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Solidarity induced by Group Contributions: The MI^k -value for Transferable Utility Games

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Abstract The most popular values in cooperative games with transferable utilities are perhaps the Shapley and the Shapley like values which are based on the notion of players' marginal productivity. The equal division rule on the other hand, is based on egalitarianism where resource is equally divided among players, no matter how productive they are. However none of these values explicitly discuss players' multilateral interactions with peers in deciding to form coalitions and generate worths. In this paper we study the effect of multilateral interactions of a player that accounts for her contributions with her peers not only at an individual level but also on a group level. Based on this idea, we propose a value called the MI^k -value and characterize it by the axioms of linearity, anonymity, efficiency and a new axiom: the axiom of MN^k -player. An MN^k -player is one whose average marginal contribution due to her multilateral interactions upto level k is zero and can be seen as a generalization of the standard null player axiom of the Shapley value. We have shown that the MI^k -value on a variable player set is asymptotically close to the equal division rule. Thus our value realizes solidarity among players by incorporating both their individual and group contributions.

Keywords: Group contributions, Multilateral Interactions, MN^k -player, MI^k -value.

Mathematics Subject Classification (2010) : 91A12

JEL Classification: C71, D63, D81

1 Introduction

"I am, somehow, less interested in the weight and convolutions of Einstein's brain than in the near certainty that people of equal talent have lived and died in cotton fields and sweatshops."

— Stephen Jay Gould

The two values which are focal in the literature on values of cooperative games with transferable utilities or simply TU games are the Shapley value and the Equal Division rule. While there is a deeper sense of meritocracy embedded in the definition of the Shapley value, the Equal Division has a flavour of socialism. We propose a new rule based on the progressive taxation which takes the middle ground and can be compared to Social Liberalism. Apart from the individual marginal contributions, our rule considers the marginal contributions of groups of varying sizes.

The evidence of preference for equality, independent of (marginal) productivity or contribution, has been found in groups where there is a sense of group identity. It has been observed in sociological and psychological experiments viz., (Deutsch, 1985; Mikula, 1980) that individuals care about not just equity but also equality when there is an association of group identity, e.g., soldiers in the same unit.

The division of resources within a family or social insurance among the caste network (Munshi and Rosenzweig, 2016) reinforces the evidence of equality at local level. Indeed, the family laws, e.g., the Hindu Undivided Family inheritance law is based on this equality principle. All the siblings in an undivided family, which can include up to several generations, have equal rights on the property of the family. The inheritance rights do not depend on the marginal contribution of the family members, see for example (Borkotokey et al., 2020). Other examples include the allocation of Health and Social care. When the health and social care is decided in any reasonable society, the marginal contribution is never the consideration.

The Shapley value does not just have axiomatic justifications, see for example (Shapley, 1953; Hart and Mas-Colell, 1989; Young, 1985) etc., but it also has the non-cooperative foundations, see for example (Gul, 1989; McQuillin and Sugden, 2016) and others. Moreover, the equivalence between the Shapley value and competitive market equilibrium has also been established, see for example (Shapley, 1964; Aumann, 1975; Aumann and Shapley, 1974).

The neoclassical market-based solutions and the solutions based on non-cooperative bargaining can lead to highly unequal outcomes. For example, the wage differential of CEOs and the average workers that we find today is under debate in recent times when the economic inequality is at its highest of all times. This wage differential is not limited to the standard corporate sector but has been rising in the academic institutions

as well.¹ Similarly, when the market allocations lead to highly unequal dividends across different professions (farmer suicide rates across countries versus investment bankers' salaries, the 99 percent campaign etc.) one is bound to think of ways of equalizing the dividends.

The support for raising the income in the lower strata has been increasing in these days². The primary challenge facing the egalitarian solution is to avoid free riding problems and keep the incentives of the agents to put efforts and contribute to their capabilities. Therefore, one needs to consider a solution which has a flavour of egalitarianism and at the same time incorporates the marginal contributions vector.

Motivated by the above discussion, we introduce a new value for TU Cooperative games: the MI^k -value, that is based on the idea of affirmative action and progressive taxation and still rewards the individual productivity. Several values are introduced in the literature through which solidarity among players is achieved. Examples include the Solidarity value (Nowak and Radzik, 1994), The Egalitarian Shapley value (Joosten, 1996), the Generalized Egalitarian Shapley value (Choudhury et al., 2020), the Solidarity value due to (Béal et al., 2017), the discounted Shapley value (Joosten, 1996) and their respective characterizations, (Casajus and Huettner, 2013; Béal et al., 2017; Kamiyo and Kongo, 2012; Radzik, 2013; Casajus and Huettner, 2014; van den Brink, 2007; van den Brink et al., 2013; van den Brink and Funaki, 2015; Weber, 1988; Yokote and Funaki, 2017; Young, 1985) etc., just to name a few. We assume here that players form groups within coalitions to indulge in interactions which we call multilateral interactions.³ Players in many instances influence and are influenced by their peers in deciding to join or leave a coalition and behave compassionately to each other. Thus sharing of the wealth they generate should also reflect their interdependence and their compassion within a group. We introduce a new way of sharing the group contributions among the players and add them to their final payoffs with their individual contributions prescribed by the Shapley value. The notion of group contributions by players however indirectly appears in the computation of all the Shapley like values. Our proposal makes an explicit and direct method to add these finer contributions to the players' payoffs. This we discuss in more details in Section 3.

The null player axiom of the Shapley value (Shapley, 1953) blocks all sorts of egalitarianism in the distribution of the wealth. Thus, combining the axioms of efficiency, symmetry and linearity⁴ with a property that replaces the null player axiom has been accepted as essential for ensuring solidarity, take for example the Average null-player property for the Solidarity value, (Nowak and Radzik, 1994), the Null player in a productive environment for the Egalitarian Shapley value, (Casajus and Huettner, 2013), the null player in a null environment for positive game for the Solidarity value (Béal et al., 2017), the null player in a non-negative environment for the Generalized Egalitarian Shapley value (Choudhury et al., 2020), the Invariance from proportional and quasi proportional deletion for the Equal Division and Solidarity value (Kamiyo and Kongo, 2012), the δ -reducing property for the discounted Shapley value (van den Brink and Funaki, 2015) etc., to name just a few.

Our MI^k -value is also characterized by efficiency, symmetry and linearity along with a new axiom: the MN^k -player replacing the null player property. It is based on the notion of "average marginal multilateral contribution" of a player. The upper suffix k denotes the level upto which multilateral interactions are allowed i.e., to what size of a group inside a coalition can make some significant contribution. Obviously, the share to a player from her group contributions is directly proportional to the size of the coalition. The idea is similar to the one proposed in (Borkotokey et al., 2020), but the two values differ by their counting process proposed therein. Unlike ours, the counting process in (Borkotokey et al., 2020) is based on the original Shapley's rule of counting, where the players are allowed to enter a room to form the grand coalition, following an order

¹ UCU members at 61 universities in the UK have recently launched a four week strike, to protest against planned changes to their pensions which they say will mean a typical lecturer will receive £10,000 a year less in retirement (The Guardian, 15 February, 2018).

² Universal basic income: renaissance for a 500-year-old idea, Martin Sandbu, Financial Times, December 8, 2017

³ A similar concept was proposed in (Borkotokey et al., 2015) for a class of fixed and flexible network allocation rules and recently in (Borkotokey et al., 2020) which introduces the notion of group contributions within coalitions.

