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Bounded Clique Cover of Some Sparse Graphs

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Abstract

We show that $f(x) = \lfloor \frac{3}{2}x \rfloor$ is a θ-bounding function for the class of subcubic graphs and that it is best possible. This generalizes a result by Henning et al. [*Independent sets and matchings in subcubic graphs*, Discrete Mathematics 312 (11) (2012) 1900-1910], who showed that $\theta(G) \leq \frac{3}{2}\alpha(G)$ for any subcubic triangle-free graph G. Moreover, we provide a θ-bounding function for the class of K_4-free graphs with maximum degree at most 4. Finally, we study the problem Clique Cover for subclasses of planar graphs and graphs with bounded maximum degree: in particular, answering a question of Cerioli et al. [*Partition into cliques for cubic graphs: Planar case, complexity and approximation*, Discrete Applied Mathematics 156 (12) (2008) 2270-2278], we show it admits a PTAS for planar graphs.

Keywords: Clique cover, θ-bounded, Bounded degree graphs, PTAS

1. Introduction

A *clique* of a graph is a set of pairwise adjacent vertices, a *clique cover* is a set of cliques such that each vertex of the graph belongs to at least one of them and an *independent set* is a set of pairwise non-adjacent vertices. We denote by $\theta(G)$ and $\alpha(G)$ the minimum size of a clique cover and the maximum size of an independent set of the graph G, respectively. Clearly, for any graph G, we have $\theta(G) \geq \alpha(G)$ and a class of graphs $\mathcal{G}$ is *θ-bounded* if there exists a function $f\colon \mathbb{N} \to \mathbb{R}$ such that for all $G \in \mathcal{G}$ and all induced subgraphs H of G, we have $\theta(H) \leq f(\alpha(H))$. Such a function f is a *θ-bounding* function for $\mathcal{G}$. The notion of θ-boundedness and its complementary χ-boundedness were introduced by Gyárfás [1] in order to provide a natural extension of the class of perfect graphs: indeed, this class is exactly the class of graphs θ-bounded by the identity function. One of the main questions formulated in [1] is the following: given a class $\mathcal{G}$, what is the smallest θ-bounding function for $\mathcal{G}$, if any? We answer this question for the class of subcubic graphs:

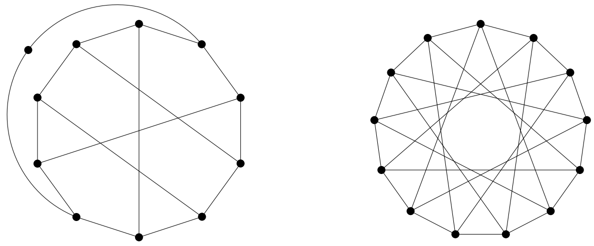
Theorem 1. *If G is a subcubic graph, then $\theta(G) \leq \frac{3}{2}\alpha(G)$. Moreover, $f(x) = \lfloor \frac{3}{2}x \rfloor$ is the smallest θ-bounding function for the class of subcubic graphs.*

Elaborating on a result by Choudum et al. [2], Pedersen conjectured that $\theta(G) \leq \frac{3}{2}\alpha(G)$, for any subcubic triangle-free graph G (see [3]). Recall that, if G is a triangle-free graph and $\alpha'(G)$ denotes the maximum size of a matching in G, then $\theta(G) = \alpha'(G) + (|V(G)| - 2\alpha'(G)) = |V(G)| - \alpha'(G)$. Pedersen's conjecture was confirmed by Henning et al. [3], who actually proved the following generalization:

Theorem 2 (Henning et al. [3]). *If G is a subcubic graph, then*

$$\frac{3}{2}\alpha(G) + \alpha'(G) + \frac{1}{2}t(G) \geq |V(G)|,$$

where $t(G)$ denotes the maximum number of vertex-disjoint triangles of G. Moreover, equality holds if and only if every component of G is in $\{K_3, K_4, C_5, G_{11}\}$ (see Figure 1).



G_{11} G_{13}

Figure 1: The graphs G_{11} and G_{13}.

Theorem 2 implies that $f(x) = \lfloor \frac{3}{2}x \rfloor$ is the smallest θ-bounding function for the class of subcubic triangle-free graphs. Consider now the class $\mathcal{C}$ containing those graphs G such that $\alpha(H) \geq \frac{|V(H)|}{3}$, for every induced subgraph H of G. Gyárfás et al. [4] showed that $f(x) = \lfloor \frac{8}{5}x \rfloor$ is the smallest θ-bounding function for the class $\mathcal{C}$. In particular, they proved the following:

Theorem 3 (Gyárfás et al. [4]). *If $G \in \mathcal{C}$, then $\theta(G) \leq \frac{8}{5}\alpha(G)$.*

By Brooks' Theorem [5], every connected subcubic graph different from K_4 belongs to $\mathcal{C}$ and so $f(x) = \lfloor \frac{8}{5}x \rfloor$ is a θ-bounding function for the class of subcubic graphs as well. On the other hand, Gyárfás et al. [6] provided evidence for the following meta-statement: the graphs for which the difference $\theta - \alpha$ is large are triangle-free. It would therefore be natural to expect that the ratio $\frac{\theta}{\alpha}$ is maximum for triangle-free graphs and Theorem 1 partially confirms this intuition.

Our proof of Theorem 1 in Section 2 is inspired by that of Theorem 3. The main idea is rather simple and it is based on the notion of θ-criticality, a graph G being *θ-critical* if $\theta(G - v) < \theta(G)$, for every $v \in V(G)$. First, we show that a minimum counterexample is connected and θ-critical. We then rely on the following result by Gallai (see [7] for a short proof and an extension):

Theorem 4 (Gallai [8]). *If v is any vertex of a connected θ-critical graph G, then G has a minimum-size clique cover in which v is the only isolated vertex. In particular, $\theta(G) \leq \frac{|V(G)|+1}{2}$.*

The final contradiction is then reached by using an appropriate lower bound for the independence number of a subcubic graph.

