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PSEUDOSPECTRA OF THE DAMPED WAVE EQUATION WITH UNBOUNDED DAMPING

AHMET ARIFOSKI AND PETR SIEGL

ABSTRACT. We analyze pseudospectra of the generator of the damped wave equation with unbounded damping. We show that the resolvent norm diverges as $\operatorname{Re} z \rightarrow -\infty$. The highly non-normal character of the operator is a robust effect preserved even when a strong potential is added. Consequently, spectral instabilities and other related pseudospectral effects are present.

1. INTRODUCTION

We consider a linear damped wave equation

$$\partial_t^2 u(t, x) + 2a(x)\partial_t u(t, x) = (\partial_x^2 - q(x))u(t, x), \quad t > 0, \quad x \in \mathbb{R}, \quad (1.1)$$

with a non-negative damping a that is unbounded at infinity and a non-negative potential q that is also possibly unbounded. As demonstrated in recent works [5, 7], new effects occur due the unboundedness of a . In particular, the new spectral features investigated in [5] is the “overdamping at infinity” reflected in the presence of the essential spectrum $(-\infty, 0]$ responsible for the loss of an exponential energy decay of solutions; for polynomial decay estimates see [7]. This paper deals with a more subtle pseudospectral analysis, which reveals highly non-normal character of the semigroup generator G , see (1.2), similarly as for Schrödinger operators with complex potentials, see *e.g.* [2, 4, 8, 10]. Thus this type of the “overdamping” is responsible also for strong spectral instabilities.

Traditionally, the second order wave equation (1.1) can be rewritten as the first order system

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ \partial_x^2 - q & -2a \end{pmatrix}}_G \begin{pmatrix} u \\ v \end{pmatrix} \quad (1.2)$$

so that the semigroup theory can be employed. Indeed, it was established in [5] that, under suitable regularity assumptions on a and q , the operator G generates a contraction semigroup in a natural Hilbert space

$$\mathcal{H} := \mathcal{W}(\mathbb{R}) \times L^2(\mathbb{R}), \quad \left\langle \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = \langle \phi_1', \psi_1' \rangle + \langle q^{\frac{1}{2}} \phi_1, q^{\frac{1}{2}} \psi_1 \rangle + \langle \phi_2, \psi_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are $L^2(\mathbb{R})$ inner product and norm, respectively, and $\mathcal{W}(\mathbb{R})$ is the completion of the pre-Hilbert space $(C_0^\infty(\mathbb{R}), (\|\partial_x \cdot\|^2 + \|q^{\frac{1}{2}} \cdot\|^2)^{\frac{1}{2}})$. Further estimates on the energy decay of solutions for unbounded damping in higher dimensions were performed in [7] and polynomial rates established.

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The main goal of this work is to study pseudospectra of G , aiming at lower estimates of $\|(G - \lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})}$ for λ in the left complex half-plane, where the numerical range of G is located. More precisely, we identify curves Γ in the left complex half-plane and construct pseudomodes on them, *i.e.* families $\{\Psi_\lambda \in \text{Dom}(G) : \lambda \in \Gamma\}$ of compactly supported smooth functions satisfying

$$\lim_{\substack{\lambda \rightarrow \infty \\ \lambda \in \Gamma}} \frac{\|(G - \lambda)\Psi_\lambda\|_{\mathcal{H}}}{\|\Psi_\lambda\|_{\mathcal{H}}} = 0. \quad (1.3)$$

Our results provide estimates on the decay rates in (1.3) as $\lambda \rightarrow \infty$, which can be turned into lower estimates of the resolvent norm, as well as a description of admissible curves Γ . Both the rates and curves depend on growth and regularity of a and q , see Theorem 3.1 and examples in Sections 2.4 and 3.5; see also Remark 3.2.

In the special case of a monomial damping $a(x) = x^{2m}$, $m \in \mathbb{N}$, and no potential, *i.e.* $q = 0$, a corollary of Theorem 3.1 reads

Theorem 1.1. *Let G be as in (1.2) where (with some $m \in \mathbb{N}$)*

$$q(x) = 0, \quad a(x) = x^{2m}.$$

Then, for every $N \in \mathbb{N}$, there exists a λ -dependent family of functions $\{\Psi_\lambda\} \subset C_0^\infty(\mathbb{R})^2$ such that

$$\|(G - \lambda)\Psi_\lambda\|_{\mathcal{H}} = \mathcal{O}(|\text{Re } \lambda|^{-N})\|\Psi_\lambda\|_{\mathcal{H}} \quad (1.4)$$

as $\text{Re } \lambda \rightarrow -\infty$, provided that $\text{Im } \lambda$ satisfies

$$|\text{Im } \lambda| \geq |\text{Re } \lambda|^{-\frac{1}{2m} + \varepsilon}, \quad (1.5)$$

for some $\varepsilon > 0$ independent of λ .

The decay estimate in (1.4) can be made more precise, taking into account also the size of $\text{Im } \lambda$, and the potential q can be added, which may affect the estimate on the shape of the pseudospectral region Ω if q is much stronger at infinity than a , see Examples 2.7, 3.9 for details.

FIGURE 1. Numerical computation of the spectrum (in black, see (1.6)) and the pseudospectra ($\log_{10}$ scale, approximation by an 800×800 matrix) of G with $a(x) = x^2$ and $q(x) = 0$, $x \in \mathbb{R}$.

Figure 1 is an illustration of Theorem 1.1 for $a(x) = x^2$ and $q(x) = 0$. In this case, the spectrum is explicit (as well as for other monomial dampings, see [5,

Prop. 6.1]), namely

$$\sigma(G) = (-\infty, 0] \dot{\cup} \left\{ 2^{\frac{1}{3}} e^{\pm i \frac{2}{3} \pi} (2k+1)^{\frac{2}{3}} \right\}_{k \in \mathbb{N}_0}. \quad (1.6)$$

Clearly, pseudospectra do not localize at a neighborhood of the discrete spectrum and the shape of the pseudospectral region Ω seems to be in agreement with the analytic result, see (1.5). We note that although the spectrum of this example (also for $a(x) = x^{2m}$) is contained in a sector, in fact lying on three rays, see (1.6), the generated semigroup cannot be analytic due to the “bad” resolvent behavior in the left-complex half-plane, see *e.g.* [3, Sec. 8.4]. Moreover, unlike the essential spectrum $(-\infty, 0]$ that can be shifted to the left (even made to disappear completely) by adding the potential q of a comparable strength to a at infinity, see [5, Sec. 7], a vast pseudospectral region is preserved even if the potential q strongly dominates the damping a , see Theorems 2.1, 3.1 and examples in Sections 2.4, 3.5.

Pseudospectral behavior, more precisely the decay rates and the shape of pseudospectral region Ω , is well-studied for Schrödinger operators with complex potentials V ,

$$H = -\partial_x^2 + V(x),$$

see [10]; results for semi-classical operators can be found for instance in [2, 4]. The occurrence of the non-trivial pseudospectral region Ω here is due to the *imaginary part* of V and it essentially depends on its behavior and size at infinity comparing to $\operatorname{Re} V$, see [10]. Our problem originates in equation (1.1) with real coefficients. Nevertheless, the associated spectral problem for G is closely related with the quadratic operator function having the form of a Schrödinger operator

$$T(\lambda) = -\partial_x^2 + q + 2\lambda a + \lambda^2, \quad \lambda \in \mathbb{C} \setminus (-\infty, 0] \quad (1.7)$$

with a “ λ -dependent potential” $q + 2\lambda a + \lambda^2$ which is *complex* in general. In more detail, the spectral equivalence between G and T

$$\lambda \in \sigma(G) \iff 0 \in \sigma(T(\lambda)), \quad \lambda \in \mathbb{C} \setminus (-\infty, 0],$$

was established in a quite general setting with possibly unbounded and singular coefficients, *cf.* [5]. On a formal level, such equivalence is immediate for eigenvalues since one can check that

$$(G - \lambda)(\psi, \phi)^t = 0 \iff \phi = \lambda\psi \quad \text{and} \quad T(\lambda)\psi = 0.$$

In fact, the link between G and T is used also for the pseudospectral analysis here. We search for the pseudomodes of G in the form

$$\Psi_\lambda := (f_\lambda, \lambda f_\lambda)^t,$$

which reduces the problem to the pseudomodes for T since then

$$\frac{\|(G - \lambda)\Psi_\lambda\|_{\mathcal{H}}^2}{\|\Psi_\lambda\|_{\mathcal{H}}^2} = \frac{\|T(\lambda)f_\lambda\|^2}{\|f'_\lambda\|^2 + \|q^{\frac{1}{2}}f_\lambda\|^2 + |\lambda|^2\|f_\lambda\|^2}. \quad (1.8)$$

Thus the construction of pseudomodes is based on a complex WKB-method applied for the quadratic operator function $T(\lambda)$, *cf.* (1.7). Technically, this goes beyond the simpler semi-classical setting, *cf.* for instance [2], due to different powers of λ , and the non-semiclassical one for Schrödinger operators, *cf.* [10], due to the “ λ -dependent potential” $q + 2\lambda a + \lambda^2$. Since more than a local behavior of the damping a is required in our approach, the spirit of the performed estimates is closer to the non-semiclassical case, nonetheless, the quadratic dependence on λ in the “potential” brings new obstacles and effects. In particular, already basic pseudomodes with one term in the phase, see (2.1), yield decay in (1.8) for various behaviors of a although it is possibly slower and for a smaller region Ω , see examples

in Sections 2.4, 3.5. Intuitively, this difference to Schrödinger operators occurs due to the large parameter λ in front of the damping a .

