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Crann, J., Kribs, D. W., Levene, R., & Todorov, I. (2020). State Convertibility in the von Neumann Algebra Framework. *Communications in Mathematical Physics*, 378, 1123–1156. <https://doi.org/10.1007/s00220-020-03803-3>

Published in:
Communications in Mathematical Physics

Document Version:
Peer reviewed version

Queen's University Belfast - Research Portal:
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STATE CONVERTIBILITY IN THE VON NEUMANN ALGEBRA FRAMEWORK

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ABSTRACT. We establish a generalisation of the fundamental state convertibility theorem in quantum information to the context of bipartite quantum systems modelled by commuting semi-finite von Neumann algebras. Namely, we establish a generalisation to this setting of Nielsen's theorem on the convertibility of quantum states under local operations and classical communication (LOCC) schemes. Along the way, we introduce an appropriate generalisation of LOCC operations and connect the resulting notion of approximate convertibility to the theory of singular numbers and majorisation in von Neumann algebras. As an application of our result in the setting of II_1 -factors, we show that the entropy of the singular value distribution relative to the unique tracial state is an entanglement monotone in the sense of Vidal, thus yielding a new way to quantify entanglement in that context. Building on previous work in the infinite-dimensional setting, we show that trace vectors play the role of maximally entangled states for general II_1 -factors. Examples are drawn from infinite spin chains, quasi-free representations of the CAR, and discretised versions of the CCR.

1. INTRODUCTION

Quantum entanglement is a central notion in quantum information theory and a key resource in the applications that are driving efforts to develop quantum technologies. While there is a growing depth of understanding of the concept and its many potential uses, the theory of quantum entanglement remains an active, challenging, and fundamental area of investigation in quantum information theory with, of particular note here, relatively little progress having been made in the general infinite-dimensional and von Neumann algebra settings.

The mathematical theory that provides the foundation for these efforts rests in many ways on an understanding of how entanglement between quantum states can be transformed through various tasks and processes. A very

Date: 18 April 2019.

2010 *Mathematics Subject Classification.* 46L10, 46L30, 46N50, 47L90, 81P40, 81P45, 81R15.

Key words and phrases. quantum state, entanglement, majorisation, local operations and classical communication, completely positive map, von Neumann algebra, singular values, trace vectors.

natural and central question is to ask whether a given state, entangled between multiple parties, can be transformed by certain restricted classes of quantum operations to other types of entangled states, with restrictions determined by theoretical or physical limitations. This is a basic question that is relevant, for instance, in the development of any quantum communication scheme, realisations of quantum algorithms, physical implementations of quantum networks, etc.

As the simplest starting point in this subject, consider the scenario in which two parties A and B each have the ability to implement all local quantum operations, described mathematically by completely positive and trace-preserving maps on the algebras of bounded linear operators on their respective system Hilbert spaces H_A , H_B , but such that the parties are limited in that they can only communicate with each other using classical communication. An initial core problem then is to start with an entangled state $\psi \in H_A \otimes H_B$ shared by the parties, and to determine what are the possible entangled states that ψ can be converted to through local operations and classical communication (LOCC) between them.

This question has a neat matrix theoretic solution in the finite-dimensional case, known as Nielsen's Theorem [37]. For every pure state $\psi \in H_A \otimes H_B$, let $\rho_\psi = \text{tr}_B(\psi\psi^*)$ be the (in general, mixed) state on H_A found by applying the partial trace map over H_B to the projection $\psi\psi^*$ with range the one-dimensional space of scalar multiples of ψ . The eigenvalues of ρ_ψ (including multiplicities) form a probability distribution and can be arranged in non-increasing order, thus giving rise to a real vector λ_ψ . Nielsen's Theorem states that ψ can be converted into another state φ by LOCC between A and B if and only if λ_ψ is majorised by λ_φ , that is, all partial sums of values from λ_ψ (respecting the non-increasing indexing) are bounded above by the corresponding partial sums of values from λ_φ . Recall that ρ_ψ is a pure (rank one) state if and only if ψ is separable. Obviously if we start with a separable state ψ , then we can only transform it to another separable state via LOCC, and this case is easily captured by the theorem with $\lambda_\psi = (1, 0, \dots, 0)$. On the other hand, given any separable state φ , any arbitrary state ψ can be transformed to it via LOCC, in particular by making use of local depolarising maps on the individual systems. The power of Nielsen's theorem lies in the fact that it gives a matrix and spectral theoretic description of which entangled states are attainable through LOCC when we start with a given entangled state. The theorem has far-reaching implications and applications throughout finite-dimensional quantum information theory; indeed, it is one of the most important and widely used results in the entire field.

While the primary focus of research in quantum information has been on challenges and applications in the finite-dimensional and qubit setting for more than two decades now, ultimately general quantum mechanics is an infinite-dimensional theory that is rooted in the theory of von Neumann algebras [50]. Thus, one can reasonably expect that continued long-term

progress in quantum information theory and its connections within theoretical physics will depend at least partly on the successful extension of central results in the field to the infinite-dimensional and general von Neumann algebra settings, with new peculiarities and connections uncovered along the way. This is clearly a desirable goal, and there has been a recent reemergence of activity in this direction, including quantum error correction and privacy (e.g. [5, 10]), entropy theory (e.g. [6, 25, 26, 35, 36]), Bell inequalities (e.g. [28]), the Connes embedding conjecture (e.g. [22, 44, 16]), and entanglement in quantum field theory (e.g. [27] and the references therein).

In this paper, we make new progress in this direction, establishing, as our main result, a generalisation of Nielsen's Theorem to the context of von Neumann algebras, and more specifically for bipartite quantum systems modelled by commuting semi-finite von Neumann algebras, say $\mathcal{A}$ and $\mathcal{B}$ [21, 45, 46]. We note that the case where $\mathcal{A}$ and $\mathcal{B}$ are separably acting factors of type I was considered in [39]. While some parts of the theory extend in a somewhat straightforward way, there are, as one would expect, significant technical challenges to overcome, well beyond the generalisation provided in [39]. En route, we introduce an appropriate generalisation of LOCC operations [9] to our context. The setting for our version of Nielsen's Theorem is provided by the theory of singular numbers [17] and majorisation [23, 24] in von Neumann algebras. We build our analysis on key aspects of operator algebra theory, such as the standard form of a von Neumann algebra, the theory of completely bounded maps, the Haagerup tensor product, and dilation theory [17, 46, 7, 20, 21, 23, 32].

We include a number of examples and applications of our results. In particular, we show that the entropy of the singular value distribution relative to the unique tracial state of a type II_1 -factor is an entanglement monotone in the sense of Vidal [49, 38], thus yielding a new way to quantify entanglement in that context. Building on previous work in the infinite-dimensional setting, we also show that trace vectors play the role of maximally entangled states for general II_1 -factors. Examples are drawn from quantum measurement theory [50], infinite spin chains [29, 30], quasi-free representations of the CAR [2, 13, 14], and discretised versions of the CCR [1, 5, 18].

This paper is organised as follows. The next section includes requisite preliminaries, focussed mainly on von Neumann algebra theory. In Section 3, we introduce the general notion of LOCC maps and derive some basic properties. In Section 4, we investigate approximate convertibility of states by LOCC maps, showing that we can restrict attention to one-way convertibility. This is used in Section 5 to establish our main result on state convertibility and majorisation; this section also includes several supporting technical results. Section 6 includes the aforementioned applications and examples. Other illustrative examples are presented throughout the paper. We finish with a brief outlook discussion on our results and the subject.

2. PRELIMINARIES

Let $\mathcal{A}$ be a von Neumann algebra. We denote by $\mathcal{A}_*$ its predual; thus, the elements of $\mathcal{A}_*$ are normal (that is, weak* continuous) linear functionals on $\mathcal{A}$. If $\mathcal{B}$ is another von Neumann algebra with $\mathcal{A} \subseteq \mathcal{B}$, a map $\Phi : \mathcal{B} \rightarrow \mathcal{B}$ is called an $\mathcal{A}$ -bimodule map if $\Phi(axb) = a\Phi(x)b$, for all $a, b \in \mathcal{A}$ and all $x \in \mathcal{B}$. We denote by $\text{CP}_{\mathcal{A}}^{\sigma}(\mathcal{B})$ the cone of all normal completely positive $\mathcal{A}$ -bimodule maps on $\mathcal{B}$. We let $\text{UCP}_{\mathcal{A}}^{\sigma}(\mathcal{B})$ stand for the convex subset of all unital maps in $\text{CP}_{\mathcal{A}}^{\sigma}(\mathcal{B})$. We set $\text{CP}^{\sigma}(\mathcal{B}) = \text{CP}_{\mathbb{C}I}^{\sigma}(\mathcal{B})$ and $\text{UCP}^{\sigma}(\mathcal{B}) = \text{UCP}_{\mathbb{C}I}^{\sigma}(\mathcal{B})$. We call the elements of $\text{UCP}^{\sigma}(\mathcal{B})$ *quantum channels* on $\mathcal{B}$. We refer the reader to [40] for basics of the theory of completely positive and completely bounded maps and some standard notation.

We denote by $S(\mathcal{A})$ the convex set of all normal states of $\mathcal{A}$, and by $S_{\text{ext}}(\mathcal{A})$ the set of all pure normal states of $\mathcal{A}$. If $\omega \in \mathcal{A}_*$ and $a \in \mathcal{M}$, define $a\omega, \omega a \in \mathcal{A}_*$ by $(a\omega)(x) = \omega(xa)$ and $(\omega a)(x) = \omega(ax)$. We sometimes use the duality pairing notation $\langle x, \omega \rangle := \omega(x)$ with angled brackets (in contrast, we use rounded parentheses for inner products in Hilbert spaces); in this notation, the bimodule action just described may be written as $\langle x, a\omega b \rangle = \langle bxa, \omega \rangle$, $a, b \in \mathcal{A}$.

As usual, we let $\mathcal{B}(H)$ be the C^* -algebra of all bounded operators on a Hilbert space H , and set $M_n = \mathcal{B}(\mathbb{C}^n)$, the algebra of $n \times n$ matrices with complex entries. We assume throughout this paper that all Hilbert spaces under consideration are separable, sometimes mentioning this explicitly for emphasis. For an element $a \in \mathcal{B}(H)$, we write $\text{Ad}(a)$ for the map given by $\text{Ad}(a)(x) = axa^*$, where a^* denotes the adjoint of a ; clearly, $\text{Ad}(a) \in \text{CP}^{\sigma}(\mathcal{B}(H))$. If $\varphi, \psi \in H$, we let $\varphi\psi^*$ be the rank one operator on H given by $(\varphi\psi^*)(\xi) = (\xi, \psi)\varphi$ (note that our inner products are linear in the first argument). Let $\mathcal{T}(H)$ be the space of all trace class operators in $\mathcal{B}(H)$ and $\text{tr} : \mathcal{T}(H) \rightarrow \mathbb{C}$ be the trace. We have a canonical identification $\mathcal{T}(H) \equiv \mathcal{B}(H)_*$, given by letting $\langle T, S \rangle = \text{tr}(TS)$, $T \in \mathcal{B}(H)$, $S \in \mathcal{T}(H)$. We denote by H_1 the set of unit vectors in a Hilbert space H . If $\varphi \in H$, we write $\omega_{\varphi} \in \mathcal{B}(H)_*$ for the (positive, normal) functional given by $\omega_{\varphi}(x) = (x\varphi, \varphi)$.

Throughout the paper, we will let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a von Neumann algebra, for some (separable) Hilbert space H , with unit 1, projection lattice $\mathcal{P}(\mathcal{A})$ and positive cone $\mathcal{A}^+$. We will mainly be interested in the case when $\mathcal{A}$ is semi-finite, equipped with a normal semi-finite faithful trace τ . Let $\tilde{\mathcal{A}}$ be the $*$ -algebra of all τ -measurable operators [17], that is, the set of all densely defined closed operators $T : \mathcal{D}(T) \rightarrow H$, (where $\mathcal{D}(T) \subseteq H$ is the domain of T), affiliated with $\mathcal{A}$, with the property that for every $\varepsilon > 0$ there exists a projection $e \in \mathcal{A}$ such that $\tau(1 - e) \leq \varepsilon$ and $eH \subseteq \mathcal{D}(T)$. For $p \geq 1$, let

$$\mathcal{A}_p = \{a \in \mathcal{A} : \tau(|a|^p) < \infty\},$$

and let $L^p(\mathcal{A}, \tau)$ be the completion of $\mathcal{A}_p$ with respect to the norm $\|\cdot\|_p$, given by $\|a\|_p = \tau(|a|^p)^{1/p}$, $a \in \mathcal{A}_p$. Set $L^{\infty}(\mathcal{A}, \tau) = \mathcal{A}$. We will extensively use

the fact that the elements of the space $L^p(\mathcal{A}, \tau)$ can be canonically identified with operators in $\tilde{\mathcal{A}}$ (see [17]).

Note that $L^2(\mathcal{A}, \tau)$ is a Hilbert space, which is separable since $\mathcal{A}$ is separably acting, with inner product given by

$$(a, b) = \tau(b^*a), \quad a, b \in \mathcal{A}_2.$$

In fact, $L^2(\mathcal{A}, \tau)$ is the Hilbert space arising from the GNS construction applied to τ . The associated (normal, faithful) $*$ -representation $\pi_\tau : \mathcal{A} \rightarrow \mathcal{B}(H)$ is given by

$$\pi_\tau(a)b = ab, \quad b \in \mathcal{A}_2, \quad a \in \mathcal{A}.$$

We will suppress the use of the notation π_τ ; in this way, we will consider $\mathcal{A}$ as a von Neumann subalgebra of $\mathcal{B}(L^2(\mathcal{A}, \tau))$. We then have that $\mathcal{A}$ is in its standard form [21]; we also say that $\mathcal{A}$ is standardly represented on $L^2(\mathcal{A}, \tau)$. Working in this standard representation, let $\mathcal{A}'$ be the commutant of $\mathcal{A}$, and let $J : L^2(\mathcal{A}, \tau) \rightarrow L^2(\mathcal{A}, \tau)$ be the associated conjugate linear isometry with the property that $\mathcal{A}' = J\mathcal{A}J$. Note that J is the (unique) extension of the adjoint map $a \mapsto a^*$ on $\mathcal{A}_2$, and for $\psi \in L^2(\mathcal{A}, \tau)$, we have that $\psi^* = J\psi$, where the left hand side in the latter identity is the adjoint of the linear operator ψ (by which we mean the element of $\tilde{\mathcal{A}}$ canonically identified with ψ). For $\xi, \eta \in L^2(\mathcal{A}, \tau)$, we have

$$(1) \quad (J\xi, J\eta) = (\xi^*, \eta^*) = \tau(\eta\xi^*) = \tau(\xi^*\eta) = (\eta, \xi).$$

The map $\pi'_\tau : \mathcal{A} \rightarrow \mathcal{B}(L^2(\mathcal{A}, \tau))$, given by

$$\pi'_\tau(a)b = ba, \quad b \in \mathcal{A}_2, \quad a \in \mathcal{A},$$

is a faithful anti- $*$ -homomorphism, satisfying $\pi'_\tau(a) = Ja^*J$, $a \in \mathcal{A}$ (see [45, Theorem V.2.22]). Let $R : \mathcal{A}' \rightarrow \mathcal{A}$ be the anti- $*$ -isomorphism given by

$$R(a') = Ja'^*J, \quad a' \in \mathcal{A}'.$$

We note that $L^1(\mathcal{A}, \tau)$ can be identified in a canonical way with the predual of $\mathcal{A}$. In fact, if $\omega \in \mathcal{A}_*$ then there exists a unique $\rho_\omega \in L^1(\mathcal{A}, \tau)$ such that

$$\omega(a) = \tau(\rho_\omega a), \quad a \in \mathcal{A}.$$

Here, and in the sequel, we use the fact that $L^1(\mathcal{A}, \tau)$ is an $\mathcal{A}$ -bimodule, that is, given $\rho \in L^1(\mathcal{A}, \tau)$ and $a, b \in \mathcal{A}$, we have that $a\rho b$ is a well-defined τ -measurable operator and belongs to $L^1(\mathcal{A}, \tau)$. Note that if $\omega \in \mathcal{A}_*^+$, then ρ_ω is a positive (in general unbounded) operator, which we call the *density operator* of ω .

If $x \in \tilde{\mathcal{A}}$ and $t > 0$, the t -th singular value $\mu_t(x)$ of x is defined by letting

$$\mu_t(x) = \mu_t(x; \mathcal{A}, \tau) = \inf\{\|xp\| : p \in \mathcal{P}(\mathcal{A}), \tau(1-p) \leq t\}.$$

The *singular value function* of x , namely $\mu(x) : (0, \infty) \rightarrow (0, \infty)$, $t \mapsto \mu_t(x)$, is decreasing and continuous from the right [17, Lemma 2.5]. If $x, y \in \tilde{\mathcal{A}}^+$,

we say that x is *majorised* by y if

$$\int_0^s \mu_t(x) dt \leq \int_0^s \mu_t(y) dt, \quad 0 < s \leq \infty;$$

we write $x \prec y$ to designate the fact that x is majorised by y and $\tau(x) = \tau(y)$. We refer to [23] for extensive details on majorisation of elements of $\tilde{\mathcal{A}}^+$ and to [17] for background on the theory of singular values.

