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Sliding Mode Control of a Class of Underactuated System With Non-integrable Momentum

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Abstract

In this paper, a sliding mode control scheme is developed to stabilise a class of nonlinear perturbed underactuated system with a non-integral momentum. In this scheme, by initially maintaining a subset of actuated variables on sliding manifolds, the underactuated system with the non-integrable momentum can be approximated by one with the integrable momentum in finite time. During sliding, a subset of the actuated variables converge to zero and a physically meaningful diffeomorphism is systematically calculated to transform the reduced order sliding motion into one in a strict feedback normal form in which the control signals are decoupled from the underactuated subsystem. Furthermore, based on the perturbed strict feedback form, it is possible to find a sliding mode control law to ensure the asymptotic stability of the remaining actuated and unactuated variables. The design efficacy is verified via a multi-link planar robot case study.

Index Terms

Sliding mode control, Underactuated System, Robotics.

I. INTRODUCTION

In recent decades there has been increasing interest in Euler-Lagrange systems [1], [2]. Underactuated Euler-Lagrange systems, which have fewer independent controls than the number of degrees of freedom (DOF) or configuration variables, have received extensively attentions in a broad range of applications including robotic and aerospace systems. For example, crane systems [3]–[5], Pendubot [6], [7], Acrabot [8], surface vessel [9]–[11], hypersonic vehicles [12], quadrotor systems [13], [14] and flexible or multi-link manipulators [15]–[18]. Although underactuated systems has a lower number of actuators compared with one associated with overactuated systems, the complexity of the control system design is increased. A state-of-the-art overview of the modelling, classification, control and application of the underactuated system can be found in [19].

Sliding mode control schemes have several unique properties, and these have sustained research interest in this area since the 1960s. The most important property is its insensitivity (at least theoretically) to matched uncertainty acting in the control input channels [20]–[22]. In conventional sliding mode systems, the order of the closed-loop system is reduced compared to the open-loop, by an amount equal to the number of input control signals. The reduced order dynamics during the sliding motion are determined by the choice of sliding surface – which is a key component of the design process. Many different approaches for the design of linear sliding surfaces for uncertain linear systems have been developed, and the area is quite mature [21], [22]. In conventional sliding modes the closed-loop behaviour has two quite distinguishable phases: a) the pre-sliding phase in which the system states are driven towards the sliding surface to create a sliding mode; b) the reduced order sliding motion that occurs once the surface is attained. As a promising method, sliding mode control schemes have been used to stabilise Euler-Lagrange systems [23], [24] and underactuated systems [25]–[28]. Some recent works have successfully stabilised underactuated systems in the face of the external disturbances [29]–[32], modelling uncertainty [33], [34] or even faults/failures [15].

To control underactuated systems, some works systematically developed a global change of coordinate which is capable of transforming an underactuated system with symmetry into one in a strict feedback normal form [35]. The main advantages of using this class of global coordinate transformation are: a) the order of the underactuated system
can be reduced; b) some conventional control design and controllability analysis approaches, such like sliding mode control, back-stepping control and adaptive control, can be applied to a system in the normal form straightforwardly; c) the control inputs are decoupled from the unactuated variables, which reduces the complexity of the control system design. As argued in [19], the key analytical tools which allow reduction of high-order underactuated systems using a global change of coordinate in explicit forms are normalized generalized momentums and their integrals. The integrability of these normalized momentums plays a fundamental role in the structure of the normal forms for high-order underactuated systems and an important property of normal forms for high-order underactuated systems is that they are physically meaningful. The global change of coordinate can be obtained from the Lagrangian of the system and the reduced order system is a well-defined reduced Lagrangian system that satisfies the Euler-Lagrange equations. Many works in the literature (e.g in [7], [25], [29]–[31], [35]) assumed that the normalised momentum conjugated to the unactuated variables is integrable. However, some high-order underactuated systems do not possess integrable normalized momentums such as the flexible robot and 3D Cart-Pole system. For an underactuated system with the non-integrable momentum, the global change of coordinate is not straightforwardly applicable because the shape-inertial matrix is not exact one [35]. In [19], a methodology, so called decomposition momentum, was used to decompose the normalised momentum as an integrable locked momentum and a non-integrable error momentum. Instead of using momentum decomposition approach to zero in finite time and a physically meaningful diffeomorphism, which has the capability of transforming the reduced order sliding motion into one in the strict feedback normal form, is systematically calculated. This paper also shows that the scheme may guarantee the asymptotic stability of both actuated and unactuated variables despite external disturbances.

The remainder of the paper is structured as follows: some preliminaries are in given in Section II; in Section III the sliding mode control scheme, which ensures the possible asymptotic stability of both actuated and unactuated variables, is discussed. The simulation results, associated with a multi-link planar robot, are presented in Section IV. Finally, Section V provides some concluding remarks.

The notation used in this paper is quite standard: in particular, the norm of a vector \( x \in \mathbb{R}^n \) is defined as \( \|x\| = \sqrt{x^T x} \) and the norm of a matrix \( A \in \mathbb{R}^{n \times n} \) is given by \( \|A\| = \sqrt{\lambda_{\text{max}}(A^T A)} \) where \( \lambda_{\text{max}}(A^T A) \) represents the maximum eigenvalue of \( A^T A \). For \( A \in \mathbb{R}^{m \times n} \), the pseudo inverse of \( A \) is denoted by \( A^\dagger \in \mathbb{R}^{n \times m} \).

**II. Preliminary**

Consider a nonlinear underactuated Euler-Lagrange system

\[
M(q_a) \ddot{q} + W(q, \dot{q}) \dot{q} + F(q, \dot{q}) = Bu + d
\]  

where

\[
B = \begin{bmatrix}
I_{n-m} \\
o_{m \times (n-m)}
\end{bmatrix}
\]  

and \( u \in \mathbb{R}^{n-m} \) denotes the system inputs. In (1), \( M(q_a) \in \mathbb{R}^{n \times n} \) represents the inertial matrix, \( W(q, \dot{q}) \in \mathbb{R}^{n \times n} \) captures the Coriolis and centrifugal forces, and \( F(q, \dot{q}) \in \mathbb{R}^n \) represents the damping and friction terms. In (1), the variable \( q = [q_a^T \ q_u^T]^T \) denotes system state variables, where \( q_a \in \mathbb{R}^{n-m} \) represents the actuated variables and \( q_u \in \mathbb{R}^m \) are the unactuated variables. The signal \( d \) is used to capture the external disturbance, and it is assumed that \( d = [d_a^T \ d_u^T]^T \) where \( d_a \in \mathbb{R}^{n-m} \) and \( d_u \in \mathbb{R}^m \) represent the disturbances affecting the actuated and unactuated channels, respectively. In this paper, it is assumed that \( \|d\| \leq \xi \).