To achieve better accessibility of the results and proofs, we first construct pseudomodes with a simple basic ansatz, *cf.* Section 2, and give examples where these results can be applied, *cf.* Section 2.4. In the second step, we employ an ansatz with a phase expansion, *cf.* Section 3 and demonstrate on examples, *cf.* Section 3.5, the improvements for the decay rates and pseudospectral region Ω comparing to the first step.

Finally, although we do not give explicit claims here, we remark that our method can be adapted in a straightforward way also to problems with a singular damping like $a(x) = 1/x^\alpha$, $\alpha > 1$, $x \in (0, 1)$. The pseudomodes in this case are constructed with shrinking supports strictly inside $(0, 1)$ and so the results are local and independent of possible boundary conditions at 0. Naturally, our results have also straightforward corollaries for higher dimensional problems allowing for separation of variables and their perturbations; we omit a discussion and explicit claims on these as well.

1.1. Notations. We fix some notations used throughout the paper. For positive and negative real numbers we write $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{R}_- := (-\infty, 0)$, respectively. To have a short expression for an "integer interval" we use the double brackets, $[[m, n]] := [m, n] \cap \mathbb{Z}$. Given an interval $I \subset \mathbb{R}$, the norm of $L^p(I)$ is denoted by $\|\cdot\|_{p,I}$. If $I = \mathbb{R}$ we abbreviate $\|\cdot\|_{p,\mathbb{R}} = \|\cdot\|_p$; for the most frequent $L^2(\mathbb{R})$ -norm we use $\|\cdot\| := \|\cdot\|_2$. To avoid many appearing constants, we employ the convention that $a \lesssim b$ if there exists a constant $C > 0$, independent of any relevant variable or parameter, such that $a \leq Cb$; the relation $a \gtrsim b$ is introduced analogously. By $a \approx b$ it is meant that $a \lesssim b$ and $b \lesssim a$.

2. BASIC ANSATZ

We recall that the pseudomodes are constructed for the associated quadratic operator function $T(\lambda)$, *cf.* (1.7), having the form of a Schrödinger operator with a λ -dependent potential. Following this point of view, it is natural that the most basic ansatz for the pseudomode has the form

$$f_\lambda := \xi g, \quad g(x) = \exp\left(\pm i \int_b^x \sqrt{-\lambda^2 - 2\lambda a(t) - q(t)} dt\right) \quad (2.1)$$

with a suitably chosen λ -dependent point b , see (2.2), (2.4), and a λ -dependent cut-off $\xi \in C_0^\infty(\mathbb{R})$, see [10] for details on Schrödinger operators.

Due to the accretivity of $-G$, see *e.g.* [5] for details, the construction is relevant only for λ in the left complex half-plane, *i.e.* in the second and third quadrant. It suffices to analyze the second quadrant as a and q are assumed to be real and the third quadrant can be reached by complex conjugation. Hence, we parametrize λ in the following way

$$\lambda = -\alpha + i\beta, \quad \alpha > 0, \quad \beta > 0 \quad (2.2)$$

and we choose $-i$ in the formula for g in (2.1). This choice guarantees that the principal complex square root, *i.e.* for $z \in \mathbb{C} \setminus \mathbb{R}_-$ we take

$$\sqrt{z} = \sqrt{\frac{|z| + \operatorname{Re}(z)}{2}} + \operatorname{sgn}(\operatorname{Im}(z))i\sqrt{\frac{|z| - \operatorname{Re}(z)}{2}}, \quad (2.3)$$

which is always used here, is continuous since, for $x \in \operatorname{supp} \xi$, the values of $-\lambda^2 - 2\lambda a - q$ stay away from $\mathbb{R}_-$, see (2.21) and (2.25).

Up to the cut-off ξ , the basic ansatz has the form $e^{h(x)}$ with a complex valued function h which we want to behave essentially as $-(x-b)^2$ with $b \rightarrow +\infty$. To this

end, we search for the point $b \in \mathbb{R}_+$ such that $\operatorname{Re} h(b) = 0$ and $\operatorname{Re} h$ does not change the sign at 0. It will be showed that the suitable choice is given by the equation

$$\alpha = a(b), \quad (2.4)$$

see (2.23). The cut-off function ξ is chosen such that

$$\begin{aligned} \xi &\in C_0^\infty(\mathbb{R}), \quad 0 \leq \xi \leq 1, \\ \xi(b+s) &= 1, \quad |s| < \frac{\delta}{2}, \\ \xi(b+s) &= 0, \quad |s| > \delta. \end{aligned} \quad (2.5)$$

Notice that ξ can be selected in such a way that

$$\|\xi^{(j)}\|_\infty \lesssim \delta^{-j}, \quad j = 1, 2. \quad (2.6)$$

For notational convenience in further sections and also as a preparation for the ansatz with the phase expansion, we introduce a function ψ_{-1} by

$$\lambda\psi_{-1}(x) = \int_b^x \sqrt{-\lambda^2 - 2\lambda a(t) - q(t)} dt, \quad x \in \operatorname{supp}(\xi) \quad (2.7)$$

and so f_λ can be written as

$$f_\lambda := \xi g = \xi \exp(-i\lambda\psi_{-1}). \quad (2.8)$$

2.1. The main result for the basic ansatz. In this section, we work under the following assumption. We construct pseudomodes supported in $\mathbb{R}_+$ and use the behavior of a and q in $\mathbb{R}_+$ only. The construction can be clearly repeated in $\mathbb{R}_-$ if the assumptions on a and q are adjusted accordingly.

Assumption I. Suppose that the functions $a \in C^2(\mathbb{R})$, $q \in C^1(\mathbb{R})$ satisfy the following conditions:

(a) a, q are non-negative for sufficiently large x :

$$\forall x \gtrsim 1, \quad a(x) \geq 0 \quad \text{and} \quad q(x) \geq 0;$$

(b) a is increasing for sufficiently large x and unbounded at infinity:

$$\forall x \gtrsim 1, \quad a'(x) > 0, \quad (2.9)$$

$$\lim_{x \rightarrow +\infty} a(x) = +\infty; \quad (2.10)$$

(c) the derivatives of a are controlled by a :

$$\exists \nu \geq -1, \quad \forall x \gtrsim 1, \quad |a^{(j)}(x)| \lesssim x^{j\nu} a(x), \quad j = 1, 2. \quad (2.11)$$

We recall that the condition (2.11) guarantees almost a constant behavior of functions a and a' on sufficiently small intervals, namely if $\Delta = o(x^{-\nu})$ as $x \rightarrow +\infty$, then

$$\frac{a^{(j)}(x + \Delta)}{a^{(j)}(x)} \approx 1, \quad j = 0, 1, \quad x \rightarrow +\infty; \quad (2.12)$$

the detailed proof can be found *e.g.* in [10, Sec. 3].

The result on pseudomodes is formulated for curves in the left-complex half-plane with $\operatorname{Re} \lambda = -\alpha \rightarrow -\infty$. As b is defined via the equation $\alpha = a(b)$, see (2.4), by assumptions (2.9) and (2.10), we can write $b \rightarrow +\infty$ instead of $\alpha \rightarrow +\infty$.

