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Triangle Packings and Transversals of Some K_4-Free Graphs

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Abstract Tuza's Conjecture asserts that the minimum number $\tau'_\Delta(G)$ of edges of a graph G whose deletion results in a triangle-free graph is at most 2 times the maximum number $\nu'_\Delta(G)$ of edge-disjoint triangles of G. The complete graphs K_4 and K_5 show that the constant 2 would be best possible. Moreover, if true, the conjecture would be essentially tight even for K_4-free graphs.

In this paper, we consider several subclasses of K_4-free graphs. We show that the constant 2 can be improved for them and we try to provide the optimal one. The classes we consider are of two kinds: graphs with edges in few triangles and graphs obtained by forbidding certain odd-wheels.

We translate an approximate min-max relation for $\tau'_\Delta(G)$ and $\nu'_\Delta(G)$ into an equivalent one for the clique cover number and the independence number of the triangle graph of G and we provide θ-bounding functions for classes related to triangle graphs. In particular, we obtain optimal θ-bounding functions for the classes *Free*$(K_5, \text{claw}, \text{diamond})$ and *Free*$(P_5, \text{diamond}, K_{2,3})$ and a χ-bounding function for the class (banner, odd-hole, $\overline{K_{1,4}}$).

Keywords Triangle packing and transversal · χ-Bounded · Triangle graphs

1 Introduction

Given a hypergraph, a *transversal* is a subset of its vertices intersecting each edge and a *packing* is a set of pairwise disjoint edges. The minimum size of a transversal of a hypergraph $\mathscr{H}$ is denoted by $\tau(\mathscr{H})$ and the maximum size of a packing of $\mathscr{H}$ is denoted by $\nu(\mathscr{H})$. The *triangle hypergraph* $\mathscr{H}(G)$ of a graph G is the hypergraph whose vertices are the edges of G and whose edges are the subsets spanning triangles (i.e. K_3 subgraphs) of G. Since $\mathscr{H}(G)$ is 3-uniform, it clearly satisfies the approximate min-max relation $\tau(\mathscr{H}(G)) \leq 3\nu(\mathscr{H}(G))$. Moreover, considering the complete graphs on 4 and 5 vertices, we have that $\tau(\mathscr{H}(G))$ may be as large as $2\nu(\mathscr{H}(G))$ and Tuza conjectured that these cases are extremal:

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Conjecture 1 **(Tuza's Conjecture [40])** *For any graph G, we have $\tau(\mathscr{H}(G)) \leq 2\nu(\mathscr{H}(G))$.*

In other words, Conjecture 1 asserts that the minimum number of edges of a graph G whose deletion results in a triangle-free graph is at most two times the maximum number of edge-disjoint triangles of G. It is sometimes convenient to stick to the underlying graph and so, denoting by $\tau'_{\Delta}(G)$ the quantity $\tau(\mathscr{H}(G))$ and by $\nu'_{\Delta}(G)$ the quantity $\nu(\mathscr{H}(G))$, we can rewrite Conjecture 1 as follows:

Conjecture 2 **(equivalent version of Conjecture 1)** *For any graph G, we have $\tau'_{\Delta}(G) \leq 2\nu'_{\Delta}(G)$.*

Despite having received considerable attention, Tuza's Conjecture is still open. To date, the best non-trivial bound is $\tau'_{\Delta}(G) \leq (3 - \frac{3}{23})\nu'_{\Delta}(G)$, as shown by Haxell [19]. Moreover, several graph classes for which it holds are known. For example, since every graph G has a bipartite subgraph with at least $|E(G)|/2$ edges (see, e.g., [41]) and since the complement of this edge set is clearly a triangle-transversal[1] of G, we have that Tuza's Conjecture holds if G has many edge-disjoint triangles, more precisely at least $|E(G)|/4$. Tuza [40] proved Conjecture 2 for planar graphs and for "dense" graphs, specifically for graphs on n vertices and with at least $\frac{7}{16}n^2$ edges. Lakshmanan et al. [27] showed that it holds for the class of triangle-3-colourable graphs, where a graph G is *triangle-3-colourable* if its edges can be coloured with three colours so that the edges of each triangle receive three distinct colours. This is a direct consequence of the case $r = 3$ of Ryser's Conjecture[2] proved by Aharoni [1]: indeed, if G is triangle-3-colourable, then the triangle hypergraph of G is clearly 3-partite. Since the class of triangle-3-colourable graphs contains that of 4-colourable graphs [27], the previous result is a generalization of the planar case mentioned above. Another generalization of the planar case was given by Krivelevich [26], who showed that Tuza's Conjecture holds for graphs with no $K_{3,3}$-subdivision. Recently, Krivelevich's result was further generalized by Puleo [34] who showed, using the discharging method, that Tuza's Conjecture holds for graphs having maximum average degree less than 7, where the *maximum average degree* of a graph G is defined as $\max\{2|E(H)|/|V(H)| : H \subseteq G\}$.

For all the classes mentioned so far, Conjecture 2 is tight, since they all contain the complete graph K_4. Therefore, a natural question arises: What happens if we forbid K_4? Haxell et al. [20] showed that the constant 2 cannot essentially be improved: for every $\varepsilon > 0$, there exists a K_4-free graph G such that $\tau'_{\Delta}(G) > (2 - \varepsilon)\nu'_{\Delta}(G)$.

In this paper, we consider certain subclasses of K_4-free graphs. We show that the constant 2 in Conjecture 2 can be improved for them and we try to provide the optimal one, in the spirit of the following problem:

Problem 1 **(Lakshmanan et al. [27])** Given a class of graphs $\mathcal{G}$ in which at least one $G \in \mathcal{G}$ is not triangle-free, determine the infimum of constants c such that $\tau'_{\Delta}(G) \leq c\nu'_{\Delta}(G)$ holds for every $G \in \mathcal{G}$.

[1] A *triangle-transversal* of G is a transversal of the triangle hypergraph of G.

[2] The long-standing open problem known as Ryser's Conjecture and formulated in [22] asserts that $\tau(\mathscr{H}) \leq (r-1)\nu(\mathscr{H})$, for any r-uniform r-partite hypergraph $\mathscr{H}$. Recall that a hypergraph is r-uniform if every edge has size r, and r-partite if the vertex set can be partitioned into r classes such that each edge contains at most one vertex for each class.

There are two quantities which is natural to consider when dealing with packings and transversals of triangles in a graph G: the number of triangles containing a certain edge and the number of edges shared by a certain triangle with other triangles. These quantities are in some sense dual to each other: the former corresponds to the degree of a vertex in the triangle hypergraph $\mathscr{H}(G)$ while the latter corresponds to the degree of a vertex in the subhypergraph of the dual[3] $\mathscr{H}(G)^*$ consisting of those edges of size greater than 1.

Let us begin by considering graphs with edges in few triangles. Suppose we are given a graph G such that each of its edges belongs to at most one triangle, i.e. G is the edge-disjoint union of triangles plus possibly some edges that do not belong to any triangle. Clearly, we have $\tau'_\Delta(G) = \nu'_\Delta(G)$. It is then natural to consider the "next case": What happens if each edge of G belongs to at most two triangles? Following [18], we refer to graphs having this property as *flat* graphs. Note that, if G is flat, each edge of a K_4 subgraph of G is in no triangle other than those contained in K_4 and so we may restrict ourselves to consider K_4-free flat graphs. In this case, Haxell et al. [18] showed that the constant 2 can be dropped to 3/2:

Theorem 1 **(Haxell et al. [18])** *If G is a K_4-free flat graph, then $\tau'_\Delta(G) \le \frac{3}{2}\nu'_\Delta(G)$. Equality holds if and only if G is the edge-disjoint union of 5-wheels plus possibly edges which are not in any triangle.*

Using Theorem 1, Haxell et al. [18] showed that the same bound holds for K_4-free planar graphs: If G is a K_4-free planar graph, then $\tau'_\Delta(G) \le \frac{3}{2}\nu'_\Delta(G)$.

Note that flatness can be easily expressed in terms of the triangle graph, where the *triangle graph* $T(G)$ of G is the graph having as vertices the triangles of G, two vertices being adjacent if the corresponding triangles share an edge. Indeed, a K_4-free graph G is flat if and only if $T(G)$ is triangle-free (see Lemma 3).