Suppose \( M(q_a) \) in (1) has the following structure

\[
M(q_a) = \begin{bmatrix}
M_{11}(q_a) & M_{12}(q_a) \\
M_{21}(q_a) & M_{22}(q_a)
\end{bmatrix}
\]  

where \( M_{11}(q_a) \in \mathbb{R}^{(n-m) \times (n-m)} \) and \( M_{22}(q_a) \in \mathbb{R}^{m \times m} \). In (1), \( W(q, \dot{q}) \) and \( F(q, \dot{q}) \) are given by

\[
W(q, \dot{q}) = \begin{bmatrix} W_1(q, \dot{q}) \\
W_2(q, \dot{q})
\end{bmatrix}, \quad F(q, \dot{q}) = \begin{bmatrix} F_1(q, \dot{q}) \\
F_2(q, \dot{q})
\end{bmatrix}
\]
where $W_2(q, \dot{q}) \in \mathbb{R}^{m \times n}$ and $F_2(q, \dot{q}) \in \mathbb{R}^m$. Define

$$N(q, \dot{q}) := W(q, \dot{q})\dot{q} + F(q, \dot{q})$$

and suppose $N(q, \dot{q})$ has the structure

$$N(q, \dot{q}) = \begin{bmatrix} N_1(q, \dot{q}) \\ N_2(q, \dot{q}) \end{bmatrix}$$

where $N_1(q, \dot{q}) \in \mathbb{R}^{n \times m}$ and $N_2(q, \dot{q}) \in \mathbb{R}^m$.

Notice that the system in (1) is with symmetry, i.e. the inertial matrix only corresponds to the actuated variables $q_a$.

**Assumption 2.1:** It is assumed that $n - m \leq m < n$, which is a realistic assumption for most of Euler-Lagrange systems (e.g. in [3]–[8], [13]).

**Property 2.1:** As argued in [1], for most of Euler-Lagrange systems, $M(q_a)$ is a symmetric and uniformly positive definite matrix which satisfies

$$0 < c_1 I_n \leq M(q_a) \leq c_2 I_n$$

**Assumption 2.2:** In this paper, it is assumed that the equilibrium points are zero.

**Remark 2.1:** Without loss of generality, by applying a suitable coordinate transformation, any known fixed equilibrium points can be shifted to zero.

From equations (1)-(4) and the definition in (5) and (6), it follows

$$\ddot{q}_a = -M^{-1}_{22}(q_a)(M_{21}(q_a)\dot{q}_a + N_2(q, \dot{q}) - d_u)$$

Define

$$M_a(q_a) = M_{11}(q_a) - M_{12}(q_a)M_{22}^{-1}(q_a)M_{21}(q_a)$$

and substitute (8) into (1) yields

$$M_a(q_a)\ddot{q}_a - M_{12}(q_a)M_{22}^{-1}(q_a)(N_2(q, \dot{q}) - d_u) + N_1(q, \dot{q}) = u + d_a$$

Now define a collocated partial linearisation law as

$$u = M_a(q_a)v + N_1(q, \dot{q}) - M_{12}(q_a)M_{22}^{-1}(q_a)N_2(q, \dot{q})$$

where $v \in \mathbb{R}^{n-m}$ represents the virtual control input to be calculated.

**Remark 2.2:** From (7) it follows that $M_a^{-1}(q_a)$ always exists. Furthermore $M_{11}(q_a)$ and $M_{22}(q_a)$ are bounded and symmetric positive definite (s.p.d) since $M(q_a)$ is s.p.d.

Now substituting the control law in (11) into (1), the Euler-Lagrange system in (1) can be written in the form of

$$\ddot{q}_a = -M_{22}^{-1}(q_a)M_{21}(q_a)v - M_{22}^{-1}(q_a)N_2(q, \dot{q}) + \sigma(\cdot)$$

$$\dot{q}_a = v + M_a^{-1}(q_a)(d_a - M_{12}(q_a)M_{22}^{-1}(q_a)d_u)$$

where the perturbed term $\sigma(\cdot)$ is given by

$$\sigma(\cdot) = -M_{22}^{-1}(q_a)M_{21}(q_a)M_a^{-1}(q_a)(d_a - M_{12}(q_a)M_{22}^{-1}(q_a)d_u) + M_{22}^{-1}(q_a)d_u$$

**Remark 2.3:** It can be seen from (12) that the virtual control input $v$ affects both the actuated and unactuated variables, which increases the complexity of control design for underactuated systems. In the sequel, a global change of coordinate will be selected to decouple $v$ from the unactuated subsystem.

Suppose the system in (1) involves the kinetic energy $\mathcal{K}$ and the dissipative energy $\mathcal{D}$, the Lagrangian-is

$$\frac{d}{dt} \left( \frac{\partial \mathcal{K}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{K}}{\partial q} + \frac{\partial \mathcal{D}}{\partial q} = Bu + d$$

where $\mathcal{K} = \frac{1}{2}q^T M(q_a)q$. In (14), $\mathcal{D} = \frac{1}{2}q^T D(q_a)q$ where $D(q_a)$ corresponds to the dissipative coefficients, e.g. the coefficients of friction of the translational or rotational motion. Furthermore, it follows $F(q, \dot{q}) = D(q_a)\dot{q}$. The relationship between the Lagrange equation and the equation of motion in (1) is established as

$$\frac{d}{dt} \left( \frac{\partial \mathcal{K}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{K}}{\partial q} + \frac{\partial \mathcal{D}}{\partial q} = \begin{bmatrix} M_{11}(q_a) & M_{12}(q_a) \\ M_{21}(q_a) & M_{22}(q_a) \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \dot{q}_a \end{bmatrix} + \begin{bmatrix} W_1(q, \dot{q}) \\ W_2(q, \dot{q}) \end{bmatrix} \dot{q} + \begin{bmatrix} F_1(q, \dot{q}) \\ F_2(q, \dot{q}) \end{bmatrix}$$

(15)
and the last $m$ equations in (15) associated with $q_u$ are
\[
\frac{d}{dt} \left( \frac{\partial K}{\partial q_u} \right) - \frac{\partial K}{\partial q_u} + \frac{\partial D}{\partial \dot{q}_u} = M_21(q_u)\dot{q}_a + M_21(q_a)\dot{q}_a + W_2(q, \dot{q})\dot{q} + F_2(q, \dot{q}) = d_a
\] (16)
where
\[
\frac{\partial D}{\partial \dot{q}_u} = F_2(q, \dot{q})
\] (17)
\[
\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}_u} \right) = M_21(q_u)\dot{q}_a + M_22(q_u)\dot{q}_a + W_2(q, \dot{q})\dot{q}
\] (18)
Since $K = \frac{1}{2}q^T M(q_a)\dot{q}$ is dependent on $\dot{q}_a$ but does not depend on $q_u$ explicitly, it can be obtained that
\[
\frac{\partial K}{\partial \dot{q}_u} = M_21(q_u)\dot{q}_a + M_22(q_u)\dot{q}_a
\] (19)
\[
\frac{\partial K}{\partial q_u} = 0
\] (20)
The above properties in (17)-(20) will be exploited in the sequel.