Let us now consider the class of graphs with maximum degree at most 4. Joos [9] relaxed the degree condition in Theorem 2 and showed that $\theta(G) \leq \frac{7}{4}\alpha(G)$, for any triangle-free graph G with $\Delta(G) \leq 4$:

Theorem 5 (Joos [9]). *If G is a triangle-free graph with $\Delta(G) \leq 4$, then $\frac{7}{4}\alpha(G) + \alpha'(G) \geq |V(G)|$. Moreover, equality holds if and only if every component C of G has order 13, $\alpha(C) = 4$ and $\alpha'(C) = 6$.*

It would be tempting to extend Theorem 5 to the class of graphs with maximum degree 4 in the same way we extend Theorem 2 to the class of subcubic graphs. Unfortunately, the method adopted in the proof of Theorem 1 does not seem to be powerful enough for this purpose and the price we have to pay is a bigger θ-bounding function (see Theorem 18), likely to be far from the optimal.

In Section 3, we consider the problem of finding a clique cover of minimum size. The decision version of this well-known NP-complete problem is formulated as follows:

Clique Cover	
Instance:	A graph G and a positive integer k.
Question:	Does $\theta(G) \leq k$ hold?

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Since any subset of a clique is again a clique, CLIQUE COVER is equivalent to the following problem:

CLIQUE PARTITION	
Instance:	A graph G and a positive integer k.
Question:	Does there exist a partition of $V(G)$ into k disjoint cliques?

Moreover, CLIQUE PARTITION is clearly equivalent to the well-known COLOURING problem on the complement graph.

Cerioli et al. [10] studied CLIQUE COVER on planar graphs and on subclasses of subcubic graphs. In particular, they showed that CLIQUE COVER is NP-complete even for planar cubic graphs and that the optimization version is MAX SNP-hard for cubic graphs. Moreover, they asked whether the problem admits a PTAS for planar cubic graphs and conjectured that it has a polynomial-time approximation algorithm with a fixed ratio for graphs with bounded maximum degree. In Section 3, we answer both questions in the affirmative. We also provide some hardness results for subclasses of planar graphs and subcubic graphs.

1.1. Notation and definitions

We assume the reader is familiar with notions of graph theory; for those not defined here, we refer to [5]. Note that we consider only finite undirected simple graphs. Given a graph G, we denote its order $|V(G)|$ by $n(G)$ and its size $|E(G)|$ by $m(G)$. A *k-vertex* is a vertex of degree k; in particular, we refer to a 3-vertex as a *cubic vertex*. The number of k-vertices of G is denoted by $n_k(G)$. The maximum degree of a vertex of G is denoted by $\Delta(G)$ and G is *subcubic* if $\Delta(G) \leq 3$, and *cubic* if each vertex is a cubic vertex. The complete graph on n vertices is denoted by K_n. A *triangle* is (a graph isomorphic to) K_3 and a *diamond* is the graph obtained from K_4 by removing an edge. For a vertex $v \in V(G)$, the *neighbourhood* $N(v)$ is the set of vertices adjacent to v in G and the *closed neighbourhood* $N[v]$ is the set $N(v) \cup \{v\}$. The *distance* $d(u, v)$ from a vertex u to a vertex v is the minimum length of a path between u and v in G. A *block* of G is a maximal connected subgraph with no cut-vertex.

A *vertex cover* of G is a set of vertices of G containing at least one endpoint for every edge of G and the minimum size of a vertex cover of G is denoted by $\beta(G)$. A *k-subdivision* of G is the graph obtained from G by adding k new vertices for each edge of G, i.e. each edge is replaced by a path of length $k+1$. The *line graph* $L(G)$ of G is the graph whose vertices are the edges of G, two vertices being adjacent if the corresponding edges share an endpoint. A graph is *planar* if it can be drawn in the plane without crossings. Given such a drawing Γ, the *faces* are the arcwise-connected open sets of $\mathbb{R}^2 \setminus \Gamma$. The *outer face* is the unbounded face. Given a planar graph G and a fixed planar drawing Γ of G, we define L_1 to be the set of vertices incident to the outer face and, for $i > 1$, L_i is defined recursively as the set of vertices on the outer face of the planar drawing obtained by deleting the vertices in $\bigcup_{j=1}^{i-1} L_j$. We call L_i the *i-th layer* of the drawing Γ. A graph is *k-outerplanar* if it has a planar drawing with at most k layers.

2. θ-bounding functions

We begin this section with a proof of Theorem 1. As mentioned in Section 1, our proof makes use of an appropriate lower bound for the independence number given by Harant et al. [11]. In order to state their result, we need the following definitions. A block of a graph is *difficult* if it is isomorphic to one of the four graphs depicted in Figure 2. Moreover, a connected graph is *bad* if its blocks are either difficult or are edges between difficult blocks. For a graph G, the number of bad components of G is denoted by $\lambda(G)$ and the maximum number of vertex-disjoint triangles of G is denoted by $t(G)$.

Theorem 6 (Harant et al. [11]). *Every subcubic K_4-free graph G has an independent set of size at least $\frac{1}{7}(4n(G) - m(G) - \lambda(G) - t(G))$.*

We also require the notion of distance between sets of vertices of a graph. Given two subsets X and Y of $V(G)$, the *distance* from X to Y is the quantity $d(X, Y) = \min_{x \in X, y \in Y} d(x, y)$, i.e. it is the minimum

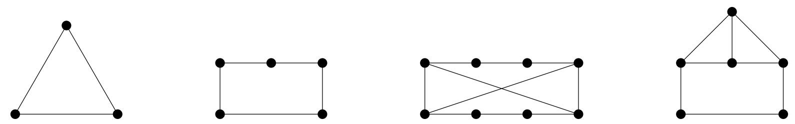

Figure 2: The difficult blocks.

length of a path between a vertex in X and a vertex in Y. With a slight abuse of notation, if T is a triangle, we write $d(T, Y)$ instead of $d(V(T), Y)$.

