Multi-Unit Assignment Under Dichotomous Preferences

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Abstract

I study the problem of allocating objects among agents without using money. Agents can receive several objects and have dichotomous preferences, meaning that they either consider objects to be acceptable or not. In this set-up, the egalitarian solution is more appealing than the competitive equilibrium with equal incomes because it is Lorenz dominant, unique in utilities, and group strategy-proof. Both solutions are disjoint.

Keywords: dichotomous preferences, multi-unit assignment, Lorenz dominance, competitive equilibrium with equal incomes.
JEL Codes: C78, D73.

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Preprint submitted to Mathematical Social Sciences *July 8, 2019*

1. Introduction

An assignment problem is an allocation problem where scarce objects are to be allocated among several agents without using monetary transfers. Assignment problems include the allocation of senators to committees, courses to students, or job interviews to applicants. In this paper, I study assignment problems in which each agent can receive more than one object, but at most one unit of each, and several identical units are available of each object. These are called multi-unit assignment problems. They include the three examples previously discussed. A U.S. senator on average participates in four committees,[1] a student can take many courses during a semester, and a job candidate can schedule many interviews. However, senators cannot have more than one seat on each committee, students cannot take a course twice for credit, and applicants cannot be interviewed more than once for the same position.

For such multi-unit assignment problems, we would like to have a systematic (probabilistic) procedure to decide fairly which agents should get which objects, which, at the same time, does not offer incentives to coalitions of agents to lie about their true preferences. In this paper, I show that the egalitarian solution achieves this purpose for multi-unit assignment problems in the dichotomous preference domain, in which objects are either considered acceptable or not, and in which agents are indifferent between all objects that they find acceptable.

The egalitarian solution is based on the well-known leximin principle. In the domain of dichotomous preferences, it is Lorenz dominant, unique in utilities, and impossible to manipulate by groups. In contrast, the celebrated competitive equilibrium with equal incomes (Hylland and Zeckhauser, 1979; Budish, 2011; Reny, 2017) fails to satisfy these three properties. Both solutions are disjoint, meaning that in general we cannot obtain the *egalitarian* solution as a competitive equilibrium when agents are endowed with *equal incomes*. This is a stark difference with the single-unit case, in which both solutions coincide (Bogomolnaia and Moulin, 2004).

Lorenz dominance is "*a ranking generally accepted as the unambiguous arbiter of inequality comparison*" (Foster and Ok, 1999) and is "*widely accepted as embodying a set of minimal ethical judgements that should be made*" (Dutta and Ray, 1989). In our set-up, the fact that a utility profile is Lorenz dominant implies that it uniquely maximizes any strictly concave utility function representing agents' preferences and is, therefore, a strong fairness prop-

[1] Source: "The many roles of a Member of Congress", *Indiana University Center on Representative Government.*

erty.

Uniqueness of the solution (in the utility profile obtained) is also a desirable property because it provides a clear recommendation of how the resources should be split. On the contrary, a multi-valued solution leaves the schedule designer with the complicated task of selecting a particular division among those suggested by the solution, thus raising the possibility of justified complaints by some agents who may argue that other allocations were also recommended by the solution that were more beneficial to them.

It is equally interesting that the egalitarian solution is group strategy-proof, implying that coalitions of agents can never profit from misrepresenting their preferences. On the contrary, the competitive solution is manipulable by groups in this set-up, as in many others. Yet, it is remarkable that even in our small dichotomous preference domain, where the possibilities to misreport are very limited, the pseudo-market solution can still be manipulated by coalitions of agents.

The fact that the egalitarian solution satisfies these three desirable properties is a strong argument for recommending its use whenever agents have dichotomous preferences.

The dichotomous preference domain is admittedly simple and not suitable for modelling problems in which objects are either complements or substitutes. However, this set-up is helpful to represent scheduling problems (such as the tennis allocation problem studied by Maher, 2016; see Table 1), in which agents are either compatible or incompatible with each object and want to maximize the number of objects they obtain, or for the aforementioned problems of assigning job interviews to candidates or seats for performances to the public, among others.[2]

Focusing on this particular domain of preferences will be helpful to show the properties of the egalitarian solution, while, at the same time, it will make the problem complicated enough to identify why the competitive equilibrium with equal incomes fails to be unique and group strategy-proof. The reason behind it is that the number of identical copies available of some objects (their supply) equals their total demand. I call these objects perfect. Because their demand is always equal to their supply, they can have a zero competitive price. However, they could also have a positive price, thus generating a large set of competitive equilibria (Table 2). Therefore, a coalition of agents can collectively misrepresent their preferences in order to make a set of objects

[2]Versions of the egalitarian solution have already been suggested and implemented in scheduling and related assignment problems (Lemaître et al., 2003; Ibing and Boche, 2007; Kurokawa et al., 2015; Maher, 2016).

Table 1: An example of a multi-unit assignment with dichotomous preferences is the tennis scheduling problem in Maher (2016). Players want to maximize the number of games they play on days in which they are available.

Players \ Days	Mon	Tues	Wed	Thurs	Fri	Times
Barry T	0	0	1	1	0	2
Tom B	1	1	0	1	0	3
Gordon B	0	0	0	0	1	1
Peter W	1	1	0	0	0	2
Colin C	1	0	0	1	0	2
Mike M	0	1	1	1	1	4
Keith I	0	1	1	0	0	2
Alan C	1	0	0	1	0	2
John S	0	1	0	0	0	1
Keith B	1	0	1	0	0	2
George StC	1	1	1	1	0	4
Michael L	0	0	1	0	0	1
Phil M	0	1	0	0	0	1
Brian F	1	1	0	0	0	2
Peter K	0	1	0	1	0	2
Willie McM	0	0	0	1	0	1
Ken L	0	1	0	0	0	1
Slots available	4	8	4	8	0	

perfect to reduce their price. Manipulating agents can benefit unambiguously from collectively reducing their demand for perfect objects (Table 4).

Structure of the Paper. Section 2 discusses the related literature. Section 3 formalizes the model. Section 4 introduces our solutions. Section 5 discusses incentives properties. Section 6 discusses the role of perfect objects. Section 7 concludes.

2. Related Literature

The model I study is closely related to two existing problems, namely:

1. *Single-unit random assignment with dichotomous preferences* by Bogomolnaia and Moulin (2004), henceforth BM04. Our model generalizes theirs in that agents can receive more than one object. They study the egalitarian and the equal income competitive solution. They show that the egalitarian solution is Lorenz dominant and can always be supported by competitive prices. Because the competitive solution is Lorenz dominant, the competitive

solution can easily be computed as the maximization of the Nash product of agents' utilities. They also prove that the egalitarian solution is group strategy-proof.

Roth et al. (2005) show that the egalitarian solution is also Lorenz dominant in assignment problems on arbitrary graphs that are not necessarily bipartite. They use dichotomous preferences to model whether a person is compatible with a particular organ for transplantation. Assignment on the dichotomous domain of preferences has been further studied by Bogomolnaia et al. (2005), Katta and Sethuraman (2006), and Bouveret and Lang (2008).

Kurokawa et al. (2015) also study a single-unit assignment problem in which agents can derive a utility equal to one or zero. This is, if an agent demands 10 objects, he obtains the same zero utility if he receives 9, 2 or 0 objects, whereas in this paper agents' utility is linear on the goods they find acceptable. They consider a broad preference domain satisfying convexity, equality, shifting allocations and optimal utilization. They show that the egalitarian solution is not Lorenz dominant in this larger domain. They allow for non individually rational allocations. In their set-up, an agent who wants an apple but who dislikes a pear may in fact get the pear. In contrast, in BM04 and the model in this paper, agents cannot receive objects they do not find acceptable, i.e. their property "*shifting allocations*" does not apply.