From (19), the normalized momentum conjugated to $q_u$ can be written as
\[
\pi = M_{22}^{-1}(q_u)\frac{\partial K}{\partial \dot{q}_u} = \dot{q}_u + \underbrace{M_{22}^{-1}(q_u)M_21(q_u)}_{\mu(q_u)} \dot{q}_a
\] (21)
where $\mu(q_u) \in \mathbb{R}^{m \times (n-m)}$ is referred as the shape-inertial matrix. If there exists a generalized configuration function $\delta = \delta(q)$ such that $\dot{\delta}(q, \dot{q}) = \nabla \delta(q)\dot{q} = \pi$, $\pi$ is an integrable normalized momentum and $\delta(q)$ is referred as the integral of $\pi$. If there does not exist a function $\delta$ which satisfies $\dot{\delta} = \pi$, $\pi$ is non-integrable.

Assumption 2.3: It is assumed that $\mu(q_u) dq_u$ is not exact one form.

Remark 2.4: As argued in [19], if $\mu(q_u) dq_u$ is not exact one form, the normalized momentum $\pi$ in (21) is non-integrable, and the diffeomorphism (e.g. used in [29]–[31]), which is capable of transforming (12) into one in the strict feedback normal form, cannot be calculated systematically.

Suppose there exists $h \leq m$ and assume $q_a$ is given by
\[
q_a = \begin{bmatrix} q_{v1}^T \ q_{v2}^T \end{bmatrix}^T
\] (22)
where $q_{v1} \in \mathbb{R}^h$ and $q_{v2} \in \mathbb{R}^{n-m-h}$. In the situation when $q_{v2} = 0$, the shape inertial matrix $\mu(q_u)$ in (21) can be written as $\mu(q_{v1}, 0)$ and this quantity will be used in the following assumption.

Assumption 2.4: Suppose the pair $(i, j)$ satisfies
\[
\frac{\partial \mu_i(q_{v1}, 0)}{\partial q_{v1,j}} = \frac{\partial \mu_j(q_{v1}, 0)}{\partial q_{v1,i}} \quad \forall \ i, j = 1, \cdots, h
\] (23)
where $\mu_i(q_{v1}, 0)$ denotes the $i$th column of $\mu(q_{v1}, 0)$ in (21) and $q_{v1,i}$ denotes $i$th component of $q_{v1}$, and it is also assumed that indices $i$ and $j$ in (23) satisfy the following relationships:
\[
\begin{cases} 
  i \neq j & \text{if } m \geq h > 1 \\
  i = j = 1 & \text{if } h = 1
\end{cases}
\] (24)
Remark 2.5: Notice that the determination of $q_{v1}$ and $q_{v2}$ is not unique. To check the availability of (23), the components in $q_a$ can be reordered.

Remark 2.6: The value $h$ will be used in the following section to select $n - m - h$ actuated variables to be maintained on-predefined sliding manifolds.

III. CONTROL OF THE UNDERACTUATED SYSTEM

In this section, a global change of coordinate in closed form will be derived, via inducing sliding for a subset of actuated variables, to transform (12) into one in the strict feedback form. Then the remaining DOF will be further exploited, based on the strict feedback system, to ensure the asymptotic stability of the left actuated and unactuated variables.
A. Global diffeomorphism with non-integrable momentum

In (11), decompose the virtual control input as \( v = [v_1^T \quad v_2^T]^T \) where \( v_1 \in \mathbb{R}^m \) and \( v_2 \in \mathbb{R}^{n-m-h} \), then from (12), \( q_1 \) and \( \dot{q}_2 \) can be written as

\[
\dot{q}_1 = v_1 + \left[ I_h \quad 0_{h \times (n-m-h)} \right] M_{a}^{-1}(q_a)(d_a - M_{12}(q_a)M_{22}^{-1}(q_a)d_u)
\]

\[
\dot{q}_2 = v_2 + \left[ 0_{(n-m-h) \times h} \quad I_{n-m-h} \right] M_{a}^{-1}(q_a)(d_a - M_{12}(q_a)M_{22}^{-1}(q_a)d_u)
\]

Clearly, from (12), \( n - m \) variables \( q_a \) are actuated. If the virtual control input \( v_2 \in \mathbb{R}^{n-m-h} \) can be selected to ensure that \( n - m - h \) actuated variables \( q_2 \) approaches to zero, only the remaining \( h \) actuated variables \( q_1 \) and \( m \) unactuated variables \( q_u \) will appear in a reduced order system. In the following part of the section, \( v_2 \) will be calculated using a sliding mode based approach.

Define a sliding manifold \( s \in \mathbb{R}^{n-m-h} \) as

\[
s = \dot{q}_2 + \Lambda \ddot{q}_2
\]

where \( \Lambda \in \mathbb{R}^{(n-m-h)\times(n-m-h)} \) is a positive definite matrix and let \( M_{21}(q_a) \in \mathbb{R}^{m \times (n-m)} \) be partitioned as

\[
M_{21}(q_a) = [M_{211}(q_a) \quad M_{212}(q_a)]
\]

The following theorem shows that a non-integrable momentum can be approximated by an integrable momentum via maintaining \( n - m - h \) actuated variables at \( s \).

**Theorem 3.1.** The virtual control law \( v_2 \) is designed as

\[
v_2 = v_{2l} + v_{2n}
\]

where the linear part is

\[
v_{2l} = -\Lambda \ddot{q}_2
\]

and the nonlinear part is defined as

\[
v_{2n} = -K(t) \frac{s}{\|s\|} \quad \text{for} \quad s \neq 0
\]

In (31) the modulation function \( K(t) \) is selected as

\[
K(t) = \|M_{a}^{-1}(q_a)\|\|(1 + \|M_{12}(q_a)M_{22}^{-1}(q_a)\|)\| \xi + \eta
\]

where \( \eta \) is a positive scalar. By applying \( v_2 \) to (12), the non-integrable momentum \( \pi \) in (21) can be approximated by an integrable momentum \( \pi_s \) defined as

\[
\pi_s = \ddot{q}_2 + M_{22}^{-1}(q_{i+1})M_{211}(q_{i+1})q_{i+1}
\]

in finite time.