⁴ We can take additivity in stead of linearity as any value that is symmetric and additive is also linear.

and then determine their contributions in groups. The standard null player axiom of the Shapley value requires zero payment to players whose marginal contribution vector is zero, see (Casajus and Huettnner, 2013). We consider the possibility that a player may be null because of things out of his control (e.g., disability) and therefore a zero allocation may not be justified. However, if the player's presence in the society is undesirable on the grounds of negative externality created by his antisocial behaviour, e.g., him being a criminal, cheater, or ill-behaved, then this individual must be awarded zero allocation and possibly separated from the society and put to correction centre. We call such a player MN^k -player. The notion of an MN^k -player and the corresponding axiom are identical with the null player and the null player axiom of the Shapley value respectively when $k = 0$. Again for large n and $k = n - 1$, the MI^k -value is asymptotically close to the equal division rule (ED). Note that in the context of the games on combinatorial structure, extensions of the Shapley value take account of the collective contributions when all the coalitions are not feasible, see for example a recent study on the concept lattice due to (Faigle et al., 2016), however our approach is completely different from theirs as can be seen from the model formulation in Section 3.

The rest of the paper is organized as follows. In Section 2 we mention some preliminary concepts of TU-games for ready reference. We formally introduce the notion of multilateral interactions in Section 3. In Section 4, we give a characterization of our proposed solution. We express the MI^k -value in terms of the Shapley values of a class of the associated games in Section 5. In Section 6, asymptotic closeness between the MI^k -value and the equal division rule are discussed. Two examples are presented in Section 7 highlighting the solidarity among the players followed by the concluding remarks in Section 8.

2 Preliminaries

Let N be the universal player set and N any subset of N . A transferable utility game (TU Cooperative game or simply a TU game) is a pair (N, v) and $v : 2^N \rightarrow \mathbb{R}$ a characteristic function satisfying $v(\emptyset) = 0$. Subsets of N are called coalitions and the value $v(S)$ for each coalition S is called its worth. Let $\mathcal{G}(N)$ denote the game space consisting of all TU games over the player set N and $\mathcal{G}$ denote the universal game space with all players of N . An intuitive assumption on $\mathcal{G}(N)$ is that the grand coalition N forms for each of its members. A solution is a function that allocates the profits or the costs due to the formation of the grand coalition among the players. A one-point solution is called a value. Thus formally a value on $\mathcal{G}(N)$ is a function that assigns a single payoff vector $\Phi(v) = (\Phi_i(v))_{i \in N} \in \mathbb{R}^n$ to every game $v \in \mathcal{G}(N)$.

Let the player set $N = \{1, 2, \dots, n\}$ be fixed. Thus the TU-game (N, v) is denoted by its characteristic function v . With some abuse of notations we denote the singleton sets without braces. Thus we write $S \cup i$ for $S \cup \{i\}$, $S \setminus i$ for $S \setminus \{i\}$ etc. The size (cardinality) of coalition S is denoted by the corresponding lower case letter s . A game v is called convex if for any $S, T \subseteq N$, $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$. A game v is called superadditive if for any $S, T \subseteq N$ with $S \cap T = \emptyset$, $v(S \cup T) \geq v(S) + v(T)$. A game v is called subadditive if for any $S, T \subseteq N$ with $S \cap T = \emptyset$, $v(S \cup T) \leq v(S) + v(T)$. A game v is monotonic if $v(T) \leq v(S)$ for every $S, T \in 2^N$ such that $T \subseteq S$.

The increase in the worth when player $i \in N$ joins coalition $S \subseteq N \setminus i$ is called the marginal contribution of player i to coalition S which is denoted by $b_i^S(v)$ and is formally given by

$$b_i^S(v) = v(S \cup i) - v(S). \quad (1)$$

The unanimity games $\hat{u}_T : 2^N \rightarrow \mathbb{R}$, the identity games $u_T : 2^N \rightarrow \mathbb{R}$ and $u_T^* : 2^N \rightarrow \mathbb{R}$, $T \subseteq N$ are respectively defined as follows.

$$\hat{u}_T(S) = \begin{cases} 1, & \text{if } T \subseteq S \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

$$u_T(S) = \begin{cases} 1, & \text{if } T = S \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

$$u_T^*(S) = \begin{cases} 1, & \text{if } T \subset S \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

The class of unanimity games and the class of identity games are bases for the linear space $\mathcal{G}(N)$. In the following we give a brief description of the Shapley value and a mention of the equal division rule (ED) for reasons already mentioned in Section 1. Among the various interpretations of the Shapley value, see for example (Branzei et al., 2008) we take the following as it relates to the interpretation of our proposed value.

Suppose that the “grand coalition” $N = \{1, 2, \dots, n\}$ forms in a way such that the players enter the coalition one by one. This order of entrance can be expressed by a permutation $\sigma : N \rightarrow N$ of the players. Let the collection of all permutations on N be denoted by $\Pi(N)$. For every $\sigma \in \Pi(N)$, let $P(\sigma, i) = \{j \in N \mid \sigma(j) < \sigma(i)\}$ be the set of players that enter before player i in the order σ . The Shapley value is the solution $\Phi^{Sh} : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ that assigns to every player her expected marginal contribution to the coalition of players that enter before him, given that every order of entrance has equal probability of $\frac{1}{n!}$ to occur and is given by,

$$\Phi_i^{Sh}(v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} [v(P(\sigma, i) \cup i) - v(P(\sigma, i))] \quad (5)$$

which after simplifications becomes,

$$\Phi_i^{Sh}(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} [v(S \cup i) - v(S)]. \quad (6)$$

The equal division rule (ED) $\Phi^{ED} : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ is given by

$$\Phi_i^{ED}(N, v) = \frac{v(N)}{n} \quad \text{for all } i \in N. \quad (7)$$

For the game $v \in \mathcal{G}(N)$, a player $i \in N$ is called a null player if for every coalition $S \subseteq N \setminus i$, we have $v(S \cup i) = v(S)$. Players $i, j \in N$ are called symmetric in $v \in \mathcal{G}(N)$ if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$.

Values satisfying the following three axioms, e.g., Efficiency, Symmetry and Linearity (Additivity) are called ESL-values.

- (a) Efficiency (*Eff*): A value $\Phi : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ is efficient if for any game $v \in \mathcal{G}(N)$:

$$\sum_{i \in N} \Phi_i(v) = v(N)$$

- (b) Symmetry (*Sym*): A value $\Phi : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ is symmetric if for all $v \in \mathcal{G}(N)$ and symmetric players $i, j \in N$ in v , it holds that $\Phi_i(v) = \Phi_j(v)$.
(c) Additivity (*Add*) and Linearity (*Lin*): A value $\Phi : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ is additive if for all games $u, w \in \mathcal{G}(N)$ and every player $i \in N$:

$$\Phi_i(u + w) = \Phi_i(u) + \Phi_i(w).$$

The value $\Phi : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ is linear if for all games $u, w \in \mathcal{G}(N)$ every pair of $\alpha, \beta \in \mathbb{R}$ and every player $i \in N$:

$$\Phi_i(\alpha u + \beta w) = \alpha \Phi_i(u) + \beta \Phi_i(w).$$

All the Shapley like values we discussed in Section 1 are ESL-values. In general they all differ by a fourth axiom in which a particular player is identified to be awarded zero payoff either because of her non-productivity or encumbrance in the coalition. Take for example, the null player axiom in case of the Shapley (Shapley, 1953; Weber, 1988), the nullifying player axiom in case of the ED (van den Brink, 2007), the A-null player axiom in case of the Solidarity value (Nowak and Radzik, 1994), the proportional null player and the quasi proportional null player in case of the ED and the Solidarity value respectively (Kamijo and Kongo, 2012).

A stronger version of *Sym* is the axiom of Anonymity (*AN*) which is defined as follows. Let $\sigma : N \rightarrow N$ be a permutation on N .