2. *The course allocation problem (CAP)* described by Brams and Kilgour (2001); Budish (2011); Budish and Cantillon (2012); Kominers et al. (2010); Krishna and Ünver (2008); and Sönmez and Ünver (2010), with some important differences. First, in CAP, students may have arbitrary preferences over the set of objects, which are considerably more complex than those I study in this paper. However, reporting combinatorial preferences is infeasible for even few alternatives, and, in practice, combinatorial mechanisms never allow agents to fully report such preferences, not only because such revelation would be complicated, but also because agents may not know their preferences in such detail. Consequently, a new strand of theory has focused on allocation mechanisms with simpler preferences (e.g. Bouveret and Lemaître, 2016; Bogomolnaia et al., 2017, 2019)). Although the dichotomous preference domain is smaller than those considered in CAP, it is not contained in any of those because CAP rules out indifferences.

Furthermore, Budish (2011) only considers deterministic assignments. I instead study randomized assignments: in practice, many allocation mechanisms use randomization to achieve a higher degree of fairness.

Two papers in this literature that focus on a smaller domain of preferences are Kojima (2009) and Heo (2014). In these two papers agents have no limits on the number of objects which they can receive. Kojima studies a multi-unit assignment problem in which the demand of objects equals its supply. Agents' preferences over objects are strict, and preferences over bundles are linear. He shows that no assignment mechanism is efficient, envy-free and weakly strategy-proof. In a similar environment, Heo characterizes the only type of mechanism that satisfies versions of efficiency, incentive compatibility, no envy and population consistency.

3. Model

I consider the allocation of m objects (each with possibly several copies of itself) to n agents. Up to q_k copies of object $k \in M$ can be assigned to the set of agents N. I refer to the integer vector $q = (q_1, \ldots, q_m)$ as **objects' capacities**.

Agents' preferences over objects are given by a $n \times m$ binary matrix R. Each entry $r_{ik} = 1$ if agent i finds object k acceptable and 0 otherwise.[3] Slightly abusing the notation, R_{iM} (resp. R_{Nk}) denotes both the i-th row (resp. k-th column) of R and the set of objects (resp. agents) for which $r_{ik} = 1$. I assume $|R_{Nk}| \geq q_k$ for each object k.[4] A pair (R, q) is called a **multi-unit assignment problem (MAP)**.

A **random assignment matrix (RAM)** for an MAP (R, q) is a matrix Z of size $n \times m$ satisfying the following conditions $\forall i \in N, k \in M$

$$\text{Feasibility} \quad \begin{cases} 0 \leq z_{ik} \leq 1 \\ \sum_{i \in N} z_{ik} \leq q_k \end{cases} \tag{1}$$

$$\text{Individual Rationality} \quad \begin{cases} z_{ik} > 0 \text{ only if } r_{ik} = 1 \end{cases} \tag{2}$$

An RAM's entries indicate what probability each agent has of obtaining one unit of each object. The feasibility conditions ensure that no agent obtains more than one unit of each object, and that the total number of units assigned of each object is less than its capacity. Similarly, individual rationality guarantees that each agent only obtains shares from acceptable objects. Throughout the paper, I only consider assignments satisfying these two properties. As before, the notation Z_{iM} (resp. Z_{Nk}) denotes both the i-th row (resp. k-th column) of Z and the set of objects (resp. players) for which $z_{ik} = 1$. The **matching size** $\nu(R, q) = \sum_{k \in M} q_k$ of an MAP represents the maximum number of object units that can be assigned. The set of RAMs

[3] R can also be understood to represent either allocation or physical constraints.
[4] This assumption is relaxed in Section 7.

for (R, q) of maximal size is denoted by

$$\mathcal{Z}(R, q) = \{Z \in \mathbb{R}^{n \times m} : \sum_{i \in N} \sum_{k \in M} z_{ik} = \nu(R, q)\} \tag{3}$$

Kojima (2009) and Budish et al. (2013) extend the well-known Birkhoff-von Neumann decomposition Theorem to prove the following Lemma.

Lemma 1. *Any RAM can be decomposed into a convex combination of binary RAMs, and can thus be implemented.*

I assume that agents are indifferent between objects that they find acceptable, and that they want to maximize the number of acceptable objects they obtain. The canonical utility function representing those preferences is

$$u_i(Z) = \sum_{k \in M} z_{ik} \tag{4}$$

for an arbitrary agent i. This function is clearly not unique but it is convenient to work with. The preference relation represented by this function is a complete order over all RAMs, and implies that an RAM Z is Pareto optimal for an MAP (R, q) if and only if $\sum_{i \in N} \sum_{k \in M} z_{ik} = \nu(R, q)$.

The set of efficient utility profiles $\mathcal{U}(R, q)$ can be described as

$$\mathcal{U}(R, q) = \{U \in \mathbb{R}^n \mid \exists\, Z \in \mathcal{Z}(R, q) : U_i = \sum_{k \in M} z_{ik},\ \forall i \in N\} \tag{5}$$

I do not distinguish between ex-ante and ex-post efficiency because in the dichotomous preference domain they coincide. This equivalence occurs because the sum of utilities is constant in all efficient assignments.[5] In our set-up, efficiency simply requires that no object is wasted.

A **welfarist solution** is a mapping Φ from (R, q) to a set of efficient utility profiles in $\mathcal{U}(R, q)$, and hence, it only focuses on the expected number of objects received by an agent and not on the exact probability distribution over deterministic assignments. Whenever a solution is single-valued I instead use the notation ϕ.

3.1. Perfect Objects and Perfect Extensions

We can partition the set of objects M into two subsets $\mathcal{P}(R, q)$ and $\mathcal{O}(R, q)$, which are called **perfect** and **over-demanded**, respectively. The

[5] Ex-ante and ex-post efficiency are equivalent in assignment problems with dichotomous preferences (BM04, Roth et al., 2005).

set of perfect objects is defined as

$$\mathcal{P}(R, q) = \{k \in M : |R_{Nk}| = q_k\} \tag{6}$$

The vectors $q_{\mathcal{P}(R,q)}$ and $q_{\mathcal{O}(R,q)}$ denote the capacities of perfect and over-demanded goods, respectively. Given a MAP (R, q), a **perfect extension** for agent i represents adding an arbitrarily perfect object k' that agent i finds acceptable. Formally, a perfect extension for agent i in a MAP (R, q) is a pair $([R\,R_{Nk'}], \overline{q})$ where $R_{Nk'}$ is a binary matrix of size $n \times 1$ such that $r_{ik'} = 1$, $[R\,R_{Nk'}]$ denotes the $n \times (m + 1)$ juxtaposition of the two matrices and $\overline{q} = (q_1, \ldots, q_m, |R_{Nk'}|)$.

4. Three Efficient Solutions

4.1. The Egalitarian Solution

An intuitive solution equalizes agents' utilities as much as possible respecting efficiency and individual rationality: this is the well-known leximin solution. I refer to it as the **Egalitarian Solution (ES)**, proposed theoretically by BM04, and applied to kidney exchange by Roth et al. (2005) and Yılmaz (2011).

To define it formally, let $\succ^l$ be the well-known lexicographic order.[6] For each $U \in \mathbb{R}^n$, let $\gamma(U) \in \mathbb{R}^n$ be the vector containing the same elements as U but sorted in ascending order, i.e. $\gamma_1(U) \leq \ldots \leq \gamma_n(U)$. The leximin order $\succ^{LM}$ is defined by $U \succ^{LM} U'$ if and only if $\gamma(U) \succ^l \gamma(U')$. The ES is defined by

$$\phi^{\text{ES}}(R, q) = \arg\max_{\succ^{LM}} \mathcal{U}(R, q) \tag{7}$$

The ES satisfies a strong fairness notion called **Lorenz dominance**, defined as follows. Define the order $\succ^{ld}$ on $\mathbb{R}^n$ so that for any two vectors U and U', $U \succ^{ld} U'$ if $\sum_{i=1}^t U_i \geq \sum_{i=1}^t U'_i \;\; \forall t \leq n$, with strict inequality for some t. We say that U Lorenz dominates U', written $U \succ^{LD} U'$, if $\gamma(U) \succ^{ld} \gamma(U')$. A vector $U \in \mathcal{U}(R, q)$ is Lorenz dominant for an MAP (R, q) if it Lorenz dominates any other vector in $\mathcal{U}(R, q)$.