**Proof:** Since by assumption \( \|d\| \leq \xi \), both \( d_a \) and \( d_u \) satisfy

\[
\|d_a\| \leq \xi \quad \text{and} \quad \|d_u\| \leq \xi
\]

From (27) it follows

\[
s^T \ddot{s} = s^T(\dot{q}_2 + \Lambda \ddot{q}_2)
\]

Substituting (26) into (35) and using the definitions in (29)-(31) yields

\[
s^T \ddot{s} = s^T(v_{2n} + h_2(\cdot))
\]

\[
\leq -K(t)\|s\| + \|s\|\|(\|M_{a}^{-1}(q_a)\|\|(1 + \|M_{12}(q_a)M_{22}^{-1}(q_a)\|)\|)\| \xi
\]

If \( K(t) \) is chosen as in (32), then

\[
s^T \ddot{s} \leq -\eta \|s\|
\]
which is a standard reachability condition and sufficient to guarantee that \( s = 0 \) is maintained [20], [21]. Integrating (37) implies that the time taken to reach \( s = 0 \)-denoted by \( t_s \)-satisfies

\[
t_s \leq (0.5\eta)^{-1}\sqrt{s^2(0)s(0)}
\]

where \( s(0) \) represents the initial value of \( s \) at \( t = 0 \) [21]. During sliding, \( \dot{q}_{v2} \rightarrow 0 \), \( q_{v2} \rightarrow 0 \), (21) approaches to (33) in finite time. In the situation when \( s = 0 \), it follows that \( \dot{s} = 0 \). Hence, \( s \) reaches \( s = 0 \) and cannot escape it, which implies the finite time stability of the origin. From (23), \( \mu_s(q_{v1}, 0) dq_{v1} \) is exact one form and the normalized momentum \( \pi_s \) in (21) is integrable [19]. This completes the proof.

Since \( \pi_s \) is integrable and the following theorem is proposed to find a global change of coordinate to transform the reduced order sliding dynamic into one in a strict feedback form.

**Theorem 3.2**: Define a global change of coordinates as

\[
\begin{align*}
q_r &= q_a + \gamma(q_{v1}) \\
p_r &= M_{22}(q_{v1})(\dot{q}_a + M_{22}^{-1}(q_{v1})M_{211}(q_{v1})\dot{q}_{v1})
\end{align*}
\]

where

\[
\gamma(q_{v1}) = \int_0^{q_{v1}} M_{22}^{-1}(\tau)M_{211}(\tau)d\tau
\]

and apply it to (12), then (12) can be transformed to one in a strict feedback normal form as

\[
\begin{align*}
\dot{q}_r &= M_{22}^{-1}(q_s) p_r \\
\dot{p}_r &= g_r(q_s, q_r, p_s, \dot{q}_r) + d_u \\
\dot{q}_s &= p_s \\
\dot{p}_s &= v_1 + h_1(q_s)
\end{align*}
\]

where \( h_1(\cdot) \) is defined in (25).

**Proof**: Since \( \pi_s \) is integrable, calculating the derivative of \( q_r \) from (39) yields

\[
\dot{q}_r = \dot{q}_a + M_{22}^{-1}(q_{v1})M_{211}(q_{v1})\dot{q}_{v1}
\]

Comparing (39) and (42) yields

\[
p_r = M_{22}(q_{v1})\dot{q}_r
\]

Since during sliding \( q_{v2} \rightarrow 0 \), it follows from (42)

\[
p_r = M_{22}(q_{v1})\dot{q}_r = M_{22}(q_{v1})\dot{q}_a + M_{211}(q_{v1})\dot{q}_{v1}
\]

Using (19) and (22), (44) can be written as

\[
p_r = M_{22}(q_a)\dot{q}_a + M_{21}(q_a)\dot{q}_a = \frac{\partial K}{\partial q_a}
\]

From (16) and (2) it follows

\[
\frac{d}{dt}\left(\frac{\partial K}{\partial q_a}\right) - \frac{\partial K}{\partial q_a} + \frac{\partial D}{\partial q_a} = d_u
\]

Then the first-order derivative of (45) is given by

\[
\dot{p}_r = \frac{d}{dt}\left(\frac{\partial K}{\partial q_a}\right) = \frac{\partial K}{\partial q_a} - \frac{\partial D}{\partial q_a} + d_u
\]

Then it follows from (20) that

\[
\dot{p}_r = -\frac{\partial D}{\partial q_a} + d_u = g_r(q_a, q_u, \dot{q}_a, \dot{q}_s) + d_u
\]

From (39), \( q_a = q_r - \gamma(q_{v1}) \), \( g_r(\cdot) \) in (48) can be written as \( g_r(q_{v1}, q_r, \dot{q}_{v1}, \dot{q}_r) \) since \( q_{v2} \rightarrow 0 \) during sliding. After defining \( q_s := q_{v1} \) and \( p_s := \dot{q}_s \), the proof ends.

**Remark 3.1**: Since the structure property in (23) is exploited, a global change of coordinate is derived without using momentum decomposition [19] which requires a complicated coordinate transformation to be found, particularly in high order systems. Furthermore, the momentum decomposition approach may generate an extra perturbed term representing the derivative of the error momentum [19].
B. Stabilization of the unactuated variables

As in [29], a sliding mode based control law will be used to calculate \( v_1 \) in (41) such that the remaining actuated variables \( q_v \) and unactuated variables \( q_u \) will asymptotically converge to the origin.

Define the error variables as

\[
\begin{align*}
e_1 &= q_r \\
e_2 &= p_r \\
e_3 &= g_r \\
E &= [e_1 \quad e_2]^T
\end{align*}
\] (49)

where \( q_r, p_r \) and \( g_r \) are defined in (41). Here, the following assumptions are made:

Assumption 3.1: The terms \( \partial g_r / \partial p_s \) is left invertible, i.e. \( \partial g_r / \partial p_s \) has full column rank. As argued in [29], if \( \partial g_r / \partial p_s \) is invertible, then \( \|d_u\| < \xi\|E\| \).