Let $\tau' : \mathcal{A}' \rightarrow \mathbb{C}$ be the functional given by $\tau'(a') = \tau(R(a'))$, $a' \in \mathcal{A}'$. Then τ' is a normal faithful semi-finite trace on $\mathcal{A}'$. Since $\mathcal{A}' \subseteq \mathcal{B}(L^2(\mathcal{A}, \tau))$, the elements of $L^1(\mathcal{A}', \tau')$ can be identified with linear densely defined operators on $L^2(\mathcal{A}, \tau)$. Given a normal functional ω' on $\mathcal{A}'$, there exists, by the preceding discussion, a (unique) element $\rho'_{\omega'} \in L^1(\mathcal{A}', \tau')$ such that $\omega'(b) = \tau'(\rho'_{\omega'} b)$, $b \in \mathcal{A}'$. The constructions described above can be performed relative to the pair $(\mathcal{A}', \tau')$; in particular, one may define the corresponding singular values $\mu'_t(x') := \mu_t(x'; \mathcal{A}', \tau')$ associated with any τ' -measurable operator x' , relative to $(\mathcal{A}', \tau')$.

We finish this section with two important examples of the previous notions.

Example 2.1. $\mathcal{A} = L^\infty(X, m)$, for a σ -finite measure space (X, m) . In this case, τ is integration by the measure m and, for any $p \geq 1$, $L^p(\mathcal{A}, \tau) = L^p(X, m)$. In particular, the standard representation is given by the pointwise action of $L^\infty(X, m)$ on $L^2(X, m)$. For a non-negative element $f \in L^1(X, m)$, its singular value function $\mu_t(f)$ satisfies

$$\mu_t(f) = \inf \{s \geq 0 \mid m(\{x \in X \mid f(x) > s\}) \leq t\};$$

in other words, $t \mapsto \mu_t(f)$ is the non-increasing rearrangement of f .

Example 2.2. $\mathcal{A} = \mathcal{B}(H)$ for a Hilbert space H . Here, the (essentially unique) normal semi-finite faithful trace τ is the canonical trace tr . In this case, for $p \geq 1$, the space $L^p(\mathcal{A}, \tau)$ coincides with the Schatten p -class $\mathcal{S}_p(H)$. In particular, the standard representation of $\mathcal{B}(H)$ is given by the left multiplication action on the Hilbert–Schmidt operators $\mathcal{S}_2(H)$. Equivalently, fixing a unitary equivalence $\mathcal{S}_2(H) \cong H \otimes \overline{H}$ (where $\overline{H}$ is the conjugate Hilbert space of H), the standard representation is the canonical action of $\mathcal{B}(H) \otimes 1_{\overline{H}}$ on $H \otimes \overline{H}$. Given a positive element $\rho \in \mathcal{T}(H) = \mathcal{S}_1(H)$, its singular value function $\mu_t(\rho)$ satisfies

$$\mu_t(\rho) = \sum_{n=1}^{\infty} \lambda_n \chi_{[n-1, n)}(t),$$

where λ_n is the n^{th} largest eigenvalue of ρ (including multiplicity).

3. LOCAL OPERATIONS AND CLASSICAL COMMUNICATION

In this section, we introduce the class of maps that realise local operations and classical communication (LOCC) in our general setting and establish some of their properties needed in the sequel. The origins of our approach lie

within the development of algebraic quantum field theory where the framework is typically encoded in two commuting $*$ -subalgebras of a larger C^* -algebra. In this paper, a bipartite quantum system is given by a Hilbert space H , together with a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(H)$, and its commutant $\mathcal{B} := \mathcal{A}'$. In the language of [30, §4], this forms a simple, bipartite system satisfying Haag duality.

The standard intuition comes from viewing $\mathcal{A}$ and $\mathcal{B}$ as the observable algebras of parties Alice and Bob, respectively, which have joint access to a quantum system modelled on the Hilbert space H . For example, when $H = H_A \otimes H_B$ for Hilbert spaces H_A and H_B , then $\mathcal{A} = \mathcal{B}(H_A) \otimes 1_{H_B}$ and $\mathcal{B} = 1_{H_A} \otimes \mathcal{B}(H_B)$ define the canonical bipartite system structure in the tensor product framework.

We now examine a suitable generalisation of local operations and classical communication (LOCC) in this general bipartite setting, inspired by the approach in [9] and related to, but slightly different from, the proposed notion in [48, §5].

Definition 3.1. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a von Neumann algebra on a Hilbert space H , and let $\mathcal{B} = \mathcal{A}'$.

- (i) A *one-way right local map relative to $\mathcal{A}$* is a normal completely positive map $\Theta : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ of the form $\Theta = \Phi \circ \Psi$, where $\Phi \in \text{UCP}_{\mathcal{A}}^{\sigma}(\mathcal{B}(H))$ and $\Psi \in \text{CP}_{\mathcal{B}}^{\sigma}(\mathcal{B}(H))$. Similarly, a *one-way left local map relative to $\mathcal{A}$* is a normal completely positive map $\Theta : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ of the form $\Theta = \Phi \circ \Psi$, where $\Phi \in \text{CP}_{\mathcal{A}}^{\sigma}(\mathcal{B}(H))$ and $\Psi \in \text{UCP}_{\mathcal{B}}^{\sigma}(\mathcal{B}(H))$.
- (ii) An *instrument* is a collection $\mathcal{I} = (\Theta_k)_{k \in \mathbb{N}}$ of normal completely positive maps on $\mathcal{B}(H)$ such that, for every $x \in \mathcal{B}(H)$, the series $\sum_{k=1}^{\infty} \Theta_k(x)$ converges in the weak* topology to a limit, say $\Theta_{\mathcal{I}}(x)$, and the map $x \rightarrow \Theta_{\mathcal{I}}(x)$ is a normal unital completely positive map. In this case, we sometimes write $\sum_{k=1}^{\infty} \Theta_k$ to denote the map $\Theta_{\mathcal{I}}$. We will identify two instruments if they differ only by a bijective relabelling of the index set.
- (iii) A *one-way right instrument relative to $\mathcal{A}$* is an instrument $\mathcal{I} = (\Theta_k)_{k \in \mathbb{N}}$, where each of the maps Θ_k is a one-way right local map relative to $\mathcal{A}$. A *one-way right LOCC map relative to $\mathcal{A}$* is a normal completely positive map $\Theta : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ of the form $\Theta = \Theta_{\mathcal{I}}$, where $\mathcal{I}$ is a one-way right instrument relative to $\mathcal{A}$.
Similarly, a *one-way left instrument relative to $\mathcal{A}$* is an instrument $\mathcal{I} = (\Theta_k)_{k \in \mathbb{N}}$, where each of the maps Θ_k is a one-way left local map relative to $\mathcal{A}$, and a *one-way left LOCC map relative to $\mathcal{A}$* is a normal completely positive map $\Theta : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ of the form $\Theta = \Theta_{\mathcal{I}}$, where $\mathcal{I}$ is a one-way left instrument relative to $\mathcal{A}$.
- (iv) An instrument $(\Gamma_j)_{j \in \mathbb{N}}$ is a *coarse-graining* of an instrument $(\Theta_k)_{k \in \mathbb{N}}$ if there is a partition $\mathbb{N} = \bigcup_{j \in \mathbb{N}} S_j$ so that $\Gamma_j = \sum_{k \in S_j} \Theta_k$ for $j \in \mathbb{N}$, where each series converges point-weak*.

- (v) An instrument $\mathcal{I}$ is called *one-way local relative to $\mathcal{A}$* if $\mathcal{I}$ is either a one-way right instrument relative to $\mathcal{A}$ or a one-way left instrument relative to $\mathcal{A}$. We say that an instrument $\mathcal{J}$ is *linked* to an instrument $\mathcal{I} = (\Theta_k)_{k \in \mathbb{N}}$ if there exist one-way instruments $(\Theta_{ki})_{i \in \mathbb{N}}$, $k \in \mathbb{N}$, such that $\mathcal{J}$ is a coarse-graining of the instrument $(\Theta_k \circ \Theta_{ki})_{i, k \in \mathbb{N}}$.
- (vi) A map $\Theta \in \text{UCP}^\sigma(\mathcal{B}(H))$ is an *LOCC map relative to $\mathcal{A}$* if there exists a sequence $(\mathcal{I}_0, \dots, \mathcal{I}_n)$ of instruments such that $\mathcal{I}_0$ is a one-way local instrument relative to $\mathcal{A}$, $\mathcal{I}_{l+1}$ is linked to $\mathcal{I}_l$, $l = 0, \dots, n-1$, and $\Theta = \Theta_{\mathcal{I}_n}$.

We denote by $\text{LOCC}(\mathcal{A})$ the set of all LOCC maps relative to $\mathcal{A}$. We also write $\text{LOCC}^r(\mathcal{A})$ and $\text{LOCC}^l(\mathcal{A})$ for the subsets of $\text{LOCC}(\mathcal{A})$ consisting of the one-way right and left LOCC maps relative to $\mathcal{A}$, respectively. Thus, any Θ in $\text{LOCC}^r(\mathcal{A})$ is given by a point weak-* convergent series

$$(2) \quad \Theta = \sum_{k \in \mathbb{N}} \Phi_k \circ \Psi_k$$

where $\Phi_k \in \text{UCP}_{\mathcal{A}}^\sigma(\mathcal{B}(H))$ and $\Psi_k \in \text{CP}_{\mathcal{B}}^\sigma(\mathcal{B}(H))$.

Example 3.2. Let H be a Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(H)$ be a von Neumann algebra, and $\{a_k \mid k \in \mathbb{N}\} \subseteq \mathcal{A}$ be a countably infinite measurement system, that is, a sequence in $\mathcal{A}$ for which $\sum_{k=1}^\infty a_k^* a_k = 1$ in the weak* topology. For each $k \in \mathbb{N}$, let $\Psi_k = \text{Ad}(a_k^*)$ on $\mathcal{B}(H)$ and let $\Phi_k \in \text{UCP}_{\mathcal{A}}^\sigma(\mathcal{B}(H))$ be a channel. Then the series $\Theta = \sum_{k=1}^\infty \Phi_k \circ \Psi_k$ defines a one-way right LOCC map relative to $\mathcal{A}$. Indeed,

$$\sum_{k=1}^N \Phi_k \circ \Psi_k(1) = \sum_{k=1}^N \Psi_k \circ \Phi_k(1) = \sum_{k=1}^N a_k^* a_k \rightarrow_{N \rightarrow \infty} 1$$

in the weak* topology. If $x \in \mathcal{B}(H)^+$ then $x \leq \|x\|1$ and hence the partial sums $\sum_{k=1}^N \Phi_k \circ \Psi_k(x)$ are dominated by $\|x\|1$. Since they form an increasing sequence, the series $\sum_{k=1}^\infty \Phi_k \circ \Psi_k(x)$ converges in the weak* topology. If $x \in \mathcal{B}(H)$ is arbitrary then, using polarisation, we can write $x = \sum_{l=1}^4 \lambda_l x_l$, where $x_l \in \mathcal{B}(H)^+$ and $\lambda_l \in \mathbb{C}$, $l = 1, 2, 3, 4$. It follows that the partial sums

$$\sum_{k=1}^N \Phi_k \circ \Psi_k(x) = \sum_{k=1}^N \sum_{l=1}^4 \lambda_l \Phi_k \circ \Psi_k(x_l) = \sum_{l=1}^4 \sum_{k=1}^N \lambda_l \Phi_k \circ \Psi_k(x_l)$$

form a weak* convergent sequence.

The predual $\Theta_* : \mathcal{T}(H) \rightarrow \mathcal{T}(H)$ of the map Θ is given by

$$\Theta_*(\rho) = \sum_{k=1}^\infty (\Phi_k)_*(a_k \rho a_k^*), \quad \rho \in \mathcal{T}(H),$$

so one can think of Θ_* as a protocol where Alice makes a measurement corresponding to the system $\{a_k \mid k \in \mathbb{N}\}$, sends the result k to Bob, who then applies $(\Phi_k)_*$. Below we show that any one-way right LOCC map relative to $\mathcal{A}$ is of this form, similar to the finite-dimensional setting.

The use of measurements with countably many outcomes is natural in an infinite-dimensional context. Indeed, even the measurement of an observable modelled by a (possibly unbounded) self-adjoint operator with continuous spectrum can, within an arbitrarily small amount of error, be modelled by an observable with countably many disjoint outcomes [50, §III.3].

Remark 3.3. Let $\Theta \in \text{LOCC}^r(\mathcal{A})$ and let Φ_k, Ψ_k be as in (2). Then $\Phi_k(a) = a\Phi_k(1) = a$, $a \in \mathcal{A}$, $k \in \mathbb{N}$, and, in the weak* topology we have

$$(3) \quad 1 = \Theta(1) = \sum_{k=1}^{\infty} \Psi_k \circ \Phi_k(1) = \sum_{k=1}^{\infty} \Psi_k(1).$$

It follows that, for every positive element $x \in \mathcal{B}(H)$, the partial sums $\sum_{i=1}^m \Psi_k(x)$ are norm bounded; since they form an increasing sequence, they converge in the weak* topology. Using polarisation, we conclude, as in Example 3.2, that the sequence $(\sum_{i=1}^m \Psi_k(x))_{m \in \mathbb{N}}$ converges in the weak* topology for every $x \in \mathcal{B}(H)$.

The locality of an LOCC operation is reflected through the bimodule structure of its implementing maps. For example, if Θ is a map of the form in (2), then Alice's local operations are modelled by the maps Ψ_k , and Bob's local operations by the channels Φ_k . Since the Φ_k are $\mathcal{A}$ -bimodule maps, they do not affect any of Alice's observables and, as shown below, they admit Kraus decompositions with operators belonging to Bob's observable algebra $\mathcal{B}$. A similar intuition is applied for Alice's local operations Ψ_k . In the case where H_A and H_B are finite dimensional Hilbert spaces, $H = H_A \otimes H_B$, $\mathcal{A} = \mathcal{B}(H_A) \otimes 1$ and $\mathcal{B} = 1 \otimes \mathcal{B}(H_B)$, Definition 3.1 reduces to the usual notions as described, for example, in [9].

A notion of LOCC operation for general bipartite systems was proposed by Verch–Werner in [48, §5]. There, a bipartite system is modelled by commuting unital C^* -subalgebras $\mathcal{A}$ and $\mathcal{B}$ of an ambient unital C^* -algebra $\mathcal{C}$. In this setting, they defined a one-way right LOCC map between bipartite systems $(\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1)$ and $(\mathcal{A}_2, \mathcal{B}_2, \mathcal{C}_2)$, where $\mathcal{A}_i$ and $\mathcal{B}_i$ are commuting C^* -subalgebras of $\mathcal{C}_i$, $i = 1, 2$, by a UCP map $\Theta : \mathcal{C}_1 \rightarrow \mathcal{C}_2$, for which there exist finitely many completely positive maps $\Psi_k : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ and UCP maps $\Phi_k : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ satisfying

$$(4) \quad \Theta(ab) = \sum_k \Psi_k(a)\Phi_k(b), \quad a \in \mathcal{A}_1, \quad b \in \mathcal{B}_1.$$

In the special case when $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{B}(H)$, $\mathcal{A}_1 = \mathcal{A}_2 =: \mathcal{A}$ is a von Neumann algebra, and $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{A}'$, our definition of a one-way right LOCC map relative to $\mathcal{A}$ satisfies this condition, albeit, allowing a countable summation over a “classical” index k . This follows from the fact that any normal completely positive $\mathcal{A}'$ -bimodule map on $\mathcal{B}(H)$ admits a Kraus decomposition with operators from $\mathcal{A}$ (see e.g. [7, 20]), and similarly for $\mathcal{B}$.

If, in addition, one assumes that $\mathcal{A}$ and $\mathcal{B}$ are injective factors, then by [8, Theorem 4.2], any completely positive map $\Psi : \mathcal{A} \rightarrow \mathcal{A}$ admits a net

of completely positive elementary operators $\Psi_i : \mathcal{A} \rightarrow \mathcal{A}$ (i.e., operators admitting finitely many Kraus operators from $\mathcal{A}$) satisfying $\|\Psi_i\|_{cb} \leq \|\Psi\|_{cb}$ and $\Psi_i \rightarrow \Psi$ in the point weak* topology of $\text{CB}(\mathcal{A})$. The maps Ψ_i admit canonical extensions to maps in $\text{CP}_{\mathcal{A}'}^\sigma(\mathcal{B}(H))$ (through their finite Kraus decompositions), so that we may approximate the (potentially non-normal) completely positive maps Ψ_k occurring in (4) by normal maps satisfying our bimodule requirements. Similar considerations hold for the maps Φ_k . Hence, in the case of injective factors, one may view the proposed definition of Verch–Werner as a “limit case” of ours. Note that by [8, Remark 4.3], when $\mathcal{A}$ is an injective factor of type II or III, it is *not* true that every *normal* completely positive map $\Psi : \mathcal{A} \rightarrow \mathcal{A}$ extends to a *normal* completely positive map $\tilde{\Psi} \in \text{CP}_{\mathcal{A}'}^\sigma(\mathcal{B}(H))$.

Proposition 3.4. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a von Neumann algebra on a separable Hilbert space H .*

- (i) *The class $\text{LOCC}^r(\mathcal{A})$ of one-way right LOCC maps is closed under finite compositions.*
- (ii) *The maps Ψ_k in (2) can be taken of the form $\Psi_k = \text{Ad}(a_k^*)$, for some $a_k \in \mathcal{A}$, $k \in \mathbb{N}$, with $\sum_{k=1}^\infty a_k^* a_k = 1$ in the weak* topology.*
- (iii) *The maps Φ_k in (2) can be taken of the form $\Phi_k = \sum_{i=1}^\infty \text{Ad}(c_{ki}^*)$, a point weak*-convergent series, for some $c_{ki} \in \mathcal{A}'$, $k, i \in \mathbb{N}$, with $\sum_{i=1}^\infty c_{ki}^* c_{ki} = 1$ in the weak* topology for every $k \in \mathbb{N}$.*

Proof. Let, as before, $\mathcal{B} = \mathcal{A}'$. We first claim that if $\Phi \in \text{CP}_{\mathcal{A}}^\sigma(\mathcal{B}(H))$ and $\Psi \in \text{CP}_{\mathcal{B}}^\sigma(\mathcal{B}(H))$, then

$$(5) \quad \Phi \circ \Psi = \Psi \circ \Phi.$$

Indeed, since H is separable, by [20] (see also [7]), there exists a bounded column operator $(a_i)_{i \in \mathbb{N}}$ with entries $\mathcal{A}$, such that

$$\Psi(x) = \sum_{i=1}^\infty a_i^* x a_i, \quad x \in \mathcal{B}(H),$$

where the series converges in the weak* topology. For every $x \in \mathcal{B}(H)$ we now have

$$\Phi \circ \Psi(x) = \Phi \left(\sum_{i=1}^\infty a_i^* x a_i \right) = \sum_{i=1}^\infty \Phi(a_i^* x a_i) = \sum_{i=1}^\infty a_i^* \Phi(x) a_i = \Psi \circ \Phi(x),$$

showing (5).