Lorenz dominance is a partial order in $\mathcal{U}(R, q)$ and therefore a Lorenz dominant utility profile need not exist. Nevertheless, the ES solution is Lorenz dominant.

Theorem 1. *The ES solution is well-defined and Lorenz dominant in the set of efficient utility profiles.*

[6] So that for any two vectors $U, U' \in \mathbb{R}^n$, $U \succ^l U'$ only if $U_t > U'_t$ for some integer $t \leq n$, and $U_p = U'_p$ for any positive integer $p < t$.

I prove Theorem 1 using Theorem 3 in Dutta and Ray (1989), which states that the core of specific cooperative games has a Lorenz dominant element. The construction of the corresponding cooperative game can be found in the Appendix.

4.2. The Constrained Competitive Equilibrium with Equal Incomes

A second solution, which is substantially more complicated, requires to balance the supply and demand for goods when agents are endowed with equal budgets. These equal budgets are often normalized to one currency unit, a normalization that I also use. This solution is known as the **Competitive Equilibrium with Equal Incomes (CEEI)** (Varian, 1974; Hylland and Zeckhauser, 1979). In MAPs, each agent can consume at most one unit of each object, hence having particular constraints on their consumption set. I use the term **Constrained Competitive Equilibrium** (**CCE**, still with equal incomes) from now on to make this distinction obvious. The CCE solution is different from the CEEI as defined in Hylland and Zeckhauser (1979) in that in our case agents never partially consume objects that have different prices (see Table 1 in their paper).[7] This distinction justifies the different terminology of CCE. The CCE constraints are justified in my view, since a rational agent should consume cheaper goods instead of expensive ones, given that she derives the same utility from both.

Definition 1. A CCE for an MAP (R, q) is a pair of an RAM Z^* and a non-negative price vector p^* such that, $\forall i \in N$, agents maximize their utilities

$$Z^*_{iM} \in \arg \max_{Z_{iM} \in \beta_i(p^*)} u_i(Z_{iM}) \tag{8}$$

where $\beta_i(p)$ is the budget set defined as $\beta_i(p) = \{Z_{iM} \mid \sum_{k \in M} z_{ik} \leq |R_{iM}| : p \cdot Z_{iM} \leq 1\}$, and the market clears, so that

$$Z^* \in \mathcal{Z}(R, q) \tag{9}$$

[7] If an apple pie has a higher price than a pear pie, and the agent values them equally, the agent either consumes the pear pie with probability one and eats the apple pie with positive probability, or the agent only consumes the pear pie with a positive probability and the apple pie with probability zero. What can never occur in a CCE is that an agent consumes both pies with positive probability smaller than one.

The optimality conditions of CCE imply

$$
\begin{aligned}
k \notin \mathcal{P}(R,q) \quad &\Longrightarrow \quad p_k^* > 0 && (10)\\
z_{ik}^*, z_{ik'}^* \in (0,1) \quad &\Longrightarrow \quad p_k^* = p_{k'}^* && (11)\\
[k,k' \in R_{iM}] \wedge [p_k^* < p_{k'}^*] \wedge [0 < z_{ik'}^*] \quad &\Longrightarrow \quad z_{ik}^* = 1 && (12)\\
\sum_k z_{ik}^* < |R_{iM}| \quad &\Longrightarrow \quad \sum_k p_k^* \cdot z_{ik}^* = 1 && (13)
\end{aligned}
$$

Condition (10) allows a zero price only for perfect objects, while expression (11) forces the same marginal benefit for every object that agents obtain partially but not fully. Condition (12) requires agents to exhaust desirable cheaper goods before they consume more expensive ones, whereas expression (13) establishes that if an agent has not consumed all the goods that she finds desirable, then she must have spent all her budget.

Given an MAP, I denote the set of pairs (Z^*, p^*) as $\mathcal{C}(R,q)$. The CCE solution is defined by

$$\Phi^{CCE}(R,q) = \{u(Z') \mid \exists\, p' : (Z',p') \in \mathcal{C}(R,q)\} \tag{14}$$

In our set-up, the CCE solution is always non-empty (and often multi-valued). We provide a direct proof using a standard fixed point argument.[8]

Theorem 2. *In any MAP (R,q), the CCE exists and thus the CCE solution is non-empty.*

4.3. The Egalitarian per Object Solution

Finally, a highly intuitive solution breaks up the allocation problem into m sub-problems of assigning q_k units of object k into R_{Nk}, distributing an equal share of object k among all agents who find it acceptable. I call this solution **Egalitarian Per Object (EPO)**. Given an MAP (R,q), the EPO solution assigns to each agent

$$\phi_i^{EPO}(R,q) = \sum_{k \in M} r_{ik} \cdot \frac{q_k}{|R_{Nk}|} \tag{15}$$

In the dichotomous preference domain, EPO is equivalent to the well-

[8]The existence of CCE can also be proven using Theorem 1 in Mas-Colell (1992), who shows the existence of a competitive equilibrium when preferences are convex, possibly satiated and continuous but not necessarily monotonic; for an application of this Theorem see p. 1853 in Bogomolnaia et al. (2017).

known random serial dictatorship (RSD, aka random priority).[9] To observe this equivalence, note that in RSD, each agent gets one unit of good k whenever she is among the first q_k agents among R_{Nk} in a random order. Thus, in expectation she gets

$$\frac{q_k}{|R_{Nk}|} \cdot 1 + \left(1 - \frac{q_k}{|R_{Nk}|}\right) \cdot 0 = \frac{q_k}{|R_{Nk}|} \tag{16}$$

and thus the total utility of agent i in RSD equals $\sum_{k \in M} r_{ik} \cdot \frac{q_k}{|R_{Nk}|}$. Note that EPO ignores the interaction between the m assignment problems corresponding to each object, a property to which we will come back to.[10]

4.4. Two Examples Showing that All Solutions Are Disjoint

Example 1 (EPO is disjoint from multi-valued CCE and ES)**.** Table 2 shows the different outcomes that our three solutions produce for a problem with $n = 6$, $m = 3$, and (R, q) given in subtable 2a. The CCE utilities are written in brackets in subtable 2b because there are CCE that support utility profiles between $(2.4, 1.4, 1)$ and $(2.25, 2, 1)$ with $0 \leq p_\gamma \leq \frac{4}{9}$.

Table 2: CCE is multi-valued and disjoint from EPO.

$N \backslash M$	α	β	γ	Total
a, b, c, d	1	1	1	3
e	1	1	0	2
f	1	0	0	1
Total	6	5	4	
q	4	4	4	

(a) Corresponding R matrix.

N	ES	CCE	EPO
a, b, c, d	2.25	[2.25 - 2.4]	2.47
e	2	[1.4 - 2]	1.47
f	1	1	0.67

(b) Utility profiles for each solution.

In EPO, every agent who desires good α gets $\frac{4}{6}$ of it, everybody who desires good β gets $\frac{4}{5}$ of it and everybody who desires good γ gets a full unit (because good γ is perfect). Thus, agent f who only desires good α obtains a utility of $\frac{4}{6} = 0.67$. However, CCE gives one unit of object α to agent f. To see this, note that

[9] EPO would not be efficient in a more general domain of preferences. The equivalence with random priority would also disappear.

[10] One could also consider other solutions discussed in the literature, in particular the probabilistic serial rule, defined by Bogomolnaia and Moulin (2001). The probabilistic serial rule is appealing in scenarios where different notions of efficiency do not coincide. This is not the case for MAPs.