Theorem 2.1. *Let Assumption I hold and let $b \in \mathbb{R}_+$ be defined by $\alpha = a(b)$. Take $\varepsilon > 0$ and define*

$$\delta := b^{-\nu-\varepsilon} \quad (2.13)$$

and

$$q_b^{(j)} := \|q^{(j)}\|_{\infty, (b-\delta, b+\delta)}, \quad j = 0, 1. \quad (2.14)$$

Suppose that there exists a b -dependent $\beta = \beta(b) > 0$ such that the following conditions hold as $b \rightarrow +\infty$

$$\forall c > 0, \quad \left(\frac{1}{\delta} + \frac{1}{\alpha + \beta} \frac{1}{\delta^2} \right) \exp \left(-c \frac{\beta}{\alpha + \beta} a'(b) \delta^2 \right) = o(1), \quad (2.15)$$

$$q_b^{(0)} = o(\alpha^2 + \beta^2), \quad (2.16)$$

$$\alpha^2 b^{2\nu} + q_b^{(1)} = o(\alpha^2 + \beta^2). \quad (2.17)$$

Let $\{f_\lambda\}$ with $\lambda = \lambda(b) = -\alpha + i\beta$, see (2.2), be a family of functions constructed as in (2.8) with ξ as in (2.5) and the choice of α , β and δ as above.

Then, for $\Psi_\lambda = (f_\lambda, \lambda f_\lambda)^t$, we have

$$\frac{\|(G - \lambda)\Psi_\lambda\|_{\mathcal{H}}}{\|\Psi_\lambda\|_{\mathcal{H}}} = o(1), \quad b \rightarrow +\infty. \quad (2.18)$$

The conditions in Theorem (2.1) have a complicated structure since they combine several competing terms with a different origin. In detail, the size of the cut-off, *i.e.* the choice of δ , the shape of the curve along which λ tends to infinity, *i.e.* the choice of β , the growth of the damping, *i.e.* the size of $a'(b)$, and the size and growth of the potential q , *i.e.* the size of q and q' , play against each other.

The conditions (2.16), (2.17) guarantee that q can be treated as a small perturbations in the estimates and that the remainder, see (2.22) below, is small. The condition (2.15) ensures that the cut-off ξ produces small terms. For simplicity, we assume that the limit in (2.15) is zero for all $c > 0$, nevertheless, it suffices to verify this only for one constant $c = c_2/8$ appearing in the proof. However, to estimate the size of c_2 , more precise information on a' would be needed, see Lemma 2.4. As we show in examples, see Section 2.4, also the stronger condition (2.15) can be verified easily. Notice also that (2.15) is always satisfied if $\nu < 0$ and when a sufficiently small $\varepsilon > 0$ is chosen, thus *e.g.* for polynomial-like functions with $\nu = -1$.

2.2. Strategy and technical lemmas. We recall that due to (1.8), we need to estimate $\|T(\lambda)f_\lambda\|$. With regard to our ansatz for f_λ , see (2.8), (2.7), we arrive at

$$\begin{aligned} T(\lambda)f_\lambda &= -\xi''g - 2\xi'g' - \xi g'' + (2\lambda a + q + \lambda^2)\xi g \\ &= -\xi''g + 2\xi' i \lambda \psi'_{-1} g + \xi [i \lambda \psi''_{-1} + \lambda^2 (\psi'_{-1})^2 g + 2\lambda a + q + \lambda^2] g. \end{aligned} \quad (2.19)$$

Recall that we aim to having the function g essentially as a Gaussian concentrated around b . Notice that the first two terms in (2.19) contain derivatives of the cut-off ξ for which we clearly have

$$\text{supp}(\xi^{(j)}) \subset [b - \delta, b - \delta/2] \cup [b + \delta/2, b + \delta], \quad j \geq 1. \quad (2.20)$$

As g is constructed such that $|g|$ is exponentially localized around b , the terms $\|\xi''g\|^2$ and $\|\xi' \lambda \psi'_{-1} g\|^2$, both divided by $|\lambda|^2 \|\xi g\|^2$, are expected to be small as $b \rightarrow +\infty$. Also the term $\|\xi \lambda \psi''_{-1} g\|^2$ is expected to be small when divided by $|\lambda|^2 \|\xi g\|^2$ due to the assumption (2.11). However, there is no reason why the remaining terms in (2.19) should be small and so we impose this by requiring

$$\zeta := \lambda^2 (\psi'_{-1})^2 = -\lambda^2 - 2\lambda a - q. \quad (2.21)$$

Hence the remainder to be controlled reads

$$R := i \xi \lambda \psi''_{-1}. \quad (2.22)$$

As mentioned before, the idea is to find a point $b \in \mathbb{R}$ such that $\text{Im } \zeta = 0$, see (2.7), (2.8) and (2.3). To this end, observe that

$$\text{Re } \zeta = -\alpha^2 + \beta^2 + 2\alpha a - q, \quad \text{Im } \zeta = 2\beta(\alpha - a), \quad (2.23)$$

which leads to the choice of b as in (2.4), *i.e.* $\alpha = a(b)$.

It is important to notice that, using (2.16), we obtain

$$\operatorname{Re} \zeta(b) = -\alpha^2 + \beta^2 + 2\alpha a(b) - q(b) = \alpha^2 + \beta^2 - q(b) \approx \alpha^2 + \beta^2 > 0.$$

The next lemma shows that this remains true also in the δ -neighborhood of b , more precisely for all $t \in \operatorname{supp}(\xi)$.

Lemma 2.2. *Let Assumption I hold, let $b \in \mathbb{R}_+$ be defined by the equation (2.4), let $\varepsilon > 0$, δ be as in (2.13) and ζ be as in (2.21). Suppose that q satisfy (2.16), i.e.*

$$q_b^{(0)} = o(\alpha^2 + \beta^2), \quad b \rightarrow +\infty. \quad (2.24)$$

Then, for all $t \in (b - \delta, b + \delta)$, we have as $b \rightarrow +\infty$ that

$$\operatorname{Re} \zeta(t) \approx \alpha^2 + \beta^2. \quad (2.25)$$

Proof. We give a detailed proof for $t > b$ only, the case $t < b$ is very similar. By the mean value theorem, the assumption (2.11), the property (2.12) and the choice of δ , see (2.13), we get for each $t \in (b, b + \delta)$ that

$$a(t) - a(b) = a'(\eta)(t - b) \lesssim a(b)b^\nu \delta = o(a(b)), \quad b \rightarrow +\infty. \quad (2.26)$$

Using (2.26), (2.12), (2.4) and the assumption on q , see (2.24), for all $t \in (b, b + \delta)$, we get that

$$\operatorname{Re} \zeta(t) = \alpha(a(t) + a(t) - \alpha) + \beta^2 - q(t) \approx \alpha^2(1 - o(1)) + \beta^2 - q(t) \approx \alpha^2 + \beta^2$$

as $b \rightarrow +\infty$. $\square$

Next, we aim to get two-sided estimates of $|g|$, for which we have to estimate $\operatorname{Re}(i\lambda\psi_{-1})$. As a preparation we have the following lemma.

Lemma 2.3. *Let the assumptions of Lemma 2.2 hold. Then for all $t \in (b - \delta, b + \delta)$*

$$(\operatorname{Re} \zeta(t))^{\frac{1}{2}} + |\operatorname{Im} \zeta(t)|^{\frac{1}{2}} \approx \alpha + \beta, \quad b \rightarrow +\infty. \quad (2.27)$$

Proof. We give a detailed proof for the case $t > b$ only, the case $t < b$ is very similar. By Lemma 2.2 we already have

$$(\operatorname{Re} \zeta(t))^{\frac{1}{2}} \approx \alpha + \beta, \quad b \rightarrow +\infty.$$

We proceed with the estimate of $|\operatorname{Im} \zeta(t)|^{\frac{1}{2}}$, see (2.23). From (2.26) and (2.4), we obtain (recall that we consider only $\beta > 0$ here)

$$|\operatorname{Im} \zeta(t)| = 2\beta|a(b) - a(t)| = o(\alpha^2 + \beta^2), \quad b \rightarrow +\infty.$$

Hence,

$$(\operatorname{Re} \zeta(t))^{\frac{1}{2}} + |\operatorname{Im} \zeta(t)|^{\frac{1}{2}} \approx (\alpha + \beta)(1 + o(\alpha + \beta)) \approx \alpha + \beta, \quad b \rightarrow +\infty. \quad \square$$

The next step is the two-sided estimate of $\operatorname{Re}(i\lambda\psi_{-1})$.