Let $\Theta \in \text{LOCC}^r(\mathcal{A})$, with corresponding maps $\Phi_k \in \text{UCP}_{\mathcal{A}}^\sigma(\mathcal{B}(H))$ and $\Psi_k \in \text{CP}_{\mathcal{B}}^\sigma(\mathcal{B}(H))$ for $k \in \mathbb{N}$, as in (2). Write

$$\Psi_k(x) = \sum_{i=1}^\infty a_{ki}^* x a_{ki}, \quad x \in \mathcal{B}(H),$$

for some bounded column operator $(a_{ki})_{i \in \mathbb{N}}$ with entries in $\mathcal{A}$ [20]. By (3), $\Psi_k(1) \leq 1$, and hence the column operator $(a_{ki})_{i \in \mathbb{N}}$ is contractive. Thus, a_{ki}

is a contraction for all $k, i \in \mathbb{N}$. Set $\Phi_{ki} = \Phi_k$ for all $k, i \in \mathbb{N}$ and note that, in the weak* topology, we have

$$\Theta(x) = \lim_{p \rightarrow \infty} \lim_{q \rightarrow \infty} \sum_{k=1}^p \sum_{i=1}^q \Phi_{ki} \circ \text{Ad}(a_{ki}^*)(x), \quad x \in \mathcal{B}(H).$$

Let $x \in \mathcal{B}(H)^+$ and $\rho \in \mathcal{T}(H)^+$. Then the double limit

$$(6) \quad \lim_{p \rightarrow \infty} \lim_{q \rightarrow \infty} \sum_{k=1}^p \sum_{i=1}^q \langle \Phi_{ki} \circ \text{Ad}(a_{ki}^*)(x), \rho \rangle$$

exists. Since the terms of the sequence in (6) are positive, the limit

$$\lim_{L \rightarrow \infty} \sum_{k,i=1}^L \langle \Phi_{ki} \circ \text{Ad}(a_{ki}^*)(x), \rho \rangle$$

exists. Thus, the partial sums $\sum_{k,i=1}^L \Phi_{ki} \circ \text{Ad}(a_{ki}^*)(x)$ converge in the weak* topology for every $x \in \mathcal{B}(H)^+$ and hence, by the polarisation identity, for every $x \in \mathcal{B}(H)$. It is now straightforward to see that the limit coincides with $\Theta(x)$. Note, moreover, that identity (3) shows that $\sum_{k,i=1}^{\infty} a_{ki}^* a_{ki} = 1$ in the weak* topology. This establishes (ii). The assertion in (iii) follows by considering the Kraus decomposition of the maps Φ_k .

To show (i), assume that $\Theta_i \in \text{LOCC}^r(\mathcal{A})$ and let $\Phi_k^{(i)}, \Psi_k^{(i)}, k \in \mathbb{N}$, be the maps as in (2), associated with $\Theta_i, i = 1, 2$. By (5), for every $x \in \mathcal{B}(H)$, we have that, in the weak* topology,

$$\begin{aligned} (\Theta_1 \circ \Theta_2)(x) &= \lim_{p \rightarrow \infty} \lim_{q \rightarrow \infty} \sum_{k=1}^p \sum_{l=1}^q \left(\Phi_k^{(1)} \circ \Psi_k^{(1)} \circ \Phi_l^{(2)} \circ \Psi_l^{(2)} \right)(x) \\ &= \lim_{p \rightarrow \infty} \lim_{q \rightarrow \infty} \sum_{k=1}^p \sum_{l=1}^q \left(\Phi_k^{(1)} \circ \Phi_l^{(2)} \circ \Psi_k^{(1)} \circ \Psi_l^{(2)} \right)(x). \end{aligned}$$

Set $\Phi_{kj} = \Phi_k^{(1)} \circ \Phi_l^{(2)}$ and $\Psi_{kj} = \Psi_k^{(1)} \circ \Psi_l^{(2)}$. An argument similar to the one in the previous paragraph now implies that

$$(\Theta_1 \circ \Theta_2)(x) = \sum_{k,l=1}^{\infty} (\Phi_{kj} \circ \Psi_{kj})(x), \quad x \in \mathcal{B}(H),$$

in the weak* topology, so $\Theta_1 \circ \Theta_2 \in \text{LOCC}^r(\mathcal{A})$. $\square$

Remark 3.5. (i) The expression of one-way right LOCC maps given in Proposition 3.4(ii) reflects the notion of fine-graining of LOCC channels described in [9].

(ii) By symmetry, observations analogous to those above for one-way right LOCC maps also hold for the one-way left LOCC maps.

4. STATE CONVERTIBILITY VIA LOCC($\mathcal{A}$)

Having established an appropriate generalisation of LOCC operations in the preceding section, we now define the corresponding notions of convertibility.

Definition 4.1. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a von Neumann algebra on a Hilbert space H , and let $\psi, \varphi \in H_1$.

- (i) We say that ψ is *convertible to φ via LOCC($\mathcal{A}$)* if there exists $\Theta \in \text{LOCC}(\mathcal{A})$ such that $\Theta_*(\omega_\psi) = \omega_\varphi$.
- (ii) We say that ψ is *approximately convertible to φ via LOCC($\mathcal{A}$)* if for every $\varepsilon > 0$ there exists $\Theta \in \text{LOCC}(\mathcal{A})$ such that $\|\Theta_*(\omega_\psi) - \omega_\varphi\| < \varepsilon$.

We also make analogous definitions with $\text{LOCC}^l(\mathcal{A})$ and $\text{LOCC}^r(\mathcal{A})$ in place of $\text{LOCC}(\mathcal{A})$.

The goal of the next few results is to show that approximate convertibility can be realised by using only one-way LOCC maps (see Corollary 4.10). This generalises to the commuting operator framework a result of Lo-Popescu [34] for finite-dimensional bipartite systems. We note that another generalisation of this theorem was established in [39], which we recover from our results by taking the special case $\mathcal{A} = \mathcal{B}(H)$ in its standard representation, for H a separable Hilbert space. The essential feature of our argument, similar to the finite-dimensional case, is the symmetry induced from the standard form of a von Neumann algebra [21], as highlighted in Section 1.

Let H be a Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(H)$ be a (semi-finite) von Neumann algebra equipped with a normal semi-finite faithful trace τ , and $\psi \in H$. To avoid double subscripts, we will write $\rho_\psi \in L^1(\mathcal{A}, \tau)$ for the density operator ρ_{ω_ψ} arising from the restriction of the vector state ω_ψ to $\mathcal{A}$. Recall that, in the case where $H = L^2(\mathcal{A}, \tau)$ and $\mathcal{A}$ is in standard form, we write $\rho'_\psi \in L^1(\mathcal{A}', \tau')$ for the density operator affiliated to $\mathcal{A}'$ satisfying

$$(a'\psi, \psi) = \omega_\psi|_{\mathcal{A}'}(a') = \tau'(\rho'_\psi a'), \quad a' \in \mathcal{A}',$$

where $\tau' = \tau \circ R$ is the canonical trace on $\mathcal{A}'$. Recall that for $t > 0$, we write $\mu'_t(\rho'_\psi) = \mu_t(\rho'_\psi; \mathcal{A}', \tau')$ for the t -th singular value of ρ'_ψ relative to $(\mathcal{A}', \tau')$.

Lemma 4.2. *Let $(\mathcal{A}, \tau)$ be a semi-finite von Neumann algebra, represented in its standard form on $L^2(\mathcal{A}, \tau)$, and let $\psi \in L^2(\mathcal{A}, \tau)$. Then*

$$J\rho'_\psi J = \rho_{\psi^*} \quad \text{and} \quad \mu'_t(\rho'_\psi) = \mu_t(\rho_{\psi^*}) = \mu_t(\rho_\psi), \quad t > 0.$$

Proof. First we show that $\rho_{\psi^*} = J\rho'_\psi J$ and $\mu'_t(\rho'_\psi) = \mu_t(\rho_{\psi^*})$. Recall that $\psi^* = J\psi$. For $a \in \mathcal{A}$, we therefore have

$$\begin{aligned} \tau(\rho_{\psi^*} a) &= (aJ\psi, J\psi) = (Ja^*J\psi, \psi) = \tau'(\rho'_\psi Ja^*J) \\ &= \tau(J(JaJ\rho'_\psi)J) = \tau(J\rho'_\psi Ja). \end{aligned}$$

Hence,

$$(7) \quad \rho_{\psi^*} = J\rho'_\psi J.$$

Recall that J is isometric, with $J^2 = 1$, and note that $R: \mathcal{A}' \rightarrow \mathcal{A}$ induces a trace-preserving bijection $\mathcal{P}(\mathcal{A}') \rightarrow \mathcal{P}(\mathcal{A})$, $p' \mapsto Jp'J$. For $t > 0$, using (7) we obtain

$$\begin{aligned}\mu'_t(\rho'_\psi) &= \inf\{\|\rho'_\psi p'\| : p' \in \mathcal{P}(\mathcal{A}'), \tau'(1 - p') \leq t\} \\ &= \inf\{\|\rho'_\psi Jp'J\| : p \in \mathcal{P}(\mathcal{A}), \tau(1 - p) \leq t\} \\ &= \inf\{\|J\rho'_\psi Jp\| : p \in \mathcal{P}(\mathcal{A}), \tau(1 - p) \leq t\} \\ &= \inf\{\|\rho_{\psi^*} p\| : p \in \mathcal{P}(\mathcal{A}), \tau(1 - p) \leq t\} = \mu_t(\rho_{\psi^*}).\end{aligned}$$

Now, by [46, Exercises IX.1.2–3], there exists a partial isometry $u \in \mathcal{A}$ such that $|\psi^*| = \psi u^*$, and $\psi^* = u^* |\psi^*|$. It follows that

$$\psi^* = u^* \psi u^* = u^* |\psi^*| = u^* J |\psi^*| = u^* J u \psi^* = u^* J u J \psi.$$

Since $u \in \mathcal{A}$, we have $\rho_{\psi^*} = \rho_{u^* J u J \psi} = u^* \cdot \rho_{J u J \psi} \cdot u$. By [17, Lemma 2.5 (vi)],

$$(8) \quad \mu_t(\rho_{\psi^*}) \leq \|u^*\| \|u\| \mu_t(\rho_{J u J \psi}) \leq \mu_t(\rho_{J u J \psi}),$$

for $t > 0$. But $J u J$ is a contraction in $\mathcal{A}'$, so $\omega_{J u J \psi}|_{\mathcal{A}} \leq \omega_\psi|_{\mathcal{A}}$, that is, $\rho_{J u J \psi} \leq \rho_\psi$. By [17, Lemma 2.5 (iii)], $\mu_t(\rho_{J u J \psi}) \leq \mu_t(\rho_\psi)$, $t > 0$. Thus, by (8), $\mu_t(\rho_{\psi^*}) \leq \mu_t(\rho_\psi)$, for $t > 0$. By symmetry, we obtain equality. $\square$

Proposition 4.3. *Let $(\mathcal{A}, \tau)$ be a semi-finite factor in its standard form on $L^2(\mathcal{A}, \tau)$, let $\mathcal{B} = \mathcal{A}'$ and let $\varepsilon > 0$. For any $(\psi, b) \in H \times \mathcal{B}$, there exist a unitary $u \in \mathcal{B}$ and partial isometries $v, w \in \mathcal{A}$ so that $\psi = v^* v \psi$ and, if $z = J b J v$, then*

$$\|b\psi\| = \|z\psi\| \quad \text{and} \quad \|b\psi - u w z \psi\| < \varepsilon.$$

Moreover, the partial isometry v can be chosen independently of b .

Proof. Let $\psi^* = v |\psi^*|$ be the polar decomposition of ψ^* [46, Exercises IX.1.2–3]; thus, $v \in \mathcal{A}$ is a partial isometry with $\psi = v^* |\psi|$, and the projections $v^* v$ and vv^* have ranges $\overline{\mathcal{B}\psi}$ and $\overline{\mathcal{B}|\psi|}$, respectively, so in particular, $\psi = v^* v \psi$. It follows that

$$(9) \quad v\psi = vv^* |\psi| = |\psi| = J |\psi| = J v \psi.$$

Note that $z \in \mathcal{A}$. We have $v^* v b \psi = b v^* v \psi = b \psi$, and hence, using (9),

$$\|b\psi\| = \|v b \psi\| = \|b v \psi\| = \|b J v \psi\| = \|J b J v \psi\| = \|z \psi\|,$$

as desired.

Note that

$$(10) \quad \tau'(\rho') = \tau(J\rho'^* J), \quad \rho' \in L^1(\mathcal{B}, \tau');$$

indeed, the formula holds by the definition of τ' in the case $\rho' \in \mathcal{B}$, and the general case follow by approximating ρ' by a sequence in $\mathcal{B}$ in the norm $\|\cdot\|_1$. Let $\alpha = z\psi$ and $\beta = b\psi$. For $c \in \mathcal{B}$, by (10), (9) and (1), we have

$$\begin{aligned}\tau(\rho_\alpha R(c)) &= (R(c) z \psi, z \psi) = (J c^* J J b J v \psi, J b J v \psi) = (J c^* b v \psi, J b v \psi) \\ &= (b v \psi, c^* b v \psi) = (c v^* v b \psi, b \psi) = (c \beta, \beta) = \tau'(\rho'_\beta c) \\ &= \tau(J \rho'_\beta J R(c)).\end{aligned}$$

Thus, by Lemma 4.2, $\rho_\alpha = J\rho'_\beta J = \rho_{\beta^*}$ and

$$\mu'_t(\rho'_\alpha) = \mu_t(\rho_\alpha) = \mu_t(\rho_{\beta^*}) = \mu'_t(\rho'_\beta), \quad t > 0.$$

Since $\mathcal{A}$ is a factor, by [23, Theorem 3.4 (1)] for every $\varepsilon > 0$ there exists a unitary $u \in \mathcal{B}$ such that

$$\|\rho'_\beta - \rho'_{u\alpha}\|_1 = \|\rho'_\beta - u\rho'_\alpha u^*\|_1 < \varepsilon^2.$$

By the continuity of Stinespring's representation [32, Theorem 1], there exist a Hilbert space K , a *-homomorphism $\pi : \mathcal{B} \rightarrow \mathcal{B}(K)$ and vectors $\xi, \eta \in K$ such that $\omega_\beta|_{\mathcal{B}} = \omega_\xi \circ \pi$ and $\omega_{u\alpha}|_{\mathcal{B}} = \omega_\eta \circ \pi$, and

$$\|\xi - \eta\| \leq \|\omega_\beta|_{\mathcal{B}} - \omega_{u\alpha}|_{\mathcal{B}}\|^{1/2} = \|\rho'_\beta - \rho'_{u\alpha}\|_1^{1/2} < \varepsilon.$$

By the uniqueness of Stinespring representations, there exist partial isometries $w_1 : L^2(\mathcal{A}, \tau) \rightarrow K$ and $w_2 : K \rightarrow L^2(\mathcal{A}, \tau)$ such that $w_1 c = \pi(c)w_1$ and $w_2 \pi(c) = c w_2$ for all $c \in \mathcal{B}$, $w_1 u \alpha = \eta$ and $w_2 \xi = \beta$. Then $w := w_2 w_1$ is a contraction in $\mathcal{A}$, and

$$\|b\psi - uwz\psi\| = \|\beta - wu\alpha\| = \|w_2 \xi - w_2 \eta\| < \varepsilon. \quad \square$$

The following estimate is straightforward and we will make use of it multiple times.