- (d) Anonymity (*AN*): For given $v \in \mathcal{G}(N)$ define the game $\sigma v \in \mathcal{G}(N)$ by $\sigma v(S) = v(\sigma^{-1}S)$ for all $S \subseteq N$. A value Φ satisfies Anonymity (*AN*) if for each permutation $\sigma : N \rightarrow N$, $\Phi_i(v) = \Phi_{\sigma(i)}(\sigma v)$.

Note that *AN* implies *Sym* but for values that satisfy *Lin* and *Eff* they are equivalent (Malawski, 2013). In the following we have a characterization theorem of the class of ESL-values proposed in (Ruiz et al., 1998) as follows.

Theorem 1 A value $\Phi : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ is an ESL value if and only if there exists a class of real numbers $\rho_s (s = 1, 2, 3, \dots, n-1)$ such that

$$\Phi_i(v) = \frac{v(N)}{n} + \sum_{S: i \in S \neq N} \rho_s \frac{v(S)}{s} - \sum_{S: i \notin S} \rho_s \frac{v(S)}{n-s}. \quad (8)$$

A fifth axiom, called the Marginality (*M*) due to (Young, 1985) is satisfied by the Shapley value. It is defined as follows.

- (e) Marginality (*M*): A value $\Phi : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ satisfies Marginality (*M*) if for each $i \in N$, $v, w \in \mathcal{G}(N)$ and $S \subseteq N \setminus i$ such that $v(S \cup i) - v(S) = w(S \cup i) - w(S)$, then $\Phi_i(v) = \Phi_i(w)$.

In (Radzik, 2013), the concepts of asymptotic equivalence of two values Φ and Ψ in $\mathcal{G}$ are discussed in details. It is proposed that the solidarity value and the equal division rule are asymptotically weakly-equivalent in any uniformly bounded class of games. The uniformly bounded class of games and the asymptotically weakly-equivalent are defined as follows.

Definition 1 A class of games $\mathcal{M} \subseteq \mathcal{G}(N)$ is called uniformly bounded if there is a real constant c such that for all $S \subseteq N$, the inequality $|v(S)| \leq c$ holds for every game $v \in \mathcal{M}$.

Definition 2 Two values Φ and Ψ are said to be asymptotically weakly-equivalent in $\mathcal{G}(N) \subseteq \mathcal{G}$, if for every sequence $\{v_n\} \in \mathcal{G}(N)$,

$$\lim_{n \rightarrow \infty} [\Phi_i(v_n) - \Psi_i(v_n)] = 0, \quad \forall i = 1, 2, 3, \dots$$

3 Group Contribution and Multilateral Interactions

Now we state the main idea of this paper which builds on both the notion of marginality and egalitarianism by considering group contributions and multilateral interactions among the players as discussed in Section 1. Take for example $N = \{1, 2, 3\}$ and any TU game $v \in \mathcal{G}(N)$. Recall from Section 2 that the Shapley value $\Phi^{Sh}(v)$ rewards player 1 for example, the equal share of each of the possible marginal contributions of the form $v(\sigma(1, 2, 3)) - v(\sigma(2, 3))$, $v(\sigma(1, 2) - v(\sigma(2)))$, $v(\sigma(1, 3) - v(\sigma(3)))$ and finally $v(\sigma(1)) - v(\emptyset)$ over all possible permutations σ on N . The group contributions of the form $v(\sigma(1, 2, 3)) - v(\sigma(2))$ (similarly $v(\sigma(1, 2, 3)) - v(\sigma(3))$, $v(\sigma(1, 2, 3)) - v(\emptyset)$ etc., which evaluate the joint contributions of player 1 and 3 respectively (similarly of player 1 and 2) are actually hidden in the marginal contributions of the individual players. Take for example, $v(\sigma(1, 2, 3)) - v(\sigma(2)) = [v(\sigma(1, 2, 3)) - v(\sigma(1, 2))] + [v(\sigma(1, 2)) - v(\sigma(2))]$ etc., which is simply the sum of the contributions of players 3 and 1. Thus player 3 and 1 get fractions of their shares directly from the contributions $v(\sigma(1, 2, 3)) - v(\sigma(1, 2))$ and $v(\sigma(1, 2) - v(\sigma(2)))$ respectively. It means that they receive whatever they produce. This is attributed to the implicit assumption that players take independent decisions to join or leave a coalition and do not expect any solidarity from their peers. However in reality, they are influenced by their peers to accomplish cooperative endeavours that involves compassion for each other. Therefore, such group influence needs to be accommodated to the sharing of the total worth. We propose an alternative sharing of these group contributions. In addition to her own marginal contributions, we assume that each player gets a share of the contributions from her peers in the group. For example, in the above example, player 1 gets shares from $v(\sigma(1, 2, 3)) - v(\sigma(2, 3))$, $v(\sigma(1, 2, 3)) - v(\sigma(2))$, $v(\sigma(1, 2, 3)) - v(\sigma(3))$ and $v(\sigma(1, 2, 3)) - v(\emptyset)$. We call the contributions of the form $v(\sigma(1, 2, 3)) - v(\sigma(2))$ etc., appearing in our proposed value “group contributions” and the standard marginal contributions $b_i^S(v)$ given by Eq.(1) “individual contributions”. Also we use the terms “multilateral” and “unilateral” to distinguish the interactions of the players that lead to group and individual contributions respectively. The way the unilateral interactions are captured by the Shapley like ESL-values, our value captures multilateral interactions at various levels where players will get shares from their

multilateral interactions at these levels. We will define an MN^k -player for each level $k \in \{0, 1, 2, \dots, n-1\}$.

The notion of multilateral interaction captures the idea of equality at local levels that we had mentioned earlier. This identifies the preference for equality by the coalitions of varying sizes that may share a sense of group identity, s.a., soldiers of the same unit (Deutsch, 1985; Mikula, 1980), or social safety under caste network (Munshi and Rosenzweig, 2016) etc.

In the following we build a more formal model of what we have discussed so far. Given $i \in N$, a coalition $S \subseteq N \setminus i$ and $T \subseteq S$, the “group contribution” of $T \cup i$ with respect to $S \cup i$ is defined by $v(S \cup i) - v(S \setminus T)$. Thus the “average group contribution” of $T \cup i$ with respect to $S \cup i$ is given by $\frac{v(S \cup i) - v(S \setminus T)}{t+1}$. Assume that i joins a group T having at most k players within a coalition $S \cup i$ from where she expects solidarity.⁵ The group formation takes place in the following two steps.

- Step 1: Choose a group J within $S \cup i$ of size j such that $0 \leq j \leq \min(s, k)$ to which player i joins. This can be done in $\binom{s}{j}$ ways.
- Step 2: Choose a position for player i to enter into the group J . There are $j+1$ such positions in $J \cup i$. Thus the total number of possible formations of $J \cup i$ is $\sum_{0 \leq j \leq \min(s, k)} (j+1) \binom{s}{j}$.

Suppose now the group $T \subseteq S$ is formed where player i can enter in $t+1$ positions. Then the probability of such formation is given by: $\frac{t+1}{\sum_{0 \leq j \leq \min(s, k)} (j+1) \binom{s}{j}}$, assuming that a player has equal probability of entering or leaving any position in a group within a coalition.⁶

Next, the “average marginal multilateral contribution” of $i \in N$ to the coalition S upto level k , $0 \leq k \leq n-1$ denoted by $m_{i,k}^S(v)$ is defined by the expectation of the average group contributions of all groups $T \cup i$ where $T \subseteq S$ and $t \leq k$ with respect to S . Formally we have,

$$\begin{aligned} m_{i,k}^S(v) &= \sum_{\substack{T \subseteq S \\ 0 \leq t \leq \min(s, k)}} \frac{t+1}{\sum_{0 \leq t \leq \min(s, k)} (t+1) \binom{s}{t}} \left\{ \frac{v(S \cup i) - v(S \setminus T)}{t+1} \right\} \\ &= \frac{1}{\sum_{0 \leq t \leq \min(s, k)} (t+1) \binom{s}{t}} \sum_{\substack{T \subseteq S \\ 0 \leq t \leq \min(s, k)}} \left\{ v(S \cup i) - v(S \setminus T) \right\} \end{aligned} \quad (9)$$

Remark 1 Observe that for $k = 0$, $m_{i,0}^S(v) = v(S \cup i) - v(S) = b_i^S(v)$, the unilateral marginal contribution of i in S .