Assumption 3.2: The terms \( \|\partial g_r / \partial p_s\| \leq \beta_1, \|\partial g_r / \partial p_r\| \leq \beta_2 \) where \( \beta_1 \) and \( \beta_2 \) are positive constants.

Assumption 3.3: \( g_r(q_v, 0, p_s, 0) = 0 \) is an asymptotically stable manifold, i.e., \( q_v \) and \( p_s \) will converge to 0 if \( g_r(q_v, 0, p_s, 0) = 0 \).

Remark 3.2: From Assumption 2.3, \( m \geq h \). Then it is possible that \( \partial g_r / \partial p_s \) is left invertible and it follows

\[
\lambda^{-1} \geq \max_{\partial g_r / \partial p_s} \leq \beta_1, \max_{\partial g_r / \partial p_r} \leq \beta_2
\] (50)

Remark 3.3: In the case Assumption 3.3 is not satisfactory, it is possible to rewrite \( d_u \) as \( d_u = d_{u_1}(-) + d_{u_2}(q_v, 0, p_s, 0) \) where \( d_{u_2} \) represents the design freedom that allows \( g_r(q_v, 0, p_s, 0) = 0 \) to be an asymptotically stable manifold.

Define a sliding manifold as

\[
\Phi = \Psi_1 e_1 + \Psi_2 e_2 + e_3
\] (51)

where \( \Psi_1 \in \mathbb{R}^{m \times m} \) and \( \Psi_2 \in \mathbb{R}^{m \times m} \) are positive definite matrices which guarantee the following matrix \( A_n(q_v) \in \mathbb{R}^{2m \times 2m} \) is Hurwitz.

\[
A_n(q_v) := \begin{bmatrix}
0 & M_{22}^{-1}(q_v) \\
-\Psi_1 & -\Psi_2
\end{bmatrix}
\] (52)

Lemma 3.1: According to Property 2.1, it is assumed that

\[
\beta_1 I_m \leq M_{22}^{-1}(q_v) \leq \beta_2 I_m
\] (53)

If \( \Psi_1 \) and \( \Psi_2 \) satisfied the following inequalities

\[
2\lambda_{\min}(\Psi_1) + \frac{\beta_1}{\beta_2} \lambda_{\min}(\Psi_2) - \lambda_{\max}(\Psi_2) - (\lambda_{\min}(\Psi_1) + \beta_2) \frac{\lambda_{\max}(\Psi_1)}{\lambda_{\min}(\Psi_2)} > 0
\] (54)

the matrix \( A_n(q_v) \) in (52) is guaranteed to be Hurwitz.

Proof: Define an Lyapunov matrix as

\[
\mathcal{P} = \begin{bmatrix}
\alpha_1 I_m & I_m \\
I_m & \alpha_2 I
\end{bmatrix}
\] (55)

where the positive scalars \( \alpha_1 \) and \( \alpha_2 \) are defined as

\[
\alpha_1 = \frac{\lambda_{\min}(\Psi_2)}{\beta_2} \quad \text{and} \quad \alpha_2 = \frac{\beta_2}{\lambda_{\min}(\Psi_2)} \left( \frac{\lambda_{\min}(\Psi_1)}{\beta_2} + 1 \right)
\] (56)

and it is easy to verify from (56) that \( \alpha_1 \alpha_2 > 1 \) and therefore \( \mathcal{P} \) is proper.

Now define

\[
Q(q_v) = -(\mathcal{P} A_n(q_v) + A_n^T(q_v) \mathcal{P})
\] (57)

and suppose \( Q(q_v) > 0 \) has the following structure

\[
Q(q_v) = \begin{bmatrix}
Q_{11} & Q_{12}(q_v) \\
Q_{12}^T(q_v) & Q_{22}(q_v)
\end{bmatrix} = \begin{bmatrix}
\Psi_1 + \Psi_1^T - \alpha_1 M_{22}^{-1}(q_v) - \Psi_2 + \alpha_2 \Psi_1 & -\alpha_1 M_{22}^{-1}(q_v) + \Psi_2 + \alpha_2 \Psi_1 \\
-2 \alpha_2 \Psi_1 + \alpha_2 \Psi_2 & -2 M_{22}^{-1}(q_v) + \alpha_2 \Psi_2
\end{bmatrix}
\] (58)
Let $q_{11}$ and $q_{12}$ represent the lower bound of $Q_{11}$ and $Q_{22}(q_{12})$, respectively, then

$$q_{11} = 2\lambda_{\text{min}}(\Psi_1)$$
$$q_{12} = -2\beta_2 + 2\alpha_2\lambda_{\text{min}}(\Psi_2)$$

(59)

Substituting the definition of $\alpha_2$ in (56) into (59), it can be obtained that

$$q_{11} = q_{12} = 2\lambda_{\text{min}}(\Psi_1)$$

(60)

Let $q_{12}$ represent the upper bound of $Q_{12}(q_{1})$ defined in (58)

$$q_{12} = -\alpha_1\beta_1 + \lambda_{\text{max}}(\Psi_2) + \alpha_2\lambda_{\text{max}}(\Psi_1)$$

(61)

Using (59) and (61) it follows

$$q_{11} - q_{12} = 2\lambda_{\text{min}}(\Psi_1) + \alpha_1\beta_1 - \lambda_{\text{max}}(\Psi_2) - \alpha_2\lambda_{\text{max}}(\Psi_1)$$

(62)

Substituting (7) into (62) yields

$$q_{11} - q_{12} = 2\lambda_{\text{min}}(\Psi_1) + \frac{\beta_1^2}{\beta_2}\lambda_{\text{min}}(\Psi_2) - \lambda_{\text{max}}(\Psi_2) - (\lambda_{\text{min}}(\Psi_1) + \beta_1)\frac{\lambda_{\text{max}}(\Psi_1)}{\lambda_{\text{min}}(\Psi_2)}$$

(63)

If (54) is satisfied, $q_{11} - q_{12} > 0$. From (60) it follows $q_{12} - q_{11} > 0$ and therefore it is easy to verify that $Q(q_{12}) > 0$. Since $P$ is proper, from (57) $A_n(q_{12})$ is Hurwitz. This ends the proof.