Lemma 4.4. *Let $\psi \in H$, $\|\psi\| \leq 1$, and let $v \in \mathcal{B}(H)$ be a contraction. If $\|\psi\| - \|v\psi\| < \varepsilon$, then $\|(1 - v^*v)^{1/2}\psi\| < \sqrt{2\varepsilon}$.*

Proof. The assumption implies

$$\begin{aligned} \|(1 - v^*v)^{1/2}\psi\|^2 &= ((1 - v^*v)\psi, \psi) = \|\psi\|^2 - \|v\psi\|^2 \\ &= (\|\psi\| + \|v\psi\|)(\|\psi\| - \|v\psi\|) \leq 2(\|\psi\| - \|v\psi\|) < 2\varepsilon. \end{aligned} \quad \square$$

Lemma 4.5. *Let $\mathcal{A}$ be a von Neumann algebra equipped with a normal faithful semi-finite trace τ , and let $(\rho_k)_{k \in \mathbb{N}}$ be an increasing sequence of hermitian elements of $L^1(\mathcal{A}, \tau)$. If $\rho_k \rightarrow_{k \rightarrow \infty} \rho$ in the weak topology, for some $\rho \in L^1(\mathcal{A}, \tau)$, then $\rho_k \rightarrow_{k \rightarrow \infty} \rho$ in norm.*

Proof. We have that $\rho_k \leq \rho$; thus, $\rho - \rho_k \geq 0$ for every k . Thus,

$$\|\rho - \rho_k\| = (\rho - \rho_k)(1) \rightarrow_{k \rightarrow \infty} 0. \quad \square$$

Lemma 4.6. *Let H be a Hilbert space and $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ be positive. Then for any self-adjoint $T \in \mathcal{B}(H)$ and any self-adjoint $\rho \in \mathcal{T}(H)$,*

$$|\langle \Phi(T), \rho \rangle| \leq \|T\| \langle \Phi(1), |\rho| \rangle.$$

Proof. Let $(e_n)_{n=1}^\infty$ be orthonormal eigenvectors of ρ with corresponding eigenvalues $(\lambda_n)_{n=1}^\infty \subseteq \mathbb{R}$. Since $-\|T\|1 \leq T \leq \|T\|1$, by positivity of $\omega_{e_n} \circ \Phi$ we have

$$-\|T\| \langle \Phi(1)e_n, e_n \rangle \leq \langle \Phi(T)e_n, e_n \rangle \leq \|T\| \langle \Phi(1)e_n, e_n \rangle,$$

so that $|\langle \Phi(T)e_n, e_n \rangle| \leq \|T\| |\langle \Phi(1)e_n, e_n \rangle|$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned} |\langle \Phi(T), \rho \rangle| &= \left| \sum_{n=1}^{\infty} \lambda_n \langle \Phi(T)e_n, e_n \rangle \right| \leq \sum_{n=1}^{\infty} |\lambda_n| |\langle \Phi(T)e_n, e_n \rangle| \\ &\leq \sum_{n=1}^{\infty} |\lambda_n| \|T\| |\langle \Phi(1)e_n, e_n \rangle| = \|T\| |\langle \Phi(1), |\rho| \rangle|. \end{aligned} \quad \square$$

Theorem 4.7. *Let $(\mathcal{A}, \tau)$ be a semi-finite factor in its standard form on $H = L^2(\mathcal{A}, \tau)$. For any $\Theta \in \text{LOCC}(\mathcal{A})$, $\psi \in H$ and $\varepsilon > 0$, there exists $\Theta_\varepsilon \in \text{LOCC}^r(\mathcal{A})$ such that $\|\Theta_*(\omega_\psi) - \Theta_{\varepsilon*}(\omega_\psi)\| < \varepsilon$.*

Proof. We may assume that $\|\psi\| \leq 1$. Any LOCC map $\Theta \in \text{LOCC}(\mathcal{A})$ is of the form $\Theta = \Theta_{\mathcal{I}_n}$ where $(\mathcal{I}_0, \dots, \mathcal{I}_n)$ is a sequence of instruments such that $\mathcal{I}_0$ is one-way local relative to $\mathcal{A}$, and $\mathcal{I}_{l+1}$ is linked to $\mathcal{I}_l$ for each $l = 0, \dots, n-1$. Since the trivial instrument $(\text{id}, 0, 0, \dots)$ is one-way right local and any one-way left local instrument is linked to it, without loss of generality, we may suppose that $\mathcal{I}_0$ is a one-way right instrument. Writing $\mathcal{I}_n = (\Theta_k^{(n)})_k$, consider the following proposition $P(n)$: for every $\varepsilon > 0$ there exists an instrument $\mathcal{I}_\varepsilon = (\Gamma_k)_k$ which is a coarse-graining of some one-way right instrument relative to $\mathcal{A}$, such that

$$\sum_{k=1}^{\infty} \left\| \Theta_{k*}^{(n)}(\omega_\psi) - \Gamma_{k*}(\omega_\psi) \right\| < \varepsilon.$$

If $P(n)$ were true for every $n \in \mathbb{N}$ then the Theorem follows with $\Theta_\varepsilon = \Theta_{\mathcal{I}_\varepsilon}$. We therefore proceed by induction on n , starting with the base case $n = 1$ (as the claim for $n = 0$ is vacuous).

Let $\varepsilon > 0$ and write $\mathcal{I}_0 = (\Theta_k)_{k \in \mathbb{N}}$. Each Θ_k is one-way right local, so by Proposition 3.4 (ii) we may write $\Theta_k = \sum_{j,l=1}^{\infty} \text{Ad}(a_{kj}^*) \circ \text{Ad}(c_{kl}^*)$, where $a_{kj} \in \mathcal{A}$ satisfy $\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj}^* a_{kj} = 1$ and $c_{kl} \in \mathcal{B}$ satisfy $\sum_{l=1}^{\infty} c_{kl}^* c_{kl} = 1$ for each $k \in \mathbb{N}$. We define $\psi_{kj} := a_{kj}\psi \in H$, for $k, j \in \mathbb{N}$.

Since $\mathcal{I}_1$ is linked to $\mathcal{I}_0$, the instrument $\mathcal{I}_1$ is a coarse-graining of an instrument of the form $(\Theta_k \circ \Theta_{ki})_{k,i}$, for a collection of one-way instruments $\mathcal{J}_k = (\Theta_{ki})_i$ indexed by $k \in \mathbb{N}$. Write $L = \{k \in \mathbb{N} \mid \mathcal{J}_k \text{ is one-way left}\}$ and $R = \{k \in \mathbb{N} \mid \mathcal{J}_k \text{ is one-way right}\}$.

Suppose first that $k \in L$. By Proposition 3.4 (ii) and Remark 3.5 (ii), we may assume that $\Theta_{ki} = \Psi_{ki}^L \circ \text{Ad}(b_{ki}^*)$, where $\Psi_{ki}^L \in \text{UCP}_{\mathcal{B}}^\sigma(\mathcal{B}(H))$, $b_{ki} \in \mathcal{B}$ and $\sum_{i=1}^{\infty} b_{ki}^* b_{ki} = 1$. Hence, in the point weak*-topology, we have

$$(11) \quad \Theta_k \circ \Theta_{ki} = \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \text{Ad}((b_{ki} c_{kl} a_{kj})^*) \circ \Psi_{ki}^L.$$

Since $\mathcal{A}$ is a factor, we can apply Proposition 4.3 to the pairs $(\psi_{kj}, b_{ki} c_{kl}) \in H \times \mathcal{B}$. We obtain unitaries $u_{kijl} \in \mathcal{B}$ and partial isometries $v_{kj}, w_{kijl} \in \mathcal{A}$ so that $\psi_{kj} = v_{kj}^* v_{kj} \psi_{kj}$ and $z_{kijl} = J b_{ki} c_{kl} J v_{kj} \in \mathcal{A}$ satisfy $\|b_{ki} c_{kl} \psi_{kj}\| =$

$\|z_{kijl}\psi_{kj}\|$ and

$$(12) \quad \|b_{ki}c_{kl}\psi_{kj} - u_{kijl}w_{kijl}z_{kijl}\psi_{kj}\| < \frac{\varepsilon}{2^{i+j+k+l+2}}$$

for $i, j, k, l \in \mathbb{N}$. Let $a_{kijl} = w_{kijl}z_{kijl} \in \mathcal{A}$, and observe that

$$\|z_{kijl}\psi_{kj}\| - \|a_{kijl}\psi_{kj}\| = \|b_{ki}c_{kl}\psi_{kj}\| - \|u_{kijl}w_{kijl}z_{kijl}\psi_{kj}\| < \frac{\varepsilon}{2^{i+j+k+l+2}}.$$

Let $\widetilde{a_{kijl}} := (1 - w_{kijl}^*w_{kijl})z_{kijl} \in \mathcal{A}$. By Lemma 4.4, the preceding inequality implies

$$(13) \quad \|\widetilde{a_{kijl}}\psi_{kj}\| = \|(1 - w_{kijl}^*w_{kijl})z_{kijl}\psi_{kj}\| < \sqrt{\frac{\varepsilon}{2^{i+j+k+l+1}}}.$$

Now suppose that $k \in R$. The map $\Theta_{\mathcal{J}_k} = \sum_{i=1}^{\infty} \Theta_{ki}$ is then a one-way right LOCC map relative to $\mathcal{A}$, with $\Theta_{ki} = \Psi_{ki}^R \circ \Phi_{ki}$, where $\Psi_{ki}^R \in \text{CP}_{\mathcal{B}}^{\sigma}(\mathcal{B}(H))$ and $\Phi_{ki} \in \text{UCP}_{\mathcal{A}}^{\sigma}(\mathcal{B}(H))$. Then

$$(14) \quad \Theta_k \circ \Theta_{ki} = \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \text{Ad}(a_{kj}^*) \circ \Psi_{ki}^R \circ (\text{Ad}(c_{kl}^*) \circ \Phi_{ki})$$

in the point weak*-topology.

Recall that v_{kj} was defined above for $k \in L$ and $j \in \mathbb{N}$; we define $v_{kj} := 1$ for $k \in R$ and $j \in \mathbb{N}$. Then

$$A := \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj}^* v_{kj}^* v_{kj} a_{kj} \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj}^* a_{kj} = 1;$$

thus, the operator $(1 - A)^{1/2}$ is a well-defined element of $\mathcal{A}$. Moreover, since $a_{kj}\psi = \psi_{kj} = v_{kj}^* v_{kj} \psi_{kj} = v_{kj}^* v_{kj} a_{kj} \psi$ for all $j, k \in \mathbb{N}$, we have $\psi = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj}^* a_{kj} \psi = A\psi$, showing that $\psi \in \text{Ker}(1 - A)$ and thus that $\psi \in \text{Ker}((1 - A)^{1/2})$. In particular,

$$(15) \quad \left(\text{Ad}((1 - A)^{1/2}) \right)_* (\omega_{\psi}) = 0.$$

Fix a probability distribution (p_n) over $\mathbb{N}$ with $p_n > 0$ for each n . For $k \in L$ and $i, j, l \in \mathbb{N}$, define $\Gamma_{kijl}^L \in \text{CP}_{\mathcal{B}}^{\sigma}(\mathcal{B}(H))$ for $i \in \mathbb{N}$ by

$$\Gamma_{kijl}^L = \left(\text{Ad}(a_{kj}^*) \circ (\text{Ad}(a_{kijl}^*) + \text{Ad}(\widetilde{a_{kijl}}^*)) + p_k p_i p_j p_l \text{Ad}((1 - A)^{1/2}) \right) \circ \Psi_{ki}^L.$$

Since

$$a_{kijl}^* a_{kijl} + \widetilde{a_{kijl}}^* \widetilde{a_{kijl}} = z_{kijl}^* z_{kijl} = v_{kj}^* J c_{kl}^* b_{ki}^* b_{ki} c_{kl} J v_{kj}$$

and $\sum_{i=1}^{\infty} \sum_{l=1}^{\infty} c_{kl}^* b_{ki}^* b_{ki} c_{kl} = 1$ for every $k \in L$, in the weak* topology we have

$$\begin{aligned} \sum_{k \in L} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \Gamma_{kijl}^L(1) &= \sum_{k \in L} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} a_{kj}^* z_{kijl}^* z_{kijl} a_{kj} + p_k p_i p_j p_l (1 - A) \\ &= \sum_{k \in L} \sum_{j=1}^{\infty} a_{kj}^* v_{kj}^* v_{kj} a_{kj} + p_k p_j (1 - A). \end{aligned}$$

For $k \in R$, define one-way right local maps Γ_{ki}^R by

$$\Gamma_{ki}^R = \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} (\text{Ad}(a_{kj}^*) \circ \Psi_{ki}^R + p_k p_i p_j \text{Ad}((1-A)^{1/2})) \circ (\text{Ad}(c_{kl}^*) \circ \Phi_{ki}).$$

Then

$$\begin{aligned} \sum_{k \in R} \sum_{i=1}^{\infty} \Gamma_{ki}^R(1) &= \sum_{k \in R} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{kj}^* \Psi_{ki}^R(1) a_{kj} + p_k p_i p_j (1-A) \\ &= \sum_{k \in R} \sum_{j=1}^{\infty} a_{kj}^* a_{kj} + p_k p_j (1-A). \end{aligned}$$

Putting things together, we get

$$\begin{aligned} &\sum_{k \in L} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \Gamma_{kijl}^L(1) + \sum_{k \in R} \sum_{i=1}^{\infty} \Gamma_{ki}^R(1) \\ &= \sum_{k \in L} \sum_{j=1}^{\infty} a_{kj}^* v_{kj}^* v_{kj} a_{kj} + p_k p_j (1-A) + \sum_{k \in R} \sum_{j=1}^{\infty} a_{kj}^* a_{kj} + p_k p_j (1-A) \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj}^* v_{kj}^* v_{kj} a_{kj} + p_k p_j (1-A) \\ &= A + (1-A) \\ &= 1. \end{aligned}$$

It follows that the series

$$\sum_{k \in L} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \text{Ad}(u_{kijl}^*) \circ \Gamma_{kijl}^L(x) + \sum_{k \in R} \sum_{i=1}^{\infty} \Gamma_{ki}^R(x)$$

is convergent in the weak* topology for every positive $x \in \mathcal{B}(H)$. By polarisation, it is convergent in the weak* topology for every $x \in \mathcal{B}(H)$. With

$$\Gamma_{ki}^L := \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \text{Ad}(u_{kijl}^*) \circ \Gamma_{kijl}^L,$$

and $\Gamma_{ki} := \Gamma_{ki}^L$ for $k \in L$, $i \in \mathbb{N}$, and $\Gamma_{ki} := \Gamma_{ki}^R$ for $k \in R$, $i \in \mathbb{N}$, it follows that $(\Gamma_{ki})_{k,i}$ is a coarse-graining of a one-way right instrument relative to $\mathcal{A}$.

By (14) and (15), for $k \in R$ we have $\Theta_{ki*}(\Theta_{k*}(\omega_\psi)) = \Gamma_{ki*}(\omega_\psi)$, $i \in \mathbb{N}$. Also, using (11), (12), (13) and (15) with the identity $\text{Ad}(a_{kj})\omega_\psi = \omega_{\psi_{kj}}$ and Lemma 4.5 consecutively, we have

$$\sum_{k \in L} \sum_{i=1}^{\infty} \|\Gamma_{ki*}(\omega_\psi) - \Theta_{ki*}(\Theta_{k*}(\omega_\psi))\|$$

$$\begin{aligned}
&\leq \sum_{k \in L} \sum_{i,j,l=1}^{\infty} \left\| \Psi_{ki*}^L \circ (\text{Ad}(u_{kijl}) \circ (\text{Ad}(a_{kijl}) + \text{Ad}(\widetilde{a_{kijl}})) - \text{Ad}(b_{ki}c_{kl})) \omega_{\psi_{kj}} \right\| \\
&\leq \sum_{k \in L} \sum_{i,j,l=1}^{\infty} \left\| u_{kijl} (a_{kijl} \omega_{\psi_{kj}} a_{kijl}^* + \widetilde{a_{kijl}} \omega_{\psi_{kj}} \widetilde{a_{kijl}}^*) u_{kijl}^* - b_{ki} c_{kl} \omega_{\psi_{kj}} c_{kl}^* b_{ki} \right\| \\
&\leq \sum_{k \in L} \sum_{i,j,l=1}^{\infty} \left\| u_{kijl} a_{kijl} \omega_{\psi_{kj}} a_{kijl}^* u_{kijl}^* - b_{ki} c_{kl} \omega_{\psi_{kj}} c_{kl}^* b_{ki} \right\| + \left\| \widetilde{a_{kijl}} \omega_{\psi_{kj}} \widetilde{a_{kijl}}^* \right\| \\
&\leq \sum_{k \in L} \sum_{i,j,l=1}^{\infty} \left\| u_{kijl} a_{kijl} \psi_{kj} - b_{ki} c_{kl} \psi_{kj} \right\| (\|a_{kijl} \psi_{kj}\| + \|b_{ki} c_{kl} \psi_{kj}\|) + \|\widetilde{a_{kijl}} \psi_{kj}\|^2 \\
&\leq 2 \sum_{k \in L} \sum_{i,j,l=1}^{\infty} \frac{\varepsilon}{2^{k+i+j+l+1}} \leq \varepsilon.
\end{aligned}$$

Since $\mathcal{I}_1$ is a coarse-graining of $(\Theta_k \circ \Theta_{ki})_{k,i}$, applying the same coarse-graining to $(\Gamma_{ki})_{k,i}$ produces an instrument $\mathcal{I}_\varepsilon$ satisfying $P(1)$.