Lemma 2.4. *Let the assumptions of Lemma 2.2 hold. Then there exist two positive constants $c_1, c_2 > 0$ such that for all $s \in (b - \delta, b + \delta)$*

$$c_2 \frac{\beta}{\alpha + \beta} a'(b)(s - b)^2 \leq \operatorname{Re}(i\lambda\psi_{-1}(s)) \leq c_1 \frac{\beta}{\alpha + \beta} a'(b)(s - b)^2, \quad b \rightarrow +\infty. \quad (2.28)$$

Proof. Rewriting $\operatorname{Re}(i\lambda\psi_{-1})(s)$ and using (2.3), we obtain

$$\begin{aligned} \operatorname{Re}(i\lambda\psi_{-1})(s) &= \int_b^s \operatorname{Re}(i\sqrt{\zeta(t)}) dt = - \int_b^s \operatorname{Im} \sqrt{\zeta(t)} dt \\ &= - \int_b^s \operatorname{sgn}(\operatorname{Im} \zeta(t)) \sqrt{\frac{|\zeta(t)| - \operatorname{Re} \zeta(t)}{2}} dt. \end{aligned}$$

Taking into account (2.9) and (2.23), we observe that (recall that $\beta > 0$)

$$\operatorname{sgn}(\operatorname{Im} \zeta(t)) = \operatorname{sgn}(2\beta(a(b) - a(t))) = \operatorname{sgn}(b - t).$$

Further we consider the case $s > b$ only, the other one is fully analogous. By straightforward manipulations and estimates, we obtain (recall that $\operatorname{Re} \zeta(t) > 0$)

$$\begin{aligned} \operatorname{Re}(i\lambda\psi_{-1})(s) &\approx \int_b^s \sqrt{\frac{|\zeta(t)|^2 - (\operatorname{Re} \zeta(t))^2}{|\zeta(t)| + \operatorname{Re} \zeta(t)}} dt = \int_b^s \frac{|\operatorname{Im} \zeta(t)|}{\sqrt{|\zeta(t)| + \operatorname{Re} \zeta(t)}} dt \\ &\approx \int_b^s \frac{|\operatorname{Im} \zeta(t)|}{(\operatorname{Re} \zeta(t))^{\frac{1}{2}} + |\operatorname{Im} \zeta(t)|^{\frac{1}{2}}} dt, \quad s > b. \end{aligned}$$

By (2.23), Lemma 2.3, the mean value theorem and (2.12), we get ($\beta, a'(t) > 0$)

$$\begin{aligned} \operatorname{Re}(i\lambda\psi_{-1})(s) &\approx \int_b^s \frac{|\beta(a(b) - a(t))|}{\alpha + \beta} dt \approx \int_b^s \frac{|\beta a'(b)(b - t)|}{\alpha + \beta} dt \\ &\approx \frac{\beta}{\alpha + \beta} a'(b) \int_b^s (t - b) dt \approx \frac{\beta}{\alpha + \beta} a'(b)(s - b)^2, \end{aligned}$$

hence (2.28) holds for $s > b$. $\square$

The next step is to estimate $\|\xi g\|^2$ from below and $\|\xi' g'\|^2, \|\xi'' g\|^2$ from above.

Lemma 2.5. *Let the assumptions of Lemma 2.2 hold, let ξ, g be as in (2.5), (2.8), respectively, and let $c_2 > 0$ be as in Lemma 2.4. Then, as $b \rightarrow +\infty$,*

$$\|\xi g\|^2 \gtrsim \delta \exp\left(-\frac{c_2}{4} \frac{\beta}{\alpha + \beta} a'(b) \delta^2\right), \quad (2.29)$$

$$\|\xi' g'\|^2 \lesssim (\alpha^2 + \beta^2) \delta^{-1} \exp\left(-\frac{c_2}{2} \frac{\beta}{\alpha + \beta} a'(b) \delta^2\right), \quad (2.30)$$

$$\|\xi'' g\|^2 \lesssim \delta^{-3} \exp\left(-\frac{c_2}{2} \frac{\beta}{\alpha + \beta} a'(b) \delta^2\right). \quad (2.31)$$

Proof. We start with the estimate of $\|\xi g\|^2$. Using the definition of ξ , see (2.5),

$$\|\xi g\|^2 = \int_{b-\delta}^{b+\delta} |\xi(s)|^2 |e^{-i\lambda\psi_{-1}(s)}|^2 ds \geq \int_{b-\frac{\delta}{2}}^{b+\frac{\delta}{2}} e^{-2\operatorname{Re}(i\lambda\psi_{-1}(s))} ds.$$

So with the upper bound of $\operatorname{Re}(i\lambda\psi_{-1}(s))$ in Lemma 2.4, we get

$$\|\xi g\|^2 \geq \int_{b-\frac{\delta}{2}}^{b+\frac{\delta}{2}} e^{-2c_1 \frac{\beta}{\alpha + \beta} a'(b)(s-b)^2} ds = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} e^{-2c_1 \frac{\beta}{\alpha + \beta} a'(b)s^2} ds.$$

Since $c_1 \geq c_2$, taking a positive k such that $k^2 := 2c_1/c_2 > 1$, we arrive at

$$\|\xi g\|^2 \geq \int_{-\frac{\delta}{2k}}^{\frac{\delta}{2k}} e^{-2c_1 \frac{\beta}{\alpha + \beta} a'(b)s^2} ds \gtrsim \delta e^{-2c_1 \frac{\beta}{\alpha + \beta} a'(b) \frac{\delta^2}{4k^2}} = \delta e^{-\frac{c_2}{4} \frac{\beta}{\alpha + \beta} a'(b) \delta^2},$$

thus (2.29) is proved.

Next we analyze $\|\xi'' g\|^2$. The estimate (2.6) and the lower bound for $\operatorname{Re}(i\lambda\psi_{-1}(s))$ in (2.28) lead to

$$\|\xi'' g\|^2 = \int_{\operatorname{supp}(\xi'')} |\xi''(s)|^2 e^{-2\operatorname{Re}(i\lambda\psi_{-1}(s))} ds \lesssim \delta^{-4} \int_{\operatorname{supp}(\xi'')} e^{-2c_2 \frac{\beta}{\alpha + \beta} a'(b)(s-b)^2} ds.$$

By symmetry and (2.20), we get

$$\|\xi'' g\|^2 \lesssim \delta^{-4} \int_{\delta/2}^{\delta} e^{-2c_2 \frac{\beta}{\alpha + \beta} a'(b)s^2} ds \lesssim \delta^{-3} e^{-\frac{c_2}{2} \frac{\beta}{\alpha + \beta} a'(b) \delta^2},$$

so (2.31) is proved.

Finally, to estimate $\|\xi'g'\|^2$, we use (2.21), (2.27) and obtain

$$\|\xi'g'\|^2 \approx \int_{\mathbb{R}} |\xi'(s)|^2 |\lambda\psi'_{-1}(s)|^2 |e^{-i\lambda\psi_{-1}(s)}|^2 ds \lesssim \frac{\alpha^2 + \beta^2}{\delta^2} \int_{\text{supp}(\xi')} |e^{-i\lambda\psi_{-1}(s)}|^2 ds,$$

which can be continued as the estimate of $\|\xi''g\|^2$. $\square$

It remains to estimate the remainder R , see (2.22).

Lemma 2.6. *Let the assumptions of Lemma 2.2 hold and let R be as in (2.22). Then*

$$\|R\|_{\infty, (b-\delta, b+\delta)}^2 \lesssim \alpha^2 b^{2\nu} + \frac{(q_b^{(1)})^2}{\alpha^2 + \beta^2}, \quad b \rightarrow +\infty. \quad (2.32)$$

Proof. From (2.21) and (2.27), we have for all $t \in (b - \delta, b + \delta)$ that

$$|\lambda\psi''_{-1}(t)| = \left| \frac{-2\lambda a'(t) - q'(t)}{2(-\lambda^2 - 2\lambda a(t) - q(t))^{\frac{1}{2}}} \right| \approx \frac{|\lambda| |2a'(t) + \frac{q'(t)}{\lambda}|}{|\zeta|^{\frac{1}{2}}} \lesssim a'(t) + \frac{|q'(t)|}{\alpha + \beta}.$$

Since

$$\|R\|_{\infty, (b-\delta, b+\delta)}^2 \leq \|\lambda\psi''_{-1}\|_{\infty, (b-\delta, b+\delta)}^2,$$

the claim follows from (2.11) and (2.14). $\square$

2.3. The proof of Theorem 2.1. Equipped with Lemmas 2.5 and 2.6, we are in position to prove the main result of this section.

Proof of Theorem 2.1. Using (2.19), the triangle inequality and the definition of the remainder R , see (2.22), we have

$$\frac{\|T(\lambda)\xi g\|^2}{\|(\xi g)'\|^2 + \|q^{\frac{1}{2}}\xi g\|^2 + |\lambda|^2 \|\xi g\|^2} \lesssim \underbrace{\frac{\|\xi''g\|^2}{|\lambda|^2 \|\xi g\|^2}}_{=:Q_1} + \underbrace{\frac{\|\xi'g'\|^2}{|\lambda|^2 \|\xi g\|^2}}_{=:Q_2} + \underbrace{\frac{\|R\|_{\infty, (b-\delta, b+\delta)}^2 \|\xi g\|^2}{|\lambda|^2 \|\xi g\|^2}}_{=:Q_3},$$

where we kept only $|\lambda|^2 \|\xi g\|^2$ in the denominator in the last step; q might be 0 and it can be showed that the term $\|(\xi g)'\|^2$ does not improve the estimate in general.