We can finally proceed to the proof of Theorem 1:

Proof of Theorem 1. Let us begin by showing that $\theta(G) \leq \frac{3}{2}\alpha(G)$, for any subcubic graph G. Suppose, by contradiction, that G is a counterexample with the minimum number of vertices. In the following, we deduce some structural properties of G and we show how they lead to a contradiction. Each claim is followed by a short proof.

Claim 7. *G is connected.*

Otherwise, G is the disjoint union of two non-empty graphs G_1 and G_2. By minimality, we have

$$\theta(G) = \theta(G_1) + \theta(G_2) \leq \frac{3}{2}(\alpha(G_1) + \alpha(G_2)) = \frac{3}{2}\alpha(G),$$

a contradiction. $\Diamond$

Claim 8. *G is θ-critical.*

Indeed, suppose there exists a vertex $v \in V(G)$ such that $\theta(G) = \theta(G - v)$. By minimality, we have

$$\theta(G) = \theta(G - v) \leq \frac{3}{2}\alpha(G - v) \leq \frac{3}{2}\alpha(G),$$

a contradiction. $\Diamond$

Claim 9. *G has minimum degree at least 2.*

Suppose there exists a 1-vertex u of G. By minimality, we have

$$\theta(G) \leq \theta(G - N[u]) + 1 \leq \frac{3}{2}\alpha(G - N[u]) + 1 \leq \frac{3}{2}(\alpha(G) - 1) + 1 < \frac{3}{2}\alpha(G),$$

a contradiction. $\Diamond$

Claim 10. *G is 2-connected.*

Since every connected subcubic bridgeless graph is 2-connected, it is enough to show that G has no cut-edges. Therefore, suppose $e = u_1u_2$ is a cut-edge and let G_1 be the component of $G - e$ containing u_1 and $G_2 = G - V(G_1)$ (therefore, $u_2 \in V(G_2)$). Clearly, $\theta(G) \leq \theta(G_1) + \theta(G_2)$. If there exists $i \in \{1, 2\}$ such that a maximum independent set of G_i avoids u_i, then $\alpha(G) \geq \alpha(G_1) + \alpha(G_2)$ and so, by minimality,

$$\theta(G) \leq \theta(G_1) + \theta(G_2) \leq \frac{3}{2}\alpha(G_1) + \frac{3}{2}\alpha(G_2) \leq \frac{3}{2}\alpha(G),$$

a contradiction. Therefore, for each $i \in \{1, 2\}$, every maximum independent set of G_i contains u_i. This means that $\alpha(G_i - u_i) = \alpha(G_i) - 1$, for each $i \in \{1, 2\}$. Moreover, denoting by I_i a maximum independent set of G_i, we have that $I_1 \cup (I_2 \setminus \{u_2\})$ is an independent set of G and so $\alpha(G) \geq \alpha(G_1) + \alpha(G_2) - 1$. But then, by minimality,

$$\begin{aligned}
\theta(G) &\leq \theta(G_1 - u_1) + \theta(G_2 - u_2) + 1 \\
&\leq \frac{3}{2}\alpha(G_1 - u_1) + \frac{3}{2}\alpha(G_2 - u_2) + 1 \\
&= \frac{3}{2}(\alpha(G_1) - 1) + \frac{3}{2}(\alpha(G_2) - 1) + 1 \\
&= \frac{3}{2}(\alpha(G_1) + \alpha(G_2)) - 2 \\
&< \frac{3}{2}\alpha(G),
\end{aligned}$$

a contradiction. $\diamondsuit$

Claim 11. *G does not contain a diamond.*

Suppose G contains a diamond and let u and v be its 2-vertices. Since G is connected and $\theta(K_4) = \alpha(K_4)$, we have $uv \notin E(G)$. Therefore, by minimality,

$$\theta(G) \leq \theta(G - N[u] - N[v]) + 3 \leq \frac{3}{2}\alpha(G - N[u] - N[v]) + 3 \leq \frac{3}{2}(\alpha(G) - 2) + 3 \leq \frac{3}{2}\alpha(G),$$

a contradiction. $\diamondsuit$

Claim 12. *$d(u, T) \geq 4$, for any 2-vertex $u \in V(G)$ and any triangle $T \subseteq G$.*

Suppose first a triangle T contains a 2-vertex. By minimality, we have

$$\theta(G) \leq \theta(G - V(T)) + 1 \leq \frac{3}{2}\alpha(G - V(T)) + 1 \leq \frac{3}{2}(\alpha(G) - 1) + 1 < \frac{3}{2}\alpha(G),$$

a contradiction. Therefore, we have $d(u, T) \geq 1$, for any 2-vertex $u \in V(G)$ and any triangle $T \subseteq G$.

Suppose $d(u, T) = 1$, for a triangle $T \subseteq G$ and a 2-vertex $u \in V(G) \setminus V(T)$. This means T contains a vertex v such that $uv \in E(G)$ and let $v' \in V(T) \setminus \{v\}$. By Claim 11, we have $uv' \notin E(G)$ and so, by minimality,

$$\theta(G) \leq \theta(G - N[u] - N[v']) + 3 \leq \frac{3}{2}\alpha(G - N[u] - N[v']) + 3 \leq \frac{3}{2}(\alpha(G) - 2) + 3 \leq \frac{3}{2}\alpha(G),$$

a contradiction.