Now, assuming $P(n)$ is true, let Θ be an LOCC map of the form $\Theta = \Theta_{\mathcal{I}_{n+1}}$ where $(\mathcal{I}_0, \dots, \mathcal{I}_{n+1})$ is a sequence of instruments such that $\mathcal{I}_0$ is a one-way local right instrument relative to $\mathcal{A}$, and $\mathcal{I}_{l+1}$ is linked to $\mathcal{I}_l$ for each $l = 0, \dots, n$. Then $\mathcal{I}_{n+1}$ is a coarse-graining of $(\Theta_k \circ \Theta_{ki})_{k,i}$, where $\mathcal{I}_n = (\Theta_k)_k$ and $\mathcal{J}_k = (\Theta_{ki})_i$ is a one-way instrument for each $k \in \mathbb{N}$. Given $\varepsilon > 0$, by $P(n)$ there exists a coarse-graining $\mathcal{I}_\varepsilon = (\Gamma_k)_k$ of some one-way right instrument such that

$$\sum_{k=1}^{\infty} \|\Theta_{k*}(\omega_\psi) - \Gamma_{k*}(\omega_\psi)\| < \frac{\varepsilon}{2}.$$

We have $\Gamma_k = \sum_{j \in S_k} \tilde{\Gamma}_j$ for some one-way right instrument $\mathcal{I}'_0 = (\tilde{\Gamma}_j)_j$ and some partition $(S_k)_k$ of $\mathbb{N}$. Let $\tilde{\Theta}_{ji} = \Theta_{ki}$ for $j \in S_k$, $k \in \mathbb{N}$. Then $(\tilde{\Theta}_{ji})_i$ is a one-way instrument for each $j \in \mathbb{N}$, and $(\Gamma_k \circ \Theta_{ki})_{k,i}$ is a coarse-graining of $\mathcal{I}'_1 = (\tilde{\Gamma}_j \circ \tilde{\Theta}_{ji})_{j,i}$, and $\mathcal{I}'_1$ is linked to $\mathcal{I}'_0$. Applying $P(1)$ to the pair $(\mathcal{I}'_0, \mathcal{I}'_1)$, we obtain an instrument $(\hat{\Gamma}_{ji})_{j,i}$ which is a coarse-graining of some one-way right instrument, and satisfies

$$\sum_{j,i=1}^{\infty} \left\| \tilde{\Theta}_{ji*}(\tilde{\Gamma}_{j*}(\omega_\psi)) - \hat{\Gamma}_{ji*}(\omega_\psi) \right\| < \frac{\varepsilon}{2}.$$

Setting $\Gamma_{ki} := \sum_{j \in S_k} \hat{\Gamma}_{ji}$, we obtain an instrument $(\Gamma_{ki})_{k,i}$ which is a coarse-graining of a one-way right instrument, and satisfies

$$\begin{aligned}
\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \|\Theta_{ki*}(\Gamma_{k*}(\omega_\psi)) - \Gamma_{ki*}(\omega_\psi)\| &\leq \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \sum_{j \in S_k} \left\| \tilde{\Theta}_{ji*}(\tilde{\Gamma}_{j*}(\omega_\psi)) - \hat{\Gamma}_{ji*}(\omega_\psi) \right\| \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\| \tilde{\Theta}_{ji*}(\tilde{\Gamma}_{j*}(\omega_\psi)) - \hat{\Gamma}_{ji*}(\omega_\psi) \right\| < \frac{\varepsilon}{2}.
\end{aligned}$$

For each $k, i \in \mathbb{N}$, $\Theta_{ki*}((\Theta_{k*} - \Gamma_{k*})(\omega_\psi)) \in \mathcal{T}(H)$ is self-adjoint, and attains its norm on self-adjoint operators in $\mathcal{B}(H)$. Hence, there exists $T_{ki} \in \mathcal{B}(H)$ with $T_{ki} = T_{ki}^*$ and $\|T_{ki}\| \leq 1$ satisfying

$$\|\Theta_{ki*}((\Theta_{k*} - \Gamma_{k*})(\omega_\psi))\| = |\langle \Theta_{ki}(T_{ki}), (\Theta_{k*} - \Gamma_{k*})(\omega_\psi) \rangle|.$$

By Lemma 4.6,

$$\begin{aligned} \|\Theta_{ki*}((\Theta_{k*} - \Gamma_{k*})(\omega_\psi))\| &\leq \|T_{ki}\| \langle \Theta_{ki}(1), |(\Theta_{k*} - \Gamma_{k*})(\omega_\psi)| \rangle \\ &\leq \langle \Theta_{ki}(1), |(\Theta_{k*} - \Gamma_{k*})(\omega_\psi)| \rangle. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{k,i=1}^{\infty} \|\Theta_{ki*}(\Theta_{k*}(\omega_\psi)) - \Theta_{ki*}(\Gamma_{k*}(\omega_\psi))\| &\leq \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \langle \Theta_{ki}(1), |(\Theta_{k*} - \Gamma_{k*})(\omega_\psi)| \rangle \\ &= \sum_{k=1}^{\infty} \|\Theta_{k*}(\omega_\psi) - \Gamma_{k*}(\omega_\psi)\| \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

By the triangle inequality we obtain

$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \|\Theta_{ki*}(\Theta_{k*}(\omega_\psi)) - \Gamma_{ki*}(\omega_\psi)\| < \varepsilon.$$

Recall that $\mathcal{I}_{n+1}$ is a coarse-graining of $(\Theta_k \circ \Theta_{k,i})_{k,i}$. Letting $\mathcal{I}_\varepsilon$ be the result of applying the same coarse-graining to $(\Gamma_{ki})_{k,i}$, the preceding inequality and the triangle inequality then show that $\mathcal{I}_\varepsilon$ satisfies $P(n+1)$. $\square$

Remark 4.8. The proof of Theorem 4.7 may seem complicated when compared to Lo and Popescu's intuitive argument for the special case $\mathcal{A} = M_n \otimes 1$. This may be explained by our approximate version of convertibility together with the additional approximation provided by Proposition 4.3, the latter not being required in the type I case.

Using the representation theory of properly infinite von Neumann algebras, we now remove the standardness assumption in the previous theorem.

Recall that a von Neumann algebra is σ -finite if every set of mutually orthogonal projections is at most countable, and that every von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ on a separable Hilbert space H enjoys this property.

Theorem 4.9. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a semi-finite factor on a separable Hilbert space H . Given $\Theta \in \text{LOCC}(\mathcal{A})$ and $\psi \in H$, for every $\varepsilon > 0$ there exists $\Xi \in \text{LOCC}^r(\mathcal{A})$ such that*

$$\|\Theta_*(\omega_\psi) - \Xi_*(\omega_\psi)\| < \varepsilon.$$

Proof. Clearly, we may assume that $\psi \in H_1$. Let K be a separable infinite-dimensional Hilbert space and fix a unit vector $\xi \in K$. Consider the factors $\tilde{\mathcal{A}} := \mathcal{A} \otimes \mathcal{B}(K) \otimes 1_K$ and $\tilde{\mathcal{B}} := \tilde{\mathcal{A}}' = \mathcal{B} \otimes 1_K \otimes \mathcal{B}(K)$, acting on $H \otimes K \otimes K$, and equip $\tilde{\mathcal{A}}$ with the trace $\tilde{\tau} = \tau \otimes \text{tr} \otimes \text{tr}$. Letting $L^\infty(\tilde{\mathcal{A}}) \subseteq \mathcal{B}(L^2(\tilde{\mathcal{A}}))$

denote the standard representation of $\tilde{\mathcal{A}}$, one sees that both $\tilde{\mathcal{B}} = \tilde{\mathcal{A}}'$ and $L^\infty(\tilde{\mathcal{A}})' \cong L^\infty(\tilde{\mathcal{A}})$ are σ -finite and properly infinite factors. Hence, by [45, Proposition V.3.1], the representations $(\tilde{\mathcal{A}}, H \otimes K \otimes K)$ and $(\tilde{\mathcal{A}}, L^2(\tilde{\mathcal{A}}))$ are unitarily equivalent, implemented by the unitary operator $U : H \otimes K \otimes K \rightarrow L^2(\tilde{\mathcal{A}})$, say.

Clearly, $\text{Ad}(U) \circ (\Theta \otimes \text{id}_{\mathcal{B}(K \otimes K)}) \circ \text{Ad}(U^*) \in \text{LOCC}(L^\infty(\tilde{\mathcal{A}}))$. Since $L^\infty(\tilde{\mathcal{A}})$ is a factor, by Theorem 4.7, for every $\varepsilon > 0$ there exists a one-way right LOCC map $\tilde{\Xi} : \mathcal{B}(L^2(\tilde{\mathcal{A}})) \rightarrow \mathcal{B}(L^2(\tilde{\mathcal{A}}))$ relative to $L^\infty(\tilde{\mathcal{A}})$ such that

$$\left\| (\text{Ad}(U) \circ (\Theta \otimes \text{id}_{\mathcal{B}(K \otimes K)}) \circ \text{Ad}(U^*))_*(\omega_{U(\psi \otimes \xi \otimes \xi)}) - \tilde{\Xi}_*(\omega_{U(\psi \otimes \xi \otimes \xi)}) \right\| < \varepsilon.$$

Then $\text{Ad}(U^*) \circ \tilde{\Xi} \circ \text{Ad}(U) : \mathcal{B}(H \otimes K \otimes K) \rightarrow \mathcal{B}(H \otimes K \otimes K)$ is a one-way right LOCC map relative to $\tilde{\mathcal{A}}$ satisfying

$$\left\| (\Theta \otimes \text{id}_{\mathcal{B}(K \otimes K)})_*(\omega_{\psi \otimes \xi \otimes \xi}) - (\text{Ad}(U^*) \circ \tilde{\Xi} \circ \text{Ad}(U))_*(\omega_{\psi \otimes \xi \otimes \xi}) \right\| < \varepsilon.$$

Let $\tilde{\Psi}_k \in \text{CP}_{\tilde{\mathcal{B}}}^\sigma(\mathcal{B}(H \otimes K \otimes K))$ and $\tilde{\Phi}_k \in \text{UCP}_{\tilde{\mathcal{A}}}^\sigma(\mathcal{B}(H \otimes K \otimes K))$ satisfy $\text{Ad}(U^*) \circ \tilde{\Xi} \circ \text{Ad}(U) = \sum_{k=1}^\infty \tilde{\Phi}_k \circ \tilde{\Psi}_k$, and let $\mathcal{E} : \mathcal{B}(H \otimes K \otimes K) \rightarrow \mathcal{B}(H)$ denote the normal unital completely positive map given by $\mathcal{E}(X) = (\text{id} \otimes \omega_\xi \otimes \omega_\xi)(X)$. Define $\Psi_k \in \text{CP}_{\mathcal{B}}^\sigma(\mathcal{B}(H))$ and $\Phi_k \in \text{UCP}_{\mathcal{A}}^\sigma(\mathcal{B}(H))$ by

$$\Psi_k = \mathcal{E} \circ \tilde{\Psi}_k|_{\mathcal{B}(H) \otimes 1_K \otimes 1_K} \circ \iota \quad \text{and} \quad \Phi_k = \mathcal{E} \circ \tilde{\Phi}_k|_{\mathcal{B}(H) \otimes 1_K \otimes 1_K} \circ \iota,$$

where $\iota : \mathcal{B}(H) \rightarrow \mathcal{B}(H) \otimes 1_K \otimes 1_K$ is the embedding $\iota(T) = T \otimes 1 \otimes 1$. Then $\Xi := \sum_{k=1}^\infty \Phi_k \circ \Psi_k$ is a one-way right LOCC map on $\mathcal{B}(H)$ relative to $\mathcal{A}$. Moreover, letting $\sigma_{r,s}$, for $r, s \in \{2, 3, 4, 5\}$ be the flip between terms r and s acting on the tensor product $H \otimes K \otimes K \otimes K \otimes K$, for every $T \in \mathcal{B}(H)$, we have

$$\begin{aligned} & \langle T, \Xi_*(\omega_\psi) \rangle \\ &= \sum_{k=1}^\infty \left\langle \mathcal{E}(\tilde{\Phi}_k(T \otimes 1 \otimes 1)) \otimes 1 \otimes 1, (\tilde{\Psi}_k)_*(\omega_\psi \otimes \omega_\xi \otimes \omega_\xi) \right\rangle \\ &= \sum_{k=1}^\infty \left\langle \tilde{\Phi}_k(T \otimes 1 \otimes 1) \otimes 1 \otimes 1, (\mathcal{E}_* \otimes \text{id} \otimes \text{id})((\tilde{\Psi}_k)_*(\omega_\psi \otimes \omega_\xi \otimes \omega_\xi)) \right\rangle \\ &= \sum_{k=1}^\infty \left\langle \tilde{\Phi}_k(T \otimes 1 \otimes 1) \otimes 1 \otimes 1, \sigma_{24}\sigma_{35}((\tilde{\Psi}_k)_*(\omega_\psi \otimes \omega_\xi \otimes \omega_\xi) \otimes \omega_\xi \otimes \omega_\xi) \right\rangle \\ &= \sum_{k=1}^\infty \left\langle \tilde{\Phi}_k(T \otimes 1 \otimes 1) \otimes 1 \otimes 1, \sigma_{35}((\tilde{\Psi}_k)_*(\omega_\psi \otimes \omega_\xi \otimes \omega_\xi) \otimes \omega_\xi \otimes \omega_\xi) \right\rangle \\ &= \sum_{k=1}^\infty \left\langle \tilde{\Phi}_k(T \otimes 1 \otimes 1) \otimes 1 \otimes 1, (\tilde{\Psi}_k)_*(\omega_\psi \otimes \omega_\xi \otimes \omega_\xi) \otimes \omega_\xi \otimes \omega_\xi \right\rangle \\ &= \left\langle T \otimes 1 \otimes 1 \otimes 1 \otimes 1, (\text{Ad}(U^*) \circ \tilde{\Xi} \circ \text{Ad}(U))_*(\omega_\psi \otimes \omega_\xi \otimes \omega_\xi) \otimes \omega_\xi \otimes \omega_\xi \right\rangle \end{aligned}$$

$$= \left\langle T \otimes 1 \otimes 1, (\text{Ad}(U^*) \circ \tilde{\Xi} \circ \text{Ad}(U))_*(\omega_\psi \otimes \omega_\xi \otimes \omega_\xi) \right\rangle,$$

where in the fourth equality we used the fact that $\tilde{\Phi}_k(T \otimes 1 \otimes 1) \otimes 1 \otimes 1 \in \mathcal{B}(H) \otimes 1 \otimes \mathcal{B}(K) \otimes 1 \otimes 1$ is symmetric under σ_{24} and in the fifth equality we used the fact that $(\tilde{\Psi}_k)_*$ acts trivially on the third leg. (These facts follow from Proposition 3.4, for example.) It follows that

$$\begin{aligned} & \|\Theta_*(\omega_\psi) - \Xi_*(\omega_\psi)\| \\ & \leq \left\| (\Theta \otimes \text{id}_{\mathcal{B}(K \otimes K)})_*(\omega_{\psi \otimes \xi \otimes \xi}) - (\text{Ad}(U^*) \circ \tilde{\Xi} \circ \text{Ad}(U))_*(\omega_{\psi \otimes \xi \otimes \xi}) \right\| < \varepsilon. \quad \square \end{aligned}$$

By left-right symmetry, Theorem 4.9 immediately yields the following corollary:

Corollary 4.10. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a semi-finite factor on a separable Hilbert space. For unit vectors $\psi, \varphi \in H$, the following are equivalent:*

- (i) ψ is approximately convertible to φ via $\text{LOCC}(\mathcal{A})$;
- (ii) ψ is approximately convertible to φ via $\text{LOCC}^r(\mathcal{A})$;
- (iii) ψ is approximately convertible to φ via $\text{LOCC}^l(\mathcal{A})$.

5. THE MAIN THEOREM

In this section we establish Theorem 5.3, a version of Nielsen's theorem for bipartite systems modelled by semi-finite, σ -finite von Neumann algebras (or by standardly represented von Neumann algebras). The next group of lemmas will help justify certain technical arguments in its proof.

Lemma 5.1. *Let $(\mathcal{A}, \tau)$ be a semi-finite von Neumann algebra. For any $\rho_1, \dots, \rho_n \in \mathcal{A}_*^+$, there exist $M_1, \dots, M_{n+1} \in \mathcal{A}$ with $\sum_{i=1}^{n+1} M_i^* M_i = 1$ such that, for $\rho = \sum_{i=1}^n \rho_i$, we have*

$$M_i \rho M_i^* = \rho_i, \quad 1 \leq i \leq n, \quad \text{and} \quad M_{n+1} \rho M_{n+1} = 0.$$

Proof. We may assume that $\mathcal{A}$ is standardly represented on $H = L^2(\mathcal{A}, \tau)$, and identify $\mathcal{A}_*$ with $L^1(\mathcal{A}, \tau)$. Let $\mathcal{B} = \mathcal{A}'$. Since ρ is a positive element of $L^1(\mathcal{A}, \tau)$, the (positive, densely defined) operator $\xi := \rho^{1/2}$ is an element of $H = L^2(\mathcal{A}, \tau)$. Let p and p' be the orthogonal projections onto $\overline{\mathcal{B}\xi}$ and $\overline{\mathcal{A}\xi}$, respectively, and note that

$$p \in \mathcal{A}, \quad p' \in \mathcal{B}, \quad Jp'J = p \quad \text{and} \quad JpJ = p'$$

(see [46, Section IX.1]; for the last two equalities, $J\xi = \xi^* = \xi$ so $JpJa\xi = Jp(JaJ\xi) = J(JaJ\xi) = a\xi$, so $p' \leq JpJ$ and similarly $p \leq Jp'J$.)

For $1 \leq i \leq n$, we have $\rho_i \leq \rho$, that is, $\omega_{\rho_i}^{1/2} \leq \omega_\xi$. By the Radon–Nikodym theorem (see [31, Proposition 7.3.5] and its proof) there exists $b_i \in \mathcal{B}^+$ such that

$$\omega_{\rho_i}^{1/2}(a) = (ab_i\xi, \xi) = \left(ab_i^{1/2}\xi, b_i^{1/2}\xi \right) = \omega_{b_i^{1/2}\xi}(a), \quad a \in \mathcal{A}.$$

By the uniqueness of the GNS representation, there exists a partial isometry $v_i \in \mathcal{B}$ such that for $c_i := v_i b_i^{1/2} \in \mathcal{B}$, we have

$$(16) \quad c_i \xi = \rho_i^{1/2}, \quad 1 \leq i \leq n.$$

Consider $M_1, \dots, M_{n+1} \in \mathcal{A}$, given by

$$M_i := J c_i J p, \quad 1 \leq i \leq n, \quad M_{n+1} := 1 - p.$$

For $1 \leq i \leq n$, using (16) we have

$$M_i \xi = J c_i J \xi = \xi c_i^* = (c_i \xi)^* = (\rho_i^{1/2})^* = \rho_i^{1/2},$$

so

$$M_i \rho M_i^* = (M_i \xi)(M_i \xi)^* = \rho_i, \quad 1 \leq i \leq n.$$

Similarly, $M_{n+1} \xi = (1 - p) \xi = 0$, so $M_{n+1} \rho M_{n+1}^* = 0$. For $a, b \in \mathcal{A}$, we have $p' a \xi = a \xi$ and $p' b \xi = b \xi$; hence, for $1 \leq i \leq n$,

$$\begin{aligned} (J M_i^* M_i J a \xi, b \xi) &= (J p J c_i^* c_i J p J a \xi, b \xi) = (c_i p' a \xi, c_i p' b \xi) \\ &= (c_i a \xi, c_i b \xi) = (a c_i \xi, b c_i \xi) = (a \rho_i^{1/2}, b \rho_i^{1/2}) \\ &= \tau(\rho_i^{1/2} b^* a \rho_i^{1/2}) = \tau(b^* a \rho_i). \end{aligned}$$

Thus, the operator $S := \sum_{i=1}^n M_i^* M_i$ satisfies $(J S J a \xi, b \xi) = \tau(b^* a \rho) = (a \xi, b \xi)$. Hence, $p' J S J p' = p'$, so

$$S = p S p = J p' J S J p' J = J p' J = p,$$

and

$$\sum_{i=1}^{n+1} M_i^* M_i = S + M_{n+1}^* M_{n+1} = p + (1 - p) = 1. \quad \square$$

The version of the following lemma for the case where $\varepsilon = 0$ is well-known. We require the following approximate extension.