**Theorem 3.3:** Let the virtual control law $v_1$ be

$$v_1 = v_{1l} + v_{1n}$$

(64)

where the components $v_{1l}$ and $v_{1n}$ are defined as

$$v_{1l} = -(\frac{\partial g_r}{\partial p_s})(\Psi_1 + \frac{\partial g_r}{\partial q_s})M_{22}^{-1}(q_s)p_r + \Psi_2g_r + \frac{\partial g_r}{\partial q_s}p_s + \frac{\partial g_r}{\partial p_r}g_r$$
$$v_{1n} = -(\frac{\partial g_r}{\partial p_s})(K(t))\frac{\Phi}{\|\Phi\|} + g\Phi) \text{ for } \Phi \neq 0$$

(65)

where $\frac{\partial g_r}{\partial p_s}$ is defined in (50), $\varphi$ is a positive scalar to be selected and the modulation function is chosen as

$$K(t) = \|\Psi_2\|\xi + \beta_2\xi + \beta_1K(t) + \eta_1$$

(66)

where $K(t)$ is defined in (32). By choosing suitable design parameters $\Psi_1$ and $\Psi_2$, all states in (41) asymptotically converge to the origin.

**Proof:** Define a candidate Lyapunov function as $V = \frac{1}{2}\Phi^T \Phi$, the derivative of $V$ is

$$\dot{V} = \Phi^T \Phi = \Phi^T (\Psi_1 \dot{e}_1 + \Psi_2 \dot{e}_2 + \dot{e}_3)$$
$$= \Phi^T (\Psi_1M_{22}^{-1}(q_s)p_r + \Psi_2g_r + \Psi_2d_u + \frac{\partial g_r}{\partial p_s}p_s + \frac{\partial g_r}{\partial q_s}p_s + \frac{\partial g_r}{\partial p_r}g_r)$$
$$+ \frac{\partial g_r}{\partial p_r}g_r + \frac{\partial g_r}{\partial p_r}d_u + \frac{\partial g_r}{\partial p_s}h_1)$$

(67)

Substituting (65) into (67) yields

$$\dot{V} = \Phi^T (\Psi_2d_u + \frac{\partial g_r}{\partial p_r}d_u + \frac{\partial g_r}{\partial p_s}h_1) - \Omega\|\Phi\|^2 - K\|\Phi\|$$
$$\leq \|\Phi\|(|\Psi_2||d_u| + \|\frac{\partial g_r}{\partial p_r}||d_u| + \|\frac{\partial g_r}{\partial p_s}||h_1|) - \varphi\|\Phi\|^2 - K\|\Phi\|$$

(68)

Using the fact that $\|d_u\| \leq \xi$ and

$$\|h_1\| \leq \|M_{22}^{-1}(q_s)\|(1 + \|M_{12}(q_s)M_{22}^{-1}(q_s)\|)\|d_u\|
\leq \|M_{22}^{-1}(q_s)\|(1 + \|M_{12}(q_s)M_{22}^{-1}(q_s)\|)\|\xi < K$$

(69)

Inequality (68) satisfies

$$\dot{V} < \|\Phi\|(|\Psi_2\|\xi + \|\frac{\partial g_r}{\partial p_r}\|\xi + \|\frac{\partial g_r}{\partial p_s}||K\|) - \varphi\|\Phi\|^2 - K\|\Phi\|$$

(70)
From Assumption 3.2,
\[ \dot{V} < \|\Phi\| \left( \|\Psi_2\| \xi + \beta_2 \xi + \beta_1 K \right) - \varrho \|\Phi\|^2 - K \|\Phi\| \] (71)

Substituting (66) into (71), it follows
\[ \dot{V} < -\varrho \|\Phi\|^2 - \eta_1 \|\Phi\| \] (72)

Clearly from (72), \( \Phi \to 0 \) and sliding occurs in finite time. During sliding
\[ e_3 = -\Psi_1 e_1 - \Psi_2 e_2 \] (73)

and it follows from (41), (49) and (73) that
\[ \dot{e}_2 = e_3 + d_u = -\Psi_1 e_1 - \Psi_2 e_2 + d_u \] (74)

According to the definition of \( A_n \) in (52)
\[ \dot{E} = A_n E + D \] (75)

where
\[ D = \begin{bmatrix} 0 \\ d_u \end{bmatrix} \in \mathbb{R}^{2m \times 1} \] (76)

Next define s.p.d matrices \( P \in \mathbb{R}^{2m \times 2m} \) and \( Q(q_{v1}) \in \mathbb{R}^{2m \times 2m} \) which satisfy
\[ A_n^T(q_{v1}) P + P A_n(q_{v1}) = -Q(q_{v1}) \] (77)

and
\[ \xi < \frac{\lambda_{\text{min}}(Q(q_{v1}))}{2\lambda_{\text{max}}(P)} \] (78)

Now consider another candidate Lyapunov equation according to \( V_1 = E^T P E \), the derivative of \( V_1 \) is
\[ \dot{V}_1 = \dot{E}^T P E + E^T P \dot{E} \] (79)

Substituting (75) into (79) yields
\[ \dot{V}_1 = (E^T A_n^T + D^T) P E + E^T P (A_n E + D) \]
\[ = E^T (A_n^T P + P A_n) E + 2E^T P D \]
\[ \leq -\lambda_{\text{min}}(P) \|E\|^2 + 2\lambda_{\text{max}}(P) \|E\|^2 \] (80)

Since \( \xi < \lambda_{\text{min}}(Q)/2\lambda_{\text{max}}(P) \), \( \dot{V}_1 < 0 \) and the origin is asymptotically stable i.e. \( e_1 \to 0 \) and \( e_2 \to 0 \). Furthermore, during sliding \( \Phi = 0 \), \( e_3 = g_r = -\Psi_1 e_1 - \Psi_2 e_2 = 0 \). From Assumption 3.3, \( q_s \) and \( p_s \) converge to zero. This completes the proof.

Remark 3.4: If Assumption 3.3 is not satisfactory, after a certain period, \( \|E\| \leq 2\|P\| \xi/\lambda_{\text{min}}(Q) \) and \( e_1, e_2 \) and \( e_3 \) will converge to a small ball containing the origin.

Remark 3.5: Due to (39) is global diffeomorphism, the stabilization of (41) only guarantees the stability of the reduced order sliding motion in which \( \dot{s} = s = 0 \).

Remark 3.6: In this paper, the stochastic processes are not taken into account. In the situation when there exists various processes with stochastic abrupt structural changes such as the component failures or the contact forces in unknown environment, the system can be modelled as Markov jump systems. Furthermore, due to the non-synchronization phenomenon between the mode of the system and mode of the controller, the novel asynchronous mode dependent sliding mode surface can be used to ensure the finite time stability of the asynchronous stochastic hybrid model [36], [37].
IV. A six-link robot case study

A. Modelling

In this paper a planar multi-link robot is used to evaluate the efficacy of the scheme. The structure of the robot is shown in Fig. 1. Here it is assumed that the robot has six links, i.e. \( n = 6 \) and is influenced and driven by the ground frictions due to the velocities of the links.