1. $p^*_\alpha = p^*_\beta$, because otherwise agents $a{:}e$ should only consume the cheaper good, or should exhaust it, by conditions (11,12). If they only consume the cheaper good, the assignment is not efficient because the expensive good is not fully assigned. If they exhaust the cheaper good, we have that more than 4 units were assigned of such good, violating feasibility.

2. By contradiction, assume $z^*_{f\alpha} < 1$. Then, by condition (13), $p^*_\alpha z^*_{f\alpha} = 1$. Because agent f cannot afford one unit of good α, neither can agents $a{:}e$, nor they can afford one unit of good β. But then the assignment is inefficient, because there are 8 units available of goods α and β, but agents $a{:}f$ can only afford 6 of them. This implies that $z^*_{f\alpha}$ was not a CCE.

In the previous example, we saw that there are no CCE prices that support the EPO outcome and thus is an argument against this solution, as competitive equilibria are considered "*essentially the description of perfect justice*" (Arnsperger, 1994), and the base of Dworkin's "*equality of resources*" (Dworkin, 1981). But interestingly, the ES solution can also produce outcomes that cannot be supported as a CCE, as I show in the following Example. Note that in the single-unit case (Theorem 1 in BM04), the ES is always supported by competitive prices.

Example 2 (ES differs from CCE)**.** I show this using a MAP with $n = 9$, $m = 6$, and (R, q) given in subtable 3.

Table 3: ES and CCE are disjoint (a RAM supporting ES appears in parenthesis).

$N\backslash M$	α	β	γ, δ	ϵ, θ	Total
a, b, c	1 (1)	1 (1)	0	0	2 (2)
d	0	1 (.5)	1 (1)	0	3 (2.5)
e	0	1 (.5)	0	1 (1)	3 (2.5)
f, g, h, i	1 (.25)	0	1 (.75)	1 (.75)	5 (3.25)
Total	7	5	5	5	
q	4	4	4	4	

If the ES solution (2, 2.5, 2.5, 3.25) could be supported as a CCE, then $p^*_\alpha = p^*_\gamma = p^*_\delta = p^*_\epsilon = p^*_\theta$ because agents $f{:}i$ obtain those objects with positive probability but do not exhaust them (condition (11)). Furthermore, agents d, e and f, g, h, i must spend their whole budget because they do not consume all the goods they desire (condition (13)), and thus

$$0.5p^*_\beta + 2p^*_\alpha = 1 \quad (17)$$

$$3.25p^*_\alpha = 1 \quad (18)$$

which yield $p^*_\alpha = \frac{4}{13}$ and $p^*_\beta = \frac{10}{13}$. However, at such prices, the ES utility for agents a:c is unaffordable: it costs $\frac{14}{13}$ currency units.

The fact that ES and CCE do not coincide is interesting: in the non constrained context, the competitive solution can be computed by maximizing the Nash product, solving what is known as the Eisenberg-Gale program (see chapter 7 in Moulin, 2003 for a textbook treatment). The Eisenberg-Gale program is otherwise a rather robust result since it extends to a large family of utility functions beyond the linear case (Jain and Vazirani, 2010), as well as to the mixed division of objects and bads (Bogomolnaia et al., 2017).

4.5. Envy

It is easy to see that our three solutions are envy-free. A solution ϕ is **envy-free** if, for any MAP (R, q) with agents i and j such that $R_{iM} \subseteq R_{jM}$, $\phi_i(R, q) \leq \phi_j(R, q)$. For the multi-valued CCE, envy-freeness holds for any selection from it.[11]

Lemma 2. *ES, CCE and EPO are envy-free.*

5. Manipulation by Groups

I consider agents' manipulation in the direct revelation mechanism associated with each solution. For this purpose, we need to know exactly how objects are assigned. A **detailed solution** ψ maps every MAP (R, q) into an RAM $Z \in \mathcal{Z}(R, q)$, specifying which share of each object is allocated to each agent, whereas a welfarist solution ϕ maps every MAP into a utility profile $U \in \mathcal{U}(R, q)$ and only tells us the expected number of objects received by each agent. Every detailed solution ψ projects onto the welfarist solution $\phi(R, q) = u(\psi(R, q))$. The direct revelation mechanism associated with a detailed solution ψ is such that all agents reveal their preferences R_{iM}, and then ψ is applied to the corresponding MAP (R, q), implementing the RAM $\psi(R, q) = Z$.

I assume that agent i with the true preferences R_{iM} can only misrepresent her preferences by understating the number of objects that she finds acceptable, i.e. by declaring a preference profile $R'_{iM} \subset R_{iM}$ (we then say that R'_{iM} is IR for R_{iM}). I use this assumption for two reasons. The first is theoretical: I have not specified the dis-utility that the consumption of an undesirable object brings to an agent, as I have only focused on individually rational assignments. I would need to specify such dis-utility to

[11]There is no efficient solution that is strongly envy-free, i.e. that for any MAP (R, q) with agents i and j such that $|R_{iM}| < |R_{jM}|$, $\phi_i(R, q) \leq \phi_j(R, q)$ (Ortega, 2016).

analyse the manipulation of a solution by exaggerating the set of acceptable objects (or alternatively allow agents to disregard shares of undesirable objects, creating inefficiencies). The second reason is that such an assumption has already been imposed in the study of scheduling problems (e.g. Koutsoupias, 2014). In many scheduling problems motivating MAPs, cancelling consumption could be strongly punished by the central clearinghouse, particularly when other agents' consumption depends on other agents exhausting their bundles (no double tennis match can be made with only 3 out of 4 players).[12]

A detailed solution ψ is **group strategy-proof** if for every MAP (R, q) and every coalition $S \subset N$, $\nexists$ R' satisfying i) $R'_{jM} = R_{jM}$ $\forall j \notin S$, and ii) R'_{SM} is IR for R_{SM}, such that

$$\forall i \in S, \quad u_i(\psi(R', q)) \geq u_i(\psi(R, q)) \tag{19}$$

with strict inequality for at least one agent in S. A welfarist solution ϕ is **group strategy-proof** if every detailed solution ψ projecting onto ϕ is group strategy-proof.

BM04 show that no deterministic solution is group strategy-proof when agents can obtain at most one object. Deterministic solutions include priority ones, i.e. those in which agents choose sequentially their most preferred available bundle according to some pre-specified order. The reason of why deterministic solutions are manipulable by groups is that the agent with the highest priority could change her report and still receive one acceptable alternative, leaving her utility unchanged and, at the same time, benefiting an agent with low priority.[13]

This argument does not extend to MAPs. Because agents can obtain multiple objects, the agent with higher priority can belong to a manipulating coalition only by claiming fewer objects. But since she has the highest priority, it is immediate that such manipulation would always give her strictly less utility, so she cannot be in the coalition. The same argument applies to all remaining agents and, consequently,

Lemma 3. *Any deterministic priority solution is group strategy-proof.*

The previous Lemma shows that group strategy-proofness is relatively

[12]BM04 allow assignments in which agents receive undesirable objects with positive probability, yet these allocations never occur ex-post due to the individual rationality constraints. Restricting the set of manipulations to understating acceptable objects makes group strategy-proofness a weaker property that is easier to satisfy.

[13]This property is known as bossiness in the literature.

easy to achieve for MAPs in the dichotomous domain. In fact, it is evident that EPO is also group strategy-proof and I show below that the ES solution is also group strategy-proof. Is CCE also group strategy-proof? There are two extensions of our group strategy-proofness definition to set valued solutions.