We start with Q_1 . By (2.29) and (2.31), see Lemma 2.5, we arrive at

$$Q_1 \lesssim \frac{\delta^{-3} e^{-\frac{c_2}{2} \frac{\beta}{\alpha+\beta} a'(b) \delta^2}}{(\alpha^2 + \beta^2) \delta e^{-\frac{c_2}{4} \frac{\beta}{\alpha+\beta} a'(b) \delta^2}} = \frac{1}{(\alpha^2 + \beta^2) \delta^4} e^{-\frac{c_2}{4} \frac{\beta}{\alpha+\beta} a'(b) \delta^2}, \quad b \rightarrow +\infty.$$

Hence the assumption (2.15) yields that $Q_1 = o(1)$ as $b \rightarrow +\infty$.

The estimate of Q_2 is similar. From (2.29) and (2.30), we get

$$Q_2 \lesssim \frac{(\alpha^2 + \beta^2) \delta^{-1} e^{-\frac{c_2}{2} \frac{\beta}{\alpha+\beta} a'(b) \delta^2}}{(\alpha^2 + \beta^2) \delta e^{-\frac{c_2}{4} \frac{\beta}{\alpha+\beta} a'(b) \delta^2}} = \frac{1}{\delta^2} e^{-\frac{c_2}{4} \frac{\beta}{\alpha+\beta} a'(b) \delta^2}, \quad b \rightarrow +\infty,$$

hence the assumption (2.15) yields again that $Q_2 = o(1)$ as $b \rightarrow +\infty$.

Finally, we estimate Q_3 using (2.32) and the assumption (2.17), namely

$$Q_3 \lesssim \frac{\alpha^2 b^{2\nu}}{\alpha^2 + \beta^2} + \frac{(q_b^{(1)})^2}{\alpha^4 + \beta^4} = o(1), \quad b \rightarrow +\infty.$$

Hence, putting all the estimate above and (1.8) together, we get (2.18). $\square$

2.4. Examples.

Example 2.7. (Polynomial-like dampings and potentials)

First we consider dampings $a \in C^2(\mathbb{R})$ and potentials $q \in C^1(\mathbb{R})$ satisfying Assumption I with $\nu = -1$ and

$$\forall x \gtrsim 1, \quad a(x) = x^p, \quad q(x) \lesssim x^r, \quad |q'(x)| \lesssim x^{r-1} \quad p, r \in \mathbb{R}_+. \quad (2.33)$$

We determine b when $b \rightarrow +\infty$ from the equation $\alpha = a(b)$, see (2.4), namely,

$$b = \alpha^{\frac{1}{p}}, \quad b \rightarrow +\infty.$$

For a sufficiently small $\varepsilon > 0$, we take $\delta = b^{1-\varepsilon}$ and start to check the conditions in Theorem 2.1. To this end, we observe that as $b \rightarrow +\infty$, we have

$$a(b) = b^p, \quad q_b^{(j)} \lesssim b^{r-j}, \quad j = 0, 1. \quad (2.34)$$

The condition (2.15) guaranteeing the successful cut-off is clearly satisfied independently of the choice of $\beta(b)$ since $\delta \rightarrow \infty$ as $b \rightarrow +\infty$. To satisfy the remaining conditions (2.16) and (2.17), we impose the following restrictions on $\beta(b)$

$$\begin{aligned} \text{if } r \geq 2p, \quad & \beta(b) \gtrsim b^s, \quad s > \frac{r}{2}, \\ \text{if } r < 2p, \quad & \beta(b) > 0. \end{aligned} \quad (2.35)$$

Recalling (2.34) and our choice of β , we indeed have

$$\frac{\alpha^2}{b^2} + q_b^{(0)} + q_b^{(1)} \lesssim b^{2p-2} + b^r + b^{r-1} = o(\alpha^2 + \beta^2), \quad b \rightarrow +\infty.$$

In summary, with the choice of $\beta = \beta(b)$ in (2.35), the statement of Theorem 2.1 holds.

Example 2.8. (Exponential dampings and potentials)

Next we consider dampings $a \in C^2(\mathbb{R})$ and potentials $q \in C^1(\mathbb{R})$ satisfying

$$\forall x \gtrsim 1, \quad a(x) = e^{x^p}, \quad q(x) \lesssim e^{x^r}, \quad |q'(x)| \lesssim x^{r-1} e^{x^r}, \quad p, r \in \mathbb{R}_+,$$

thus Assumption I holds with $\nu = p - 1$. We further suppose that

$$r \leq p. \quad (2.36)$$

From (2.4), we have

$$b = (\ln \alpha)^{\frac{1}{p}}, \quad b \rightarrow +\infty.$$

With a sufficiently small $\varepsilon > 0$, we take $\delta = b^{-(p-1)-\varepsilon}$ and obtain that as $b \rightarrow +\infty$

$$a(b) = e^{b^p}, \quad a'(b) \approx b^{p-1} e^{b^p}, \quad q_b^{(j)} \lesssim e^{(1+o(1))b^r}, \quad j = 0, 1.$$

With regard to the conditions (2.16) and (2.17), it follows from (2.36) that

$$q_b^{(0)} + q_b^{(1)} \lesssim e^{(1+o(1))b^r} = o(\alpha^2), \quad b \rightarrow +\infty,$$

thus no restrictions on β are imposed. On the other hand, the first term in (2.17) behaves as

$$a(b)^2 b^{2\nu} = b^{2(p-1)} e^{2b^p}, \quad b \rightarrow +\infty,$$

thus we obtain the following restrictions on β

$$\begin{aligned} \text{if } p \geq 1, \quad & \beta(b) \gtrsim b^s e^{b^p}, \quad s > p - 1, \\ \text{if } p < 1, \quad & \beta(b) > 0. \end{aligned} \quad (2.37)$$

Finally, we can verify that with this choice of β , also the condition (2.15) is satisfied. Indeed, for every $c > 0$, we have

$$\left(b^{2(p-1)+2\varepsilon} + \frac{b^{4(p-1)+4\varepsilon}}{\alpha^2 + \beta^2} \right) e^{-c \frac{\beta}{\alpha + \beta} a'(b) b^{-2(p-1)-2\varepsilon}} = o(1), \quad b \rightarrow +\infty$$

since, in the non-obvious case $p \geq 1$, we have at least exponential decay due to

$$\frac{\beta}{\alpha + \beta} a'(b) b^{-2(p-1)-2\varepsilon} \approx b^{-p+1-2\varepsilon} e^{b^p}, \quad b \rightarrow +\infty.$$

In summary, with $\beta = \beta(b)$ as in (2.37), the statement of Theorem 2.1 holds.

Example 2.9. (Logarithmic dampings and potentials)

Finally, we consider dampings and potentials $a \in C^2(\mathbb{R})$ and $q \in C^1(\mathbb{R})$ satisfying

$$\forall x \gtrsim 1, \quad a(x) = \ln(x), \quad q(x) + |q'(x)| \lesssim \ln(x),$$

which satisfy Assumption I with $\nu = -1$. From (2.4), we immediately have that

$$b = e^\alpha, \quad b \rightarrow +\infty$$

and for a sufficiently small $\varepsilon > 0$, we take $\delta = b^{1-\varepsilon}$. It follows that, as $b \rightarrow +\infty$,

$$a(b) = \ln(b), \quad a'(b) = \frac{1}{b}, \quad q_b^{(j)} \lesssim \ln(b), \quad j = 0, 1.$$

Hence the condition (2.17) holds without any restrictions on $\beta > 0$ as

$$\frac{a(b)^2}{b^2} + q_b^{(0)} + q_b^{(1)} \lesssim \ln(b) = o(\alpha^2), \quad b \rightarrow +\infty.$$

Since $\delta \rightarrow +\infty$ as $b \rightarrow +\infty$, the condition (2.15) holds also without any restrictions on β .

In summary, the statement of Theorem 2.1 holds with any choice of $\beta(b) > 0$.