Lemma 5.2. *Let H be a Hilbert space, $\varphi \in H_1$ and $(\omega_k)_{k=1}^\infty$ be a sequence in $\mathcal{T}(H)^+$ such that $\sum_{k=1}^\infty \omega_k$ converges weakly to an element in the closed unit ball of $\mathcal{T}(H)$. Set $\alpha_k = \langle \varphi \varphi^*, \omega_k \rangle$, $k \in \mathbb{N}$. If $\varepsilon > 0$ and*

$$(17) \quad \left\| \omega_\varphi - \sum_{k=1}^\infty \omega_k \right\| < \varepsilon,$$

then $\sum_{k=1}^\infty \|\omega_k - \alpha_k \omega_\varphi\| < 2\sqrt{\varepsilon} + \varepsilon$.

Proof. Let tr denote the canonical trace on $\mathcal{T}(H)$ and, for simplicity, write $p = \varphi \varphi^*$ and $p^\perp = 1 - p$. Observe that $p^\perp \omega_\varphi p^\perp = 0$. Thus, by (17),

$$\begin{aligned} (18) \quad \sum_{k=1}^\infty \|p^\perp \omega_k p^\perp\| &= \sum_{k=1}^\infty \text{tr}(p^\perp \omega_k p^\perp) = \text{tr}\left(p^\perp \left(\sum_{k=1}^\infty \omega_k\right) p^\perp\right) \\ &= \left\| p^\perp \left(\sum_{k=1}^\infty \omega_k\right) p^\perp \right\| < \varepsilon. \end{aligned}$$

By the Cauchy–Schwarz inequality, for any $T \in \mathcal{B}(H)$, we have

$$\begin{aligned} \left| p\omega_k p^\perp(T) \right| &= \left| \omega_k(p^\perp T p) \right| \leq \omega_k \left(p^\perp T T^* p^\perp \right)^{1/2} \omega_k(p)^{1/2} \\ &\leq \|T\| \left\| p^\perp \omega_k p^\perp \right\|^{1/2} \omega_k(p)^{1/2}. \end{aligned}$$

Hence, $\|p\omega_k p^\perp\| \leq \|p^\perp \omega_k p^\perp\|^{1/2} \omega_k(p)^{1/2}$. Applying the Cauchy–Schwarz inequality once again and using (18), we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \|p\omega_k p^\perp\| &\leq \sum_{k=1}^{\infty} \omega_k(p)^{1/2} \|p^\perp \omega_k p^\perp\|^{1/2} \\ &\leq \left(\sum_{k=1}^{\infty} \omega_k(p) \right)^{1/2} \left(\sum_{k=1}^{\infty} \|p^\perp \omega_k p^\perp\| \right)^{1/2} < \sqrt{\varepsilon}. \end{aligned}$$

Since $\|p^\perp \omega_k p^\perp\| = \|(p\omega_k p^\perp)^*\| = \|p\omega_k p^\perp\|$, we have $\sum_{k=1}^{\infty} \|p^\perp \omega_k p^\perp\| < \sqrt{\varepsilon}$. Decomposing $\omega_k = p\omega_k p + p\omega_k p^\perp + p^\perp \omega_k p + p^\perp \omega_k p^\perp$, and noting $p\omega_k p = \omega_k(p)\omega_\varphi = \alpha_k \omega_\varphi$, we see that

$$\sum_{k=1}^{\infty} \|\omega_k - \alpha_k \omega_\varphi\| \leq \sum_{k=1}^{\infty} \left(\|p\omega_k p^\perp\| + \|p^\perp \omega_k p\| + \|p^\perp \omega_k p^\perp\| \right) < 2\sqrt{\varepsilon} + \varepsilon. \square$$

We are now in a position to prove the main result of the paper. It provides a version of Nielsen’s theorem for bipartite systems without any explicit (spatial) tensor product structure.

Theorem 5.3. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a semi-finite factor on a separable Hilbert space H . For unit vectors $\psi, \varphi \in H$, the following are equivalent:*

- (i) ψ is approximately convertible to φ via LOCC($\mathcal{A}$);
- (ii) $\rho_\psi \prec \rho_\varphi$.

Proof. (i) $\Rightarrow$ (ii) Let $\varepsilon > 0$ and $\delta > 0$ be such that $2(\delta + \sqrt{\delta}) < \frac{1}{2}\varepsilon$. By Corollary 4.10, there exists a one-way right LOCC map Θ relative to $\mathcal{A}$ such that

$$(19) \quad \|\omega_\varphi - \Theta_*(\omega_\psi)\| < \delta.$$

By Proposition 3.4 (ii), we may write

$$\Theta(x) = \sum_{k=1}^{\infty} \Phi_k(a_k^* x a_k), \quad x \in \mathcal{B}(H),$$

for some $\Phi_k \in \text{UCP}_{\mathcal{A}}^\sigma(\mathcal{B}(H))$ and $a_k \in \mathcal{A}$, $k \in \mathbb{N}$, with

$$(20) \quad \sum_{k=1}^{\infty} a_k^* a_k = 1,$$

where the series converge in the weak* topology. Let

$$\omega_k = \Phi_{k*}(a_k \omega_\psi a_k^*) = a_k \Phi_{k*}(\omega_\psi) a_k^* \in \mathcal{T}(H)^+;$$

then the series $\sum_{k=1}^{\infty} \omega_k$ is weakly convergent to $\Theta_*(\omega_\psi)$. Let us write $\alpha_k := \langle \varphi \varphi^*, \omega_k \rangle$, $k \in \mathbb{N}$. Since $\langle \varphi \varphi^*, \omega_\varphi \rangle = 1$, the bound (19) implies

$$(21) \quad \sum_{k=1}^{\infty} \alpha_k = \langle \varphi \varphi^*, \Theta_*(\omega_\psi) \rangle \in (1 - \delta, 1].$$

By Lemma 5.2, $\sum_{k=1}^{\infty} \|\omega_k - \alpha_k \omega_\varphi\| < 2\sqrt{\delta} + \delta$. Taking restrictions to $\mathcal{A}$, and using the fact that $\Phi_k|_{\mathcal{A}}$ coincides with the identity map, we obtain

$$(22) \quad \sum_{k=1}^{\infty} \|a_k \rho_\psi a_k^* - \alpha_k \rho_\varphi\|_1 < 2\sqrt{\delta} + \delta.$$

For each $a \in \mathcal{A}$, by (20) we have

$$\begin{aligned} \sum_{k=1}^l \left\langle a, \rho_\psi^{1/2} a_k^* a_k \rho_\psi^{1/2} \right\rangle &= \sum_{k=1}^l \tau(a \rho_\psi^{1/2} a_k^* a_k \rho_\psi^{1/2}) = \sum_{k=1}^l \tau(a_k^* a_k \rho_\psi^{1/2} a \rho_\psi^{1/2}) \\ &= \sum_{k=1}^l \left\langle a_k^* a_k, \rho_\psi^{1/2} a \rho_\psi^{1/2} \right\rangle \xrightarrow{l \rightarrow \infty} \left\langle 1, \rho_\psi^{1/2} a \rho_\psi^{1/2} \right\rangle = \langle a, \rho_\psi \rangle. \end{aligned}$$

Hence, the series $\sum_{k=1}^{\infty} \rho_\psi^{1/2} a_k^* a_k \rho_\psi^{1/2}$ converges weakly to ρ_ψ . By Lemma 4.5, the convergence is in norm. Choose $L \in \mathbb{N}$ so that

$$(23) \quad \left\| \rho_\psi - \sum_{k=1}^L \rho_\psi^{1/2} a_k^* a_k \rho_\psi^{1/2} \right\|_1 < \frac{1}{2} \varepsilon.$$

By the right polar decomposition, there exists a partial isometry $v_k \in \mathcal{A}$ such that $a_k \rho_\psi^{1/2} = (a_k \rho_\psi a_k^*)^{1/2} v_k^*$, $k \in \mathbb{N}$. Writing $\alpha_0 = 1 - \sum_{k=1}^{\infty} \alpha_k$, we have by (21) that $\alpha_0 < \delta$. Setting $v_0 = 1$ and using (22) and (23), we see that

$$\begin{aligned} \left\| \rho_\psi - \sum_{k=0}^L \alpha_k v_k \rho_\varphi v_k^* \right\|_1 &< \delta + \left\| \sum_{k=1}^L \rho_\psi^{1/2} a_k^* a_k \rho_\psi^{1/2} - \alpha_k v_k \rho_\varphi v_k^* \right\|_1 + \frac{1}{2} \varepsilon \\ &= \delta + \left\| \sum_{k=1}^L v_k (a_k \rho_\psi a_k^* - \alpha_k \rho_\varphi) v_k^* \right\|_1 + \frac{1}{2} \varepsilon \\ &\leq \delta + \sum_{k=1}^L \|a_k \rho_\psi a_k^* - \alpha_k \rho_\varphi\|_1 + \frac{1}{2} \varepsilon \\ &< 2(\delta + \sqrt{\delta}) + \frac{1}{2} \varepsilon < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, it follows from [23, Theorem 2.5 (3)] that $\rho_\psi \prec \rho_\varphi$.

(ii) $\Rightarrow$ (i) Suppose $\rho_\psi \prec \rho_\varphi$, and fix $\varepsilon > 0$. Pick $\delta > 0$ such that $4\sqrt{\delta} < \varepsilon$. Since $\mathcal{A}$ is a factor, by [23, Theorem 2.5], there exist a family $(u_i)_{i=1}^n$ of unitary operators in $\mathcal{A}$ and a probability distribution $(p_i)_{i=1}^n$, such that, if

$\widetilde{\rho}_\psi = \sum_{i=1}^n p_i u_i \rho_\varphi u_i^*$, then $\|\rho_\psi - \widetilde{\rho}_\psi\|_1 < \delta$. Set $m = n + 1$. By Lemma 5.1, there exist $M_1, \dots, M_m \in \mathcal{A}$ with $\sum_{i=1}^m M_i^* M_i = 1$, such that

$$(24) \quad M_i \widetilde{\rho}_\psi M_i^* = p_i \rho_\varphi \text{ for } 1 \leq i \leq n, \quad \text{and} \quad M_m \widetilde{\rho}_\psi M_m^* = 0.$$

Let $e_1, \dots, e_m$ be the standard basis of $\mathbb{C}^m$, and consider the UCP maps $\Psi, \Phi : \mathcal{B}(H \otimes \mathbb{C}^m) \rightarrow \mathcal{B}(H)$ given by

$$\Psi(T) = \sum_{i=1}^m M_i^* (\text{id} \otimes \omega_{e_i})(T) M_i \quad \text{and} \quad \Phi(T) = \sum_{i=1}^n p_i (\text{id} \otimes \omega_{e_i})(T)$$

for $T \in \mathcal{B}(H \otimes \mathbb{C}^m)$. We have that

$$(25) \quad \Psi_*(\rho) = \sum_{i=1}^m M_i \rho M_i^* \otimes e_i e_i^* \quad \text{and} \quad \Phi_*(\rho) = \sum_{i=1}^n p_i \rho \otimes e_i e_i^*,$$

for $\rho \in \mathcal{T}(H)$. Letting $V, W : H \rightarrow H \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be the isometries given by

$$V\eta = \sum_{i=1}^m M_i \eta \otimes e_i \otimes e_i \quad \text{and} \quad W\eta = \sum_{i=1}^n \sqrt{p_i} \eta \otimes e_i \otimes e_i, \quad \eta \in H,$$

we have Stinespring representations

$$(26) \quad \Psi(T) = V^*(T \otimes 1)V \quad \text{and} \quad \Phi(T) = W^*(T \otimes 1)W, \quad T \in \mathcal{B}(H \otimes \mathbb{C}^m).$$

Consider the states $\nu_\psi, \nu_\varphi : \mathcal{A} \otimes M_m \rightarrow \mathbb{C}$ given by

$$\nu_\psi = \omega_\psi \circ \Psi|_{\mathcal{A} \otimes M_m} \quad \text{and} \quad \nu_\varphi = \omega_\varphi \circ \Phi|_{\mathcal{A} \otimes M_m}.$$

By (25) and (24), we have

$$\begin{aligned} \|\nu_\psi - \nu_\varphi\|_{cb} &= \|\nu_\psi - \nu_\varphi\| \\ &= \|(\Psi|_{\mathcal{A} \otimes M_m})_*(\rho_\psi) - (\Phi|_{\mathcal{A} \otimes M_m})_*(\rho_\varphi)\| \\ &= \left\| \sum_{i=1}^m M_i \rho_\psi M_i^* \otimes e_i e_i^* - M_i \widetilde{\rho}_\psi M_i^* \otimes e_i e_i^* \right\| \\ &= \|(\Psi|_{\mathcal{A} \otimes M_m})_*(\rho_\psi - \widetilde{\rho}_\psi)\| < \delta. \end{aligned}$$

Let $\theta : \mathcal{A} \otimes M_m \rightarrow \mathcal{B}(H \otimes \mathbb{C}^m \otimes \mathbb{C}^m)$ be the $*$ -homomorphism given by $\theta(X) = X \otimes 1$, $X \in \mathcal{A} \otimes M_m$. By (26), the maps ν_ψ and ν_φ have Stinespring representations

$$\nu_\psi = \omega_{V\psi} \circ \theta \quad \text{and} \quad \nu_\varphi = \omega_{W\varphi} \circ \theta.$$

By the continuity of the Stinespring representation [32, Theorem 1] there exist a Hilbert space K , a $*$ -homomorphism $\pi : \mathcal{A} \otimes M_m \rightarrow \mathcal{B}(K)$, and vectors $\eta_1, \eta_2 \in K$ yielding Stinespring representations

$$\nu_\psi = \omega_{\eta_1} \circ \pi \quad \text{and} \quad \nu_\varphi = \omega_{\eta_2} \circ \pi$$

with

$$(27) \quad \|\eta_1 - \eta_2\| < \sqrt{\delta}.$$

By the uniqueness of Stinespring representations, there exist partial isometries $U_1 : H \otimes \mathbb{C}^m \otimes \mathbb{C}^m \rightarrow K$ and $U_2 : K \rightarrow H \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ satisfying $U_1 V \psi = \eta_1$, $U_2 \eta_2 = W \varphi$,

$$U_1(X \otimes 1) = \pi(X)U_1 \quad \text{and} \quad U_2\pi(X) = (X \otimes 1)U_2, \quad X \in \mathcal{A} \otimes M_m.$$

Let $\mathcal{B} = \mathcal{A}'$. The preceding relations imply that the contraction $U := U_2 U_1$ satisfies

$$U \in (\mathcal{A} \otimes M_m \otimes 1)' = \mathcal{A}' \otimes 1 \otimes M_m = \mathcal{B} \otimes 1 \otimes M_m;$$

moreover, by (27),

$$(28) \quad \|UV\psi - W\varphi\| < \sqrt{\delta}.$$

Since $U \in \mathcal{B} \otimes 1 \otimes M_m$, we have $U = \sum_{k,l=1}^m b_{kl} \otimes 1 \otimes e_k e_l^*$ for some $b_{kl} \in \mathcal{B}$. Set $b_i = b_{ii}$, $1 \leq i \leq m$. Then b_i is a contraction in $\mathcal{B}$. Let $\Phi_i \in \text{UCP}_{\mathcal{A}}^{\sigma}(\mathcal{B}(H))$ be the channel given by

$$\Phi_i(x) = b_i^* x b_i + (1 - b_i^* b_i)^{1/2} x (1 - b_i^* b_i)^{1/2}, \quad x \in \mathcal{B}(H),$$

and define $\Theta \in \text{LOCC}(\mathcal{A})$ by

$$\Theta(x) = \sum_{i=1}^m \Phi_i(M_i^* x M_i), \quad x \in \mathcal{B}(H).$$

We claim that $\|\Theta_*(\omega_{\psi}) - \omega_{\varphi}\| < \varepsilon$, which will complete the proof. To see this, let $P : \mathbb{C}^m \otimes \mathbb{C}^m \rightarrow \mathbb{C}^m \otimes \mathbb{C}^m$ denote the orthogonal projection onto $\text{span}\{e_i \otimes e_i \mid 1 \leq i \leq m\}$, and consider the contraction

$$\tilde{U} = (1 \otimes P)U \in \mathcal{B}(H \otimes \mathbb{C}^m \otimes \mathbb{C}^m).$$

A calculation shows that

$$(29) \quad \tilde{U}V\psi = \sum_{i=1}^m b_i M_i \psi \otimes e_i \otimes e_i.$$