The first requires that for every MAP (R, q), there is no competitive equilibrium of the manipulated MAP (R', q) that is weakly better than every competitive equilibria of the original problem (R, q), for every member of the manipulating coalition S. A stronger extension is that there is at least one competitive equilibrium of (R, q) which yields a weakly higher utility than some competitive equilibrium of (R', q), with strict inequality for at least one member of the deviating coalition S. It turns out that CCE violates both conditions. The reason for this is that a group can coordinate to make several objects perfect, thus allowing those objects to have a zero price.

Theorem 3. *ES and EPO are group strategy-proof but CCE is not.*

The proof of ES being group strategy-proof can be found in the Appendix, but I show that CCE is unambiguously manipulable by groups below.

Example 3 (CCE not group strategy-proof)**.** Let $n = 7$, $m = 4$, and (R, q) given by Table 4.

Table 4: CCE not group strategy-proof.

$N \backslash M$	α	β	γ	δ	Φ^{CCE}
$\boldsymbol{a}$	1	**1**	1	1	**2.5**
$\boldsymbol{b}$	1	1	**1**	1	**2.5**
$\boldsymbol{c}$	1	1	1	**1**	**2.5**
d	1	0	1	1	2.5
e	1	1	0	1	2.5
f	1	1	1	0	2.5
g	1	0	0	0	1
Total	7	5	5	5	
q	4	4	4	4	

(a) True preferences R.

α	β	γ	δ	Φ^{CCE}
1	**0**	1	1	**[2.5 - 2.57]**
1	1	**0**	1	**[2.5 - 2.57]**
1	1	1	**0**	**[2.5 - 2.57]**
1	0	1	1	[2.5 - 2.57]
1	1	0	1	[2.5 - 2.57]
1	1	1	0	[2.5 - 2.57]
1	0	0	0	[0.57 - 1]
7	4	4	4	
q 4	4	4	4	

(b) Misreport R' for $S = \{a, b, c\}$.

Consider the coalition $S = \{a, b, c\}$. When agents submit their real preferences, there exists a unique CCE that supports the ES solution: agents a, b, and c obtain 2.5 expected objects. By changing their report each for a different object, as in subtable 4b, they make objects β, γ and δ perfect, consequently enlarging the set of CCE solutions, which includes utilities that are always weakly above 2.5 and up to 2.57. By misrepresenting and creating artificially perfect objects, they allow those to be priced at 0, weakly increasing the number of expected objects received in any competitive equilibria of (R', q), at the expense of agents with limited acceptable objects, in this case g.

I do not discuss strategy-proofness (manipulation by individuals on their own) since it is immediate that ES, CCE and EPO are strategy-proof. For CCE, we can construct a selection of it that is strategy-proof, since reducing the total demand for an object either reduces its price, relatively increasing the price of other objects, or leaves its price unchanged.

Efficiency, fairness, and non-manipulability are standard goals in the design of resource allocation mechanisms. Before concluding, I discuss a new goal that arises naturally for MAPs.

6. Independence of Perfect Objects

Some solutions do not depend on the number of perfect objects desired by an agent. If an agent finds a new perfect object to be acceptable, we could expect that she would always receive one extra expected unit. This is what our following property captures.

A solution ϕ is **independent of perfect objects (IPO)** if, for every MAP, every $i \in N$ and for any of its perfect extensions $([R\ R_{Nk'}], \overline{q})$,

$$\phi_i(R, q) + 1 = \phi_i([R\ R_{Nk'}], \overline{q}) \tag{20}$$

IPO is a desirable property for two reasons. First, perfect objects belong unambiguously to agents who find them acceptable, so they can argue that they should obtain them fully, irrespective of the share they obtain from over-demanded objects. Second, if the clearinghouse uses a solution that was not IPO, the set of agents who find perfect objects acceptable could avoid reporting their demand for perfect objects and obtain them fully outside the centralized mechanism, a concern for scheduling applications in which agents may organize teamwork activities on their own.[14]

[14]IPO bears some resemblance with the *composition up* axiom in the claims problem in that both require some consistency of the solution when more resources become available (Young, 1988; Moulin, 2000). Important differences between these axioms are that IPO does not increase the number of units available of each good, but rather the number of goods available. Second, the outcome of a claims problem is an allocation profile that can be used as a claims profile too, whereas the outcome of a MAP is a utility profile that cannot be used as a preference matrix R.

As we discussed before, EPO ignores the interaction between the assignment of each object and clearly satisfies IPO. CCE also (partially) satisfies this requirement.

Lemma 4. *EPO satisfies IPO but ES does not satisfy IPO. There exists a selection of CCE that satisfies IPO.*

The proof of Lemma 4 makes evident that *Lorenz dominance is incompatible with IPO*. To see this, consider a simple MAP with just one good, that is both over-demanded and acceptable to every agent. Lorenz dominance requires that, after adding a perfect object to the original MAP, the shares of the over-demanded good increase for the agents who do not find the new perfect object acceptable. However, this is a violation of IPO, which requires that such shares remain constant.

Lemma 4 also highlights that CCE can always assign a zero price to all perfect objects: this is how we construct the selection of CCE that satisfies IPO. But it may also assign a zero price to some perfect objects only, or to no perfect object at all. The designer has a high degree of flexibility in choosing the equilibrium prices.

The selection problem extends to Budish's (2011) competitive mechanism for CAP in which students reveal their preferences to a centralized clearinghouse which announces a corresponding equilibrium allocation. Budish argues that this mechanism is transparent, meaning that students can verify that the allocation is an equilibrium. But the mechanism can be "manipulated from the inside", selectively assigning zero prices to hand-picked courses, while at the same time rightly arguing that it produces a competitive allocation.

If IPO must be achieved (a decision depending on the context and the designer's objectives), we would like to have a solution that, at the same time, avoids the multiplicity problem of the CCE, while being envy-free and as fair as possible. Such a solution exists: we call it the **refined egalitarian solution (ES*)**. To define it, we use the partition of M into $\mathcal{P}(R,q)$ and $\mathcal{O}(R,q)$, and split the original MAP (R,q) into two independent problems $(R_{N\mathcal{P}(R,q)}, q_{\mathcal{P}(R,q)})$ and $(R_{N\mathcal{O}(R,q)}, q_{\mathcal{O}(R,q)})$, which correspond to the independent MAPs with perfect and over-demanded objects, respectively. ES* is given by

$$\phi_i^{\text{ES}^*}(R,q) = \phi_i^{\text{ES}}(R_{N\mathcal{O}(R,q)}, q_{\mathcal{O}(R,q)}) + \left|R_{i\,\mathcal{P}(R,q)}\right| \tag{21}$$

ES* takes the egalitarian solution for the MAP with over-demanded objects only, and adds the number of perfect objects desired by the agent. ES* is close to a suggestion in Budish (2011). Noting that some courses may be in excess supply, he proposes to run the allocation mechanism only on the set of over-demanded courses: "*if some courses are known to be in substantial excess supply, we can reformulate the problem as one of allocating only the potential scarce courses*". ES* formalizes this suggestion. It also satisfies several desiderata.

Lemma 5. *The ES* solution is well-defined and single-valued, efficient, IPO, envy-free, and Lorenz dominant for the problem* $(R_{N\mathcal{O}(R,q)}, q)$.

It is immediate that ES* is single-valued, efficient and IPO. The remaining properties are straightforward modifications of the proofs of Lemmas 1 and 2 and Theorem 1. Unfortunately, the properties in Lemma 5 come at a cost: ES* is not group strategy-proof.[15] ES* can be manipulated by groups reducing their demand in order to make some objects perfect. Therefore, the members of the manipulating coalition obtain those objects fully, while also obtaining an egalitarian fraction of the remaining over-demanded problem.