The bound in Theorem 1 actually holds for the following superclass of K_4-free flat graphs: the class of K_4-free graphs G such that each triangle of G shares its edges with at most three other triangles. We can succintly express this property by invoking the triangle graph: each triangle of G shares its edges with at most three other triangles if and only if $T(G)$ is subcubic. In Section 3, we show the following generalization of Theorem 1:

Theorem 2 *If G is a K_4-free graph such that $T(G)$ is subcubic, then $\tau'_\Delta(G) \le \frac{3}{2}\nu'_\Delta(G)$.*

The proof of Theorem 2 consists of two steps. Lakshmanan et al. [27] showed that if G is a K_4-free graph, we can translate an approximate min-max relation for τ'_Δ and ν'_Δ into an equivalent one for the clique cover number θ and the independence number α of the triangle graph $T(G)$. More precisely, they showed that the following holds: If G is K_4-free, then

$$\nu'_\Delta(G) = \alpha(T(G)) = \omega(\overline{T(G)}) \quad \text{and} \quad \tau'_\Delta(G) = \theta(T(G)) = \chi(\overline{T(G)}). \qquad (1)$$

On the other hand, we showed in [33] that $\theta(G) \le \frac{3}{2}\alpha(G)$, for every subcubic graph G. Therefore, combining these two results, we obtain Theorem 2. The

[3] Recall that the set of vertices of the dual hypergraph $\mathscr{H}(G)^*$ is $\{y_S : S \in \mathscr{H}(G)\}$, where the y_S are pairwise distinct. Moreover, for each vertex x of $\mathscr{H}(G)$, the set $\{y_S : S \in \mathscr{H}(G),\ x \in S\}$ is an edge of $\mathscr{H}(G)^*$.

reduction of Tuza's Conjecture to a more manageable statement about bounded clique covers of triangle graphs is in fact the recurrent theme of this paper (see Section 3).

Let us now consider the second quantity mentioned above: the number of edges shared by a triangle with other triangles. Suppose we are given a graph G and a maximal family of edge-disjoint triangles. If every triangle of G shares (at most) one edge with other triangles, it is enough to take one edge for each triangle in the family in order to obtain a triangle-transversal of G. Therefore, in this case, we have $\tau'_{\Delta}(G) = \nu'_{\Delta}(G)$. Moreover, if every triangle of G shares at most two edges with other triangles[4], the reasoning above gives us $\tau'_{\Delta}(G) \leq 2\nu'_{\Delta}(G)$. Quite surprisingly, this trivial observation guarantees an essentially tight bound for the class in question: for each $\varepsilon > 0$, there exists a graph G such that $\tau'_{\Delta}(G) > (2-\varepsilon)\nu'_{\Delta}(G)$ and each of its triangles shares at most two edges with other triangles (see the construction in Lemma 14). Nevertheless, if in addition each edge belongs to at most four triangles, the constant 2 can be dropped to 3/2, as we show in Section 3.1 (see Figure 4 for a picture of the graph G_{13}):

Theorem 3 *If G is a graph such that each triangle shares at most two of its edges with other triangles and each edge belongs to at most four triangles (or, equivalently, G is a K_4-free graph such that $T(G)$ is (K_5, claw)-free), then $\tau'_{\Delta}(G) \leq \frac{3}{2}\nu'_{\Delta}(G)$. Equality holds if and only if each component of $T(G)$ is either C_5 or $L(G_{13})$.*

For the classes of graphs in Theorems 1 to 3, we have that the constant c in Problem 1 equals 3/2. In general, for the class of all graphs, Tuza's Conjecture claims that $c = 2$ and we certainly have $c \geq 2$ for every class containing K_4. Moreover, we show that even for $\mathcal{G} = Free(W_3, \ldots, W_j)$, where $j \geq 3$ is fixed, we cannot expect $c < 2$ (Lemma 14). On the other hand, forbidding all odd-wheels could lead to a $c < 2$. Lakshmanan et al. [27] showed that Tuza's Conjecture holds for the class of odd-wheel-free graphs and they noticed that the odd-wheel-free graph $\overline{C_7}$ implies $c \geq 4/3$. This result was generalized by Puleo [34], who showed that if G is a graph with no subgraph isomorphic to any odd-wheel W_n, for $n \geq 5$, then $\tau'_{\Delta}(G) \leq 2\nu'_{\Delta}(G)$.

In this paper, motivated by the previous discussion, we are also interested in graphs without the odd-wheels W_3 and W_5 (note that $W_3 = K_4$). As a partial result towards a proof of Tuza's Conjecture for this class, we obtain the following (see Figure 1 for a picture of the co-banner):

Theorem 4 *If G is a (W_3, W_5)-free graph such that $T(G)$ is co-banner-free (or, equivalently, G is a K_4-free graph such that $T(G)$ is (C_5, co-banner)-free), then $\tau'_{\Delta}(G) \leq \frac{10}{7}\nu'_{\Delta}(G)$.*

Consider now the class of K_4-free graphs with odd-hole-free triangle graphs. It is easy to see that this is a subclass of odd-wheel-free graphs. Using the Strong Perfect Graph Theorem [5], Lakshmanan et al. [27] showed that if G is a K_4-free graph such that $T(G)$ is C_{2k+1}-free for all $k \geq 2$, then $\tau'_{\Delta}(G) = \nu'_{\Delta}(G)$. This follows from the fact that if G is K_4-free, then $T(G)$ is diamond-free (see Lemma 7) and so $\overline{C_{2k+1}}$-free, for any $k \geq 3$. It is then enough to recall (1).

[4] Note that a graph G with this property is necessarily K_4-free.

The previous result implies in particular that the equality $\tau'_\Delta = \nu'_\Delta$ holds for the K_4-free graphs with a P_4-free triangle graph. In Section 3.3, we show the following tight bound for the "next case":

Theorem 5 *If G is a K_4-free graph such that $T(G)$ is P_5-free, then $\tau'_\Delta(G) \leq \frac{3}{2}\nu'_\Delta(G)$.*

As already mentioned, our results on Tuza's Conjecture are obtained by providing θ-bounding functions for classes related to triangle graphs. To this end, Section 2 is devoted to triangle graphs: we summarize their basic properties and provide a (partial) list of forbidden induced subgraphs ($\{K_{1,4}, K_{2,3}, \text{diamond}, \text{twin-}C_5\}$) that comes in handy for the proofs of our results in Section 3. In fact, in Section 2, we consider more generally the class of k-line graphs, a common generalization of line graphs and triangle graphs.

We conclude this section with a table summarizing the results we prove in Section 3. The bounds followed by an asterisk are tight.

G	$T(G)$	Upper bound for $\tau'_\Delta(G)/\nu'_\Delta(G)$	Reference
K_4-free	subcubic	3/2 *	Theorem 2
K_4-free	K_4-free, maximum degree 4	193/98	Theorem 9
K_4-free	(K_5, claw)-free	3/2 *	Theorem 3
K_4-free	$(C_5, \text{co-banner})$-free	10/7	Theorem 4
K_4-free	P_5-free	3/2 *	Theorem 5

It is easy to see that the classes above are mutually incomparable with respect to set inclusion.

1.1 Notation and definitions

We assume the reader is familiar with notions of graph theory; for those not defined here, we refer to [41]. Note that we consider only finite undirected simple graphs.

Given a graph G, we denote its order $|V(G)|$ by $n(G)$ and its size $|E(G)|$ by $m(G)$. A *k-vertex* is a vertex of degree k; in particular, we refer to a 3-vertex as a *cubic vertex*. The maximum degree of a vertex of G is denoted by $\Delta(G)$ and G is *subcubic* if $\Delta(G) \leq 3$, and *cubic* if each vertex is a cubic vertex. For a vertex $v \in V(G)$, the *neighbourhood* $N(v)$ is the set of vertices adjacent to v in G and the *closed neighbourhood* $N[v]$ is the set $N(v) \cup \{v\}$.

The subgraph of G induced by a set of vertices S is denoted by $G[S]$. If a graph does not contain induced subgraphs isomorphic to graphs in a set Z, then it is *Z-free* and the set of all Z-free graphs is denoted by *Free*(Z). A class of graphs is *hereditary* if it is closed under deletions of vertices. It is well-known and easy to see that a class of graphs X is hereditary if and only if it can be defined by a set of forbidden induced subgraphs, i.e. $X = \mathit{Free}(Z)$ for some set of graphs Z.