3. EXPANSION OF THE PHASE

In Section 2, only the simplest form of the pseudomode was used. In detail, the function g in (2.8) has the form

$$g = \exp(-i\lambda\psi_{-1}).$$

If a and q are more regular, more terms in the exponent can be considered, namely we employ a general WKB expansion

$$g := \exp\left(-i\lambda\psi_{-1} - \sum_{k=0}^{n-1} \lambda^{-k} \psi_k\right); \quad (3.1)$$

the functions ψ_k are determined by a standard procedure briefly summarized in Section 3.2 below.

The basic ansatz in Section 2 works already for several important examples, see Section 2.4. Nonetheless, by taking more terms in g , we obtain faster decay rates in the main statement, see (2.18), which we make quantitative this time, see (3.7) and *e.g.* the obtained rates (3.25) in the example with a polynomial damping. The expansion allows also to achieve a larger set of curves along which we have a decay in the main statement (3.7). In other words, we can relax restrictions on the choice of $\beta = \beta(b)$, see examples in Sections 2.4 and 3.5.

As in the previous case, we need to employ a λ -dependent cut-off $\xi \in C_0^\infty(\mathbb{R})$ to construct a suitable pseudomode $f_\lambda := \xi g$; the choice of ξ is the same as in Section 2, see (2.5).

3.1. The main result. In this section, we assume the following basic regularity and growth assumptions on the damping and potential and state the main result of the paper.

Assumption II. Suppose that the functions $a, q \in C^{n+1}(\mathbb{R})$ with $n \in \mathbb{N}$ satisfy the following conditions:

(a) a, q are non-negative for sufficiently large x :

$$\forall x \gtrsim 1, \quad a(x) \geq 0 \quad \text{and} \quad q(x) \geq 0;$$

(b) a is increasing for sufficiently large x and unbounded at infinity:

$$\begin{aligned} \forall x \gtrsim 1, \quad a'(x) &> 0, \\ \lim_{x \rightarrow +\infty} a(x) &= +\infty; \end{aligned}$$

(c) the derivatives of a are controlled by a :

$$\exists \nu \geq -1, \quad \forall m \in [[1, n+1]], \quad \forall x \gtrsim 1 \quad |a^{(m)}(x)| \lesssim x^{m\nu} a(x). \quad (3.2)$$

Theorem 3.1. *Let Assumption II hold and let $b \in \mathbb{R}_+$ be defined by $\alpha = a(b)$. Take $\varepsilon > 0$ and define*

$$\delta := b^{-\nu-\varepsilon}$$

and

$$q_b^{(j)} := \|q^{(j)}\|_{\infty, (b-\delta, b+\delta)}, \quad j \in [[0, n]].$$

Suppose that there exists a b -dependent $\beta = \beta(b) > 0$ such that the following conditions hold as $b \rightarrow +\infty$

$$\forall c > 0, \quad \left(\frac{1}{\delta} + \frac{1}{\alpha + \beta} \frac{1}{\delta^2} \right) \exp \left(-c \frac{\beta}{\alpha + \beta} a'(b) \delta^2 \right) =: \kappa_1(b, c) = o(1), \quad (3.3)$$

$$q_b^{(0)} = o(\alpha^2 + \beta^2), \quad (3.4)$$

$$\forall j \in [[1, n]], \quad q_b^{(j)} = \mathcal{O}(\alpha(\alpha + \beta)b^{j\nu}), \quad (3.5)$$

$$\frac{b^\nu}{\alpha + \beta} = \mathcal{O}(1). \quad (3.6)$$

Let $\{f_\lambda\}$ with $\lambda = \lambda(b) = -\alpha + i\beta$, see (2.2), be a family of functions constructed as in (2.8) with ξ as in (2.5), g as in (3.1) and the choice of α, β and δ as above.

Then there exists a positive $C > 0$ such that for $\Psi_\lambda = (f_\lambda, \lambda f_\lambda)^t$, we have

$$\frac{\|(G - \lambda)\Psi_\lambda\|_{\mathcal{H}}}{\|\Psi_\lambda\|_{\mathcal{H}}} \lesssim \kappa_1(b, C) + \kappa_2(b), \quad b \rightarrow +\infty, \quad (3.7)$$

where

$$\kappa_2(b) := \frac{\alpha b^{\nu(n+1)}}{(\alpha + \beta)^n} + \sum_{k=1}^{n-1} \frac{b^{\nu(n+k+1)} \alpha^2}{(\alpha + \beta)^{n+1+k}}. \quad (3.8)$$

Analogously to the remarks below Theorem 2.1, the condition (3.3) is actually too strong. It suffices to satisfy it for one sufficiently small constant $C = c_4/8 > 0$, which also enters $\kappa_1(b, C)$ and which can be estimated with more detailed information on a' . As examples show, the decay of κ_1 is typically much faster than of κ_2 which then determines the final decay rate estimate.

Remark 3.2. It might appear that the dependence of the rates in (3.7) as well as the conditions on β , determining the curves along which we have a decay in (3.7), are just a limitation of the method. However, several examples and results for Schrödinger operators, see [2, 1, 11, 6, 9], suggest that this dependence is fundamental and indeed reflecting the regularity of coefficients and their behavior at infinity. A more detailed discussion can be found in the introduction in [9].

3.2. WKB expansion. We follow the standard WKB procedure, for details see *e.g.* [2] or in particular [10, Sec. 2.4] where only minor modifications (mainly notational, one should set $V := 2\lambda a + q$ and the new spectral parameter is $-\lambda^2$ instead of λ) are needed. We obtain that

$$T(\lambda)g = \left(\sum_{k=n-1}^{2(n-1)} \lambda^{-k} \phi_{k+1} \right) g =: r_n g, \quad (3.9)$$

where functions ϕ_k are defined as (with some $c_{\omega, \chi} \in \mathbb{C}$ with $|c_{\omega, \chi}| = 1$)

$$\psi_k'' - \sum_{\substack{\omega + \chi = k \\ \omega, \chi \neq -1}} c_{\omega, \chi} \psi_\omega' \psi_\chi' =: \phi_{k+1}, \quad (3.10)$$

with the convention that $\psi_\omega = 0$ whenever $\omega \geq n$ or $\omega \leq -2$ and ψ_k satisfy

$$\begin{aligned} \psi_{-1}' &= \left(\frac{-\lambda^2 - 2\lambda a - q}{\lambda^2} \right)^{\frac{1}{2}}, \\ \psi_{k+1}' &= \frac{1}{2\psi_{-1}'} \left(\psi_k'' - \sum_{\substack{\omega + \chi = k \\ \omega, \chi \neq -1}} c_{\omega, \chi} \psi_\omega' \psi_\chi' \right), \quad k \in [[-1, n-2]], \end{aligned} \quad (3.11)$$

again with the same convention for ψ_ω . For the function ψ_0' , one gets in particular

$$\psi_0' = -\frac{1}{4} \frac{2\lambda a' + q'}{-\lambda^2 - 2\lambda a - q}. \quad (3.12)$$

For the forthcoming estimates, it is crucial to understand the structure of the functions ψ_k' and remainders r_n , which is the content of the following two lemmas; detailed proofs (with minor modifications in notations) are in [10, Appendix].

Lemma 3.3. *Let $n \in \mathbb{N}_0$, $a, q \in C^{n+1}(\mathbb{R})$ and functions $\{\psi_k'\}_{k \in [[-1, n-1]]}$ be determined by (3.11). Then*

$$\psi_k^{(m)} = \frac{(-\lambda^2)^{\frac{k}{2}}}{(-\lambda^2 - 2\lambda a - q)^{\frac{k}{2}}} \sum_{j=0}^{k+m} \frac{T_j^{k+m, k+m+1-j}}{(-\lambda^2 - 2\lambda a - q)^j}, \quad m \in [[1, n+1-k]], \quad (3.13)$$

where (with some $c_\omega \in \mathbb{C}$)

$$T_j^{r,s} := \sum_{\omega \in \mathcal{I}_j^{r,s}} c_\omega ((2\lambda a + q)^{(1)})^{\omega_1} ((2\lambda a + q)^{(2)})^{\omega_2} \cdot \dots \cdot ((2\lambda a + q)^{(s)})^{\omega_s}, \quad (3.14)$$

and

$$\mathcal{I}_j^{r,s} := \left\{ \omega \in \mathbb{N}_0^s : \sum_{i=1}^s i\omega_i = r \quad \& \quad \sum_{i=1}^s \omega_i = j \right\}. \quad (3.15)$$

Lemma 3.4. *Let $n \in \mathbb{N}_0$, $a, q \in C^{n+1}(\mathbb{R})$ and functions $\{\psi_k'\}_{k \in [[-1, n-1]]}$ be determined by (3.11), $\{\phi_k\}_{k \in [[-1, 2n-1]]}$ be as in (3.10) and r_n as in (3.9). Then*

$$|r_n| \lesssim \frac{|(2\lambda a + q)^{(n+1)}|}{|\lambda^2 + 2\lambda a + q|^{\frac{n+1}{2}}} + \sum_{k=0}^{n-1} \frac{1}{|\lambda^2 + 2\lambda a + q|^{\frac{n-1+k}{2}}} \sum_{j=2}^{n+1+k} \frac{T_j^{n+1+k, n}}{|\lambda^2 + 2\lambda a + q|^j},$$

where $T_j^{r,s}$ are as in (3.14).