Suppose now $d(u, T) = 2$, for a triangle $T \subseteq G$ and a 2-vertex $u \in V(G) \setminus V(T)$. This means T contains a vertex v such that u and v are linked by an induced path of length 2 with inner vertex not in $V(T)$. By minimality,

$$\theta(G) \leq \theta(G - N[u] - N[v]) + 3 \leq \frac{3}{2}\alpha(G - N[u] - N[v]) + 3 \leq \frac{3}{2}(\alpha(G) - 2) + 3 \leq \frac{3}{2}\alpha(G),$$

a contradiction.

Finally, suppose $d(u, T) = 3$, for a triangle $T \subseteq G$ and a 2-vertex $u \in V(G) \setminus V(T)$. This means T contains a vertex v such that u and v are linked by an induced path of length 3 with no inner vertex in $V(T)$. By minimality,

$$\theta(G) \leq \theta(G - N[u] - N[v]) + 3 \leq \frac{3}{2}\alpha(G - N[u] - N[v]) + 3 \leq \frac{3}{2}(\alpha(G) - 2) + 3 \leq \frac{3}{2}\alpha(G),$$

a contradiction. $\diamondsuit$

Claim 13. *$d(u,v) \geq 3$, for any two distinct 2-vertices u and v of G.*

Suppose first there exist two adjacent 2-vertices u and v and let $u' \in N(u) \setminus \{v\}$ and $v' \in N(v) \setminus \{v\}$. By Claim 12, we have $u' \neq v'$. If there exists a vertex $w \in V(G)$ adjacent to both u' and v' then, by minimality,

$$\theta(G) \leq \theta(G - N[u] - N[w]) + 3 \leq \frac{3}{2}\alpha(G - N[u] - N[w]) + 3 \leq \frac{3}{2}(\alpha(G) - 2) + 3 \leq \frac{3}{2}\alpha(G),$$

a contradiction. Therefore, no vertex of G is adjacent to both u' and v'.

Consider now the graph G' obtained from G by deleting $\{u,v\}$ and by adding, if necessary, the edge $u'v'$. The graph G' is clearly simple and subcubic. Since a maximum independent set I' of G' is also an independent set of $G - \{u,v\}$ and I' contains at most one of the vertices u' and v', we have $\alpha(G) \geq \alpha(G') + 1$. Moreover, we claim that $\theta(G) \leq \theta(G') + 1$. Indeed, consider a minimum clique cover C' of G'. If no clique in C' contains $\{u', v'\}$, then $C' \cup \{u,v\}$ is a clique cover of G of size $\theta(G') + 1$. On the other hand, by the paragraph above, if a clique in C' contains $\{u', v'\}$, then it must be of size 2. Therefore, $(C' \setminus \{u', v'\}) \cup \{u', u\} \cup \{v', v\}$ is a clique cover of G of size $\theta(G') + 1$ and we have established our claim. But then, again by minimality, we have

$$\theta(G) \leq \theta(G') + 1 \leq \frac{3}{2}\alpha(G') + 1 \leq \frac{3}{2}(\alpha(G) - 1) + 1 < \frac{3}{2}\alpha(G),$$

a contradiction.

Suppose now there exist two 2-vertices u and v such that $d(u,v) = 2$. Since $uv \notin E(G)$ then, by minimality, we have

$$\theta(G) \leq \theta(G - N[u] - N[v]) + 3 \leq \frac{3}{2}\alpha(G - N[u] - N[v]) + 3 \leq \frac{3}{2}(\alpha(G) - 2) + 3 \leq \frac{3}{2}\alpha(G),$$

a contradiction. $\Diamond$

Claim 14. *$d(T_1, T_2) \geq 3$, for any two distinct triangles T_1 and T_2 of G.*

By Claim 11 and since G is subcubic, we have $d(T_1, T_2) \geq 1$. Suppose first there exist two triangles T_1 and T_2 at distance 1 and let $u_1 \in V(T_1)$ and $u_2 \in V(T_2)$ be such that $d(u_1, u_2) = 1$. Moreover, consider a vertex $u_2' \in V(T_2) \setminus \{u_2\}$. By minimality,

$$\theta(G) \leq \theta(G - N[u_1] - N[u_2']) + 3 \leq \frac{3}{2}\alpha(G - N[u_1] - N[u_2']) + 3 \leq \frac{3}{2}(\alpha(G) - 2) + 3 \leq \frac{3}{2}\alpha(G),$$

a contradiction.

Finally, suppose there exist two triangles T_1 and T_2 at distance 2 and let $u_1 \in V(T_1)$ and $u_2 \in V(T_2)$ be such that $d(u_1, u_2) = 2$. In particular, $u_1 u_2 \notin E(G)$ and so, by minimality,

$$\theta(G) \leq \theta(G - N[u_1] - N[u_2]) + 3 \leq \frac{3}{2}\alpha(G - N[u_1] - N[u_2]) + 3 \leq \frac{3}{2}(\alpha(G) - 2) + 3 \leq \frac{3}{2}\alpha(G),$$

a contradiction. $\Diamond$

Claim 15. *Each cycle of G on four vertices contains only cubic vertices.*

Indeed, suppose there exists a cycle $C \subseteq G$ on four vertices containing a 2-vertex u of G. Let $v \in V(C)$ be such that $d(u,v) = 2$. By minimality and since $uv \notin E(G)$ (Claim 11),

$$\theta(G) \leq \theta(G - N[u] - N[v]) + 3 \leq \frac{3}{2}\alpha(G - N[u] - N[v]) + 3 \leq \frac{3}{2}(\alpha(G) - 2) + 3 \leq \frac{3}{2}\alpha(G),$$

a contradiction. $\Diamond$

Claim 16. *$n_2(G) > 0$.*

Suppose this is not the case. By Claim 9 and Claim 10, G is a cubic bridgeless graph and the well-known Petersen's Theorem implies it has a perfect matching. Moreover, since G is connected and different from K_4 (Claim 11), Brooks' Theorem implies it is 3-colourable. But then $\theta(G) \leq \frac{n(G)}{2} \leq \frac{3}{2}\alpha(G)$, a contradiction. $\Diamond$

Claim 17. $6t(G) \leq n_3(G) - 6$.