The complete graph on n vertices is denoted by K_n. The complete bipartite graph with partition classes of size n and m is denoted by $K_{n,m}$. The path of order n is denoted by P_n and the cycle of order n is denoted by C_n. A *triangle* is the graph K_3, a *claw* is the graph $K_{1,3}$ and a *diamond* is the graph obtained from K_4 by deleting an edge. For $n \geq 3$, an *n-wheel* W_n is the graph $C_n \vee K_1$

obtained from C_n by adding a vertex adjacent to all the vertices of the cycle. An *odd-wheel* is a graph W_n with n odd. Figure 1 depicts two other graphs appearing in the paper: the banner and co-banner graphs. Note that the prefix "co-" denotes the complement of a certain graph: for example, the co-banner is the complement of the banner. A *hole* in a graph G is an induced subgraph isomorphic to C_n, for $n \geq 4$. An *antihole* in G is an induced subgraph isomorphic to the complement of C_n, for $n \geq 4$. An *odd-hole* is a hole isomorphic to C_n, with n odd. Similarly, an *odd-antihole* is an antihole isomorphic to $\overline{C_n}$, with n odd.

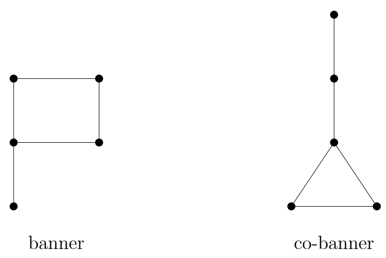

Fig. 1: The banner and co-banner graphs.

An *independent set* of a graph is a set of pairwise non-adjacent vertices. The maximum size of an independent set of G is the *independence number* $\alpha(G)$. A *clique* of a graph is a set of pairwise adjacent vertices. The *clique number* $\omega(G)$ is the maximum size of a clique of G. A *colouring* of a graph G is a partition of $V(G)$ into independent sets and the minimum number of partition classes is the *chromatic number* $\chi(G)$. A *clique cover* of a graph is a set of cliques such that each vertex of the graph belongs to at least one of them. The minimum size of a clique cover of G is denoted by $\theta(G)$. A *matching* of a graph is a set of pairwise non-incident edges and the *matching number* $\alpha'(G)$ is the maximum size of a matching of G. A *vertex cover* of a graph is a subset of vertices containing at least one endpoint of every edge. The minimum size of a vertex cover of G is denoted by $\beta(G)$. A *vertex triangle-transversal* of G is a subset $T \subseteq V(G)$ such that $G - T$ is triangle-free. We denote by $\tau_{\Delta}(G)$ the minimum size of a vertex triangle-transversal of G. Similarly, an *edge triangle-transversal* of G is a subset $T \subseteq E(G)$ such that $G - T$ is triangle-free and we denote by $\tau'_{\Delta}(G)$ the minimum size of an edge triangle-transversal of G. Whenever the context is clear, we simply refer to triangle-transversals. We denote by $\nu_{\Delta}(G)$ the maximum number of vertex-disjoint triangles of G and by $\nu'_{\Delta}(G)$ the maximum number of edge-disjoint triangles of G.

Let A and B be disjoint subsets of $V(G)$ and let $b \in V(G) \setminus A$. The vertex b is *complete* to A if b is adjacent to every vertex of A and b is *anticomplete* to A if b is non-adjacent to every vertex of A. If every vertex of A is complete to B, then A is *complete* to B. Similarly, if every vertex of A is anticomplete to B, then A is *anticomplete* to B. A *module* in G is a subset $M \subseteq V(G)$ such that every vertex in $V(G) \setminus M$ is either complete or anticomplete to M. A *homogeneous set* in G is a module in G properly contained in $V(G)$ and containing at least two vertices.

2 k-Line graphs

In this section we state some basic properties of triangle graphs. Our motivation comes from the fact that Tuza's Conjecture restricted to K_4-free graphs translates into the following equivalent assertion: $\theta(T(G)) \leq 2\alpha(T(G))$, for any K_4-free graph G. It is therefore useful to study the structure of triangle graphs and in this context, we provide a (partial) list of forbidden induced subgraphs which will be extensively used in Section 3. In fact, it is natural to consider k-line graphs, a common generalization of line graphs and triangle graphs.

Recall that the *line graph* $L(G)$ of a graph G is the graph having as vertices the edges of G, two vertices being adjacent if the corresponding edges intersect. What is arguably a cornerstone in the theory of graph classes is Beineke's characterization in terms of forbidden induced subgraphs:

Theorem 6 (Beineke [3]) *A graph is a line graph if and only if it does not contain any of the graphs depicted in Figure 2 as an induced subgraph.*

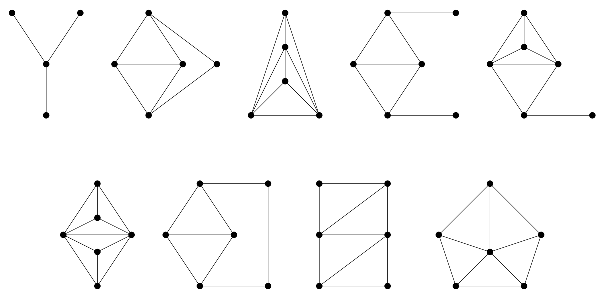

Fig. 2: The minimal forbidden induced subgraphs for the class of line graphs.

Theorem 6 immediately implies that line graphs can be recognized in polynomial time[5].

One possible generalization of line graphs is given by the following construction: For an integer $k \geq 2$, the *k-line graph* $L_k(G)$ of a graph G is the graph having as vertices the cliques of G of size k, two vertices being adjacent if the corresponding cliques intersect in a clique of size $k-1$. This notion has been introduced independently and with different motivations by several authors [9; 7; 8]. Clearly, 2-line graphs are the usual line graphs, whereas 3-line graphs are exactly triangle graphs. Unlike line graphs, the class of k-line graphs with $k \geq 3$ is not hereditary, as will become evident in the next paragraph. Nevertheless, it is still of interest to find forbidden induced subgraphs for this class. In particular, it follows directly from the definition that every k-line graph is $K_{1,k+1}$-free and in this section we expand this list.

Two interesting classes of spanning subgraphs of k-line graphs are defined as follows. The *k-Gallai graph* $\Gamma_k(G)$ of a graph G is the graph having as vertices the cliques of G of size k, two vertices being adjacent if the corresponding cliques

[5] The trivial algorithm was improved by Roussopoulos [36] and Lehot [32], who showed that recognition is possible in linear time.

intersect in a clique of size $k-1$ but their union is not a clique of size $k+1$. Conversely, the *anti-k-Gallai graph* $\Delta_k(G)$ of G is the graph having as vertices the cliques of G of size k, two vertices being adjacent if the union of the corresponding cliques is a clique of size $k+1$. Clearly, $\Delta_k(G)$ is the complement of $\Gamma_k(G)$ in $L_k(G)$. 2-Gallai graphs are simply known as *Gallai graphs* and were introduced by Gallai [14] in his work on comparability graphs. Anti-2-Gallai graphs are also known as *anti-Gallai graphs* or *triangular line graphs* [2]. The classes of k-Gallai graphs and anti-k-Gallai graphs are not hereditary: for every graph G, the k-Gallai graph $\Gamma_k(\overline{G} \vee K_{k-1})$ has a component isomorphic to G and the anti-k-Gallai graph $\Delta_k(G \vee K_{k-1})$ contains G as an induced subgraph [30]. Anand et al. [2] showed that recognizing anti-Gallai graphs is NP-complete. In fact, they showed that even deciding whether a connected graph is the anti-Gallai graph of some K_4-free graph is NP-complete. This was recently used by Lakshmanan et al. [28] in order to show that, for every fixed $k \geq 3$, deciding whether a given graph is the k-line graph of a K_{k+1}-free graph is NP-complete. Quite surprisingly, this is in sharp contrast with the case of line graphs mentioned above. Moreover, they completed the picture about generalized anti-Gallai graphs by showing that, for every fixed $k \geq 3$, recognizing anti-k-Gallai graphs is NP-complete. On the other hand, the recognition of k-Gallai graphs remains a major open problem.