Remark 2 For coalitions S with $s \leq k$, $m_{i,k}^S(v)$ is only averaging group contributions for groups T with $t \leq k-1$, but not all groups.

Next, we introduce the notion of an “average null player upto k level of multilateral interactions” or simply an MN^k -player, $k \in \{0, 1, 2, \dots, n-1\}$. As the name suggests, the average marginal multilateral contribution of an MN^k -player i given by Eq (9) is zero.

Definition 3 A player $i \in N$ is called an MN^k -player if for each $S \subseteq N \setminus i$, $m_{i,k}^S(v) = 0$.

It follows from Eq.(9) that the condition for the MN^k -player $i \in N$ is equivalently put as:

$$v(S \cup i) = \frac{1}{\sum_{p=0}^{\min\{s, k\}} \binom{s}{p}} \sum_{\substack{T \subseteq S \\ 0 \leq t \leq \min\{s, k\}}} v(S \setminus T). \quad (10)$$

⁵ An exactly same formulation can be made for player i when she leaves the coalition S with a group $T \subseteq S$.

⁶ Take for example, $S = \{2, 3, 4\}$, $i = 1$ and $k = 2$. Any group J of size $0 \leq j \leq 2$ can be chosen in $\binom{3}{j}$ ways. Player 1 can join J in $j+1$ ways. Thus the total number of possible formations of $J \cup \{1\}$ is $(0+1) \cdot \binom{3}{0} + (1+1) \cdot \binom{3}{1} + (2+1) \cdot \binom{3}{2}$. In particular, let the group $\{2, 3\}$ be formed. Then, player 1 can join $\{2, 3\}$ in 3 positions: before 2, after 3 or in between 2 and 3. Therefore, the probability of $\{1, 2, 3\}$ being formed is $\frac{3}{(0+1) \cdot \binom{3}{0} + (1+1) \cdot \binom{3}{1} + (2+1) \cdot \binom{3}{2}}$.

Eq.(10) implies that for $k = 0$, the MN^0 -player is simply the standard null player. If no ambiguity on k arises, now onwards we consider a fixed k with $0 \leq k \leq n - 1$ and assume that all interactions upto the k -level are accounted for.

Note that the MN^k -player on an average has zero marginal multilateral contribution to the interactions in a coalition. That is, if the MN^k -player contributes negatively if not zero, then there is another situation where she has a positive role to play as well. If we implement the Shapley value in this setting, the MN^k -player will be awarded negative or zero payoff. This effectively highlights the negative contributions of the MN^k -player in the game more. In contrast, we recognize both the positives and the negatives in an unbiased manner. In view of this discussion, one observes that the MN^k -player should be awarded with zero payoff. Consequently our proposal highlights the positive contributions of the MN^k -player relatively more. Thus formally we have the following axiom.

- (f) MN^k -player (MNP^k): A value $\Phi : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ satisfies MN^k -player (MNP^k) if for a fixed $k \in \{0, 1, \dots, n - 1\}$ and $v \in \mathcal{G}(N)$ such that $i \in N$ is an MN^k -player, then $\Phi_i(v) = 0$.

4 Characterization

In this section we introduce and characterize our proposed value for TU games based on a set of standard axioms. In the following, we add our axioms successively to finally obtain a new value which we eventually call the multilateral value as it is based on the multilateral interactions among the players. Lemma 1 below is due to (Weber, 1988).

Lemma 1 (Theorem 1 in (Weber, 1988), p. 6) Let Φ_i be a value for $\mathcal{G}(N)$ that satisfies *Lin* for each $i \in N$. Then there exist real constants α_S^i for all $S \subseteq N$ such that for every $v \in \mathcal{G}(N)$,

$$\Phi_i(v) = \sum_{\emptyset \neq S \subseteq N} \alpha_S^i v(S). \quad (11)$$

Lemma 2 Let the value Φ_i satisfy *Lin* and MNP^k . Then for all $i \in N$, there exist real constants $\gamma_{S,i}^{(k)}$ for all $S \subseteq N \setminus i$ such that for every $v \in \mathcal{G}(N)$,

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{\substack{T \subseteq S \\ 0 \leq t \leq k}} \gamma_{S,i}^{(k)} \left[v(S \cup i) - v(S \setminus T) \right].$$

Proof When Φ_i satisfies *Lin*, we have from Eq.(11)

$$\begin{aligned} \Phi_i(v) &= \sum_{\emptyset \neq S \subseteq N} \alpha_S^i v(S) \\ &= \sum_{S \subseteq N \setminus i} \left\{ \alpha_{S \cup i}^i v(S \cup i) + \alpha_S^i v(S) \right\}. \end{aligned} \quad (12)$$

After rearranging the coefficients α_S^i in Eq.(12), we obtain

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \left\{ \alpha_{S \cup i}^i v(S \cup i) + \sum_{\substack{T \subseteq S \\ 0 \leq t \leq k}} \beta_{S \setminus T}^i v(S \setminus T) \right\} \quad (13)$$

where,

$$\beta_S^i = \frac{1}{\sum_{p=0}^{\min(n-s-1, k)} \binom{n-s-1}{p}} \alpha_S^i, \quad \forall S \subseteq N \setminus i.$$

Let Φ satisfy MNP^k . If i is an MN^k -player for v , then

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \left\{ \sum_{\substack{T \subseteq S \\ 0 \leq t \leq k}} \left(\frac{1}{\sum_{p=0}^{\min(s,k)} \binom{s}{p}} \alpha_{S \cup i}^i + \beta_{S \setminus T}^i \right) v(S \setminus T) \right\} = 0. \quad (14)$$

Eq.(14) holds for any TU-game v such that i is an MN^k -player for v . For each $S \subset N$ $S \neq \emptyset$, define a game w_S as follows.

$$w_S(P) = \begin{cases} \lambda_p^s & \text{if } S \subseteq P \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

where

$$\lambda_p^s = \begin{cases} 1 & \text{if } s = p \\ \frac{\sum_{t=0}^{\min(p-s-1,k)} \binom{p-s-1}{t} \lambda_{p-t-1}^s}{\sum_{t=0}^{\min(p-1,k)} \binom{p-1}{t}}, & \text{if } s < p. \end{cases} \quad (16)$$

Then the set $\{w_S : S \neq \emptyset\}$ is a basis for $G(N)$. Also for any $i \notin S$, i is an MN^k -player in w_S . Thus Eq.(14) holds for all games w_S in particular with a suitable choice of S such that i is an MN^k -player for w_S . Let $S = N \setminus i$. Then $\Phi_i(w_{N \setminus i}) = 0$ implies,

$$\frac{1}{\sum_{p=0}^{\min(n-1,k)} \binom{n-1}{p}} \alpha_N^i + \beta_{N \setminus i}^i = 0 \quad (17)$$