Let u be a 2-vertex of G. We first show that the set $S = \{v \in V(G) : d(v, u) = 2\}$ of vertices at distance 2 from u has size 4. Indeed, by Claim 13, the neighbours u' and u'' of u are cubic vertices and, by Claim 12, $u'u'' \notin E(G)$. Moreover, by Claim 15, no neighbour of u' different from u is also a neighbour of u'' and so $|S| = 4$. Note that, by Claim 13, each $v \in S$ is a cubic vertex.

Consider now the triangles of G. By Claim 12 and Claim 11, each vertex belonging to a triangle has a neighbour not in the triangle and any two such neighbours are distinct. Moreover, by Claim 14, the set of neighbours of T_1 does not intersect the set of neighbours of T_2, for any two (vertex-disjoint) triangles T_1 and T_2 of G. Finally, by Claim 12, no vertex in $S \cup \{u', u''\}$ belongs to a triangle or is a neighbour of a triangle and each neighbour of a triangle is a cubic vertex. Therefore, we have $6t(G) \leq n_3(G) - 6$. $\Diamond$

We are finally in a position to conclude our proof. By Claim 10, G is 2-connected and since no graph in Figure 2 is a counterexample, we have $\lambda(G) = 0$. Therefore, by Theorem 6 and recalling that $n(G) = n_3(G) + n_2(G)$ and $6t(G) \leq n_3(G) - 6$, we get

$$\begin{aligned}
\alpha(G) &\geq \left\lceil \frac{1}{7}\Big(4n(G) - m(G) - t(G)\Big) \right\rceil \\
&= \left\lceil \frac{1}{7}\Big(4n_3(G) + 4n_2(G)\Big) - \frac{1}{7}\Big(\frac{3}{2}n_3(G) + n_2(G)\Big) - \frac{1}{7}t(G) \right\rceil \\
&\geq \left\lceil \frac{1}{7}\Big(4n_3(G) + 4n_2(G)\Big) - \frac{1}{7}\Big(\frac{3}{2}n_3(G) + n_2(G)\Big) - \frac{1}{42}(n_3(G) - 6) \right\rceil \\
&= \left\lceil \frac{1}{3}n_3(G) + \frac{3}{7}n_2(G) + \frac{1}{7} \right\rceil. \qquad (1)
\end{aligned}$$

We now claim that

$$\left\lceil \frac{1}{3}n_3(G) + \frac{3}{7}n_2(G) + \frac{1}{7} \right\rceil \geq \frac{1}{3}(n_3(G) + n_2(G) + 1). \qquad (2)$$

This can be easily seen if $n_2(G) \geq 2$. Therefore, suppose $n_2(G) = 1$ and let $n_3(G) = 3k + a$, for some integer $0 \leq a \leq 2$. Inequality (2) is then equivalent to

$$\left\lceil \frac{7a + 12}{21} \right\rceil \geq \frac{a + 2}{3},$$

which clearly holds for $0 \leq a \leq 2$.

On the other hand, by Claim 7 and Claim 8, G is a connected θ-critical graph and so, by Theorem 4, we have $\theta(G) \leq \frac{n(G)+1}{2}$. Therefore, combining this with (1) and (2), we have

$$\frac{3}{2}\alpha(G) \geq \frac{3}{2}\left\lceil \frac{1}{3}n_3(G) + \frac{3}{7}n_2(G) + \frac{1}{7} \right\rceil \geq \frac{1}{2}(n_3(G) + n_2(G) + 1) = \frac{1}{2}(n(G) + 1) \geq \theta(G),$$

a contradiction. This concludes the proof of the first statement in Theorem 1.

As for the second statement, we need to show that, for each integer $x \geq 0$, there exists a subcubic graph G such that $\alpha(G) = x$ and $\theta(G) = \lfloor \frac{3}{2}\alpha(G) \rfloor$. If x is even, we construct G as the disjoint union of $\frac{x}{2}$ copies of C_5. On the other hand, if x is odd, it is enough to construct G as the disjoint union of $\lfloor \frac{x}{2} \rfloor$ copies of C_5 together with an isolated vertex. $\square$

Let us now consider the class of graphs with maximum degree at most 4. Joos [9] relaxed the degree condition in Theorem 2 and showed that $\theta(G) \leq \frac{7}{4}\alpha(G)$, for any triangle-free graph G with $\Delta(G) \leq 4$.

Following the intuition expressed in Section 1, it would be natural to expect that the previous inequality holds for any graph with maximum degree at most 4. Unfortunately, the method adopted in the proof of Theorem 1 gives an upper bound for the ratio $\frac{\theta(G)}{\alpha(G)}$ which gets larger as the size of a maximum clique of G increases. Since we believe the maximum is attained by triangle-free graphs, we present only the bound for K_4-free graphs, which is already substantially larger than the expected $\frac{7}{4}$.