3.3. Technical lemmas. The first lemma enables us to treat the terms with $k \geq 0$ in the expansion of g , see (3.1).

Lemma 3.5. *Let Assumptions II hold, let $\{\psi'_k\}_{k \in [[-1, n-1]]}$ be determined by (3.11) and let, as $b \rightarrow +\infty$,*

$$\begin{aligned} q_b^{(0)} &= o(\alpha^2 + \beta^2), \\ \forall j \in [[1, n]], \quad q_b^{(j)} &= \mathcal{O}(\alpha(\alpha + \beta)b^{j\nu}). \end{aligned} \quad (3.16)$$

Then, for all $k \in [[0, n-1]]$ and for all $t \in (b - \delta, b + \delta)$

$$|\lambda^{-k}\psi'_k(t)| \lesssim \frac{b^{\nu(k+1)}}{(\alpha + \beta)^k} \sum_{j=1}^{k+1} \frac{\alpha^j}{(\alpha + \beta)^j}, \quad b \rightarrow +\infty. \quad (3.17)$$

Proof. The essential ingredient of the proof is Lemma 3.3. We omit writing the argument t , but we always use that $t \in (b - \delta, b + \delta)$. From (3.13), the definition of $T_j^{r,s}$, see (3.14), and $|\lambda^2 + 2\lambda a + q| = |\zeta| \approx \alpha^2 + \beta^2 \approx |\lambda|^2$, see Lemma 2.3, we get

$$\begin{aligned} |\lambda^{-k}\psi'_k| &\lesssim \sum_{j=1}^{k+1} \frac{|T_j^{k+m, k+2-j}|}{|\lambda^2 + 2\lambda a + q|^{j+\frac{k}{2}}} \\ &\lesssim \sum_{j=1}^{k+1} \sum_{\omega \in \mathcal{I}_j^{k+1, k+2-j}} \frac{|2\lambda a' + q'|^{\omega_1} \cdots |2\lambda a^{(k+2-j)} + q^{(k+2-j)}|^{\omega_{k+2-j}}}{|\lambda|^{2j+k}}. \end{aligned}$$

The assumptions (3.2) and (3.16) give further that

$$|\lambda^{-k}\psi'_k| \lesssim \sum_{j=1}^{k+1} \sum_{\omega \in \mathcal{I}_j^{k+1, k+2-j}} \frac{|\lambda \alpha b^{\nu}|^{\omega_1} \cdots |\lambda \alpha b^{\nu(k+2-j)}|^{\omega_{k+2-j}}}{|\lambda|^{2j+k}}.$$

The definition of $\mathcal{I}_j^{r,s}$, see (3.15), yields $\sum_{i=1}^{k+2-j} i\omega_i = k+1$ and $\sum_{i=1}^{k+2-j} \omega_i = j$, thus (3.17) follows. $\square$

The next aim is to estimate $|g|$. It turns out that with the assumptions above the result for the basic pseudomode with $n = 0$ remains valid (with possibly different constants), see Lemma 2.4.

Lemma 3.6. *Let the assumptions of Lemma 3.5 hold and suppose in addition that*

$$\frac{b^\nu}{\alpha + \beta} = \mathcal{O}(1), \quad b \rightarrow +\infty. \quad (3.18)$$

Let g be defined as in (3.1). Then there exist two positive constants $c_3, c_4 > 0$ such that for all $s \in (b - \delta, b + \delta)$, we have, as $b \rightarrow +\infty$,

$$\exp\left(c_3 \frac{\beta}{\alpha + \beta} a'(b)(s - b)^2\right) \lesssim |g(s)| \lesssim \exp\left(c_4 \frac{\beta}{\alpha + \beta} a'(b)(s - b)^2\right). \quad (3.19)$$

Proof. First we deal with the terms with $k > 0$ in the expansion, the case $k = 0$ is treated separately and differently. With Lemma 3.5 and assumption (3.18) we get

$$\left| \sum_{k=1}^{n-1} \int_b^s \lambda^{-k} \psi'_k(t) dt \right| \lesssim \sum_{k=1}^{n-1} \sum_{j=1}^{k+1} \frac{b^{\nu k - \varepsilon} \alpha^j}{(\alpha + \beta)^{j+k}} \lesssim b^{-\varepsilon}, \quad b \rightarrow +\infty. \quad (3.20)$$

In the case $k = 0$, we use the formula for ψ'_0 , see (3.12), which leads to

$$|\exp(-\psi_0(s))| = \left| \exp\left(-\int_b^s \psi'_0(t) dt\right) \right| = \left| \frac{\lambda^2 + 2\lambda a(s) + q(s)}{\lambda^2 + 2\lambda a(b) + q(b)} \right|^{\frac{1}{4}} = \left| \frac{\zeta(s)}{\zeta(b)} \right|^{\frac{1}{4}}.$$

With Lemma 2.3 and (2.21), we get that for all $s \in (b - \delta, b + \delta)$

$$|\exp(-\psi_0(s))| \approx 1, \quad b \rightarrow +\infty. \quad (3.21)$$

Now we are ready to estimate $|g|$. Using (3.21) and (3.20), we get, as $b \rightarrow +\infty$,

$$|g| = \exp \left(\operatorname{Re} \left(-i\lambda\psi_{-1} - \sum_{k=0}^{n-1} \lambda^{-k} \psi_k \right) \right) = |\exp(-\psi_0 + o(1))| |\exp(\operatorname{Re}(-i\lambda\psi_{-1}))|,$$

which holds for all $s \in (b - \delta, b + \delta)$. Thus (3.19) follows from Lemma 2.4. $\square$

The next step is to estimate $\|\xi g\|^2$ from below and $\|\xi' g'\|^2$, $\|\xi'' g\|^2$ from above. In fact, we show that under our assumptions, these estimates remain the same as for the basic pseudomode, see Lemma 3.7.

Lemma 3.7. *Let the assumptions of Lemma 3.6 hold. Let ξ , g be defined as in (2.5), (3.1), respectively. Then, as $b \rightarrow +\infty$,*

$$\begin{aligned} \|\xi g\|^2 &\gtrsim \delta \exp \left(-\frac{c_4}{4} \frac{\beta}{\alpha + \beta} a'(b) \delta^2 \right), \\ \|\xi' g'\|^2 &\lesssim (\alpha^2 + \beta^2) \delta^{-1} \exp \left(-\frac{c_4}{2} \frac{\beta}{\alpha + \beta} a'(b) \delta^2 \right), \\ \|\xi'' g\|^2 &\lesssim \delta^{-3} \exp \left(-\frac{c_4}{2} \frac{\beta}{\alpha + \beta} a'(b) \delta^2 \right). \end{aligned} \quad (3.22)$$

Proof. Using the previously proved lemmas, we reduce the proof to the estimates obtained in Lemma 2.5 for the basic pseudomode; with regard to Lemma 3.6, this is immediate for the estimates of $\|\xi g\|$ and $\|\xi'' g\|$. The remaining term $\|\xi' g'\|$ is estimated using Lemmas 2.3, 3.5 and assumption (3.18). Recalling (2.6) and the size of $\operatorname{supp} \xi'$, see (2.20), we obtain

$$\begin{aligned} \|\xi' g'\|^2 &\lesssim \int_{\operatorname{supp} \xi'} \|\xi'\|_\infty^2 |g'(s)|^2 ds \\ &\lesssim \delta^{-2} \int_{\operatorname{supp} \xi'} \left(\underbrace{|\lambda \psi'_{-1}(s)|^2}_{=|\zeta| \approx (\alpha + \beta)^2} + \sum_{k=0}^{n-1} |\lambda^{-k} \psi'_k(s)|^2 \right) |g(s)|^2 ds \\ &\lesssim \delta^{-2} \int_{\operatorname{supp} \xi'} \left((\alpha + \beta)^2 + b^{2\nu} \right) |g(s)|^2 ds, \\ &\lesssim \delta^{-2} \int_{\operatorname{supp} \xi'} (\alpha + \beta)^2 (1 + \mathcal{O}(1)) |g(s)|^2 ds, \quad b \rightarrow +\infty. \end{aligned}$$

Thus (3.22) follows from Lemma 2.5 as well. $\square$

Finally, we estimate the remainder r_n , see (3.9).