Since $W\varphi$ lies in the range of $1 \otimes P$, the bound (28) implies

$$(30) \quad \|\tilde{U}V\psi - W\varphi\| = \|(1 \otimes P)(UV\psi - W\varphi)\| < \sqrt{\delta},$$

and so

$$(31) \quad \|\omega_{\tilde{U}V\psi} - \omega_{W\varphi}\| \leq 2 \|\tilde{U}V\psi - W\varphi\| < 2\sqrt{\delta}.$$

Observe that for $x \in \mathcal{B}(H)$, equation (29) yields

$$\left\langle x, \left(\sum_{i=1}^m \omega_{b_i M_i \psi} \right) - \omega_{\varphi} \right\rangle = \left\langle x \otimes 1 \otimes 1, \omega_{\tilde{U}V\psi} - \omega_{W\varphi} \right\rangle$$

so, in particular, using (31), we have

$$(32) \quad \left\| \sum_{i=1}^m b_i M_i \omega_\psi M_i^* b_i^* - \omega_\varphi \right\| = \left\| \left(\sum_{i=1}^m \omega_{b_i M_i \psi} \right) - \omega_\varphi \right\| \\ \leq \left\| \omega_{\tilde{U} V \psi} - \omega_{W \varphi} \right\| < 2\sqrt{\delta}.$$

Since V and W are isometries, we have $\|V\psi\| = 1 = \|W\varphi\|$. Using (30), we thus have

$$\|V\psi\| - \|\tilde{U} V \psi\| = \|W\varphi\| - \|\tilde{U} V \psi\| < \sqrt{\delta}.$$

By Lemma 4.4,

$$\sum_{i=1}^m \left\| (1 - b_i^* b_i)^{1/2} M_i \psi \right\|^2 = \sum_{i=1}^m \|M_i \psi\|^2 - \|b_i M_i \psi\|^2 = \|V\psi\|^2 - \|\tilde{U} V \psi\|^2 \\ = \left\| (1 - \tilde{U}^* \tilde{U})^{1/2} V \psi \right\|^2 < 2\sqrt{\delta}.$$

Thus, using (32), we have

$$\|\Theta_*(\omega_\psi) - \omega_\varphi\| \\ = \left\| \sum_{i=1}^m b_i M_i \omega_\psi M_i^* b_i^* + (1 - b_i^* b_i)^{1/2} M_i \omega_\psi M_i^* (1 - b_i^* b_i)^{1/2} - \omega_\varphi \right\| \\ \leq \left\| \sum_{i=1}^m b_i M_i \omega_\psi M_i^* b_i^* - \omega_\varphi \right\| + \sum_{i=1}^m \left\| (1 - b_i^* b_i)^{1/2} M_i \omega_\psi M_i^* (1 - b_i^* b_i)^{1/2} \right\| \\ < 2\sqrt{\delta} + \sum_{i=1}^m \left\| (1 - b_i^* b_i)^{1/2} M_i \psi \right\|^2 < 4\sqrt{\delta} < \varepsilon. \quad \square$$

Remark 5.4. The question of convertibility for purely infinite factors becomes trivial. For instance, if $\mathcal{A}$ is a factor of type III_1 with separable predual then, by [46, Theorem XII.5.12], for all normal states ω_1, ω_2 on $\mathcal{A}$, we have

$$\inf\{\|u^* \omega_1 u - \omega_2\| \mid u \in \mathcal{U}(\mathcal{A})\} = 0.$$

Hence, given states ψ, φ in the representation space H of $\mathcal{A}$, for every $\varepsilon > 0$ there exists a unitary $u \in \mathcal{A}$ such that $\|u^* \omega_\psi|_{\mathcal{A}} u - \omega_\varphi|_{\mathcal{A}}\| < \varepsilon$. Appealing to continuity and uniqueness of Stinespring representations (as in Proposition 4.3) one can build a channel $\Theta \in \text{LOCC}(\mathcal{A})$ for which

$$\|\Theta_*(\omega_\psi) - \omega_\varphi\| < 2\sqrt{\varepsilon}.$$

Hence, ψ is approximately convertible to φ via $\text{LOCC}(\mathcal{A})$ and vice-versa.

Remark 5.5. It is natural to ask if the statement of Theorem 5.3 holds in the case of general semi-finite von Neumann algebras. Such an extension would require a treatment of integral decompositions of normal completely positive maps, and is left for a further study. Here we only include an illustration involving a typical non-factor case. Let $\mathcal{D}$ be a maximal abelian

selfadjoint algebra with separable predual, acting on a Hilbert space H . We may assume, without loss of generality, that (X, μ) is a probability measure space such that $H = L^2(X, \mu)$ and $\mathcal{D} = \{M_a : a \in L^\infty(X, \mu)\}$, where, for $a \in L^\infty(X, \mu)$, we have let $M_a \in \mathcal{B}(H)$ be the operator of multiplication by a . We equip $\mathcal{D}$ with the trace τ given by $\tau(M_a) = \int_X a d\mu$. Note that, since $\mathcal{D} = \mathcal{D}'$, we have $\text{LOCC}^r(\mathcal{D}) = \text{LOCC}^l(\mathcal{D}) = \text{LOCC}(\mathcal{D})$, and these sets consist of all unital positive Schur multipliers relative to (X, μ) , that is, the maps $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ of the form

$$(33) \quad \Phi(T) = \sum_{i=1}^{\infty} M_{a_i}^* T M_{a_i}, \quad T \in \mathcal{B}(H),$$

where $a_i \in L^\infty(X, \mu)$, $i \in \mathbb{N}$ and

$$\sum_{i=1}^{\infty} |a_i(s)|^2 = 1 \quad \text{for almost all } s \in X.$$

Let $\psi, \varphi \in H$. We claim that the following are equivalent:

- (i) there exists $\Phi \in \text{LOCC}(\mathcal{D})$ such that $\Phi_*(\omega_\psi) = \omega_\varphi$;
- (ii) ψ is approximately convertible to φ via $\text{LOCC}(\mathcal{D})$;
- (iii) $|\psi| = |\varphi|$ almost everywhere.

Indeed, the implication (i) $\Rightarrow$ (ii) is trivial. Assuming (ii), fix $\varepsilon > 0$ and let $\Phi \in \text{LOCC}(\mathcal{D})$ be such that $\|\omega_\varphi - \omega_\psi \circ \Phi\| < \varepsilon$. Writing Φ in the form (33), we have

$$\begin{aligned} \sup_{\|c\|_\infty \leq 1} \left| \int_X c(|\varphi|^2 - |\psi|^2) d\mu \right| &= \sup_{\|c\|_\infty \leq 1} \left| \int_X c(|\varphi|^2 - \left(\sum_{i=1}^{\infty} |a_i|^2 \right) |\psi|^2) d\mu \right| \\ &= \sup_{\|c\|_\infty \leq 1} |\omega_\varphi(M_c) - \omega_\psi(\Phi(M_c))| \\ &\leq \|\omega_\varphi - \omega_\psi \circ \Phi\| < \varepsilon. \end{aligned}$$

Thus, $\| |\varphi|^2 - |\psi|^2 \|_1 < \varepsilon$. Hence $|\varphi|^2 = |\psi|^2$ in $L^1(X, \mu)$, and (iii) follows. Finally, assuming (iii), let $\theta : X \rightarrow \mathbb{C}$ be a unimodular function such that $\varphi = \theta\psi$, and let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ be the map given by $\Phi(T) = M_\theta^* T M_\theta$. Then $\Phi \in \text{LOCC}(\mathcal{D})$ and $\Phi_*(\omega_\psi) = \omega_\varphi$.

6. TRACE VECTORS AND ENTANGLEMENT IN II_1 -FACTORS

This section is dedicated to some examples and applications of our convertibility result from Section 5. In its first part, we consider a generalisation of maximally entangled vectors to the commuting von Neumann algebra setting, while in its second part we show that entropy of states, relative to the trace, is an entanglement monotone in the sense of [49].

6.1. Trace vectors.

Definition 6.1. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a finite factor on a Hilbert space. A unit vector $\psi \in H$ is said to be a *trace vector* for $\mathcal{A}$ if $\omega_\psi|_{\mathcal{A}} = \tau$, the unique (normal) tracial state on $\mathcal{A}$.

Remark 6.2. Since $\omega_\psi|_{\mathcal{A}}(a) = \tau(\rho_\psi a)$ for $a \in \mathcal{A}$, we see that ψ is a trace vector if and only if $\rho_\psi = 1_{\mathcal{A}}$.

It follows from Nielsen's theorem [37] that the maximally entangled state $\psi = \frac{1}{\sqrt{n}} \sum_{i=1}^n e_n \otimes e_n \in \mathbb{C}^n \otimes \mathbb{C}^n$ is LOCC-convertible (that is, convertible via $\text{LOCC}(M_n \otimes 1)$) to any other state $\varphi \in \mathbb{C}^n \otimes \mathbb{C}^n$. Notice that $\omega_\psi|_{M_n \otimes 1} = \frac{1}{n} \text{tr}$, the normalised trace on M_n . Hence, ψ is a trace vector for $M_n \otimes 1_n \subseteq \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^n)$. The next proposition shows that trace vectors play the role of maximally entangled states relative to II_1 -factors, and provides additional evidence for viewing maximal entanglement through the lens of tracial states [29, §V.A].

Proposition 6.3. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a II_1 -factor on a separable Hilbert space H . If $\psi \in H$ is a trace vector for $\mathcal{A}$, then ψ is approximately convertible to φ via $\text{LOCC}(\mathcal{A})$ for any $\varphi \in H_1$. Conversely, if there exists a trace vector $\psi_0 \in H$ for $\mathcal{A}$, and $\psi \in H_1$ is approximately convertible to any $\varphi \in H_1$ via $\text{LOCC}(\mathcal{A})$, then ψ is a trace vector for $\mathcal{A}$.*

Proof. Suppose that ψ is a trace vector for $\mathcal{A}$. By Remark 6.2, $\rho_\psi = 1_{\mathcal{A}}$. The map on $\mathcal{A}$, given by $a \mapsto \tau(a)1_{\mathcal{A}}$, is doubly stochastic (i.e., it is positive, normal, unital and trace-preserving) and its extension to $L^1(\mathcal{A}, \tau)$ maps ρ_φ to $1_{\mathcal{A}} = \rho_\psi$, since $\tau(\rho_\varphi) = \omega_\varphi(1_{\mathcal{A}}) = (\varphi, \varphi) = 1$. It follows from [23, Theorem 4.5] that $\rho_\psi \prec \rho_\varphi$, hence, ψ is approximately convertible to φ via $\text{LOCC}(\mathcal{A})$ by Theorem 5.3.

For the converse statement, suppose $\psi_0 \in H_1$ is a trace vector for $\mathcal{A}$, and that $\psi \in H_1$ is approximately convertible to every $\varphi \in H_1$ via $\text{LOCC}(\mathcal{A})$. Then ψ is approximately convertible to ψ_0 and vice-versa. By Theorem 5.3 and Remark 6.2, $\rho_\psi \prec 1_{\mathcal{A}}$ and $1_{\mathcal{A}} \prec \rho_\psi$, that is, ρ_ψ and $1_{\mathcal{A}}$ are spectrally equivalent in the sense of [23, §3]. By [23, Theorem 3.4 (2)], for every $\varepsilon > 0$, there exists a unitary $u \in \mathcal{A}$ such that

$$\|\rho_\psi - 1_{\mathcal{A}}\|_1 = \|\rho_\psi - u \cdot \rho_{\psi_0} \cdot u^*\|_1 < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have $\rho_\psi = 1_{\mathcal{A}}$; by Remark 6.2, ψ is a trace vector for $\mathcal{A}$. $\square$

Amongst II_1 -factors, the hyperfinite (i.e., approximately finite dimensional) II_1 -factor is best suited for applications in mathematical physics. In that context, it typically appears through an infinite tensor product construction, an algebra of canonical commutation/anti-commutation relations, or an irrational rotation algebra. We now present examples of maximally entangled states relative to the hyperfinite II_1 -factor in each of the three aforementioned manifestations.

Example 6.4. This example is based on [30, §4.2]. Consider an infinite spin chain consisting of infinitely many qubits arranged on a one-dimensional lattice, say $\mathbb{Z}$. The underlying C^* -algebra of the system is given by the infinite tensor product $A = \bigotimes_{\mathbb{Z}} M_2$, that is, the inductive limit of the system $A_F = \bigotimes_{n \in F} M_2$, with canonical inclusion maps, where F ranges through the finite subsets of $\mathbb{Z}$. For $n \in \mathbb{Z}$, let ψ_n be the maximally entangled state on $A_{\{-n, n+1\}}$. Then $\omega = \bigotimes_{n \in \mathbb{Z}} \psi_n \psi_n^*$ defines a state on A . Let $\mathcal{A} = \pi_\omega(A_{(-\infty, 0)})'' \subseteq \mathcal{B}(H_\omega)$ be the von Neumann algebra generated by the left half-chain in the cyclic GNS-representation $(H_\omega, \pi_\omega, \psi_\omega)$ of ω . Then $\mathcal{B} := \mathcal{A}' = \pi_\omega(A_{[0, \infty)})''$ is the von Neumann algebra generated by the right half-chain. Both $\mathcal{A}$ and $\mathcal{B}$ are II_1 -factors, and by construction it follows that ψ_ω is a trace vector for $\mathcal{A}$. Thus, by Proposition 6.3, ψ_ω is approximately convertible to any state $\varphi \in H_\omega$ via $\text{LOCC}(\mathcal{A})$. Naturally, one may view ψ_ω as a state representing infinitely many pairs of entangled qubits.

Example 6.5. Let K be a real Hilbert space and $H = K \oplus iK$ its complexification. Let $\mathcal{F}_a(H)$ denote the anti-symmetric Fock space over H , given by

$$\mathcal{F}_a(H) = \bigoplus_{n \geq 0} \wedge^n H,$$

where $\wedge^n H$ is the anti-symmetric subspace of $H^n := \bigotimes_{k=1}^n H$ for $n \geq 1$ and $\wedge^0 := \mathbb{C}$. For $\psi \in H$, let $a(\psi)^*$ and $a(\psi)$ denote the Fock creation and annihilation operators, namely the bounded [12] linear maps $\mathcal{F}_a(H) \rightarrow \mathcal{F}_a(H)$ given by

$$a(\psi)^* \varphi = \sqrt{n+1} P_a^{n+1}(\psi \otimes \varphi), \quad a(\psi) \varphi = \sqrt{n} P_a^n(\psi^* \otimes \text{id}) \varphi,$$

where $n \geq 1$, $\varphi \in \wedge^n H$ and $P_a^n : H^n \rightarrow \wedge^n H$ is the canonical projection. Let $S \in \mathcal{B}(\mathcal{F}_a(H))$ denote the parity operator defined by $S = \bigoplus_{n \geq 0} (-1)^{\otimes n}$. Letting $B(\psi) := a(\psi)^* + a(\psi)$ represent the corresponding (self-adjoint) Fermionic field operators, it follows that $\mathcal{A} := \{B(\psi) \mid \psi \in K\}''$ is a II_1 -factor associated to a real-wave representation of the canonical anti-commutation relations [13, §13] whose commutant satisfies $\mathcal{B} := \mathcal{A}' = \{SB(i\psi) \mid \psi \in K\}''$.

It is known that the vacuum vector $\Omega = (1, 0, 0, \dots) \in \mathcal{F}_a(H)$ is a quasi-free trace vector for $\mathcal{A}$, with

$$(34) \quad (B(\psi)B(\varphi)\Omega, \Omega) = (\psi, \varphi), \quad \psi, \varphi \in K.$$

More generally, given an anti-symmetric tensor $c \in H \wedge H$ (seen as a Hilbert-Schmidt operator from $\overline{H}$ to H), the Fermionic Gaussian vector associated with c is given by

$$\Omega_c = \det(1 + c^*c)^{-\frac{1}{4}} e^{-\frac{1}{2}a^*(c)} \Omega,$$

where $\det(\cdot)$ is the Fredholm determinant and $a^*(c)$ is the two particle creation operator defined by

$$a^*(c)\psi = \sqrt{(n+2)(n+1)} P_a^{n+2}(c \otimes \psi), \quad \psi \in \wedge^n H, \quad n \geq 1.$$

Such vectors occur in the Hartree–Fock–Bogoliubov method for approximating Fermionic systems (see e.g. [14, §4]), which is related to the Bardeen–Cooper–Schrieffer theory of superconductivity [2]. For every c there exists an orthogonal transformation O_c on K , and a unitary U_c on $\mathcal{F}_a(H)$ satisfying $\Omega_c = U_c^* \Omega$ and $U_c B(\psi) U_c^* = B(O_c \psi)$, $\psi \in K$ [13]. Thus, $\omega_{\Omega_c}|_{\mathcal{A}} = \omega_{\Omega} \circ \text{Ad}(U_c)$, and it follows from (34) that Ω_c is also a trace vector for $\mathcal{A}$. Thus, by Proposition 6.3, any of the Fermionic Gaussian vectors Ω_c may be converted into any Fock state $\varphi \in \mathcal{F}_a(H)$ by means of local operations and classical communication relative to the real-wave representation $\mathcal{A}$ of the CAR. In particular, the vectors Ω_c display properties of maximal entanglement relative to $\mathcal{A}$ and its commutant.

We present one more instance of Proposition 6.3, based on the example from [5, §7], which in turn was partly motivated by [18]. This example is a particular realization of the irrational rotation algebra and is related to discretised CCR relations, whose relevance to numerical analysis of quantum systems was advocated by Arveson [1].