Similarly take $S = N \setminus \{i, j\}$. Then $\Phi_i(w_{N \setminus \{i, j\}}) = 0$ and Eq.(17) implies,

$$\begin{aligned} & \frac{1}{\sum_{p=0}^{\min(n-1,k)} \binom{n-1}{p}} \alpha_N^i + \beta_{N \setminus i}^i + \frac{1}{\sum_{p=0}^{\min(n-2,k)} \binom{n-2}{p}} \alpha_{N \setminus j}^i + \beta_{N \setminus \{i, j\}}^i = 0 \\ & \Rightarrow \frac{1}{\sum_{p=0}^{\min(n-2,k)} \binom{n-2}{p}} \alpha_{N \setminus j}^i + \beta_{N \setminus \{i, j\}}^i = 0. \end{aligned}$$

Thus, by choosing appropriate w_S where $i \notin S$, we can prove that

$$\frac{1}{\sum_{p=0}^{\min(s,k)} \binom{s}{p}} \alpha_{S \cup i}^i = -\beta_{S \setminus T}^i, \quad \forall \emptyset \neq S \subseteq N \setminus i, T \subseteq S \text{ and } 0 \leq t \leq \min(s, k).$$

Setting $\gamma_{S,i}^{(k)} := \frac{1}{\sum_{p=0}^{\min(s,k)} \binom{s}{p}} \alpha_{S \cup i}^i = -\beta_{S \setminus T}^i \quad \forall \emptyset \neq S \subseteq N \setminus i, T \subseteq S$ such that

$0 \leq t \leq \min(s, k)$ in Eq.(13), we obtain

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{\substack{T \subseteq S \\ 0 \leq t \leq k}} \gamma_{S,i}^{(k)} \left[v(S \cup i) - v(S \setminus T) \right]. \quad (18)$$

This completes the proof. $\square$

Note that Lemma 2 is similar to Theorem 2 in (Weber, 1988), but is different in that we use MNP^k along with Lin instead of the dummy axiom considered in (Weber, 1988). Further observe that Eq.(18) builds on *group contributions* of the players upto a group size of k whereas its corresponding expression in (Weber, 1988) builds on *individual contributions* discussed in Section 2. Thus, in particular, when $k = 1$, Lemma 2 coincides with Theorem 2 in (Weber, 1988).

The next axiom AN requires that the coefficients $\gamma_{S,i}^{(k)}$, $i \in N$ and $S \subseteq N$ should be identical for coalitions of the same size. We have the following lemma.

Lemma 3 Under Lin , MNP^k and AN for all $i \in N$, there exist real constants $\gamma_s^{(k)}$ for all $S \subseteq N \setminus i$ such that for every $v \in \mathcal{G}(N)$,

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{\substack{T \subseteq S \\ 0 \leq t \leq k}} \gamma_s^{(k)} \left[v(S \cup i) - v(S \setminus T) \right]$$

Proof The proof proceeds exactly as in (Weber, 1988) and so is omitted. $\square$
The next axiom is the Eff .

Lemma 4 Under Lin , MNP^k , AN and Eff the value $\Phi(v)$ is given by

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{\substack{T \subseteq S \\ 0 \leq t \leq k}} \gamma_s^{(k)} \left[v(S \cup i) - v(S \setminus T) \right] \quad (19)$$

where ⁷

$$\gamma_{n-1}^{(k)} = \frac{1}{n \left[\sum_{t=0}^{\min(s,k)} \binom{n-1}{t} \right]} \quad (20)$$

$$\gamma_{n-2}^{(k)} = \frac{1}{(n-1) \left[\sum_{t=0}^{\min(n-2,k)} \binom{n-2}{t} \right]} \gamma_{n-1}^{(k)} \quad (21)$$

$$\vdots \quad \vdots \quad \vdots$$

$$\gamma_s^{(k)} = \frac{n-s-1}{(s+1) \left[\sum_{t=0}^{\min(s,k)} \binom{s}{t} \right]} \sum_{t=0}^{\min(n-s-2,k)} \binom{n-s-2}{t} \gamma_{s+t+1}^{(k)}, \text{ for } s < n; \quad (22)$$

Eq.(20) - (22) give recursive formulae for the coefficients $\gamma_s^{(k)}$'s. In particular when $k = n-1$,

$$\gamma_{n-1}^{(k)} = \frac{1}{n \cdot 2^{n-1}}$$

$$\gamma_s^{(k)} = \frac{n-s-1}{(s+1) \cdot 2^s} \sum_{t=0}^{\min\{n-s-2,k\}} \binom{n-s-2}{t} \gamma_{s+t+1}^{(k)}, \text{ for } s < n;$$

Proof Under Lin , MNP^k , AN and by Lemma 3, for all $i \in N$, there exist real constants $\gamma_s^{(k)}$ for all $S \subseteq N \setminus i$ such that for every $v \in \mathcal{G}(N)$,

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{\substack{T \subseteq S \\ 0 \leq t \leq k}} \gamma_s^{(k)} \left[v(S \cup i) - v(S \setminus T) \right]$$

7

$$\gamma_{n-3}^{(k)} = \frac{2}{(n-2) \left[\sum_{t=0}^{\min\{n-3,k\}} \binom{n-3}{t} \right]} \left[\gamma_{n-2}^{(k)} + \gamma_{n-1}^{(k)} \right]$$

$$\gamma_{n-4}^{(k)} = \frac{3}{(n-3) \left[\sum_{t=0}^{\min\{n-4,k\}} \binom{n-4}{t} \right]} \left[\gamma_{n-3}^{(k)} + 2\gamma_{n-2}^{(k)} + \gamma_{n-1}^{(k)} \right]$$

$$\gamma_{n-5}^{(k)} = \frac{4}{(n-4) \left[\sum_{t=0}^{\min\{n-5,k\}} \binom{n-5}{t} \right]} \left[\gamma_{n-4}^{(k)} + 3\gamma_{n-3}^{(k)} + 3\gamma_{n-2}^{(k)} + \gamma_{n-1}^{(k)} \right]$$

Using *Eff*, we have

$$\begin{aligned}
v(N) &= \sum_{i \in N} \Phi_i(v) \\
&= \sum_{i \in N} \left(\sum_{S \subseteq N \setminus i} \sum_{\substack{T \subseteq S \\ 0 \leq t \leq k}} \gamma_s^{(k)} \{v(S \cup i) - v(S \setminus T)\} \right) \\
&= \sum_{\emptyset \neq S \subseteq N} v(S) \left\{ \sum_{i \in S} \sum_{t=0}^{\min(s-1, k)} \binom{s-1}{t} \gamma_{s-1}^{(k)} - \sum_{i \in N \setminus S} \sum_{i \in S} \sum_{t=0}^{\min(n-s-1, k)} \binom{n-s-1}{t} \gamma_{s+t}^{(k)} \right\} \\
&= \sum_{\emptyset \neq S \subseteq N} v(S) \left\{ s \sum_{t=0}^{\min(s-1, k)} \binom{s-1}{t} \gamma_{s-1}^{(k)} - (n-s) \sum_{t=0}^{\min(n-s-1, k)} \binom{n-s-1}{t} \gamma_{s+t}^{(k)} \right\}.
\end{aligned}$$

Consider the identity games in particular, which yield the following recursive relations.

$$\begin{aligned}
\gamma_{n-1}^{(k)} &= \frac{1}{n \left[\sum_{t=0}^{\min(s, k)} \binom{n-1}{t} \right]} \\
\gamma_s^{(k)} &= \frac{n-s-1}{(s+1) \left[\sum_{t=0}^{\min(s, k)} \binom{s}{t} \right]} \sum_{t=0}^{\min(n-s-2, k)} \binom{n-s-2}{t} \gamma_{s+t+1}^{(k)}.
\end{aligned}$$

This completes the proof. $\square$

Combining Lemmas 1, 2, 3 and 4, we have the following theorem.