Lemma 3.8. *Let the assumptions of Lemma 3.6 hold and let r_n be as in (3.9). Then*

$$\|r_n\|_{\infty, (b-\delta, b+\delta)} \lesssim \frac{\alpha b^{\nu(n+1)}}{(\alpha + \beta)^n} + \sum_{k=1}^{n-1} \frac{b^{\nu(n+k+1)} \alpha^2}{(\alpha + \beta)^{n+1+k}} = \kappa_2(b), \quad b \rightarrow +\infty. \quad (3.23)$$

Proof. Lemma 3.4 on the structure of r_n , Lemma 2.3 on the size of ζ and the assumption (3.16) yield

$$\begin{aligned} |r_n| &\lesssim \frac{|(2\lambda a + q)^{(n+1)}|}{|\lambda^2 + 2\lambda a + q|^{\frac{n+1}{2}}} + \sum_{k=0}^{n-1} \sum_{j=2}^{n+1+k} \frac{|T_j^{n+1+k, n}|}{|\lambda^2 + 2\lambda a + q|^{\frac{n-1+k}{2} + j}} \\ &\lesssim \frac{\alpha b^{\nu(n+1)}}{|\lambda|^n} + \sum_{k=0}^{n-1} \sum_{j=2}^{n+1+k} \frac{|T_j^{n+1+k, n}|}{|\lambda|^{n-1+k+2j}}. \end{aligned} \quad (3.24)$$

From the definition of $T_j^{n+1+k,n}$, see (3.14) and (3.15), and assumptions (3.2) and (3.16), we get for all $s \in (b - \delta, b + \delta)$

$$\begin{aligned} |T_j^{n+1+k,n}(s)| &\lesssim \sum_{\omega \in \mathcal{I}_j^{n+1+k,n}} |2\lambda a'(s) + q'(s)|^{\omega_1} \cdot \dots \cdot |2\lambda a^{(n)}(s) + q^{(n)}(s)|^{\omega_n} \\ &\lesssim \sum_{\omega \in \mathcal{I}_j^{n+1+k,n}} (\alpha|\lambda|b^\nu)^{\omega_1} \cdot \dots \cdot (\alpha|\lambda|b^{\nu n})^{\omega_n} \\ &\lesssim (\alpha|\lambda|)^j b^{\nu(n+1+k)}. \end{aligned}$$

Inserting this into (3.24) and observing that $\alpha/|\lambda| \approx \alpha/(\alpha + \beta) \leq 1$, which enables us to skip the term with $k = 0$ in the sum, we obtain the estimate (3.23) in the claim. $\square$

3.4. The proof of the main Theorem 3.1. Lemmas 3.6 and 3.8 are analogues of Lemmas 2.4 and 2.6 and so the proof of the main Theorem 3.1 becomes a direct analogue of the one of Theorem 2.1.

Proof of Theorem 3.1. As in the proof of Theorem 2.1, the estimates are split into three parts. Using (2.19) we arrive at (neglecting $\|(\xi g)'\|^2 + \|q^{\frac{1}{2}}\xi g\|^2$ as before in the proof of Theorem 2.1)

$$\frac{\|T(\lambda)\xi g\|^2}{\|(\xi g)'\|^2 + \|q^{\frac{1}{2}}\xi g\|^2 + |\lambda|^2\|\xi g\|^2} \lesssim \underbrace{\frac{\|\xi''g\|^2}{|\lambda|^2\|\xi g\|^2}}_{=:Q_1} + \underbrace{\frac{\|\xi'g'\|^2}{|\lambda|^2\|\xi g\|^2}}_{=:Q_2} + \underbrace{\frac{\|r_n\|_{\infty,(b-\delta,b+\delta)}^2\|\xi g\|^2}{|\lambda|^2\|\xi g\|^2}}_{=:Q_3}.$$

The terms Q_1, Q_2, Q_3 are estimated in the completely same way as in the proof of Theorem 2.1, nevertheless, appropriate replacements of technical steps, *i.e.* Lemmas 3.6 and 3.8, are employed and definitions of κ_1, κ_2 are used, see (3.3), (3.8). $\square$

3.5. Examples.

Example 3.9. (Polynomial-like dampings and potentials – continued)

We consider non-negative dampings $a \in C^{n+1}(\mathbb{R})$ and potentials $q \in C^{n+1}(\mathbb{R})$ with $n > 1$ satisfying, similarly as in (2.33),

$$\forall x \gtrsim 1, \quad a(x) = x^p, \quad |q^{(j)}(x)| \lesssim x^{r-j} \quad p, r \in \mathbb{R}_+, \quad j \in [[0, n]].$$

It is easy to see that Assumption II holds with $\nu = -1$.

Notice that the decay of $\kappa_1(b, c)$, see (3.3), is exponential for every $c > 0$ if $\beta = \beta(b) \gtrsim b^s$ with $s > -1$, thus the decay rate in (3.7) comes from the remainder, *i.e.* from $\kappa_2(b)$. For smaller β , one needs to compare the decay rates of κ_1 and κ_2 ; we omit discussing such cases in the following.

Choosing $\beta = \beta(b)$ similarly in (2.35), namely,

$$\begin{aligned} &\text{if } r \geq 2p, \quad \beta(b) \gtrsim b^s, \quad s > r - p, \\ &\text{if } r < 2p, \quad \beta(b) \gtrsim b^s, \quad s > -1, \end{aligned}$$

we can easily check that all conditions (3.4), (3.5) and (3.6) are indeed satisfied.

Finally, we determine the decay rates of κ_2 in (3.7). Ignoring even the contributions of β , we get (in fact from the first term in (3.8))

$$\kappa_2(b) = \mathcal{O}(b^{-(n-1)(p+1)-2}), \quad b \rightarrow +\infty. \quad (3.25)$$

Example 3.10. (Exponential dampings and potentials – continued).

Similarly as in Example 2.8, we consider non-negative $a, q \in C^{n+1}(\mathbb{R})$ with $n > 1$ of the form

$$\forall x \gtrsim 1, \quad a(x) = e^{x^p}, \quad |q^{(j)}(x)| \lesssim x^{r-j} e^{x^r}, \quad p, r \in \mathbb{R}_+, \quad r \leq p, \quad j \in [[0, n]],$$

which satisfy Assumption II with $\nu = p - 1$.

It is straightforward to check that the condition (3.3) holds if we choose $\beta = \beta(b)$ in the following way

$$\begin{aligned} &\text{if } p \geq 1, \quad \beta(b) \gtrsim b^s, \quad s > p - 1, \\ &\text{if } p < 1, \quad \beta(b) > 0. \end{aligned} \tag{3.26}$$

The remaining conditions (3.4), (3.5) and (3.6) are satisfied due to $r \leq p$.

The second term in (3.7) can be estimated as

$$\kappa_2(b) = \mathcal{O}(b^{(p-1)(n+1)} e^{(1-n)b^p}), \quad b \rightarrow +\infty.$$

If β is small, then the estimate of $\kappa_1(b, C)$ might be lengthier and the rate can be effectively slower than of $\kappa_2(b)$. Nonetheless, for *e.g.* $\beta \gtrsim \alpha$, we have

$$\kappa_1(b, C) = \mathcal{O}(e^{-\tilde{C}e^{b^p}}), \quad b \rightarrow +\infty,$$

with some $\tilde{C} > 0$, *i.e.* a much faster decay than in κ_2 .

Notice that the set of suitable curves along which λ can tend to infinity, *i.e.* the restrictions on β , see (3.26), is substantially enlarged for $p \geq 1$ comparing to (2.37).

Example 3.11. (Logarithmic dampings and potentials – continued).

Finally, we consider non-negative $a, q \in C^{n+1}(\mathbb{R})$ with $n > 1$ of the form

$$\forall x \gtrsim 1, \quad a(x) = \ln(x), \quad |q^{(j)}(x)| \lesssim x^{-j} \ln x, \quad j \in [[0, n]],$$

which satisfies Assumption II with $\nu = -1$. Similarly as in Example 2.9, the conditions (3.3), (3.4), (3.5) and (3.6) hold without any restriction on $\beta = \beta(b) > 0$. If β is not too small, *e.g.* $\beta(b) \gtrsim b^s$ with $s > -1$, the term $\kappa_1(b, C)$ exhibits much faster decay (exponential in b) than κ_2 , for which we obtain

$$\kappa_2(b) = \mathcal{O}((\ln b)^{1-n} b^{-n-1}), \quad b \rightarrow +\infty.$$

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