In fact, these three properties are incompatible, namely restricted Lorenz dominance in the overdemanded problem, group strategy-proofness and IPO. To see this, let ϕ be a solution that is restricted Lorenz dominant and IPO, and consider the original manipulated preference matrices R and R' in Example 3. By restricted Lorenz dominance and IPO, $\phi(R,q) = (2.5, 2.5, 2.5, 2.5, 2.5, 2.5, 1)$ and $\phi(R',q) = (2.57, 2.57, 2.57, 2.57, 2.57, 2.57, 0.57)$. But then the coalition of agents $\{a, b, c\}$ have incentives to misrepresent their preferences and thus any solution that is restricted Lorenz dominant and IPO is manipulable by groups.

However, group strategy-proofness and IPO are compatible. EPO satisfies them both (plus uniqueness in utilities), and thus is a reasonable alternative to ES if independence of perfect objects is more desirable than Lorenz dominance. Table 5 below summarizes the properties of the solutions discussed in this paper.

7. Conclusion

For multi-unit assignment problems with dichotomous preferences, the egalitarian solution is Lorenz dominant, single-valued and group strategy-proof. For these reasons, it stands as a more reasonable solution than the celebrated competitive equilibrium with equal incomes, which fails these three desirable properties.

The egalitarian solution, however, fails to be independent of perfect objects. This property is incompatible with either Lorenz dominance or with

[15] For an example, use the MAP and manipulation R' illustrated in Example 4.

Table 5: Summary of the properties of each solution.

	ES	CCE	EPO	ES*
Lorenz Dominant	Yes	No	No	+
Single-Valued	Yes	No	Yes	Yes
Group Strategy-Proof	Yes	No	Yes	No
Independent of Perfect Objects	No	++	Yes	Yes
Envy-Free	Yes	Yes	Yes	Yes

+: ES* is Lorenz dominant for the corresponding MAP with over-demanded objects only.
++: there exists a selection of CCE that satisfies IPO.

the combination of group strategy-proofness and Lorenz dominance in the overdemanded part of the problem only.

I conclude with a brief discussion of two natural extensions.

1. *Some agents have restricted demands*, i.e. they find many objects acceptable but only want some of them. This extension leaves efficiency and fairness properties unchanged but makes ES and priority solutions fail group strategy-proofness. The reason is that both solutions become bossy: an agent who has his demand fulfilled does not care exactly which objects she receives, and can modify her demand to help another agent, as in BM04.

2. *Some objects are in excess supply.* If we dispose of over-supplied copies of such objects so to make them perfect, the efficiency and fairness properties are unaffected. A new form of manipulation emerges as groups of agents may reduce their demand to make some objects under-demanded and thus reducing the matching size of the MAP. But the extra copies will be disposed of and the object will become perfect. Lemma 6 case 3 shows that such manipulation is never successful, and thus the ES remains group strategy-proof. EPO trivially remains group strategy-proof.

Acknowledgements

This paper is based on the second chapter of my PhD thesis supervised by Hervé Moulin, to whom I am indebted for his valuable suggestions. I have also received helpful comments from anonymous referees, Anna Bogomolnaia, Bram Driesen, Georg Nöldeke, Fedor Sandomirskiy, Erel Segal-Halevi, Olivier Tercieux, and audiences at the Winter Meeting of the Econometric Society in Barcelona and the Tenth Conference on Economic Design at York. Sarah Fox and Christina Loennblad proof-read the paper. Errors are mine alone.

References

Arnsperger, C. (1994): "Envy-freeness and distributive justice," *Journal of Economic Surveys*, 8, 155–186.

Bogomolnaia, A. and H. Moulin (2001): "A new solution to the random assignment problem," *Journal of Economic Theory*, 100, 295 – 328.

——— (2004): "Random matching under dichotomous preferences," *Econometrica*, 72, 257–279.

Bogomolnaia, A., H. Moulin, F. Sandomirskiy, and E. Yanovskaia (2019): "Dividing bads under additive utilities," *Social Choice and Welfare*, 52, 395–417.

Bogomolnaia, A., H. Moulin, F. Sandomirskiy, and E. Yanovskaya (2017): "Competitive division of a mixed manna," *Econometrica*, 85, 1847–1871.

Bogomolnaia, A., H. Moulin, and R. Stong (2005): "Collective choice under dichotomous preferences," *Journal of Economic Theory*, 122, 165 – 184.

Bouveret, S. and J. Lang (2008): "Efficiency and envy-freeness in fair division of indivisible goods: logical representation and complexity," *Journal of Artificial Intelligence Research*, 32, 525–564.

Bouveret, S. and M. Lemaître (2016): "Characterizing conflicts in fair division of indivisible goods using a scale of criteria," *Autonomous Agents and Multi-Agent Systems*, 30, 259–290.

Brams, S. and M. Kilgour (2001): "Competitive fair division," *Journal of Political Economy*, 109, 418–443.

Budish, E. (2011): "The combinatorial assignment problem: approximate competitive equilibrium from equal incomes," *Journal of Political Economy*, 119, 1061–1103.

Budish, E. and E. Cantillon (2012): "The multi-unit assignment problem: theory and evidence from course allocation at Harvard," *American Economic Review*, 102, 2237–71.

Budish, E., Y.-K. Che, F. Kojima, and P. Milgrom (2013): "Designing random allocation mechanisms: theory and applications," *American Economic Review*, 103, 585–623.

Debreu, G. (1982): "Existence of competitive equilibrium," in *Handbook of Mathematical Economics*, ed. by K. Arrow and M. Intrilligator, Elsevier, vol. 2, 697 – 743.

Dutta, B. and D. Ray (1989): "A concept of egalitarianism under participation constraints," *Econometrica*, 57, 615–635.

Dworkin, R. (1981): "What is equality? Part 2: equality of resources," *Philosophy & Public Affairs*, 10, 283–345.

Foster, J. and E. Ok (1999): "Lorenz dominance and the variance of logarithms," *Econometrica*, 67, 901–907.

Heo, E. J. (2014): "Probabilistic assignment problem with multi-unit demands: A generalization of the serial rule and its characterization," *Journal of Mathematical Economics*, 54, 40 – 47.

Hylland, A. and R. Zeckhauser (1979): "The efficient allocation of individuals to positions," *Journal of Political Economy*, 87, 293–314.

Ibing, A. and H. Boche (2007): "Fairness vs. efficiency: Comparison of game theoretic criteria for OFDMA scheduling," in *41st Asilomar Conference on Signals, Systems and Computers*, 275–279.

Jain, K. and V. Vazirani (2010): "Eisenberg – Gale markets: algorithms and game-theoretic properties," *Games and Economic Behavior*, 70, 84 – 106.

Katta, A. and J. Sethuraman (2006): "A solution to the random assignment problem on the full preference domain," *Journal of Economic Theory*, 131, 231 – 250.

Kojima, F. (2009): "Random assignment of multiple indivisible objects," *Mathematical Social Sciences*, 57, 134 – 142.

Kominers, S., M. Ruberry, and J. Ullman (2010): "Course allocation by proxy auction," in *Proceedings of the Sixth International Workshop on Web and Internet Economics, WINE 2010*, 551–558.

Koutsoupias, E. (2014): "Scheduling without payments," *Theory of Computing Systems*, 54, 375–387.

Krishna, A. and U. Ünver (2008): "Improving the efficiency of course bidding at business schools: field and laboratory studies," *Marketing Science*, 27, 262–282.

KUROKAWA, D., A. PROCACCIA, AND N. SHAH (2015): "Leximin allocations in the real world," in *Proceedings of the Sixteenth ACM Conference on Economics and Computation*, ACM, 345–362.

LEMAÎTRE, M., G. VERFAILLIE, H. FARGIER, J. LANG, N. BATAILLE, AND J.-M. LACHIVER (2003): "Equitable allocation of earth observing satellites resources," in *5th ONERA-DLR Aerospace Symposium.*

MAHER, M. (2016): "A tennis assignment algorithm," *Mathematics Today*, 52, 130–131.

MAS-COLELL, A. (1992): *Equilibrium theory with possibly satiated preferences*, London: Palgrave Macmillan UK, 201–213.