Let us now translate Tuza's Conjecture in terms of the triangle graph. By the very definition of triangle graph, we have the following:

Lemma 1 *For any graph G, we have $\nu'_{\Delta}(G) = \alpha(T(G))$.*

This is the "higher dimensional" analogue of the bijection between the matchings of a graph and the independent sets of its line graph. But what about $\tau'_{\Delta}(G)$? Continuing with the analogy, the case $k = 2$ gives the following well-known fact:

Lemma 2 **(Folklore)** *If G is a triangle-free graph, then $\beta(G) = \theta(L(G))$.*

The proof of Lemma 2 follows by noticing that there are two types of cliques in a line graph. Moreover, if G is the line graph of a triangle-free graph H, then a clique of G corresponds to the edges incident to a fixed vertex of H. The same situation occurs for $k \geq 3$:

Lemma 3 **(Lakshmanan et al. [28])** *Every n-clique of a k-line graph $L_k(G)$ either corresponds to n k-cliques of G sharing a fixed $(k-1)$-clique or to n k-cliques contained in a common $(k+1)$-clique.*

Lemma 3 tells us that if G is a K_4-free graph, then a clique of $T(G)$ corresponds to a set of triangles of G sharing a common edge and so a clique cover of $T(G)$ corresponds to an edge triangle-transversal of G:

Lemma 4 **(Lakshmanan et al. [27])** *If G is a K_4-free graph, then $\tau'_{\Delta}(G) = \theta(T(G))$.*

Lemmas 1 and 4 will be repeatedly used in Section 3 in order to reduce Tuza's Conjecture to a more manageable statement about bounded clique covers. In this context, it is useful to see which graphs cannot appear as induced subgraphs in a triangle graph. The same can be asked, more generally, for k-line graphs. As mentioned before, the definition immediately implies the following:

Lemma 5 *Every k-line graph is $K_{1,k+1}$-free.*

In addition, Le and Prisner [31] showed that every k-line graph is $K_{2,3}$-free:

Lemma 6 (Le and Prisner [31]) *If c_1 and c_2 are two non-adjacent vertices of a k-line graph G, then $G[N(c_1) \cap N(c_2)]$ is an induced subgraph of C_4. In particular, every k-line graph is $K_{2,3}$-free.*

Clearly, any two triangles of a graph share at most one edge and so Lemma 3 implies the following:

Lemma 7 *If G is a K_4-free graph, then two maximal cliques of $T(G)$ cannot have more than one vertex in common. In particular, $T(G)$ is diamond-free.*

Lemma 7 shows once again that the class of triangle graphs is not hereditary: a diamond is not a triangle graph, whereas it is an induced subgraph of $T(K_2 \vee \overline{P_3})$.

We now show that the graph depicted in Figure 3 does not appear as an induced subgraph:

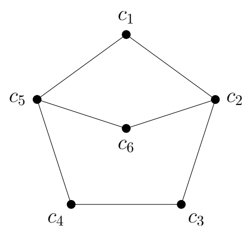

Fig. 3: Twin-C_5.

Lemma 8 *Every k-line graph is twin-C_5-free.*

Proof Suppose $G = L_k(H)$ is a k-line graph containing an induced twin-C_5 with vertex set $\{c_1, \ldots, c_6\}$, as depicted in Figure 3. For $1 \le i \le 6$, let Q_i be the k-clique of H corresponding to c_i. Since c_1 and c_6 are both adjacent to c_5 and c_2 but $c_1c_6 \notin E(G)$ and $c_2c_5 \notin E(G)$, it is easy to see that $Q_1 = \{u_1, u_2, a_1, \ldots, a_{k-2}\}$, $Q_6 = \{v_1, v_2, a_1, \ldots, a_{k-2}\}$, $Q_5 = \{u_1, v_1, a_1, \ldots, a_{k-2}\}$ and $Q_2 = \{u_2, v_2, a_1, \ldots, a_{k-2}\}$, where $\{u_1, u_2\} \cap \{v_1, v_2\} = \varnothing$.

Suppose now the clique Q_4 does not contain some a_i, say a_{k-2}. Since $|Q_4 \cap Q_5| = k-1$ and $|Q_4 \cap Q_2| = k-2$, we have either $Q_4 = \{u_1, v_1, u_2, a_1, \ldots, a_{k-3}\}$ or $Q_4 = \{u_1, v_1, v_2, a_1, \ldots, a_{k-3}\}$. In either case we obtain a contradiction to the fact that $c_4c_1 \notin E(G)$ and $c_4c_6 \notin E(G)$. Therefore, we have $\{a_1, \ldots, a_{k-2}\} \subseteq Q_4$. On the other hand, since $c_4c_1 \notin E(G)$ and $c_4c_6 \notin E(G)$, then $Q_4 \cap \{u_1, u_2, v_1, v_2\} = \varnothing$, a contradiction to the fact that $c_4c_5 \in E(G)$. $\square$

As noticed in the proof of Lemma 8, an induced C_4 in a triangle graph $T(G)$ corresponds to a W_4 in G. It is easy to see that a similar situation occurs for C_5 (see also [29]):

Lemma 9 *An induced C_5 in the triangle graph of a K_4-free graph G corresponds to an induced W_5 in G.*

Note that if G is not K_4-free, then an induced C_5 in $T(G)$ may also correspond to a K_5 in G. For general C_n, the situation becomes even more complicated. Nevertheless, Lakshmanan et al. [29] provided a forbidden subgraph characterization of graphs with C_n-free triangle graphs, for any specified $n \geq 3$.

Let us now consider anti-Gallai graphs. Recall that the anti-Gallai graph $\Delta(G)$ of G is the graph having as vertices the edges of G, two vertices being adjacent if the corresponding edges are incident and span a triangle in G. It directly follows from the definition that every edge of an anti-Gallai graph belongs to at least one triangle. Moreover, if G is K_4-free, it is easy to see that every edge of $\Delta(G)$ belongs to at most one triangle. Recall that a graph G is *locally linear* if each edge of G belongs to exactly one triangle or, equivalently, if $G[N(v)]$ is 1-regular for every $v \in V(G)$ (see [12]). Therefore, the following holds:

Lemma 10 *If G is a K_4-free graph, then $\Delta(G)$ is locally linear.*

If G is K_4-free, we can also interpret edge triangle-transversals and edge-disjoint triangles of G in terms of the anti-Gallai graph of G. Indeed, by Lemma 10, the map f which sends a triangle of G with edge set $\{e_1, e_2, e_3\}$ to the triangle of $\Delta(G)$ with vertex set $\{e_1, e_2, e_3\}$ is a bijection between the triangles of G and those of $\Delta(G)$. This implies that $\nu'_\Delta(G) = \nu_\Delta(\Delta(G))$ and $\tau'_\Delta(G) = \tau_\Delta(\Delta(G))$.

Recall now that the *clique graph* $K(G)$ of G is the graph having as vertices the maximal cliques of G, two vertices being adjacent if the corresponding cliques share at least one vertex. If G is K_4-free, Lemma 10 implies that $\Delta(G)$ is K_4-free as well and, if every edge of G belongs to a triangle, the triangles of $\Delta(G)$ are exactly its maximal cliques. Therefore, the map f introduced above gives a bijection between the vertices of $T(G)$ (i.e. the triangles of G) and the vertices of $K(\Delta(G))$ which clearly preserves adjacency. This implies that the following holds:

Lemma 11 *If G is a K_4-free graph such that each edge belongs to a triangle, then $T(G) \cong K(\Delta(G))$.*

In fact, Lakshmanan et al. [28] showed that the converse holds as well: a connected graph F is the anti-Gallai graph of a K_4-free graph G such that every edge of G belongs to a triangle if and only if $K(F) \cong T(G)$. This was used in order to reduce the recognition of anti-Gallai graphs of K_4-free graphs to that of triangle graphs. As already mentioned, the former is NP-complete [2] and so they showed that recognizing triangle graphs is NP-complete as well.

We conclude this section with the following observation, which will be used in Section 3.1:

Lemma 12 **(Lakshmanan et al. [28])** *If G is the k-line graph of a K_{k+1}-free graph, then it is also the k'-line graph of a $K_{k'+1}$-free graph, for any $k' > k$.*

3 Tuza's Conjecture and θ-bounding functions

In this section we prove the main results of the paper. As mentioned in Section 1, our statements are obtained by providing θ-bounding functions for specific classes of graphs and then by relying on Lemmas 1 and 4. It is therefore useful to recall the following notions.

