin q_{21} - E_1) \sin \alpha - E_2 \cos \alpha) \cos^2 q_{21} + \\ & ((\sin \alpha S_z + (d_{A_1} - 2d_{A_2} + S_y) \cos \alpha - h_y + d_{B_2}) \cos^2 q_{22} + \\ & \cos \alpha \sin q_{21} E_3 \cos q_{22} + (-\sin q_{21} E_2 - S_z) \sin \alpha + (E_1 \sin q_{21} + \\ & d_{A_2} - S_y) \cos \alpha - d_{B_2} + h_y) \cos q_{21} + (((d_{A_1} - d_{A_2}) \sin q_{21} + \\ & E_1) \cos q_{22} - \sin q_{22} S_x + E_3) \sin \alpha \cos q_{22}) \sin \theta + \\ & \sin(q_{22})(\sin \alpha h_z - ((h_y - d_{B_2}) \cos \alpha + d_{A_2} - S_y) \cos \theta) \cos q_{22} \end{aligned}$$

$$\begin{aligned} K_7 := & ((\sin \alpha(d_{A_1} - d_{A_2}) \cos^3 q_{21} - \cos \alpha \sin q_{21}(d_{A_1} - d_{A_2}) \cos^2 q_{21} \\ & - \sin \alpha(d_{A_1} - d_{A_2}) \cos q_{21} + (s_z \sin \alpha + (-d_{A_2} + s_y) \cos \alpha \\ & - h_y + d_{B_2}) \sin q_{21} - \cos \alpha E_1) \cos^2 q_{22} + (-\cos \alpha \cos^2 q_{21} E_3 \\ & - \sin \alpha \sin q_{21} E_3 \cos q_{21} + s_x \cos \alpha \sin q_{22}) \cos q_{22} - \sin \alpha(d_{A_1} \\ & - d_{A_2}) \cos^3 q_{21} + (\cos \alpha \sin q_{21}(d_{A_1} - d_{A_2}) - \cos \alpha E_1 + \\ & \sin \alpha E_2) \cos^2 q_{21} + ((-E_2 \cos \alpha - \sin \alpha E_1) \sin q_{21} + \sin \alpha(d_{A_1} \\ & - d_{A_2})) \cos q_{21} + (-s_z \sin \alpha + (d_{A_2} - s_y) \cos \alpha + h_y \\ & - d_{B_2}) \sin q_{21} + \cos \alpha E_1 - \sin \alpha E_2) \sin \theta - \sin q_{22} \cos q_{22}((h_y \\ & - d_{B_2}) \cos \theta \sin \alpha + \cos \alpha h_z - s_z \cos \theta) \end{aligned}$$

$$\begin{aligned} K_8 := & ((\cos \alpha(d_{B_2} - h_y) - d_{A_2} + s_y) \cos \theta + \sin \alpha(-\sin \theta s_x + \\ & h_z))(d_{A_1} - d_{A_2})(\cos q_{22} + 1)(\cos q_{22} - 1) \cos^3 q_{21} + (-d_{A_1} \\ & - d_{A_2}) \sin q_{21}(((h_y - d_{B_2}) \sin \alpha - s_z) \cos \theta + \cos \alpha(-\sin \theta s_x + \\ & h_z)) \cos^2 q_{22} + (E_3(\sin \alpha(d_{B_2} - h_y) + s_z) \cos \theta + (((d_{A_2} \\ & - s_y)(d_{A_1} - d_{A_2}) \sin q_{22} + E_3 s_x) \sin \theta - h_z E_3) \cos \alpha \\ & - \sin \theta \sin q_{22}(\sin \alpha s_z - h_y + d_{B_2})(d_{A_1} - d_{A_2})) \cos q_{22} + \\ & (E_2(d_{B_2} - h_y) \cos \alpha - (d_{A_1} - d_{A_2})(\sin \alpha(d_{B_2} - h_y) + \\ & s_z) \sin q_{21} + E_1(d_{B_2} - h_y) \sin \alpha - E_2 d_{A_2} + s_z E_1 + E_2 s_y) \cos \theta + \\ & (((d_{A_1} - d_{A_2}) \sin q_{21} - E_1) \cos \alpha + \sin \alpha E_2)(-\sin \theta s_x + \\ & h_z)) \cos^3 q_{21} + (-((\cos \alpha(d_{B_2} - h_y) - d_{A_2} + s_y) \cos \theta + \\ & \sin \alpha(-\sin \theta s_x + h_z))(d_{A_1} - d_{A_2}) \cos^2 q_{22} - ((\cos \alpha(d_{B_2} - h_y) \\ & - d_{A_2} + s_y) \cos \theta + \sin \alpha(-\sin \theta s_x + h_z)) \sin q_{21} E_3 \cos q_{22} + \\ & ((d_{B_2} - h_y)(-E_1 \sin q_{21} + d_{A_1} - d_{A_2}) \cos \alpha + (E_2(d_{B_2} \\ & - h_y) \sin \alpha + E_1 d_{A_2} - s_y E_1 + s_z E_2) \sin q_{21} - (d_{A_2} - s_y)(d_{A_1} \\ & - d_{A_2})) \cos \theta + (-\sin q_{21} E_2 \cos \alpha + \sin \alpha(-E_1 \sin q_{21} + d_{A_1} \\ & - d_{A_2}))(-\sin \theta s_x + h_z) \cos q_{21} + ((d_{A_1} - d_{A_2}) \sin q_{21} + \\ & E_1)((h_y - d_{B_2}) \sin(\alpha) - s_z) \cos \theta + \cos(\alpha)(-\sin \theta s_x + \end{aligned}$$