MOULIN, H. (2000): "Priority rules and other asymmetric rationing methods," *Econometrica*, 68, 643–684.

——— (2003): *Fair division and collective welfare*, Cambridge, MA, USA: MIT Press.

ORTEGA, J. (2016): "Playing tennis without envy," *Mathematics Today*, 52, 268–269.

RENY, P. (2017): "Assignment problems," *Journal of Political Economy*, 125, 1903–1914.

ROTH, A., T. SÖNMEZ, AND U. ÜNVER (2005): "Pairwise kidney exchange," *Journal of Economic Theory*, 125, 151 – 188.

SÖNMEZ, T. AND U. ÜNVER (2010): "Course bidding at business schools," *International Economic Review*, 51, 99–123.

VARIAN, H. (1974): "Equity, envy, and efficiency," *Journal of Economic Theory*, 9, 63 – 91.

YILMAZ, Ö. (2011): "Kidney exchange: An egalitarian mechanism," *Journal of Economic Theory*, 146, 592 – 618.

YOUNG, P. (1988): "Distributive justice in taxation," *Journal of Economic Theory*, 44, 321–335.

Appendix: Proofs

Theorem 1 *The ES solution is well-defined and Lorenz dominant in the set of efficient utility profiles.*

Proof. I follow the construction of the proof of Theorem 1 in Bogomolnaia and Moulin (2004, p. 274) in which the original result of Dutta and Ray regarding convex games is presented using concave games instead. Fix a MAP (R, q). Consider the concave cooperative game (N, μ) where $\mu : 2^N \to \mathbb{R}$ is a function that assigns, to each subset of agents, the maximum number of objects that they can obtain together. To formalize this intuitive function, given a coalition $S \subset N$, let us partition the set of objects M into $M^+(S)$ and $M^-(S)$, defined as

$$M^+(S) = \{k \in M : |R_{Sk}| < q_k\} \tag{22}$$

The function μ is given by

$$\mu(S) = \sum_{k\in M^+(S)} \sum_{i\in S} r_{ik} + \sum_{k\in M^-(S)} q_k \tag{23}$$

This function is clearly submodular, i.e. for any two subsets $T, S \subset N$

$$\mu(S) + \mu(T) \geq \mu(S \cup T) + \mu(S \cap T) \tag{24}$$

The "core from above" is defined as the following set of profiles

$$C(R, q) = \{x \in \mathbb{R}^n \mid \sum_{i\in N} x_i = \nu(R, q) \text{ and } \nexists\, S \subset N : \sum_{i\in S} x_i > \mu(S)\} \tag{25}$$

It follows from Theorem 3 in Dutta and Ray (1989) that the set $C(R, q)$ has a Lorenz dominant element and is the egalitarian solution. But by construction of the "core from above", $\mathcal{U}(R, q) \subset C(R, q)$, the ES solution is also Lorenz dominant in the set of efficient utility profiles $\mathcal{U}(R, q)$. □

Theorem 2 *In any MAP (R, q), the CCE exists and thus the CCE solution is non-empty.*

Proof. Fix a MAP (R, q). Let $p \in \mathbb{R}^m_+$ be an arbitrary price vector such that $p \cdot q = n$, and use the notation $y_i = R_{iM}$ to denote the optimal consumption bundle for agent $i \in N$, and $y_N = (|R_{N1}|, \ldots, |R_{Nm}|)$. Note that

$$p \cdot y_N \geq p \cdot q \tag{26}$$

Let λ be a non-negative constant. Define the vector $\vec{\lambda}$ as

$$\vec{\lambda}(p) = (\lambda_1, \ldots, \lambda_n) = \text{UNIF}\{p \cdot y_i; n, \lambda\} \tag{27}$$

where UNIF denotes the uniform rationing rule: a mapping that gives to every agent the money needed to buy her preferred bundle of objects as long as it is less than λ, chosen so that $p \cdot \vec{\lambda} = n$. Define the sets of satiated and non-satiated agents

$$N_0(p) \;=\; \{i \in N \mid \lambda_i = p \cdot y_i\} \tag{28}$$
$$N_+(p) \;=\; \{i \in N \mid \lambda_i < p \cdot y_i\} \tag{29}$$

so that $\lambda_i = \lambda\, \forall i \in N_+$. Define the demand correspondence $d_i(p)$ as

$$d_i(p) = \arg \max_{Z_{iM} \in \mathcal{I}(R_{iM})} \{p \cdot Z_{iM} \leq \lambda_i\} \tag{30}$$

where $\mathcal{I}(R_{iM})$ denotes the set of individually rational assignments for R_{iM}. Note that $d_i(p) = \{y_i\}$ for every $i \in N_0(p)$, while for agents in $N_+(p)$, any vector $z_i \in d_i(p)$ satisfies $p \cdot z_i = \lambda$. By Berge's maximum Theorem, the demand correspondence is upper hemi-continuous and convex valued. The excess demand correspondence for the whole society, which inherits the properties of d_i, is given by

$$e(p) = d_N(p) - q \tag{31}$$

where $d_N(p)$ denotes the aggregate demand correspondence for each object. Using the Gale-Nikaido-Debreu Theorem (Theorem 7 in pp. 716-718 of Debreu (1982)), we know that there exists both a price vector $p^* \in R_+$ and an excess demand vector $x^* \in e(p^*)$ for which the following two conditions are satisfied

$$x^* \;=\; \vec{0} \tag{32}$$
$$p^* \cdot x^* \;=\; 0 \tag{33}$$

where Walras' law in equation (33) holds by construction of $\vec{\lambda}$ and d. Finally, $\forall i \in N$

$$Z^*_{iM} = d_i(p^*) \tag{34}$$

so that the corresponding $Z^* \in \mathcal{Z}(R, q)$ by equation (32), concluding the proof of existence of CCE and the non-emptiness of the CCE solution. □

Lemma 2 *ES, CCE and EPO are envy-free.*

Proof. For an arbitrary MAP, let $\phi^{\text{ES}}(R, q) = (U_1, \ldots, U_i, U_j, \ldots, U_n)$, and assume agent i is envious of j, which means that $R_{jM} \subseteq R_{iM}$ and that there exists a Pigou-Dalton transfer ϵ so that the utility profile $U' = (U_1, \ldots, U_i + \epsilon, U_j - \epsilon, \ldots, U_n) \in \mathcal{U}(R, q)$. But U' Lorenz dominates $\phi^{\text{ES}}(R, q)$, so $\phi^{\text{ES}}(R, q)$ was not the ES solution, a contradiction.

EPO is also clearly envy-free. Otherwise, agent i is envious of j, which means that $R_{jM} \subseteq R_{iM}$ and $\phi_j^{EPO}(R, q) > \phi_i^{EPO}(R, q)$. But then EPO establishes that $\phi_i^{EPO}(R, q) = \sum_{k \in M} r_{ik} \cdot \frac{q_k}{|R_{Nk}|} \geq \sum_{k \in M} r_{jk} \cdot \frac{q_k}{|R_{Nk}|} = \phi_j^{EPO}(R, q)$, a contradiction.

Any selection of the CCE solution is envy-free because of the standard argument: if there is any agent who is envious, she could afford the schedule of the agent she envies. □

Theorem 3 *ES and EPO are group strategy-proof but CCE is not.*
I have shown in the main text that CCE is not group strategy-proof and that EPO is group strategy-proof. To show that ES is group strategy-proof, I start with a few preliminaries. Let $\mathcal{Z}^{\text{ES}}$ denote the set of all feasible RAMs supporting the egalitarian solution, i.e.

$$\mathcal{Z}^{\text{ES}} = \{Z \in \mathcal{Z}(R, q) \mid \forall i \in N : \sum_{k \in M} z_{ik} = \phi_i^{\text{ES}}(R, q)\} \tag{35}$$

A rule is non-bossy if no agent can affect someone else's allocation without changing her own utility. That is, a solution ϕ is **non-bossy** if, for every MAP (R, q), $\forall i \in N$, and any manipulation R' such that 1) $\forall j \neq i, R_{jM} = R'_{jM}$, and 2) $R'_{iM} \subsetneq R_{iM}$, we have

$$\phi_i(R, q) = \phi_i(R', q) \quad \text{only if} \quad \phi(R, q) = \phi(R', q) \tag{36}$$

We prove a useful auxiliary Lemma below.