A class of graphs $\mathcal{G}$ is *θ-bounded* if there exists a function $f\colon \mathbb{N} \to \mathbb{R}$ such that for all $G \in \mathcal{G}$ and all induced subgraphs H of G, we have $\theta(H) \leq f(\alpha(H))$. Such a function f is a *θ-bounding function* for $\mathcal{G}$. The notion of θ-boundedness and its complementary χ-boundedness[6] were introduced by Gyárfás [15] in order to provide a natural extension of the class of perfect graphs. In [15], Gyárfás formulated the following meta-question: given a class $\mathcal{G}$, what is the smallest θ-bounding function for $\mathcal{G}$, if any?

It is easy to see that $\theta(G) \leq k\alpha(G)$, for any graph G with maximum degree at most k. Indeed, given a maximal independent set I of G, we have that the edges incident to I constitute a clique cover of G and their number is at most $k\alpha(G)$. On the other hand, for $k = 3$, we showed in [33] that this bound is far from optimal:

Theorem 7 (Munaro [33]) *If G is a subcubic graph, then $\theta(G) \leq \frac{3}{2}\alpha(G)$. Moreover, $f(x) = \lfloor \frac{3}{2}x \rfloor$ is the smallest θ-bounding function for the class of subcubic graphs.*

Theorem 7 extends the following result by Henning et al. [23], who actually characterized the cases of equality for triangle-free graphs:

Theorem 8 (Henning et al. [23]) *If G is a subcubic triangle-free graph, then $\theta(G) \leq \frac{3}{2}\alpha(G)$. Moreover, equality holds if and only if every component of G is in $\{C_5, G_{11}\}$ (see Figure 4).*

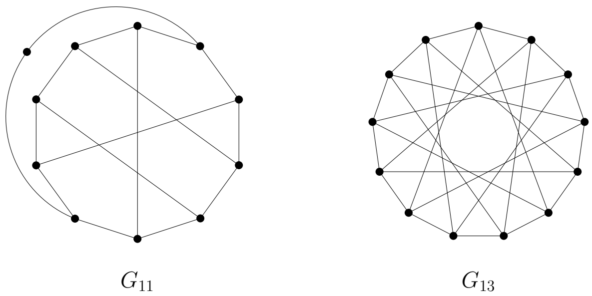

Fig. 4: The graphs G_{11} and G_{13}.

Let us now come back to the packing and transversal setting and see once again how the results above are meaningful for Problem 1 in the case of K_4-free graphs having edges in few triangles. The equality $\tau'_{\Delta}(G) = \nu'_{\Delta}(G)$ trivially holds if every edge of G is in at most one triangle and Haxell et al. [18] showed that $\tau'_{\Delta}(G) \leq \frac{3}{2}\nu'_{\Delta}(G)$, for any K_4-free graph such that each edge is in at most two triangles. This was generalized by Theorem 2, which we now restate:

Theorem 2 *If G is a K_4-free graph such that $T(G)$ is subcubic, then $\tau'_{\Delta}(G) \leq \frac{3}{2}\nu'_{\Delta}(G)$.*

[6] By substituting θ with χ and α with ω, we obtain the notion of χ-boundedness and the two are complementary, in the sense that G is χ-bounded if and only if $\overline{G}$ is θ-bounded.

Proof By Lemmas 1 and 4 and Theorem 7, we have $\tau'_\Delta(G) = \theta(T(G)) \leq \frac{3}{2}\alpha(T(G)) = \frac{3}{2}\nu'_\Delta(G)$. □

An edge-disjoint union of 5-wheels (plus possibly edges not in triangles) shows that the constant 3/2 is optimal. Note that we do not have a characterization of the subcubic graphs G such that $\theta(G) = \frac{3}{2}\alpha(G)$ and we conjectured in [33] that every component of such an extremal G is either C_5 or G_{11} (see Figure 4). On the other hand, it is easy to see that the graph G_{11} contains an induced copy of a twin-C_5 and so, by Lemma 8, it cannot appear as an induced subgraph of a triangle graph. Therefore, it is natural to conjecture the following:

Conjecture 3 *If G is a K_4-free graph such that $T(G)$ is subcubic, then $\tau'_\Delta(G) = \frac{3}{2}\nu'_\Delta(G)$ if and only if G is the edge-disjoint union of 5-wheels plus possibly edges which are not in any triangle.*

What about if each edge belongs to at most three triangles? In this case, each triangle shares its edges with at most six other triangles. If we allow each triangle to share its edges only with at most four other triangles, we can state the following:

Theorem 9 *If G is a K_4-free graph such that each edge belongs to at most three triangles and each triangle shares its edges with at most four other triangles (or, equivalently, $T(G)$ is K_4-free and has maximum degree 4), then $\tau'_\Delta(G) \leq \frac{193}{98}\nu'_\Delta(G)$.*

The proof immediately follows from this result:

Theorem 10 (Munaro [33]) *If G is a K_4-free graph with maximum degree at most 4, then $\theta(G) \leq \frac{193}{98}\alpha(G)$.*

Proof of Theorem 9 The equivalence of the two classes in the statement is immediate. By Lemmas 1 and 4 and Theorem 10, we have $\tau'_\Delta(G) = \theta(T(G)) \leq \frac{193}{98}\alpha(T(G)) = \frac{193}{98}\nu'_\Delta(G)$. □

We believe the constant in Theorem 9 is far from optimal, but it seems difficult even to formulate a conjecture on the optimal value. In fact, we do not have any example for which the ratio is greater than 3/2. In [33] we conjectured that if G is a graph with maximum degree 4, then $\theta(G) \leq \frac{7}{4}\alpha(G)$, with equality if and only if every component of G is G_{13} (see Figure 4). On the other hand, since G_{13} contains an induced $K_{1,4}$, it cannot be a triangle graph.

The rest of this section is devoted to the proofs of Theorems 3 to 5. We follow the reasoning illustrated by the two previous results and provide along the way optimal θ-bounding functions for the classes *Free*(K_5, claw, diamond) and *Free*(P_5, diamond, $K_{2,3}$) and a χ-bounding function for the class (banner, odd-hole, $\overline{K_{1,4}}$). These results might be of independent interest.

3.1 Proof of Theorem 3

Let us begin by restating Theorem 3:

Theorem 3 *If G is a graph such that each triangle shares at most two of its edges with other triangles and each edge belongs to at most four triangles (or, equivalently, G is a K_4-free graph such that $T(G)$ is (K_5, claw)-free), then $\tau'_\Delta(G) \leq \frac{3}{2}\nu'_\Delta(G)$. Equality holds if and only if each component of $T(G)$ is either C_5 or $L(G_{13})$.*

Note that, by Lemma 12, $L(G_{13})$ is indeed a triangle graph of a K_4-free graph: for example, we have $L(G_{13}) = T(K_1 \vee G_{13})$.

The crucial observation for the proof of Theorem 3 is that the triangle graph is in fact a line graph. Indeed, by Lemma 7, the triangle graph of a K_4-free graph is diamond-free and the following lemma tells us that $(K_5, \text{claw}, \text{diamond})$-free graphs are exactly line graphs of triangle-free graphs with maximum degree 4.

Lemma 13 **(Folklore)** *A graph G is the line graph of a triangle-free graph with maximum degree at most 4 if and only if it is (K_5, claw, diamond)-free.*

Proof If G is the line graph of a triangle-free graph with maximum degree at most 4, then it is clearly $(K_5, \text{claw}, \text{diamond})$-free.