Theorem 18. *If G is a K_4-free graph with maximum degree at most 4, then $\theta(G) \leq \frac{193}{98}\alpha(G)$.*

As mentioned above, our proof of Theorem 18 resembles that of Theorem 1: a counterexample G with minimum order must be connected and θ-critical; Theorem 4 then guarantees the existence of a clique cover of size at most $\frac{n(G)+1}{2}$ and by using an appropriate lower bound on the independence number, we derive a contradiction, assuming the value of $\alpha(G)$ is large enough. On the other hand, if the value of $\alpha(G)$ is small, then $n(G)$ is small and it is useful to consider several results on the *covering gap* (the difference between the minimum size of a clique cover and the maximum size of an independent set) of small graphs, as stated in the following theorem.

Theorem 19 (Gyárfás et al. [6]). *The following hold, for any graph G:*

- *If $n(G) \leq 22$, then $\theta(G) - \alpha(G) \leq 5$;*
- *If $n(G) \leq 19$, then $\theta(G) - \alpha(G) \leq 4$;*
- *If $n(G) \leq 16$, then $\theta(G) - \alpha(G) \leq 3$;*
- *If $n(G) \leq 12$, then $\theta(G) - \alpha(G) \leq 2$;*
- *If $n(G) \leq 9$, then $\theta(G) - \alpha(G) \leq 1$.*

We can finally proceed to the proof of Theorem 18:

Proof of Theorem 18. Suppose, by contradiction, that G is a counterexample with the minimum number of vertices. As in the proof of Theorem 1, it is easy to see that G is connected and θ-critical. On the other hand, Locke and Lou [12] showed that, for any connected K_4-free graph G with $\Delta(G) \leq 4$, we have $\alpha(G) \geq \frac{7n(G)-4}{26}$. Combining this with Theorem 4, and if $\alpha(G) \geq 7$, we get

$$\theta(G) \leq \frac{n(G)+1}{2} \leq \frac{26\alpha(G)+11}{14} \leq \frac{193}{98}\alpha(G).$$

For the small values of $\alpha(G)$, we rely on Theorem 19. If $\alpha(G) = 6$, then $n(G) \leq 22$ and so $\theta(G) \leq 11$. If $\alpha(G) = 5$, then $n(G) \leq 19$ and so $\theta(G) \leq 9$. If $\alpha(G) = 4$, then $n(G) \leq 15$ and so $\theta(G) \leq 7$. If $\alpha(G) = 3$, then $n(G) \leq 11$ and so $\theta(G) \leq 5$. If $\alpha(G) = 2$, then $n(G) \leq 8$ and so $\theta(G) \leq 3$. Finally, if $\alpha(G) \leq 1$, then G is complete and $\theta(G) = \alpha(G)$. It immediately follows that, for all the values of $\alpha(G)$, we have $\theta(G) \leq \frac{193}{98}\alpha(G)$, a contradiction. □

It is worth noticing that slightly modifying the proof of Theorem 18, we can obtain a short proof of Theorem 5:

Proof of Theorem 5. Suppose, by contradiction, that G is a counterexample with the minimum number of vertices. As we have seen above, G is a connected θ-critical graph. On the other hand, Fraughnaugh Jones [13] showed that $\alpha(G) \geq \frac{4}{13}n(G)$, for any triangle-free graph G with $\Delta(G) \leq 4$. Combining this with Theorem 4, and if $\alpha(G) > 4$, we get

$$\theta(G) \leq \frac{n(G)+1}{2} \leq \frac{13\alpha(G)+4}{8} < \frac{7}{4}\alpha(G).$$

For the remaining values of $\alpha(G)$, we use again Theorem 19. If $\alpha(G) = 4$, then $n(G) \leq 13$ and so $\theta(G) \leq 7$. If $\alpha(G) = 3$, then $n(G) \leq 9$ and so $\theta(G) \leq 4$. If $\alpha(G) = 2$, then $n(G) \leq 6$ and so $\theta(G) \leq 3$.

Finally, if $\alpha(G) \leq 1$, then G is complete and $\theta(G) = \alpha(G)$. It immediately follows that $\theta(G) \leq \frac{7}{4}\alpha(G)$, a contradiction.

Note that, if G is connected and θ-critical, then equality holds only if $\alpha(G) = 4$ and $n(G) = 13$. But the graph G_{13} in Figure 1 (also known as the $(3,5)$-Ramsey graph) is the only graph G such that $\omega(G) = 2$, $\omega(\overline{G}) = \alpha(G) = 4$ and $n(G) = 13$ (see [14]). On the other hand, it is easy to see that equality cannot hold if G is not θ-critical. □

We conclude this section with two conjectures on the extremal role of triangle-free graphs.

Conjecture 20. If G is a subcubic graph, then $\theta(G) = \frac{3}{2}\alpha(G)$ if and only if every component of G is either C_5 or G_{11}.

Conjecture 21. If G is a graph with maximum degree at most 4, then $\theta(G) \leq \frac{7}{4}\alpha(G)$, with equality if and only if every component of G is G_{13}.

3. The Clique Cover problem

Clique Cover is a well-known NP-complete problem polynomially equivalent to Colouring. Cerioli et al. [10] showed that the optimization version of Clique Cover is MAX SNP-hard for cubic graphs. We begin this section with an inapproximability gap result for Clique Cover restricted to subcubic line graphs. We first need the following auxiliary lemma, which implies that the well-known Vertex Cover problem restricted to triangle-free graphs polynomially reduces to Clique Cover for line graphs.

Lemma 22 (Folklore). *For any graph G, we have $\theta(L(G)) \leq \beta(G)$. Moreover, if G is triangle-free, then equality holds.*

Proof. Let Q be a minimum-size vertex cover of G. With each $v \in Q$, we can associate the clique $C_v \subseteq V(L(G))$ corresponding to the edges incident to v. But then $\bigcup_{v \in Q} C_v$ is a clique cover of $L(G)$ and so $\theta(L(G)) \leq \beta(G)$.