Example 6.6. Suppose Alice and Bob have access to a quantum system represented by the Hilbert space $L^2(\mathbb{R})$. Let q and p denote the self-adjoint operators corresponding to position and momentum:

$$q\psi(x) = x\psi(x), \quad p\psi(x) = i \frac{d}{dx} \psi(x),$$

where ψ belongs to a common dense domain for q and p . Suppose that Alice can measure periodic functions of position and momentum, with periods t_q and t_p , respectively. Such functions are given (respectively) by integer powers of the unitary operators

$$U := e^{i\omega_q q} \quad \text{and} \quad V := e^{i\omega_p p},$$

where, following [5], we let $\omega_q := \frac{2\pi}{t_q}$ and $\omega_p := \frac{2\pi}{t_p}$. The operators U and V satisfy

$$UV = e^{2\pi i \theta} VU,$$

where $\theta := \frac{\omega_q \omega_p}{2\pi}$. In what follows, we assume that $\omega_q \omega_p > 4\pi$ and that θ is irrational.

The algebra describing Alice’s measurement statistics is the von Neumann subalgebra $\mathcal{A}$ of $\mathcal{B}(L^2(\mathbb{R}))$ generated by U and V , and is known to be a type II_1 -factor. The C^* -algebra generated by U and V is known as the irrational rotation algebra corresponding to θ . The von Neumann algebra describing Bob’s measurement statistics, $\mathcal{B} = \mathcal{A}'$, is generated by

$$U' := e^{i \frac{\omega_q}{\theta} q}, \quad \text{and} \quad V' := e^{i \frac{\omega_p}{\theta} p},$$

and is also a type II_1 -factor.

Let $\psi = \frac{1}{\sqrt{2t_q}} \chi_{[-t_q, t_q]} \in L^2(\mathbb{R})$. If $x \in [-t_q, t_q]$ and $m \in \mathbb{Z}$ then, since $\omega_p > \frac{4\pi}{\omega_q} = 2t_q$, it follows that $x + m\omega_p \in [-t_q, t_q]$ if and only if $m = 0$.

Hence, for all $n, m \in \mathbb{Z}$ we have

$$\begin{aligned} (U^n V^m \psi, \psi) &= \frac{1}{2t_q} \int_{\mathbb{R}} e^{in \frac{2\pi}{t_q} x} \chi_{[-t_q, t_q]}(x + m\omega_p) \chi_{[-t_q, t_q]}(x) dx \\ &= \delta_{m,0} \frac{1}{2t_q} \int_{-t_q}^{t_q} e^{in \frac{2\pi}{t_q} x} dx \\ &= \delta_{m,0} \delta_{n,0} 1. \end{aligned}$$

By [11, Corollary VI.1.2], $\omega_\psi|_{\mathcal{A}}$ is the unique normal tracial state τ on $\mathcal{A}$. By Proposition 6.3, we have that ψ is approximately convertible to any unit vector $\varphi \in L^2(\mathbb{R})$ via LOCC($\mathcal{A}$).

One can think of ψ as representing the state of a particle whose position is uniformly distributed over the interval $[-t_q, t_q]$. This uniformity is playing the role of maximal entanglement relative to the bipartite system $(\mathcal{A}, \mathcal{B})$.

6.2. Entanglement monotones. The practical importance of quantifying the degree of entanglement present in a given state cannot be overestimated. In the standard finite-dimensional tensor product framework, this quantification is studied through notions of entanglement measures. Since entanglement at its very core is a form of non-local quantum correlation, any reasonable entanglement measure ought to be monotonic with respect to local operations and classical communication. The term entanglement monotone has since emerged for such a measure, and it was argued by Vidal [49] that monotonicity under LOCC is the *only* natural requirement for measures of entanglement. As an application of our main result, we show that the entropy of the singular value distribution satisfies this requirement for pure states relative to II_1 -factors, thus yielding an entanglement monotone.

Let $\mathcal{A}$ be a II_1 -factor with unique tracial state τ . Given a normal state $\rho \in \mathcal{S}(\mathcal{A})$, we define the *entropy of ρ relative to τ* by

$$H_\tau(\rho) := H(\mu(\rho)) = - \int_0^1 \mu_t(\rho) \log(\mu_t(\rho)) dt.$$

By splitting the entropy function $\eta(x) = -x \log(x)$ into $\chi_{[0,1]}\eta + \chi_{(1,\infty)}\eta$, and applying [17, Remark 3.3] to the non-negative Borel functions $\chi_{[0,1]}\eta$ and $-\chi_{(1,\infty)}\eta$, it follows that

$$H_\tau(\rho) = \tau(\eta(\rho)) = -\tau(\rho \log(\rho)) = -S(\rho, \tau),$$

whenever $|H_\tau(\rho)| < \infty$, where $S(\cdot, \cdot)$ is the relative entropy between normal states of $\mathcal{A}$ (see e.g. [38, §5]). As such, we see that $H_\tau(\rho) \leq 0$ and $H_\tau(\rho) = 0$ if and only if $\rho = \tau$. In particular, if $\mathcal{A} \subseteq \mathcal{B}(H)$ and $\psi \in H_1$, then $H_\tau(\rho_\psi) = 0$ if and only if ψ is a trace vector for $\mathcal{A}$.

Proposition 6.7. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a II_1 -factor with tracial state τ . The function*

$$H_\tau : \mathcal{S}(\mathcal{A}) \ni \rho \mapsto H_\tau(\rho) \in [-\infty, 0]$$

is non-increasing under approximate convertibility by LOCC($\mathcal{A}$), when restricted to states of the form ρ_ψ , $\psi \in H_1$.

Proof. First note that for any state $\rho \in \mathcal{S}(\mathcal{A})$,

$$H_\tau(\rho) = -S(\rho, \tau) = -S(\mu(\rho), \chi_{[0,1]}),$$

where $S(\mu(\rho), \chi_{[0,1]})$ is the relative entropy of the density $\mu(\rho): t \mapsto \mu_t(\rho)$ on $[0, 1]$ with respect to the uniform distribution. Given $\psi, \varphi \in H_1$, if ψ is approximately convertible to φ via $\text{LOCC}(\mathcal{A})$ then by Theorem 5.3 we have $\rho_\psi \prec \rho_\varphi$, meaning that $\mu_t(\rho_\psi) \prec \mu_t(\rho_\varphi)$ as probability densities on $[0, 1]$. By [47, Theorem 10], it follows that

$$S(\mu_t(\rho_\psi), \chi_{[0,1]}) \leq S(\mu_t(\rho_\varphi), \chi_{[0,1]}).$$

Hence, $H_\tau(\rho_\psi) \geq H_\tau(\rho_\varphi)$. $\square$

Remark 6.8. In the proof of Proposition 6.7 we could instead appeal to [23, Theorem 4.7(1)] for the connection between majorisation and double stochasticity together with monotonicity of the relative entropy [38, Theorem 5.3].

Recall that any density matrix $\rho \in M_n$ satisfies

$$S(\rho) = -S(\rho, \tau) + \log(n),$$

where $S(\cdot)$ is the von Neumann entropy and $\tau = \frac{1}{n} \text{tr}$ is the maximally mixed state (the $\log(n)$ factor would disappear if we used the unnormalised trace tr). It is known that the restriction of any entanglement monotone to pure states $\psi \in \mathbb{C}^n \otimes \mathbb{C}^n$ is a concave function of the reduced density $\rho_\psi = (\text{id} \otimes \text{tr})(|\psi\rangle\langle\psi|)$ [49, Theorem 3]. The entanglement monotone $-S(\rho_\psi, \tau)$ is equivalent (up to the translational factor $\log(n)$) to the common choice of $S(\rho_\psi)$, and is the finite-dimensional analogue of our proposed monotone above. Note that $-S(\rho_\psi, \tau) \in [-\log(n), 0]$, with the largest value of 0 occurring for maximally entangled ψ , and the lowest value of $-\log(n)$ occurring for separable ψ .

7. OUTLOOK

Several natural lines of investigation arise from this work. First, we intend to study the generalisation of our main result to the non-factor setting in connection with [24], as mentioned at the end of Section 5. This could be useful for the study of entanglement in hybrid systems [33, 15, 3, 4]. Second, a rectangular version of our main theorem, describing convertibility between states $\psi \in H$ and $\varphi \in K$, with respect to distinct bipartite systems $(\mathcal{A}_1, \mathcal{B}_1)$ in $\mathcal{B}(H)$ and $(\mathcal{A}_2, \mathcal{B}_2)$ in $\mathcal{B}(K)$, would be desirable. Among other things, this could have applications to the structure of quantum correlation matrices and values of certain non-local games [41]. We also plan to explore notions of distillability/dilution of entanglement for general bipartite systems in connection with this and previous work [48, 30]. In that direction it would be interesting to explore uniqueness of entanglement monotones in the asymptotic regime for pure states, analogous to the finite-dimensional setting [42]. Finally, we observe that while many of the von Neumann algebras

of interest in algebraic quantum field theory are of type III [19], and hence not semi-finite, it is conceivable that semi-finite von Neumann algebras may be relevant in the discretisation of space-time via tensor networks—a current area of research—analogue to those arising from discretisations of the canonical commutation relations [43].

Acknowledgements. J.C. was partially supported by the NSERC Discovery Grant RGPIN-2017-06275. D.W.K. was partly supported by NSERC Discovery Grant 400160 and by a Royal Society grant allowing research visits to Queen’s University Belfast. R.H.L. was partly supported by UCD Seed Funding.

REFERENCES

- [1] W. ARVESON, *Discretized CCR algebras*, J. Operator Thy. 26 (1991), no. 2, 225–239.
- [2] J. BARDEEN, L. N. COOPER AND J. R. SCHRIEFFER, *Theory of superconductivity*, Phys. Rev. 108 (1957), 1175.
- [3] C. BÉNY, A. KEMPF, AND D. W. KRIBS, *Generalization of quantum error correction via the Heisenberg picture*, Phys. Rev. Lett. 98 (2007), 100502.
- [4] C. BÉNY, A. KEMPF, AND D. W. KRIBS, *Quantum error correction of observables*, Phys. Rev. A 76 (2007), 042303, 22 pp.
- [5] C. BÉNY, A. KEMPF, AND D. W. KRIBS, *Quantum error correction on infinite-dimensional Hilbert spaces*, J. Math. Phys. 50 (2009), no. 6, 062108, 24 pp.
- [6] M. BERTA, F. FURRER, AND V. B. SCHOLZ, *The smooth entropy formalism for von Neumann algebras*, J. Math. Phys. 57 (2016), no. 1, 015213, 25 pp.
- [7] D. P. BLECHER AND R. R. SMITH, *The dual of the Haagerup tensor product*, J. London Math. Soc. (2) 45 (1992), 126–144.
- [8] A. CHATTERJEE AND R. R. SMITH, *The central Haagerup tensor product and maps between von Neumann algebras*, J. Funct. Anal. 112 (1993), no. 1, 97–120.
- [9] E. CHITAMBAR, D. LEUNG, L. MANCINSKA, M. OZOLS AND A. WINTER, *Everything you always wanted to know about LOCC (but were afraid to ask)*, Comm. Math. Phys. 328 (2014), no. 1, 303–326.
- [10] J. CRANN, D.W. KRIBS, R.H. LEVENE AND I.G. TODOROV, *Private algebras in quantum information and infinite-dimensional complementarity*, J. Math. Phys. 57 (2016), no. 1, 015208.
- [11] K. R. DAVIDSON, *C*-algebras by Example*, Fields Institute Monographs, 1996.
- [12] J. DEREZIŃSKI, *Introduction to representations of the canonical commutation and anticommutation relations*, in *Large Coulomb Systems: Lecture Notes on Mathematical Aspects of QED*, Springer Berlin Heidelberg, 2006. pp 63–143.
- [13] J. DEREZIŃSKI AND C. GÉRARD, *Mathematics of quantization and quantum fields*, Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2013.
- [14] J. DEREZIŃSKI, K. A. MEISSNER, AND M. NAPIÓRKOWSKI, *On the energy-momentum spectrum of a homogeneous Fermi gas*, Ann. Henri Poincaré 14 (2013), no. 1, 1–36.
- [15] I. DEVETAK, AND P.W. SHOR, *The capacity of a quantum channel for simultaneous transmission of classical and quantum information*, Comm. Math. Phys. 256 (2005), no. 2, 287–303.
- [16] K. DYKEMA, V. PAULSEN, AND J. PRAKASH, *Non-closure of the set of quantum correlations via graphs*, Comm. Math. Phys., to appear.
- [17] T. FACK AND H. KOSAKI, *Generalised s-numbers of τ -measurable operators*, Pacific J. Math. 123 (1986), no. 2, 269–300.

- [18] L. FADDEEV, *Discrete Heisenberg-Weyl group and modular group*, Lett. Math. Phys. 34 (1995), 249.
- [19] R. HAAG, *Local quantum physics. Fields, particles, algebras*, Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.
- [20] U. HAAGERUP, *Decomposition of completely bounded maps on operator algebras*, unpublished manuscript.
- [21] U. HAAGERUP, *The standard form of von Neumann algebras*. Math. Scand. 37 (1975), 271–283.
- [22] U. HAAGERUP AND M. MUSAT, *As asymptotic property of factorizable completely positive maps and the Connes embedding problem*, Comm. Math. Phys. 338 (2015), 721–752.
- [23] F. HIAI, *Majorization and stochastic maps in von Neumann algebras*, J. Math. Anal. Appl. 127 (1987), 18–48.
- [24] F. HIAI, *Spectral relations and unitary mixing in semifinite von Neumann algebras*, Hokkaido Math. J. 17 (1988), no. 1, 117–137.
- [25] F. HIAI, *Quantum f -divergences in von Neumann algebras I. Standard f -divergences*, arXiv:1805.02050.
- [26] F. HIAI, *Quantum f -divergences in von Neumann algebras II. Maximal f -divergences*, arXiv:1807.03118.
- [27] S. HOLLANDS AND K. SANDERS, *Entanglement measures and their properties in quantum field theory*, arXiv:1702.04924
- [28] M. JUNGE AND C. PALAZUELOS, *Large violation of Bell inequalities with low entanglement*, Comm. Math. Phys. 306 (2011), 695–746.
- [29] M. KEYL, D. SCHLINGEMANN AND R. F. WERNER, *Infinitely entangled states*, Quantum Inf. Comput. 3 (2003), no. 4, 281–306.
- [30] M. KEYL, T. MATSUI, D. SCHLINGEMANN, AND R. F. WERNER, *Entanglement Haag-duality and type properties of infinite quantum spin chains*, Rev. Math. Phys. 18 (2006), no. 9, 935–970.
- [31] R. V. KADISON AND J. R. RINGROSE, *Fundamentals of the theory of operator algebras, Volume II: Advanced theory*, AMS Grad. Studies in Math. vol. 16, 1997.
- [32] D. KRETSCHMANN, D. SCHLINGEMANN AND R. F. WERNER, *A continuity theorem for Stinespring’s dilation*, J. Funct. Anal. 255 (2008), 1889–1904.
- [33] G. KUPERBERG, *The capacity of hybrid quantum memory*, IEEE Trans. Inf. Thy. 49 (2003), no. 6, 1465–1473.
- [34] H.-K. LO AND S. POPESCU, *Concentrating entanglement by local actions: Beyond mean values*, Phys. Rev. A 63 (2001), no. 2, 022301.
- [35] R. LONGO, *On Landauer’s principle and bound for infinite systems*, Comm. Math. Phys. 363 (2018), no. 2, 531–560.
- [36] R. LONGO AND F. XU, *Relative entropy in CFT*, Adv. Math. 337 (2018), 139–170.
- [37] M. NIELSEN, *Conditions for a class of entanglement transformations*, Phys. Rev. Lett. 83 (1999), 436.
- [38] M. OHYA AND D. PETZ, *Quantum entropy and its use*, Texts and Monographs in Physics, Springer-Verlag, Berlin-Heidelberg 1993.
- [39] M. OWARI, S. L. BRAUNSTEIN, K. NEMOTO AND M. MURAO, *ε -convertibility of entangled states and extension of Schmidt rank in infinite-dimensional systems*, Quantum Inf. & Comp. 8 (2008), no. 1 & 2, 30–52.
- [40] V. I. PAULSEN, *Completely bounded maps and operator algebras*, Cambridge University Press, 2002.
- [41] V. I. PAULSEN, S. SEVERINI, D. STAHLKE, I. G. TODOROV AND A. WINTER, *Estimating quantum chromatic numbers*, J. Funct. Anal. 270 (2016), no. 6, 2188–2222.
- [42] S. POPESCU AND D. ROHRICH, *Thermodynamics and the measure of entanglement*, Phys. Rev. A (3) 56 (1997), no. 5, R3319–R3321.

- [43] J. SLAWNY, *On factor representations and the C^* -algebra of canonical commutation relations*, Comm. Math. Phys. 24 (1972), 151–170.
- [44] W. SLOFSTRA, *The set of quantum correlations is not closed*, preprint (2017), arXiv:1703.08618.
- [45] M. TAKESAKI, *Theory of operator algebras I*, Encyclopedia of Mathematical Sciences 124, Springer-Verlag Berlin-Heidelberg-New York, 2002.
- [46] M. TAKESAKI, *Theory of operator algebras II*, Encyclopedia of Mathematical Sciences 125, Springer-Verlag Berlin-Heidelberg-New York, 2003.
- [47] T. VAN ERVEN AND P. HARREMOËS, *Rényi Divergence and majorization*, 2010 IEEE International Symposium on Information Theory, Conference Proceedings DOI:10.1109/ISIT.2010.5513784.
- [48] R. VERCH AND R. F. WERNER, *Distillability and positivity of partial transposes in general quantum field systems*, Rev. Math. Phys. 17 (2005), no. 5, 545–576.
- [49] G. VIDAL, *Entanglement monotones*, J. Modern Opt. 47 (2000), no. 2–3, 355–376.
- [50] J. VON NEUMANN, *Mathematical foundations of quantum mechanics*, Princeton University Press, 1955.

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