Theorem 2 For each level $k, k \in \{0, 1, \dots, n-1\}$ there is a unique value $\Phi : \mathcal{G}(N) \rightarrow \mathbb{R}^n$ satisfying *Lin*, *MNP^k*, *AN* and *Eff* given by

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{\substack{T \subseteq S \\ 0 \leq t \leq k}} \gamma_s^{(k)} \left[v(S \cup i) - v(S \setminus T) \right]$$

where the coefficients $\gamma_s^{(k)}$'s are given by Eq.(20)- Eq.(22).

In the Appendix we provide an alternative proof of Theorem 2 using Theorem 1.

Now onwards, we denote by $\Phi^{(k)}(v)$ the value $\Phi(v)$ given in Eq.(19) to highlight the level k of interactions and call it a k -lateral interactive value or simply MI^k -value for $v \in \mathcal{G}(N)$.

Remark 3 Following (Weber, 1988) and the axioms of *AN* and *Eff* we can interpret the coefficient $\gamma_s^{(k)}$ in Eq.(20)- Eq.(22) as the conditional probability of forming a sub-coalition of any size not larger than k from the given coalition S when it is assumed that coalition S has already formed. Note that here we assume equal probabilities of occurrences for the sub-coalitions of all sizes of a given coalition.

It is evident that unlike the Shapley value which is the average of the marginal contributions (unilateral) of the players entering a room one by one described in Section 2, the MI^k -value is the weighted average of the group contributions of the players each entering into the room with a coalition of any size $\leq k$ of the player set N where weights are determined by the conditional probabilities as described in Remark 3.

The independence of the axioms of Theorem 2 can be seen from the following alternative solutions for $v \in \mathcal{G}(N)$.

- (i) Let $\Phi^1(v) = (0, 0, \dots, 0)$. Then $\Phi^1(v)$ satisfies all axioms except *Eff*.
- (ii) The Shapley value $\Phi^{Sh}(v)$ given by Eq.(6) satisfies all the axioms other than *MNP^k*.
- (iii) Consider the basis $W = \{w_S : S \subseteq N, S \neq \emptyset\}$ defined in Eq.(15). Let $i = \min_{j \in S} j$. Define the value Φ^2 on W by $\Phi_i^2(w_S) = w_S(N)$ and $\Phi_j^2(w_S) = 0$ for all $j \neq i, j \in N$. Extend Φ^2 linearly to $\mathcal{G}(N)$. Then $\Phi^2(v)$ satisfies all the axioms except *AN*.

(iv) Define $\Phi_i^3(v)$ as follows. Fix an $\alpha > 0$. For any $k \in \{1, 2, \dots, n-1\}$

$$\Phi_i^3(v) = \begin{cases} \frac{v(N)}{n}, & \text{when } i \text{ is not an } MN^k\text{-player and } v(N) > \alpha \\ 0, & \text{when } i \text{ is an } MN^k\text{-player and } v(N) > \alpha \\ \Phi_i^{(k)}(v), & \text{when } v(N) \leq \alpha. \end{cases}$$

Thus, $\Phi^3(v)$ satisfies all the properties except *Lin*.

5 Representation of the the MI^k -value in terms of the Shapley value

In what follows, we express the MI^k -value of a TU game in terms of the Shapley values of a class of associated games. For any $T \subseteq N$ denote by $[T]$ a single hypothetical player (Fujimoto, 2014) that represents the collective action of the players in T . Let $N \subseteq \mathbb{N}$ as usual and fix a $T \subseteq N$. Set $\bar{N} = (N \setminus T) \cup [T]$. Then for any $\bar{S} \subseteq \bar{N}$, there is an $S \subseteq N \setminus T$ such that $\bar{S} = S \cup [T]$ or $\bar{S} = S$. Let with an abuse of notation $2^{\bar{N}}$ denote the power set of all coalitions of the player set $\bar{N}$ that includes the players in T as a single hypothetical player.

Define the TU game $\bar{V}_{[T]} : 2^{\bar{N}} \rightarrow \mathbb{R}$ as follows. For any $\bar{S} \subseteq \bar{N}$

$$\bar{V}_{[T]}(\bar{S}) = \begin{cases} \bar{\gamma}_{s+t-1}^{(k)} v(S \cup T), & \text{if } [T] \in \bar{S} \\ \bar{\gamma}_{s+t-1}^{(k)} v(S), & \text{otherwise.} \end{cases} \quad (23)$$

where $\bar{\gamma}_{s+t-1}^{(k)} = \frac{\gamma_{s+t-1}^{(k)}}{\alpha_s^t}$ with $\alpha_s^t = \frac{(s-t+1)!(n-s-1)!}{(n-t+1)!}$.

Following Branzai et al. (2008) a possible interpretation of the class of games of the form $\bar{V}_{[T]}$ is given below.

The quantity $\bar{\gamma}_{s+t-1}^{(k)}$ may be taken as the conditional probability that a coalition of size $\bar{s}$ ($= s+t-1$) is chosen from $\bar{N}$ given that a subset $T \subseteq S$ is always present in S . Here α_s^t represents the probability of choosing a coalition from $\bar{N}$ with size s varying from 0 to $(n-t+1)$ where players in T behave as a single hypothetical player. Therefore $\bar{V}_{[T]}$ in Eq.(23) is the game that estimates the worth of the coalitions with respect to v and the MI^k -value in which players in T act as a single player. A unilateral interaction of each of the players in $N \setminus T$ with $[T]$ is indeed a multilateral interaction with the players in T . This suggests that we level each such interaction cardinally and call the game $\bar{V}_{[T]}$ the associated game at interaction level t .

The following theorem states that the payoff to a player under the MI^k -value is the sum of her payoffs obtained from the Shapley values of the associated games up to the k -lateral interactions.

Theorem 3 *The MI^k -value $\Phi_i^{(k)}$ for each $i \in N$ is the sum of the Shapley values $\Phi_{[T]}^{Sh}$ of $[T] : i \in T$ for the games $\bar{V}_{[T]}$ given by,*

$$\Phi_i^{(k)}(v) = \sum_{\substack{[T] \in \bar{N} : i \in T \\ 1 \leq t \leq k+1}} \Phi_{[T]}^{Sh}(\bar{V}_{[T]}).$$

Proof Using Eq.(23), it follows from Eq.(19) that,

$$\begin{aligned}
\Phi_i^{(k)}(v) &= \sum_{S \subseteq N \setminus i} \sum_{\substack{T \subseteq S \\ 0 \leq t \leq k}} \gamma_s^{(k)} \left[v(S \cup i) - v(S \setminus T) \right] \text{ where } \gamma_s^{(k)} \text{ is given by Eq.(20)-(22)} \\
&= \sum_{\substack{T \subseteq N \setminus i \\ 0 \leq t \leq k}} \sum_{S \subseteq N \setminus i : T \subseteq S} \frac{(n-t+1)!}{(s-t+1)!(n-s-1)!} \gamma_s^{(k)} \frac{(s-t+1)!(n-s-1)!}{(n-t+1)!} \\
&\quad \left[v(S \cup i) - v(S \setminus T) \right] \\
&= \sum_{\substack{[T] \in \bar{N} : i \in T \\ 1 \leq t \leq k+1}} \sum_{\bar{S} \subseteq \bar{N} : [T] \subseteq \bar{S}} \frac{(\bar{s}-1)!(\bar{n}-\bar{s})!}{\bar{n}!} \left[\bar{V}_{[T]}(\bar{S}) - \bar{V}_{[T]}(\bar{S} \setminus [T]) \right] \\
&= \sum_{\substack{[T] \in \bar{N} : i \in T \\ 1 \leq t \leq k+1}} \Phi_{[T]}^{S_h}(\bar{V}_{[T]})
\end{aligned}$$

where $\bar{s} = s - t + 1$, $\bar{n} = n - t + 1$. $\square$

Remark 4 Theorem 3 suggests that the MI^k -value of a game aggregates the Shapley values of its associated games where players in the associated games interact among themselves unilaterally at all possible levels. Thus it is a decomposition theorem that connects the MI^k -value with the Shapley values of the associated games.