Suppose now G is triangle-free and let Q be a minimum-size clique cover of $L(G)$. Each clique $C \in Q$ corresponds to the edges incident to a vertex $v_C \in V(G)$ and $\bigcup_{C \in Q} v_C$ is a vertex cover of G. Therefore, we have $\theta(L(G)) = \beta(G)$. □

The following theorem implies that it is NP-hard to approximate Clique Cover for subcubic line graphs within $\frac{391}{390}$.

Theorem 23. Clique Cover *is not approximable within $\frac{391}{390}$, unless* P = NP*, even when restricted to line graphs of 2-subdivisions of cubic triangle-free graphs.*

Proof. Chlebík and Chlebíková [15] showed that it is NP-hard to approximate Vertex Cover within $\frac{391}{390}$, even for 2-subdivisions of cubic graphs: they construct a gap-preserving reduction from Vertex Cover restricted to cubic graphs, for which they provided an NP-hard gap in [16], just by a 2-subdivision of the input graph. Since their NP-hard gap result for Vertex Cover in [16] holds for cubic triangle-free graphs as well, it follows that it is NP-hard to approximate Vertex Cover within $\frac{391}{390}$, even for 2-subdivisions of cubic triangle-free graphs. Therefore, given a 2-subdivision of a cubic triangle-free graph G, we simply construct its line graph $L(G)$. Since $\theta(L(G)) = \beta(G)$, the conclusion immediately follows. □

We now turn to the decision version of Clique Cover when restricted to planar line graphs with maximum degree at most 4:

Theorem 24. Clique Cover *is* NP*-complete even for line graphs of 2-subdivisions of planar cubic triangle-free graphs.*

Proof. Note that, given a graph G and a 2-subdivision G' of G, we have $\beta(G') = \beta(G) + |E(G)|$ (see [15] for a proof). Since Vertex Cover is NP-hard for planar cubic triangle-free graphs [17], then it is NP-hard for 2-subdivisions of planar cubic triangle-free graphs and, by Lemma 22, we can easily obtain the claimed NP-hardness of Clique Cover. $\square$

Cerioli et al. [10] showed that Clique Cover is NP-hard for planar cubic graphs. Not surprisingly, it remains NP-hard even for planar 4-regular graphs, as implied by the following theorem. The proof immediately follows by the result in [17] mentioned above.

Theorem 25. Clique Cover *remains* NP*-complete even when restricted to line graphs of planar cubic triangle-free graphs.*

It is therefore natural to look for approximation algorithms for Clique Cover when restricted to graphs having bounded maximum degree or to planar graphs. Cerioli et al. [10] showed that Clique Cover admits a polynomial-time $\frac{5}{4}$-approximation algorithm for subcubic graphs and they conjectured it has a polynomial-time approximation algorithm with a fixed ratio for graphs with bounded maximum degree. This can be easily verified once we notice the close relation between θ-boundedness of a certain class of graphs and approximation algorithms for Clique Cover for that class. Indeed, since $\alpha(G) \leq \theta(G)$, if we could show "algorithmically" that a class of graphs is θ-bounded by a linear function, we would obtain a constant-factor approximation algorithm for Clique Cover. Note that it is not clear whether the proofs of Theorem 1 and Theorem 18 can be turned into "algorithmic" ones. Nevertheless, the following holds:

Theorem 26. Clique Cover *admits a linear-time k-approximation algorithm for graphs with maximum degree at most k.*

Proof. Consider the following greedy algorithm: first, find a maximal independent set I of the input graph G and set $C = \varnothing$; then, for each $v \in I$, add the edges incident to v to the set C and return C. Clearly, the algorithm runs in linear time and it returns a clique cover of G. Moreover, we have

$$|C| \leq k \cdot |I| \leq k \cdot \alpha(G) \leq k \cdot \theta(G),$$

and so the greedy algorithm is indeed a k-approximation algorithm for graphs with maximum degree at most k. $\square$

Theorem 23 shows that Clique Cover admits no PTAS, even for subcubic graphs. Cerioli et al. [10] asked whether this could be possible in the special case of planar cubic graphs. In the rest of this section, using Baker's well-known technique [18], we show that Clique Cover indeed admits a PTAS even for planar graphs. The idea of Baker's technique is the following: partition the planar graph into k-outerplanar graphs, solve the problem optimally for each k-outerplanar graph and finally show that the union of these solutions is in fact a "near-optimal solution" for the original graph.

Bodlaender [19, 20] showed that k-outerplanar graphs have tree-width at most $3k - 1$. Moreover, using dynamic programming, it is possible to determine $\theta(G)$ in polynomial time for any graph G of bounded tree-width (see, e.g., [21]). In fact, many other problems are solvable in polynomial time for graphs of bounded tree-width (see [21]) and this led Baker [18] to design the mentioned technique, which allows a PTAS for some of them (for example, Independent Set and Vertex Cover) when restricted to planar graphs. From the sketch of the technique we gave above, it should be clear that the problems which can be treated are those for which the local solutions can be combined into a global solution. We now show that Clique Cover is indeed one of them:

Theorem 27. Clique Cover *admits a PTAS for planar graphs.*

Proof. Given a planar drawing Γ of the input graph G, we construct the layers L_i as defined in Section 1.1. Note that the neighbours of a vertex $v_i \in L_i$ must be in $L_{i-1} \cup L_i \cup L_{i+1}$. Finally, given $\varepsilon > 0$, we set $k = \lceil \frac{2}{\varepsilon} \rceil$.