In Fig. 1, \( \theta_i \) denotes the angle between the \( i \)th link and the global \( x \)-axis. The length of the link is \( 2l \) and the variable \( \phi_i = \theta_i+1 - \theta_i, \forall i = 1, \ldots, 5 \) represents the relative angle of \( i \)th joint. The torque acting on the \( i \)th joint is denoted by \( u_i, \forall i = 1, \ldots, 5 \). The middle point of the \( i \)th link is denoted by \( (x_i, y_i) \). It is assumed that the weight and moment of inertial of each link are denoted by \( \tilde{m} \), \( \tilde{m} \) and \( \tilde{J} = 1/3\tilde{m}l^2 \), respectively. The constants \( c_t \) and \( c_n \) represent the tangential viscous friction coefficient and the normal viscous friction coefficient, respectively. This model contains six configuration variables \( q = \begin{bmatrix} q_1^T \ q_6^T \end{bmatrix}^T \) and

\[
q_a = [\phi_1, \ldots, \phi_5]^T \quad \text{and} \quad q_u = \theta_6
\]

where \( q_a \) denotes five actuated relative joint angles and \( \theta_6 \) captures one unactuated heading angle.

For the \( i \)th link, the force balance equation is given by

\[
\tilde{m} \ddot{x}_i = f_{x,i} + \varepsilon_{x,i} - \varepsilon_{x,i-1}
\]

\[
\tilde{m} \ddot{y}_i = f_{y,i} + \varepsilon_{y,i} - \varepsilon_{y,i-1}
\]

where \( f_{x,i} \) and \( f_{y,i} \) (i.e. the \( i \)th component of \( f_x \) and \( f_y \)) represent friction acting on the \( i \)th link along \( x \)-axis and \( y \)-axis respectively. The variables \( \varepsilon_{x,i} \) and \( \varepsilon_{y,i} \) represent joint constraint forces on link \( i \) from link \( i+1 \) along \( x \)-axis and \( y \)-axis respectively. The variables \( \varepsilon_{x,i-1} \) and \( \varepsilon_{y,i-1} \) represent joint constraint forces on link \( i \) from link \( i-1 \) along \( x \)-axis and \( y \)-axis respectively.

The torque balance equation for the \( i \)th link can be written as

\[
J\dot{\theta}_i = u_i - u_{i-1} - l \sin \theta_i (\varepsilon_{x,i} + \varepsilon_{x,i-1}) + l \cos \theta_i (\varepsilon_{y,i} + \varepsilon_{y,i-1})
\]

Suppose the anisotropic viscous friction forces acting on all links are given by

\[
\begin{bmatrix} f_x \\ f_y \end{bmatrix} = - \begin{bmatrix} c_t C_{\theta}^2 + c_n S_{\theta}^2 & (c_t - c_n) S_{\theta} C_{\theta} \\ (c_t - c_n) S_{\theta} C_{\theta} & c_t S_{\theta}^2 + c_n C_{\theta}^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}
\]

where \( S_{\theta} \in \mathbb{R}^{6 \times 6} \) and \( C_{\theta} \in \mathbb{R}^{6 \times 6} \) are defined as

\[
S_{\theta} = \text{diag}(\sin(\theta_1), \ldots, \sin(\theta_6))
\]

\[
C_{\theta} = \text{diag}(\cos(\theta_1), \ldots, \cos(\theta_6))
\]
Define

\[ X = L^T (HH^T)^{-1} H \]  

where matrices \( L \in \mathbb{R}^{5 \times 6} \) and \( H \in \mathbb{R}^{5 \times 6} \) are in the forms of

\[
L = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 & 1 \\
0 & 1 & \cdots & 1 & 1 & 1 \\
0 & 0 & 1 & \cdots & 1 & 1 \\
\vdots & & & & & \\
0 & 0 & 0 & \cdots & 1 
\end{bmatrix}
\]

and

\[
H = \begin{bmatrix}
1 & -1 & \cdots & -1 & -1 & -1 \\
-1 & 1 & \cdots & -1 & -1 & -1 \\
\vdots & & & & & \\
-1 & -1 & \cdots & 1 & 1 & 1 \\
-1 & -1 & \cdots & 1 & 1 & 1 \\
\vdots & & & & & \\
-1 & -1 & \cdots & 1 & 1 & 1 \\
\end{bmatrix}
\]

Combining (82)-(85) for all six links yields

\[
\ddot{M}(\theta)\dot{\theta} + \ddot{W}(\theta)\dot{\theta}^2 + \dot{F}(\theta, \dot{\theta}) = H^T u
\]

where

\[
\ddot{M}(\theta) = J_1 n + \tilde{m}l^2(S_0 V S_0 + C_0 V C_0) \\
\ddot{W}(\theta) = \tilde{m}l^2(S_0 V C_0 - C_0 V S_0) \\
\dot{F}(\theta, \dot{\theta}) = -I S_0 X f_x + I C_0 X f_y
\]

In (90) \( f_x \) and \( f_y \) are defined in (85) and

\[
V = L^T (HH^T)^{-1} L
\]

To formulate (89) into one with symmetry, a coordinate transformation is defined as

\[
\theta = Rq
\]

where \( R \in \mathbb{R}^{6 \times 6} \) is defined as

\[
R = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
\vdots & & & & & \\
0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix}
\]

and in the new coordinate, (89) can be written as

\[
M(q_a)\ddot{q} + W(q)q^2 + F(q, \dot{q}) = Bu
\]

where \( B \) is defined in (2) and

\[
M(q_a) = R^T \ddot{M}(Rq)R \\
W(q)q^2 = R^T \ddot{W}(Rq)\text{diag}(R\dot{q})R\dot{q} \\
F(q, \dot{q}) = R^T \dot{F}(Rq, R\dot{q})
\]

Notice that (94) has a similar structure as in (1) and therefore the proposed scheme is applicable to (94).

B. Simulation results

Here the design and simulation results are presented. The length of a link is chosen to be \( l = 0.1 \text{m} \). The mass of each link is assumed to be \( \tilde{m} = 1 \text{kg} \) and the corresponding moment of inertia is \( J = 0.0016 \text{kg} \cdot \text{m}^2 \). The friction coefficients are selected as \( c_t = 0.5 \) and \( c_n = 10 \). The modulation function defined in (31) is chosen as \( K = 1.5 \). The modulation function \( K \) and the scalar \( \bar{\theta} \) defined in (65) is selected as \( K = 20 \) and \( \bar{\theta} = 3 \), respectively. In (27), the design freedom \( \Lambda \) is selected as \( \Lambda = 5I_4 \). During the simulation, the sampling time is chosen to be 0.01 and the solver is chosen as ode1 for the purpose of simplifying the future implementation.