Notice that $\Phi^{(k)}(v)$ coincides with $\Phi^{Sh}(v)$ when $k = 0$ for any $v \in \mathcal{G}(N)$.

6 Asymptotic equivalence between the MI^k -value and the ED

In this section we show that for a sufficiently large number of players and equally large level of multilateral interactions, the MI^k -value approaches to the ED. In (Radzik, 2013), asymptotic closeness between the solidarity value and the ED are discussed in details. However, here our formulation of the MI^k -value demands a different approach to prove a similar result. The idea is simple: the difference between the payoffs to any two players limits to zero for sufficiently large n . The basic and intuitive assumption here is that players' individual and group contributions from all possible coalitions are bounded. This may be the case when we assume that all the players in a game with sufficiently large player set get finite payoffs due to *Eff*.

Theorem 4 For any uniformly bounded class of games $\mathcal{M} \subseteq \mathcal{G}$ with $n \in \mathbb{N}$ players, $\Phi^{(n-1)}$ is asymptotically equivalent to the ED in the following sense.

$$\lim_{n \rightarrow \infty} |\Phi_i^{(k)}(v) - \Phi_j^{(k)}(v)| = 0;$$

for every pair of distinct players i and j and $v \in \mathcal{M}$.

Proof Due to Eq.(20) and Eq.(22), we have

$$\begin{aligned}
&\left| \Phi_i^{(n-1)}(v) - \Phi_j^{(n-1)}(v) \right| \\
&= \left| \sum_{S \subseteq N \setminus \{i,j\}} \left(\sum_{t=0}^{n-s} \binom{n-s}{t} \gamma_{s+t+1}^{(k)} \right) \{v(S \cup i) - v(S \cup j)\} \right| \\
&\leq \sum_{S \subseteq N \setminus \{i,j\}} \left(\sum_{t=0}^{n-s} \binom{n-s}{t} \gamma_{s+t+1}^{(k)} \right) |v(S \cup i) - v(S \cup j)|.
\end{aligned} \tag{24}$$

Now, for any $v \in \mathcal{M}$, there exists $M \in \mathbb{R}$ such that for all $S \subseteq N \setminus i$, $|v(S \cup i) - v(S)| \leq \frac{M}{2}$ and $|v(S \cup j) - v(S)| \leq \frac{M}{2}$ so that $|v(S \cup i) - v(S \cup j)| \leq M$ for all $S \subseteq N \setminus \{i,j\}$. Thus

Eq.(24) implies

$$\begin{aligned}
& |\Phi_i^{(n-1)}(v) - \Phi_j^{(n-1)}(v)| \\
& \leq M \sum_{S \subseteq N \setminus \{i,j\}} \left(\sum_{t=0}^{n-s-2} \binom{n-s-2}{t} \gamma_{s+t+1}^{(n-1)} + 2^s \gamma_s^{(n-1)} \right) \\
& = M \sum_{s=0}^{n-2} \binom{n-2}{s} \left(\sum_{t=0}^{n-s-2} \binom{n-s-2}{t} \gamma_{s+t+1}^{(n-1)} + 2^s \gamma_s^{(n-1)} \right).
\end{aligned} \tag{25}$$

Next, we claim that,

$$\gamma_{n-s}^{(n-1)} \leq \frac{s^3}{n \cdot (n-s) \cdot 2^{2n-9}}. \tag{26}$$

Our claim is proved using mathematical induction on the size s of any arbitrary coalition S as follows. From Eq.(22) we have

$$\gamma_{n-1}^{(n-1)} = \frac{1}{n \cdot 2^{n-1}}, \quad \gamma_{n-2}^{(n-1)} = \frac{1}{n \cdot (n-1) \cdot 2^{2n-3}}.$$

Thus, it can be easily verified that Eq.(26) is true for $s = 1$. For $s = k$, suppose that $\gamma_s^{(n-1)}$ satisfies Eq.(26). From Eq.(22) we obtain

$$\begin{aligned}
\gamma_{n-k-1}^{(n-1)} &= \frac{k}{(n-k) \cdot 2^{n-k-1}} \sum_{t=0}^{k-1} \binom{k-1}{t} \gamma_{n-k+t}^{(n-1)} \\
&\leq \frac{k}{(n-k) \cdot 2^{n-2}} \gamma_{n-1}^{(n-1)} \\
&\leq \frac{k}{n(n-1)(n-k) \cdot 2^{3n-11}} \\
&< \frac{(k-1)^3}{n(n-k-1) \cdot 2^{2n-9}}.
\end{aligned}$$

Thus for $s = k + 1$, $\gamma_s^{(n-1)}$ satisfies Eq.(26).

Using Eq.(26) in Eq.(25), we have

$$\begin{aligned}
& |\Phi_i^{(n-1)}(v) - \Phi_j^{(n-1)}(v)| \\
& \leq M \left(\frac{(n-2)!}{n \cdot 2^{2n-9}} \sum_{s=0}^{n-2} \frac{(\frac{n-s}{2})^3 (n-s-2)}{s! \left(\frac{(n-s-2)}{2}\right)!^2 \frac{n+s}{2}} + \frac{1}{n} \sum_{s=0}^{n-2} \frac{(n-3)^3}{s \cdot 2^{2n-s-9}} \right) \\
& \leq M \left(\frac{(n-3)!(n-2)^2 \left(\frac{n+2}{2}\right)^3 \frac{n-2}{2}}{n \cdot 2^{2n-9} \left(\left(\frac{n-2}{2}\right)!\right)^3 \frac{3n-2}{4}} + \frac{(n-3)^3}{n \cdot 2^{n-7}} \right) \\
& \leq M \left(\frac{(n-3)!(n-2)^3 (n+2)^3}{n \cdot 2^{2n-11} \left(\left(\frac{n-2}{2}\right)!\right)^3 (3n-2)} + \frac{(n-3)^3}{n \cdot 2^{n-7}} \right) \rightarrow 0 \quad \text{whenever } n \rightarrow \infty.
\end{aligned} \tag{27}$$

This completes the proof. $\square$

7 Examples of the MI^k -value

Example 1 Let us consider the game v with $N = \{1, 2, 3, 4\}$ defined as in Table 1 below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1, 2\}$
$v(S)$	4	6	4	0	7
S	$\{1, 3\}$	$\{1, 4\}$	$\{2, 3\}$	$\{2, 4\}$	$\{3, 4\}$
$v(S)$	8	2	5	3	2
S	$\{1, 2, 3\}$	$\{1, 2, 4\}$	$\{1, 3, 4\}$	$\{2, 3, 4\}$	$\{1, 2, 3, 4\}$
$v(S)$	10	4.25	4	3.75	5.5

Table 1

Note that here 4 is an MN^3 -player. The MI^k -values for $(k = 0, 1, 2, 3)$ are respectively given below.

$$\begin{aligned}\Phi^{(0)}(v) &= \Phi^{Sh}(v) = (2.7083, 2.9583, 2.2083, -2.375) \\ \Phi^{(1)}(v) &= (2.1041, 2.1805, 1.7569, -0.5486) \\ \Phi^{(2)}(v) &= (1.851, 2.172, 1.67, -0.196) \\ \Phi^{(3)}(v) &= (1.7916, 2.07291, 1.6354, 0).\end{aligned}\tag{28}$$

Observe that, in this example the individual contributions of player 4 in all the coalitions are negative and therefore the Shapley value considers her sole responsible for reducing the worths of the coalitions. Therefore she receives negative payoff under the Shapley value. However the groups containing player 4 are seen to have both positive and negative contributions in different coalitions. Take for example, $v(1, 2, 3, 4) - v(1, 2, 3) = -4.5$, $v(1, 2, 4) - v(1, 2) = -2.75$ etc. and $v(1, 2, 3, 4) - v(1) = 1.5$; $v(1, 2, 4) - v(1) = 0.25$ etc. Therefore, player 4 is not so negative as deemed by the Shapley value and is entitled to some solidarity from her peers or at least should not be penalized. Since her overall multilateral contribution is zero she should not get any positive payoff either. Thus she receives zero payoff under the MI^k -value.