A *slice* G_{ij} is an induced subgraph defined as follows. For $1 \leq i \leq k$, we denote by G_{i0} the subgraph of G induced by the vertices which belong to the consecutive layers between the first and the i-th. Moreover, for a fixed $1 \leq i \leq k$ and a $j \geq 1$, we denote by G_{ij} the subgraph of G induced by the vertices which belong to the k consecutive layers whose indices range between $(j-1)(k-1)+i$ and $j(k-1)+i$ (note that j runs until each vertex of G belongs to at least one G_{ij}). By definition, each slice is k-outerplanar and so we can determine in polynomial time a minimum-size clique cover C_{ij} of G_{ij}, for each $1 \leq i \leq k$ and $j \geq 0$. Finally, for each i, we set $C_i = \bigcup_{j \geq 0} C_{ij}$. By construction, $\bigcup_{j \geq 0} V(G_{ij}) = V(G)$ and so each C_i is a clique cover of G. We return the one with minimum size.

Let now Q denote a minimum-size clique cover of G and, for $0 \leq i \leq k-1$, denote by Q_i the set of cliques in Q which contain at least one vertex in $\bigcup_{j \equiv i \pmod k} L_j$. Clearly, $\bigcup Q_i = Q$ and each clique in Q belongs to at most two distinct Q_i's. But then there exists an index ℓ such that

$$|Q_\ell| \leq \frac{2}{k}|Q| \leq \varepsilon|Q|.$$

Let $Q_{\ell j}$ denotes the set of cliques in Q containing at least one vertex in $V(G_{\ell j})$ (if $\ell = 0$, we set $G_{\ell j} = G_{kj}$). Since $C_{\ell j}$ is a minimum-size clique cover of $G_{\ell j}$, then $|C_{\ell j}| \leq |Q_{\ell j}|$, for each $j \geq 0$. Consider now the sum $\sum_j |Q_{\ell j}|$. Each clique is counted exactly once, except those which contain vertices from layers L_j with $j \equiv \ell \pmod k$ (i.e. those in Q_ℓ), which are counted exactly twice. But then $\sum_j |Q_{\ell j}| = |Q| + |Q_\ell|$. Summarizing, we have

$$|C_\ell| \leq \sum_j |C_{\ell j}| \leq \sum_j |Q_{\ell j}| = |Q| + |Q_\ell| \leq |Q| + \varepsilon|Q| = (1+\varepsilon)|Q|.$$

Therefore, the algorithm above is a polynomial-time approximation scheme. □

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References

[1] A. Gyárfás, Problems from the world surrounding perfect graphs, Zastosowania Matematyki 19 (3-4) (1987) 413–441.

[2] S. A. Choudum, T. Karthick, M. A. Shalu, Linear chromatic bounds for a subfamily of $3K_1$-free graphs, Graphs and Combinatorics 24 (5) (2008) 413–428.

[3] M. A. Henning, C. Löwenstein, D. Rautenbach, Independent sets and matchings in subcubic graphs, Discrete Mathematics 312 (11) (2012) 1900–1910.

[4] A. Gyárfás, Z. Li, R. Machado, A. Sebő, S. Thomassé, N. Trotignon, Complements of nearly perfect graphs, Journal of Combinatorics 4 (3) (2013) 299–310.

[5] D. B. West, Introduction to Graph Theory, Prentice Hall, 2 edn., 2001.

[6] A. Gyárfás, A. Sebő, N. Trotignon, The chromatic gap and its extremes, Journal of Combinatorial Theory, Series B 102 (5) (2012) 1155–1178.

[7] M. Stehlík, Critical graphs with connected complements, Journal of Combinatorial Theory, Series B 89 (2) (2003) 189–194.

[8] T. Gallai, Kritische graphen II, A Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei 8 (1963) 373–395.

[9] F. Joos, Independence and matching number in graphs with maximum degree 4, Discrete Mathematics 323 (2014) 1–6.

[10] M. R. Cerioli, L. Faria, T. O. Ferreira, C. A. J. Martinhon, F. Protti, B. Reed, Partition into cliques for cubic graphs: Planar case, complexity and approximation, Discrete Applied Mathematics 156 (12) (2008) 2270–2278.

[11] J. Harant, M. A. Henning, D. Rautenbach, I. Schiermeyer, The independence number in graphs of maximum degree three, Discrete Mathematics 308 (23) (2008) 5829–5833.

[12] S. C. Locke, F. Lou, Finding independent sets in K_4-free 4-regular connected graphs, Journal of Combinatorial Theory, Series B 71 (1) (1997) 85–110.

[13] K. Fraughnaugh Jones, Independence in graphs with maximum degree four, Journal of Combinatorial Theory, Series B 37 (3) (1984) 254–269.

[14] S. P. Radziszowski, Small Ramsey Numbers, Electronic Journal of Combinatorics DS1.

[15] M. Chlebík, J. Chlebíková, The complexity of combinatorial optimization problems on d-dimensional boxes, SIAM Journal on Discrete Mathematics 21 (1) (2007) 158–169.

[16] M. Chlebík, J. Chlebíková, Complexity of approximating bounded variants of optimization problems, Theoretical Computer Science 354 (3) (2006) 320–338.

[17] R. Uehara, NP-complete problems on a 3-connected cubic planar graph and their applications, Tech. Rep. 4, Tokyo Woman's Christian University, 1996.
[18] B. S. Baker, Approximation algorithms for NP-complete problems on planar graphs, J. ACM 41 (1) (1994) 153–180.
[19] H. L. Bodlaender, Planar graphs with bounded treewidth, Tech. Rep. RUU-CS-88-14, Rijksuniversiteit Utrecht, 1988.
[20] H. L. Bodlaender, A partial k-arboretum of graphs with bounded treewidth, Theoretical Computer Science 209 (1) (1998) 1–45.
[21] F. Gurski, A comparison of two approaches for polynomial time algorithms computing basic graph parameters, CoRR abs/0806.4073, URL http://arxiv.org/abs/0806.4073.