Suppose now G is $(K_5, \text{claw}, \text{diamond})$-free. By Theorem 6, we have that $G = L(H)$, for some graph H. Consider the graph H' obtained from H by replacing each component isomorphic to K_3 with a claw. Clearly, $G = L(H) = L(H')$. Since G is K_5-free, H' has maximum degree at most 4. Moreover, H' is triangle-free. Indeed, if H' contains a triangle T, then there exists a vertex $v \notin V(T)$ adjacent to a vertex of T and so there exists an induced diamond in $L(H')$, a contradiction. □

By the discussions above, we know it would be enough to show that $\theta(G) \leq \frac{3}{2}\alpha(G)$, for any $(K_5, \text{claw}, \text{diamond})$-free graph G. But this is an easy corollary of the following result by Joos [25]:

Theorem 11 **(Joos [25])** *If G is a triangle-free graph with $\Delta(G) \leq 4$, then $\alpha(G) + \frac{3}{2}\alpha'(G) \geq |V(G)|$. Equality holds if and only if every component C of G is either in $\{K_1, C_5\}$ or has order 13, $\alpha(C) = 4$ and $\alpha'(C) = 6$.*

Corollary 1 *If G is a (K_5, claw, diamond)-free graph, then $\theta(G) \leq \frac{3}{2}\alpha(G)$. Equality holds if and only if every component of G is either C_5 or $L(G_{13})$.*

Proof By Lemma 13, G is the line graph of a triangle-free graph H with maximum degree at most 4. Clearly, $\alpha(G) = \alpha'(H)$ and, by Lemma 2, we have that $\theta(G) = \beta(H)$. Therefore, by Theorem 11 and since the complement of a vertex cover is an independent set, we have

$$\theta(G) = \beta(H) = |V(H)| - \alpha(H) \leq \frac{3}{2}\alpha'(H) = \frac{3}{2}\alpha(G).$$

Let us now consider the cases of equality. As remarked in [35], the $(3,5)$-Ramsey graph G_{13} is the only graph on 13 vertices with parameters $\omega = 2$ and $\alpha = 4$. If equality holds then, by Theorem 11, each component of H is in $\{K_1, C_5, G_{13}\}$. This implies that each component of $G = L(H)$ is either C_5 or $L(G_{13})$. The converse clearly holds. □

We can finally proceed to the proof of Theorem 3:

Proof of Theorem 3 The equivalence of the two classes in the statement is immediate. Moreover, since $T(G)$ is diamond-free, Lemmas 1 and 4 and Corollary 1 imply that $\tau'_\Delta(G) = \theta(T(G)) \leq \frac{3}{2}\alpha(T(G)) = \frac{3}{2}\nu'_\Delta(G)$. The characterization of equality follows again by Corollary 1. □

3.2 Proof of Theorem 4

As we have already mentioned in Section 1, Haxell et al. [20] showed that the constant 2 in Tuza's Conjecture cannot essentially be improved for K_4-free graphs. Clearly, a K_4 can be viewed as the 3-wheel and we now show that the situation does not change even if we further forbid a fixed number of wheels. We follow the same reasoning as Haxell et al. [20].

Lemma 14 *For each $\varepsilon > 0$ and each fixed $j \geq 3$, there exists a $(W_3, \dots, W_j)$-free graph G such that $\tau'_\Delta(G) > (2 - \varepsilon)\nu'_\Delta(G)$.*

Proof Erdős [10] showed that, for any fixed k and sufficiently large n, there exists a graph on n vertices with girth greater than k and independence number smaller than $n^{\frac{2k}{2k+1}}$. For $k = j$, let G_n be such a graph. We construct G by adding a new vertex v_0 adjacent to all the vertices of G_n. Clearly, since G_n has girth greater than j, the graph G is $(W_3, \dots, W_j)$-free. Moreover, every triangle of G contains v_0 and so $\nu'_\Delta(G) = \alpha'(G_n) \leq \frac{n}{2}$. We now claim there exists a minimum-size triangle-transversal of G containing only edges incident to v_0. Indeed, if T is a triangle-transversal containing $uv \in E(G_n)$, we have that $(T \setminus \{uv\}) \cup \{v_0v\}$ is a triangle-transversal of G having size at most $|T|$. On the other hand, it is easy to see that a subset $F \subseteq V(G_n)$ is a vertex cover of G_n if and only if the set $\{v_0v : v \in F\}$ is a triangle-transversal of G. Therefore, we have $\tau'_\Delta(G) = n - \alpha(G_n) > n - n^{\frac{2j}{2j+1}}$. But then, for each $\varepsilon > 0$, we can find an n such that $n - n^{\frac{2j}{2j+1}} > (2 - \varepsilon)\frac{n}{2}$. □

On the other hand, we show in this section that the following holds:

Theorem 4 *If G is a (W_3, W_5)-free graph such that $T(G)$ is co-banner-free (or, equivalently, G is a K_4-free graph such that $T(G)$ is $(C_5$, co-banner$)$-free), then $\tau'_\Delta(G) \leq \frac{10}{7}\nu'_\Delta(G)$.*

We have seen that an induced C_5 in $T(G)$ corresponds to an induced W_5 in G (Lemma 9). Moreover, $T(G)$ is diamond-free and so it is in fact odd-antihole-free. For our purposes it will be convenient to work with the complement graph and our proof of Theorem 4 relies on a characterization of (banner, odd-hole)-free graphs given by Hoàng [24]:

Theorem 12 **(Hoàng [24])** *If G is a (banner, odd-hole)-free graph, then either G is perfect, or $\alpha(G) \leq 2$, or every odd-antihole of G belongs to a homogeneous set M in G such that $G[M]$ is co-triangle-free.*

Another ingredient in the proof of Theorem 4 is the following result:

Theorem 13 *If G is a subcubic (K_3, C_5)-free graph, then $\theta(G) \leq \frac{10}{7}\alpha(G)$.*

We have seen in Theorem 8 that there are exactly two connected subcubic triangle-free graphs for which $\theta \leq \frac{3}{2}\alpha$ holds with equality: C_5 and G_{11}. Both of them clearly contain an induced C_5 and Theorem 13 tells us that by further forbidding C_5 the constant 3/2 in Theorem 8 can be dropped to 10/7.

Our proof of Theorem 13 is inspired by similar results in [17; 33]. The main idea is based on the notion of θ-criticality, a graph G being *θ-critical* if $\theta(G-v) < \theta(G)$, for every $v \in V(G)$. First, we show that a minimum counterexample is connected and θ-critical. We then rely on the following result by Gallai (see [39] for a short proof and an extension):

Theorem 14 **(Gallai [13])** *If v is any vertex of a connected θ-critical graph G, then G has a minimum-size clique cover in which v is the only isolated vertex. In particular, $\theta(G) \leq \frac{n(G)+1}{2}$.*

The final contradiction is then reached by using an appropriate lower bound for the independence number of a subcubic graph.

For a subcubic triangle-free graph G, Staton [38] showed that $\alpha(G) \geq \frac{5}{14}n(G)$ and Heckman [21] showed that there are exactly two connected graphs attaining equality. They both have 14 vertices and contain an induced C_5 (one such graph is the so-called generalized Petersen graph $P(7,2)$). Fraughnaugh and Locke [11] showed that, for graphs with more than 14 vertices, a better lower bound is possible: $\alpha(G) \geq \frac{11}{30}n(G) - \frac{2}{15}$, for any connected subcubic triangle-free graph G. Moreover, they showed that if G is not cubic and does not belong to a certain family, the previous bound can be further improved. Let us now define this special family.

We denote by $\mathcal{F}_{11}$ the class of graphs obtained by the following construction. Given any tree T with maximum degree at most four, we first obtain a graph G by replacing each vertex of T of degree at least two with a copy of G_8 (see Figure 5) and each vertex of degree one with a copy of either G_8 or G_{11} (see Figure 4). We then add a matching between the vertices of degree two of G such that the edges in the matching with an endpoint in G_v (where $G_v \in \{G_8, G_{11}\}$ is the graph replacing v) bijectively correspond to the edges of T with endpoint v. Clearly, every graph $G \in \mathcal{F}_{11}$ thus obtained is subcubic and triangle-free and it is not difficult to see that $m(G) - 7n(G) + 15\alpha(G) = -1$.

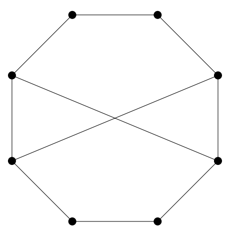

Fig. 5: The graph G_8.