Example 2 Consider another game v with the player set $N = \{1, 2, 3\}$, given in Table 2.

S	$\{1\}$	$\{2\}$	$\{3\}$
$v(S)$	0	0	1
S	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$
$v(S)$	0	0	0
S	$\{1, 2, 3\}$		
$v(S)$	10		

Table 2

The MI^k -values are respectively given below.

$$\begin{aligned}\Phi^{(0)}(v) &= \Phi^{Sh}(v) = (3.16, 3.16, 3.68) \\ \Phi^{(1)}(v) &= (3.19, 3.19, 3.62) \\ \Phi^{(2)}(v) &= (3.22, 3.22, 3.56).\end{aligned}\tag{29}$$

A close look at the values taken by v and the corresponding MI^k -values reveal that players 1 and 2 are individually non-productive and also their unilateral interactions among themselves and with player 3 fail to produce any non-zero worths. However when all three of them join together they generate a sufficiently large worth which player 3 can never generate alone. The Shapley value ($k = 0$ in the MI^k -value) considers only the individual contributions of the players in the coalitions and awards player 1 and 2 more than the MI^k -value at levels $k = 1$ and $k = 2$. It follows that when we start considering the interactions at higher levels the MI^k -value awards player 3 less because it accounts for the multilateral interactions of player 3 with 1 and 2 together. Even if we increase the worths of player 3 till 10, we see the same phenomenon. For example if we keep everything same and take $v(3) = 6$, the MI^k -values are found to be $\Phi^{Sh}(v) = \Phi^{(0)}(v) = (2.33, 2.33, 5.34)$; $\Phi^{(1)}(v) = (2.5, 2.5, 5)$; $\Phi^{(2)}(v) = (2.7, 2.7, 4.58)$. Both these examples show that the MI^k -value is more egalitarian.

8 Conclusion

This paper proposes a new value for TU Cooperative games that accounts for group contributions of the players. The characterization includes the standard Shapley like axioms of linearity, anonymity, efficiency along with a new axiom: the axiom of MN^k -player. It is shown that the notion of multilateral interactions attributing to the group contributions extends the ordinary interactions whom we call here the unilateral interactions. An MN^k -player is one whose average marginal contribution due to her multilateral interactions upto level k is zero and can also be seen as a generalization of the standard

null player axiom of the Shapley value. We show that for large number of players and equally large level of multilateral interactions the MI^k -value approaches to the equal division rule. Thus due to the players' interaction levels MI^k -value is the Shapley value on one extreme and the equal division rule on the other extreme over a variable player set. This distinguishes the MI^k -value from similar values viz., the egalitarian Shapley value, the discounted Shapley value etc., found in the literature, see for example (van den Brink et al., 2013; van den Brink and Funaki, 2015). The egalitarian Shapley value is a convex combination of the Shapley and the Equal Division rule. The null and nullifying player are crucial in obtaining the Shapley value and the Equal Division. The Weak null player out property replaces the null and nullifying player property in the characterization of the egalitarian Shapley value which builds on the condition that deleting a null player does not change the payoffs of the other players. The discounted Shapley value on the other hand penalizes those players who reduce the worth of a coalition by $\delta < 1$. Note that, the axioms of null player or the null player out take one player into account at a time and so is based on the unilateral interaction while the nullifying player accounts for all at once. Thus they are the two extremes of the notion of multilateral interactions. Consequently the notion of multilateral interaction gives a more general approach to the existing notions.

Moreover the idea of multilateral interactions does not have a precise *a priori* organizational structure, on the contrary it explores the possibilities of mutual influence among the players. Therefore it is different from a Clan game (Potters et al., 1989) where some players have veto power or games with permission structure (Gilles et al., 1992) where the organization has a hierarchical structure. Alternative characterizations of the multilateral value and its applications to networks are kept for our future study.

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9 Appendix

In the following we present an alternative proof of Theorem 2. Suppose that a value Φ satisfies Eff, AN, and Lin. Since AN implies Sym, Φ has the form given by Eq.(8) in Theorem 1. Also by Lemma 3, there exist real constants λ_s for all $S \subseteq N \setminus i$ such that for every $v \in \mathcal{G}(N)$,

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \sum_{\substack{T \subseteq S \\ 0 \leq i \leq k}} \lambda_s \left[v(S \cup i) - v(S \setminus T) \right]. \quad (30)$$

We take the value Φ in Eq.(30) such that Φ satisfies MNP^k . Rewrite Eq.(30) as follows.

$$\begin{aligned}
\Phi_i(v) &= \lambda_{n-1} \left(\sum_{t=0}^k \binom{n-1}{t} \right) v(N) - \lambda_{n-1} \sum_{T \subseteq N \setminus i: 0 \leq t \leq k} v(N \setminus T \cup i) \\
&\quad + \sum_{S \subsetneq N \setminus i} \lambda_s \sum_{T \subset S: 0 \leq t \leq k} \left[v(S \cup i) - v(S \setminus T) \right] \\
&= \lambda_{n-1} \left(\sum_{t=0}^k \binom{n-1}{t} \right) v(N) + \sum_{S: i \in S \neq N} \lambda_{s-1} \left(\sum_{t=0}^{\min(s-1, k)} \binom{s-1}{t} \right) v(S) \\
&\quad - \lambda_{n-1} \sum_{T \subseteq N \setminus i: 0 \leq t \leq k} v(N \setminus T \cup i) - \sum_{S \subsetneq N \setminus i} \lambda_s \sum_{T \subset S: 0 \leq t \leq k} v(S \setminus T) \\
&= \lambda_{n-1} \left(\sum_{t=0}^k \binom{n-1}{t} \right) v(N) + \sum_{S: i \in S \neq N} \lambda_{s-1} \left(\sum_{t=0}^{\min(s-1, k)} \binom{s-1}{t} \right) v(S) \\
&\quad - \sum_{S: i \notin S} \sum_{t=0}^{\min(n-s-1, k)} \lambda_{s+t} \binom{n-s-1}{t} v(S). \tag{31}
\end{aligned}$$

Now, Eq.(31) is identical to Eq.(8), therefore equating the corresponding terms, we have

$$\begin{aligned}
\lambda_{n-1} \left(\sum_{t=0}^k \binom{n-1}{t} \right) &= \frac{1}{n} \\
\frac{\rho_s}{s} &= \lambda_{s-1} \left(\sum_{t=0}^{\min(s-1, k)} \binom{s-1}{t} \right) \\
\frac{\rho_s}{n-s} &= \sum_{t=0}^{\min(n-s-1, k)} \lambda_{s+t} \binom{n-s-1}{t}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda_{n-1} &= \frac{1}{n \left(\sum_{t=0}^k \binom{n-1}{t} \right)} \\
\lambda_{s-1} &= \left(\frac{n-s}{s} \right) \frac{\sum_{t=0}^{\min(n-s-1, k)} \lambda_{s+t} \binom{n-s-1}{t}}{\sum_{t=0}^{\min(s-1, k)} \binom{s-1}{t}}.
\end{aligned}$$

These recursive relations coincide with the recursive relations of $\gamma_s^{(k)}$ in Theorem 2. This completes the proof.