It is easy to be verified from Assumption 2.3 that this example is non-integrable due to the shape inertial matrix is not in the exact one form. Since there exists one unactuated variable in this example, \( h \) can be selected as \( h = 1 \). Then from (24) it follows \( i = j = 1 \). Clearly, the left side of and right side of (23) are equal in the situation when \( i = j = 1 \), which verifies Assumption 2.4. Since \( h = 1 \), the virtual control law \( v_2 \) is calculated to maintain four
actuated variables (i.e. \( q_{v2} = [\phi_1, \cdots, \phi_4]^T \)) on the sliding manifolds defined in (27), such that \( q_{v2} \rightarrow 0 \) in finite time. Since during sliding on \( s \), \( q_{v2} = 0 \), and it can be obtained from (94) that

\[
M_{22}(q_{v1}) = \frac{1}{12} \cos(q_{v1}) + \frac{191}{300} \quad \text{and} \quad M_{211}(q_{v1}) = \frac{\cos(\phi_5) + 15}{24} \tag{96}
\]

and therefore the function \( \gamma(q_{v1}) \) in (39) is

\[
\gamma(q_{v1}) = \int_0^{\phi_5} \frac{25 \cos(\tau) + 375}{50 \cos(\tau) + 382} d\tau \tag{97}
\]

From (96) and (97), the global change of coordinate in (39) is calculated to transform the reduced order sliding motion to one in a strict feedback norm form where only \( q_{v1} = \phi_5 \) and \( q_{u} = \theta_6 \) are contained.

From the structure of \( M_{22}(q_{v1}) \) defined in (96), the upper and lower bounds of \( M_{22}^{-1}(q_{v1}) \) is known and the scalars \( \beta_1 \) and \( \beta_2 \) in Lemma 3.1 can thus be selected as \( \beta_1 = 1.38 \) and \( \beta_2 = 1.58 \), respectively. Furthermore, the design parameters \( \Psi_1 \) and \( \Psi_2 \) defined in (51) are chosen to be \( \Psi_1 = 3 \) and \( \Psi_2 = 30 \), respectively, and Lemma 3.1 can then be verified. In this paper the external disturbances are friction associated with the rotational motion of the link, i.e. \( d = -c_n J q^2 \).

The initial values of \( q \) is chosen to be

\[
q(0) = [0.35 \quad -0.1 \quad 0.2 \quad -0.7 \quad -0.2 \quad -0.6] \tag{98}
\]

The discontinues terms in (31) and (65) are approximated to any level of accuracy using

\[
v_{2n} = -K(t) \frac{s}{\|s\| + 0.01} \tag{99}
\]

and

\[
v_{1n} = -\left( \frac{\partial g_r}{\partial p_s} \right)^I(K(t) - 1) \frac{\Phi}{\|\Phi\| + 0.01} + \rho \Phi \tag{100}
\]

Under anisotropic friction conditions, the friction forces \( f_x \) and \( f_y \), acting in the tangential and normal direction of the links, are shown in Fig. 2 and Fig. 3, respectively. Clearly the friction forces are vanishing due to the regulation process of the joint angles.

The relative joint angles \( q_{v2} \), those are \( \phi_1, \phi_2, \phi_3 \) and \( \phi_4 \), are shown in Fig. 4. Clearly they approach to zero in finite time despite non-zero initial relative joint angles and external disturbances. The sliding manifolds \( s \) in (27), associated with \( q_{v2} \), are shown in Fig. 5. It is clear from Fig. 5 that the sliding can be induced and maintained afterwards. In Fig. 6, the blue curve represents the heading angle \( \theta_6 \) and the red curve presents the remaining one actuated variables \( \phi_5 \). It can be seen from Fig. 6 that both \( \phi_5 \) and \( \theta_6 \) converge to the origin asymptotically despite their non-zero initial values and external disturbances. The sliding manifold \( \Phi \) in (51) is shown in Fig. 7. The signals \( v_{1l} \) and \( v_{1n} \), corresponding to continuous part and discontinuous part of \( v_1 \), are shown in Fig. 8 and Fig. 9, respectively. The function \( g_r(\cdot) \) in the reduced order system (41) is shown in Fig. 10. Since \( g_r(\cdot) \) is a function of \( \phi_5, \theta_6 \) and their first order derivatives-which approach to zero as shown in Fig. 6, \( g_r \) is also vanishing.

Finally, the control inputs \( u_i \) for all \( i = 1, \cdots, 5 \) are shown in Fig. 11. Clearly chattering does not appears in control signals and the required torques or control effect are realistic for the implementation purpose.

Notice that it can be seen from the nonlinear dynamic equation (89) that there always exists a trade-off between the number of the links (i.e. the adaptation level of the system) and the computational load. During the practical implementation, if the number of the links is large and the associated computational load is not acceptable, we could use the Tayler expansion method to simplify the full nonlinear dynamics equations.

V. CONCLUSION

In this paper, a sliding mode control scheme was developed to stabilise a class of nonlinear perturbed underactuated system with a non-integral momentum. In this scheme, a subset of the actuated variables were initially selected to be maintained on sliding surfaces. During sliding, the system with a non-integrable momentum was approximated by one with an integrable momentum, and a global change of coordinate was found to transform reduced order sliding dynamic into one in the strict feedback normal form. This scheme also contained a sliding mode control law which is derived from the strict feedback form and allows the remaining actuated and unactuated variables to converge to the origin. The design efficacy was verified via a six-link planar robot case study. The future works
Fig. 2. The friction acting in the tangential direction $f_x$

Fig. 3. The friction acting in the normal direction $f_y$

Fig. 4. The relative joint angles $q_{v2}$
Fig. 5. The sliding manifolds $s$

Fig. 6. $q_{v1}$ and $\theta_6$

Fig. 7. The sliding manifold $\Phi$
Fig. 8. The virtual control $v_{1l}$

Fig. 9. The virtual control $v_{1n}$

Fig. 10. The function $g_r$
Fig. 11. The torque inputs $u$ include: a) a consideration of the situation in which the underactuated systems are not symmetry and the reduced order system is in a nontriangular form; b) the application of the design scheme to the practical multi-link robotic platform, e.g. the autonomous snake robot; c) taking into account the stochastic actuator failures and developing the sliding mode control within an asynchronous Markov jump formulation.

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