Lemma 6. *ES is non-bossy.*

Proof. We proceed by way of contradiction. Let R' be as specified in the previous definition. The manipulation may come from a reduction of demand for three types of objects:

1. $k \in \mathcal{O}(R, q)$ and $\exists Z \in \mathcal{Z}^{\text{ES}}$ such that $z_{ik} = 0$, so that after agent i misreported, the set of efficient utility profiles is a subset of the original one. Since the original ES utility profile can still be achieved in the problem with misreported preferences, it is still Lorenz dominant and hence the solution does not change.

2. $k \in \mathcal{O}(R, q)$ and $z_{ik} > 0 \; \forall Z \in \mathcal{Z}^{\text{ES}}$, so clearly agent i's utility changes, so she cannot be bossy.

3. $k \in \mathcal{P}(R, q)$, but if agent i reduces the number of perfect goods, she always reduces the utility she obtains (as I prove below), so her utility is not constant and she cannot be bossy.

Now I prove that reducing the number of perfect objects which agent i desires always strictly reduces her utility. The certain loss of the perfect object(s) must be exactly compensated by an increase of the shares she gets from all over-demanded objects, which is constant in any $Z \in \mathcal{Z}^{\text{ES}}$. Agent i was not getting full shares of those objects (as otherwise we obtain a contradiction) so another agent(s) j must be obtaining shares for those objects, implying $\phi_j^{\text{ES}}(R, q) \leq \phi_i^{\text{ES}}(R, q)$ (because otherwise the ES would give those shares to agent i). Some of the shares obtained by agent j in $\phi(R, q)$ must be transferred to agent i in $\phi(R', q)$: this is a Pigou-Dalton transfer because if agent i did not obtain a lower utility in the misrepresented problem then he would not obtain the shares of j. Moreover,

$$\phi_i^{\text{ES}}(R, q) - 1 < \phi_j^{\text{ES}}(R, q) \leq \phi_i^{\text{ES}}(R, q) \tag{37}$$

as otherwise j does not transfer any shares to i when i reduces the number of perfect objects. Let γ be the Pigou-Dalton transfer from j to i required so that the utility of i is kept constant. We have

$$\phi_i^{\text{ES}}(R', q) = \phi_i^{\text{ES}}(R, q) - 1 + \gamma = \phi_j^{\text{ES}}(R, q) - \gamma < \phi_i^{\text{ES}}(R, j) \tag{38}$$

showing that indeed reducing the number of perfect objects always yields lower utility, and thus concluding the proof that ES is non-bossy. □

We are now ready to prove that ES is group strategy-proof. We will do this by showing that nobody can join a manipulating coalition.

Proof. By way of contradiction, assume there exists a MAP (R, q), a coalition $S \subsetneq N$, and a manipulation R' such that, for all $i \in S$ $\phi_i^{\text{ES}}(R', q) \geq \phi_i^{\text{ES}}(R, q)$, and for some $j \in S$ $\phi_j^{\text{ES}}(R', q) > \phi_j^{\text{ES}}(R, q)$.

Let $\phi^{\text{ES}}(R, q) = U^{\text{ES}}$ and order the agents such that $U_1^{\text{ES}} \leq \ldots \leq U_n^{\text{ES}}$. We will show by induction on the order of agents the following property

$$i \notin S \tag{39}$$

There are two cases in which an agent i can be in S. Case 1) either she gets more utility, $\phi_i^{\text{ES}}(R', q) > \phi_i^{\text{ES}}(R, q)$, or case 2) she gets the same utility but she changes her reported preferences to help another member of S. This is ruled out by the non-bossiness of ES so we focus on case 1) only.

We prove it for $i = 1$ first, i.e. the agent with lowest utility. Agent 1 gets a strictly higher number of objects with the new profile R', which must come from a set of objects $K \subseteq \mathcal{O}(R, q)$ from which he was not getting full shares ($K = \{k \in M \mid \exists Z \in \mathcal{Z}^{\text{ES}} : 0 < z_{ik} < 1\}$), and which agents $2, \ldots, t$ also desire and $U_1^{\text{ES}} = U_2^{\text{ES}} = \ldots = U_t^{\text{ES}}$. Those agents exhaust q_k entirely; i.e. $\forall k \in K, \forall Z \in \mathcal{Z}^{\text{ES}}, \sum_1^t z_{ik} = q_k$.

Let $T = \{1, \ldots, t\} \cap S$. For any preference matrix R'_{TM} that is individually rational for R_{TM}, the objects $\{k \in K \mid R_{Nk} \neq R'_{Nk}\}$ become less over-demanded for agents $\{1, \ldots, t\} \setminus T$, and therefore the agents in T get less objects as a whole. Therefore there must be at least one agent in T who is worst off, and the coalition S is not viable. Therefore $1 \notin S$.

Now we assume that $i \notin S$ for agent $i = h - 1$ and we show it holds for agent h. We must have that $U_h^{\text{ES}} < |R_{hM}|$. We assume $\phi_1^{\text{ES}}(R, q)_1^{\text{ES}} < \phi_h^{\text{ES}}(R, q)$ as otherwise our argument for agent 1 works exactly the same.

If agent $h \in S$, it must be that there exists a manipulation R' so that $\phi_h(R', q) > \phi_h(R, q)$. The increase in her utility must come from more object shares on over-demanded objects which she was not obtaining fully, i.e. $K^h = \{k \in M \mid \exists Z \in \mathcal{Z}^{\text{ES}} : 0 < z_{hk} < 1\}$. Some of these objects are exhausted by agents $1, \ldots, h-1$. There is no way agent h could get more shares from any of those objects because $\{1, \ldots, h-1\} \cap S = \emptyset$ by our induction step.

Therefore, the increase must come from objects that are not exhausted by $\{1, \ldots, h-1\}$. Those objects become less over-demanded for $\{h, \ldots, n\} \setminus S$, and therefore agents in S get less object shares as a whole. It follows that there must be a agent in S who gets less utility, so coalition S is not viable. Therefore $h \notin S$, and this concludes the proof. $\square$

Lemma 4 *EPO satisfies IPO but ES does not satisfy IPO. There exists a selection of CCE that satisfies IPO.*

Proof. EPO is clearly IPO, as discussed in the main text. To show that ES is not IPO, let $n = 5, M = \{\alpha\}, q = 4$, and $R^\top = [1\ 1\ 1\ 1\ 1]$. $\phi_i^{\text{ES}}(R, q) = 0.8$ for any agent, but adding a perfect object k' with capacity 4 for any agent i changes $\phi_i^{\text{ES}}([R\, R_{Nk'}], (4, 4)) = 1.75 \neq 2$.

To show that there is a selection of Φ^{CCE} that is IPO, let (Z^*, p^*) be a CCE of (R, q) and $([R\, R_{Nk'}], \overline{q})$ be a perfect extension of (R, q) . Then fix $p^*_{k'} = 0$ and, for every $i \in N$ let $z^*_{ik'} = 1$ if $r_{ik'} = 1$, and 0 otherwise. The pair $([Z^*\, Z^*_{Nk'}], (p^*_1, \ldots, p^*_n, 0))$ is a CCE of the perfect extension $([R\, R_{Nk'}], \overline{q})$, because everybody interested in the perfect object is able to afford it, and the demand for k' equals its supply, because the new object k' is perfect. $\square$