$$\begin{aligned}
& h_z)) \cos^2 q_{22} + (-E_3(\sin \alpha(d_{B_2} - h_y) + s_z) \cos \theta + \\
& (-\sin \theta \sin q_{22} E_1(d_{A_1} - s_y) \sin q_{21} + ((d_{A_2}^2 + (-d_{A_1} \\
& - s_y)d_{A_2} + d_{A_1}s_y - E_1^2) \sin q_{22} - E_3s_x) \sin \theta + h_z E_3) \cos \alpha + \\
& \sin \theta \sin q_{22} (\sin \alpha s_z - h_y + d_{B_2})(E_1 \sin q_{21} + d_{A_1} \\
& - d_{A_2})) \cos q_{22} + E_2((h_y - d_{B_2}) \cos \alpha + d_{A_2} - s_y) \cos \theta \\
& - \sin q_{22} \sin \theta ((d_{A_2} - s_y) \sin q_{21} + E_1) E_3 \cos \alpha + \\
& \sin \theta \sin q_{22} E_3 (\sin \alpha s_z - h_y + d_{B_2}) \sin q_{21} - \sin \alpha E_2 (h_z \\
& - s_x \sin \theta)
\end{aligned}$$

and

$$\begin{aligned}
K_9 := & (d_{A_1} - d_{A_2})(\cos^2 q_{22} - 1)((h_y - d_{B_2}) \sin \alpha - s_z) \cos \theta + \\
& \cos \alpha (-\sin \theta s_x + h_z)) \cos^3 q_{21} + ((- (d_{A_1} - d_{A_2}) ((h_y \\
& - d_{B_2}) \cos(\alpha) + d_{A_2} - s_y) \sin q_{21} \cos^2 q_{22} - ((h_y \\
& - d_{B_2}) \cos \alpha + d_{A_2} - s_y) E_3 \cos q_{22} + (d_{A_1} - d_{A_2}) ((h_y \\
& - d_{B_2}) \cos \alpha + d_{A_2} - s_y) \sin q_{21} - E_2(d_{B_2} - h_y) \sin \alpha + \\
& E_1(d_{B_2} - h_y) \cos \alpha - E_1 d_{A_2} + s_y E_1 - s_z E_2) \cos \theta + \\
& \sin \alpha \sin q_{21} (d_{A_1} - d_{A_2}) (-\sin \theta s_x + h_z) \cos^2 q_{22} + \\
& \sin \alpha ((d_{A_2} (d_{A_1} - d_{A_2}) \sin q_{22} - E_3 s_x) \sin \theta + h_z E_3) \cos q_{22} \\
& - (\sin \alpha \sin q_{21} (d_{A_1} - d_{A_2}) - E_2 \cos \alpha - \sin \alpha E_1) (-s_x \sin \theta + \\
& h_z)) \cos^2 q_{21} + (((\sin \alpha (d_{B_2} - h_y) + s_z) (d_{A_1} \\
& - 2d_{A_2}) \cos^2 q_{22} + (\sin \alpha (d_{B_2} - h_y) + s_z) \sin q_{21} E_3 \cos q_{22} + \\
& (E_1 (d_{B_2} - h_y) \sin \alpha + E_2 (d_{B_2} - h_y) \cos \alpha - E_2 d_{A_2} + s_z E_1 + \\
& E_2 s_y) \sin q_{21} + d_{A_2} (\sin \alpha (d_{B_2} - h_y) + s_z)) \cos \theta - \cos \alpha (d_{A_1} \\
& - 2d_{A_2}) (-\sin \theta s_x + h_z) \cos^2 q_{22} + ((-\sin \theta \sin q_{22} s_z (d_{A_1} \\
& - d_{A_2}) \sin \alpha + ((-s_y (d_{A_1} - d_{A_2}) \sin q_{22} + E_3 s_x) \sin \theta \\
& - h_z E_3) \cos \alpha - \sin \theta \sin q_{22} (d_{B_2} - h_y) (d_{A_1} - d_{A_2})) \sin q_{21} \\
& - \sin \theta \sin q_{22} E_1 (d_{B_2} - h_y + \sin \alpha s_z + \cos \alpha s_y)) \cos q_{22} \\
& - (E_1 \cos \alpha - E_2 \sin \alpha) (-\sin \theta s_x + h_z) \sin q_{21} \\
& - E_3 \sin \alpha \sin \theta \sin q_{22} s_z + ((-E_3 \sin q_{22} s_y + d_{A_2} s_x) \sin \theta \\
& - h_z d_{A_2}) \cos \alpha - \sin \theta \sin q_{22} E_3 (d_{B_2} - h_y)) \cos q_{21} - \\
& (d_{A_2} \sin q_{21} + E_1) (\cos q_{22} + 1) ((h_y - d_{B_2}) \cos \alpha + d_{A_2} \\
& - s_y) (\cos q_{22} - 1) \cos \theta + \sin \alpha ((d_{A_2} \sin q_{21} + \\
& E_1) (-s_x \sin \theta + h_z) \cos^2 q_{22} - \sin \theta \sin q_{22} (d_{A_1} E_1 \sin q_{21} + \\
& d_{A_2} d_{A_1} - d_{A_2}^2 + E_1^2) \cos q_{22} - (d_{A_2} \sin q_{21} + E_1) ((\sin q_{22} E_3 \\
& - s_x) \sin \theta + h_z))
\end{aligned}$$

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