Denoting by $\gamma(G)$ the quantity $m(G) - 7n(G) + 15\alpha(G)$, Fraughnaugh and Locke [11] showed the following:

Theorem 15 **(Fraughnaugh and Locke [11])** *Let G be a connected subcubic triangle-free graph. If G is cubic, then $\gamma(G) \geq -2$. If $G \in \mathcal{F}_{11}$, then $\gamma(G) = -1$. Finally, if G is not cubic and $G \notin \mathcal{F}_{11}$, then $\gamma(G) \geq 0$.*

The following corollary is immediate:

Corollary 2 *Let G be a connected subcubic triangle-free graph. If G is cubic, then $\alpha(G) \geq \frac{11n(G)-4}{30}$, while if G is not cubic and $G \notin \mathcal{F}_{11}$, then $\alpha(G) \geq \frac{11n(G)+1}{30}$.*

Note that if G is C_5-free, then $G \notin \mathcal{F}_{11}$, as both G_8 and G_{11} contain induced copies of C_5. We can finally proceed to the proof of Theorem 13.

Proof of Theorem 13 Suppose, by contradiction, that G is a counterexample with the minimum number of vertices. In the following, we deduce some structural properties of G and we show how they lead to a contradiction. Each claim is followed by a short proof.

Claim 1 *G is connected.*

Otherwise, G is the disjoint union of two non-empty graphs G_1 and G_2. By minimality, we have

$$\theta(G) = \theta(G_1) + \theta(G_2) \leq \frac{10}{7}(\alpha(G_1) + \alpha(G_2)) = \frac{10}{7}\alpha(G),$$

a contradiction. $\diamondsuit$

Claim 2 *G is θ-critical.*

Indeed, suppose there exists a vertex $v \in V(G)$ such that $\theta(G) = \theta(G - v)$. By minimality, we have

$$\theta(G) = \theta(G - v) \leq \frac{10}{7}\alpha(G - v) \leq \frac{10}{7}\alpha(G),$$

a contradiction. $\diamondsuit$

Claim 3 *G is 2-connected.*

Since every connected subcubic bridgeless graph is 2-connected, it is enough to show that G has no cut-edges. Therefore, suppose $e = u_1u_2$ is a cut-edge and let G_1 be the component of $G - e$ containing u_1 and $G_2 = G - V(G_1)$ (therefore, $u_2 \in V(G_2)$). Since G is triangle-free, we have that $G - e$ is triangle-free and C_5-free as well. Moreover, $\theta(G) \leq \theta(G_1) + \theta(G_2)$. If there exists $i \in \{1, 2\}$ such that a maximum independent set of G_i avoids u_i, then $\alpha(G) \geq \alpha(G_1) + \alpha(G_2)$ and so, by minimality,

$$\theta(G) \leq \theta(G_1) + \theta(G_2) \leq \frac{10}{7}\alpha(G_1) + \frac{10}{7}\alpha(G_2) \leq \frac{10}{7}\alpha(G),$$

a contradiction. Therefore, for each $i \in \{1, 2\}$, every maximum independent set of G_i contains u_i. This means that $\alpha(G_i - u_i) = \alpha(G_i) - 1$, for each $i \in \{1, 2\}$. Moreover, denoting by I_i a maximum independent set of G_i, we have that $I_1 \cup$

$(I_2 \setminus \{u_2\})$ is an independent set of G and so $\alpha(G) \geq \alpha(G_1) + \alpha(G_2) - 1$. But then, by minimality,

$$\begin{aligned}
\theta(G) &\leq \theta(G_1 - u_1) + \theta(G_2 - u_2) + 1 \\
&\leq \frac{10}{7}\alpha(G_1 - u_1) + \frac{10}{7}\alpha(G_2 - u_2) + 1 \\
&= \frac{10}{7}(\alpha(G_1) - 1) + \frac{10}{7}(\alpha(G_2) - 1) + 1 \\
&= \frac{10}{7}(\alpha(G_1) + \alpha(G_2)) - \frac{13}{7} \\
&< \frac{10}{7}\alpha(G),
\end{aligned}$$

a contradiction. $\diamondsuit$

Claim 4 *G is cubic.*

Suppose it is not. Since G is C_5-free, we have that $G \notin \mathcal{F}_{11}$ and so, by Corollary 2, we have $\alpha(G) \geq \frac{11n(G)+1}{30}$. Combining this with Theorem 14 and if $\alpha(G) \geq 7$, we obtain

$$\theta(G) \leq \frac{n(G)+1}{2} \leq \frac{30\alpha(G)+10}{22} \leq \frac{10}{7}\alpha(G).$$

Let us now consider the remaining values of $\alpha(G)$ and use again Theorem 14:

- If $\alpha(G) = 6$, then $n(G) \leq 16$ and so $\theta(G) \leq 8 < 10\alpha(G)/7$.
- If $\alpha(G) = 5$, then $n(G) \leq 13$ and so $\theta(G) \leq 7 < 10\alpha(G)/7$.
- If $\alpha(G) = 4$, then $n(G) \leq 10$ and so $\theta(G) \leq 5 < 10\alpha(G)/7$.
- If $\alpha(G) = 3$, then $n(G) \leq 8$ and so $\theta(G) \leq 4 < 10\alpha(G)/7$.
- If $\alpha(G) = 2$, then $n(G) \leq 5$ and so $\theta(G) \leq 3$. Suppose now that $\theta(G) = 3$. This means that $\theta(G) = \frac{3}{2}\alpha(G)$ and Theorem 8 implies $G \cong C_5$, a contradiction. Therefore, we have $\theta(G) = 2$.
- Finally, if $\alpha(G) \leq 1$, then G is complete and again $\theta(G) = \alpha(G)$.

In all the cases we have $\theta(G) \leq \frac{10}{7}\alpha(G)$, a contradiction. $\diamondsuit$

We are now in a position to conclude our proof. Since G is cubic and 2-connected, Petersen's Theorem implies it has a perfect matching. Moreover, by Corollary 2, we have that $\alpha(G) \geq \frac{11n(G)-4}{30}$. Therefore, if $\alpha(G) \geq 3$, we obtain

$$\theta(G) = \frac{n(G)}{2} \leq \frac{30\alpha(G)+4}{22} < \frac{10}{7}\alpha(G).$$

If $\alpha(G) = 2$, then $n(G) \leq 5$ and since $G \ncong C_5$, we have $\theta(G) = 2$. Finally, if $\alpha(G) \leq 1$, then G is complete and $\theta(G) = \alpha(G)$. In all the cases we have $\theta(G) < \frac{10}{7}\alpha(G)$, a contradiction. This concludes the proof. $\square$

Finally, the following result gives us the desired χ-bounding function for the proof of Theorem 4:

Lemma 15 *If G is a (banner, odd-hole, $\overline{K_{1,4}}$)-free graph, then $\chi(G) \leq \frac{10}{7}\omega(G)$.*

Proof We proceed by induction on the number of vertices. If G is perfect, then $\chi(G) = \omega(G)$. If $\alpha(G) \leq 2$, then $\overline{G}$ is triangle-free and so, since it is $K_{1,4}$-free as well, it must be subcubic. Moreover, $\overline{G}$ is by assumption C_5-free and so Theorem 13 implies that

$$\chi(G) = \theta(\overline{G}) \leq \frac{10}{7}\alpha(\overline{G}) = \frac{10}{7}\omega(G).$$

Therefore, in view of Theorem 12, we may assume G contains an odd-antihole $\overline{C_{2k+1}}$, with $k \geq 3$, which belongs to a homogeneous set M. In particular, M contains a triangle. Now let $A \cup B$ be a partition of $V(G) \setminus M$ such that A is complete to M and B is anticomplete to M. If $A = \varnothing$, then G is the disjoint union of $G[M]$ and $G[B]$, and we immediately conclude by the induction hypothesis. Moreover, if $B = \varnothing$, then G is the join of $G[A]$ and $G[M]$ and by the induction hypothesis we have

$$\chi(G) = \chi(G[A]) + \chi(G[M]) \leq \frac{10}{7}\omega(G[A]) + \frac{10}{7}\omega(G[M]) = \frac{10}{7}\omega(G).$$

Therefore, we may assume both A and B are non-empty. Since M contains a triangle and G is $\overline{K_{1,4}}$-free, it is easy to see that A must be complete to B. But then G is the join of $G[A]$ and $G[M \cup B]$ and we conclude as above. □

The proof of Theorem 4 is now an immediate consequence of Lemma 15:

Proof of Theorem 4 Since G is K_4-free, we have that $\tau'_\Delta(G) = \chi(\overline{T(G)})$ and $\nu'_\Delta(G) = \omega(\overline{T(G)})$ (see (1)). By Lemma 9, $T(G)$ is C_5-free. Moreover, since $T(G)$ is diamond-free as well, we have that $\overline{T(G)}$ is odd-hole-free. Finally, since $T(G)$ is $K_{1,4}$-free, Lemma 15 implies that $\tau'_\Delta(G) \leq \frac{10}{7}\nu'_\Delta(G)$. □

We conclude this section with some remarks and open problems concerning Theorem 13 and Lemma 15. Note first that we do not have any example of a graph attaining equality in Theorem 13. In fact, considering C_7, we believe the optimal constant is $\frac{4}{3}$:

Conjecture 4 *If G is a subcubic (K_3, C_5)-free graph, then $\theta(G) \leq \frac{4}{3}\alpha(G)$.*

Recently, Scott and Seymour [37] proved a conjecture by Gyárfás [15] on χ-bounded classes: they showed that the class of odd-hole-free graphs is χ-bounded by the function $f(x) = 2^{2^{x+2}}$. It is likely that this exponential function can be improved, although they provided a series of examples showing that a linear bounding function is not possible. Using Theorem 12, Hoàng [24] showed that (banner, odd-hole)-free graphs are 2-divisible. Recall that a graph G is *k-divisible* if the vertex set of each induced subgraph H of G with at least one edge can be partitioned into k sets none of which contains a clique of size $\omega(H)$. An easy induction shows that $\chi(G) \leq k^{\omega(G)-1}$, for any k-divisible graph G. Therefore, the bound $\chi \leq 2^{2^{\omega+2}}$ can be improved to $\chi \leq 2^{\omega-1}$ for the subclass *Free*(banner, odd-hole). Lemma 15 then shows that by further forbidding $\overline{K_{1,4}}$, we can obtain a linear χ-bounding function.

On the other hand, we do not have any example of a graph attaining equality in Lemma 15. If true, the previous conjecture together with the reasoning in the proof of Lemma 15 would imply that the optimal constant is $4/3$. As a side remark, note that Chudnovsky et al. [6] showed that (odd-hole, K_4)-free graphs are 4-colourable, with $\overline{C_7}$ being 4-chromatic.

3.3 Proof of Theorem 5

Suppose G is a K_4-free graph such that $T(G)$ is P_5-free. Clearly, 5-wheels show that the ratio τ'_Δ/ν'_Δ is at least 3/2 and Theorem 5 tells us that this is an extremal case:

Theorem 5 *If G is a K_4-free graph such that $T(G)$ is P_5-free, then $\tau'_\Delta(G) \le \frac{3}{2}\nu'_\Delta(G)$.*

In order to prove Theorem 5, we use a structural characterization of (P_5, diamond)-free graphs by Brandstädt [4]. Before stating his result, we need to introduce the following classes of graphs which constitute the basic graphs in the characterization.

A graph is a:

- *thin spider* if it is partitionable into a clique C and an independent set I, with $|C| = |I|$ or $|C| = |I|+1$, such that the edges between C and I form a matching and at most one vertex in C is not covered by the matching;
- *matched co-bipartite graph* if it is partitionable into two cliques C_1 and C_2, with $|C_1| = |C_2|$ or $|C_1| = |C_2| + 1$, such that the edges between C_1 and C_2 form a matching and at most one vertex in C_1 and C_2 is not covered by the matching;
- *bipartite chain graph* if it is bipartite, with bipartition $X_1 \cup X_2$, and each X_i forms a chain, i.e. for $1 \le i \le 2$, we have that $\{N(x) : x \in X_i\}$ is linearly ordered with respect to set inclusion;
- *co-bipartite chain graph* if it is the complement of a bipartite chain graph;
- *enhanced co-bipartite chain graph* if it is partitionable into a co-bipartite chain graph with cliques C_1 and C_2 and three additional vertices a, b and c (a and c optional) such that $N(a) = C_1 \cup C_2$, $N(b) = C_1$ and $N(c) = C_2$;
- *enhanced bipartite chain graph* if it is the complement of an enhanced co-bipartite chain graph.

In the following, a graph G is *co-connected* if its complement $\overline{G}$ is connected.

Theorem 16 **(Brandstädt [4])** *If G is a connected and co-connected (P_5, diamond)-free graph, then either G contains a homogeneous set (inducing a P_3-free subgraph) or one of the following holds: G is a matched co-bipartite graph or a thin spider or an enhanced bipartite chain graph or it has at most 9 vertices.*

The strategy is clear: we reduce the statement in Theorem 5 into an equivalent one for the triangle graph and we use the fact that this graph is diamond-free and $K_{2,3}$-free (see Section 2).

Theorem 17 *If G is a (P_5, diamond, $K_{2,3}$)-free graph, then $\theta(G) \le \frac{3}{2}\alpha(G)$.*

Proof We proceed by induction on the number of vertices. If G is not connected, then it is the disjoint union of two non-empty graphs G_1 and G_2. By the induction hypothesis,

$$\theta(G) = \theta(G_1) + \theta(G_2) \le \frac{3}{2}\alpha(G_1) + \frac{3}{2}\alpha(G_2) = \frac{3}{2}\alpha(G).$$

Similarly, if $\overline{G}$ is not connected, then it is the disjoint union of two non-empty graphs G_1 and G_2 and, by the induction hypothesis,

$$\theta(G) = \chi(\overline{G}) = \max\{\chi(G_1), \chi(G_2)\} \leq \frac{3}{2}\max\{\omega(G_1), \omega(G_2)\} = \frac{3}{2}\omega(\overline{G}) = \frac{3}{2}\alpha(G).$$

Therefore, we may assume G to be connected and co-connected.

Suppose now G contains a homogeneous set M. By definition, $|M| \geq 2$ and $M \neq V(G)$. Moreover, $G[M]$ is P_3-free, i.e. it is a disjoint union of cliques. Let $A \cup B$ be a partition of $V(G) \setminus M$, where A is complete to M and B is anticomplete to M. Suppose first $|A| \leq 1$. This implies that M contains a simplicial vertex v, i.e. a vertex whose neighbourhood is a clique. If $N[v] = V(G)$, then $\theta(G) = \alpha(G)$. Otherwise, by the induction hypothesis,

$$\theta(G) \leq \theta(G - N[v]) + 1 \leq \frac{3}{2}\alpha(G - N[v]) + 1 \leq \frac{3}{2}(\alpha(G) - 1) + 1 < \frac{3}{2}\alpha(G).$$

Therefore, we may assume $|A| \geq 2$. Since G is diamond-free, we have that either A and M are both independent sets or they are both cliques. But if the latter holds, then M contains a simplicial vertex and we conclude as in the previous paragraph. Therefore, we may further assume that A and M are both independent sets and since G is $K_{2,3}$-free, we have $|A| = |M| = 2$. This implies that $\theta(G) \leq \theta(G - A - M) + 2$. On the other hand, $\alpha(G) \geq \alpha(G - A - M) + 2$ and so, by the induction hypothesis, we have

$$\theta(G) \leq \theta(G - A - M) + 2 \leq \frac{3}{2}\alpha(G - A - M) + 2 < \frac{3}{2}\alpha(G).$$

Therefore, we may assume G does not contain a homogeneous set. By Theorem 16, G is either a matched co-bipartite graph or a thin spider or an enhanced bipartite chain graph or it has at most 9 vertices. It is easy to see that in all the first three cases $\theta(G) = \alpha(G)$. Moreover, if G has at most 9 vertices, then Gyárfás et al. [16] showed that $\theta(G) - \alpha(G) \leq 1$. If $\alpha(G) \leq 1$, then G is complete and $\theta(G) = \alpha(G)$. Otherwise, $\theta(G) \leq \alpha(G) + 1 \leq \frac{3}{2}\alpha(G)$. □

Proof of Theorem 5 We have seen that triangle graphs are $K_{2,3}$-free (Lemma 6). Moreover, since G is K_4-free, $T(G)$ is diamond-free. Therefore, Theorem 17 implies that $\tau'_\Delta(G) = \theta(T(G)) \leq \frac{3}{2}\alpha(T(G)) = \frac{3}{2}\nu'_\Delta(G)$. □

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