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Ullah, Z., Kaczmarczyk, L., Zhou, X., Falzon, B., & Pearce, C. (2020). Hierarchical finite element-based multi-scale modelling of composite laminates. *Composites Part B: Engineering*.
<https://doi.org/10.1016/j.compositesb.2020.108321>

Published in:
Composites Part B: Engineering

Document Version:
Peer reviewed version

Queen's University Belfast - Research Portal:
[Link to publication record in Queen's University Belfast Research Portal](#)

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Hierarchical finite element-based multi-scale modelling of composite laminates

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Abstract

This paper presents a hierarchic finite element-based computational framework for the multi-scale modelling of composite laminates. Hierarchic finite elements allow changing the approximation order locally or globally without changing the underlying finite element mesh. Both micro- and macro-level structures are discretised with these elements. The macro-level structures of composite laminates are divided into several blocks during the pre-processing stage, and approximation orders are assigned to each block independently. Due to a sharp increase in the interlaminar stresses, higher approximation orders are used in the vicinity of free edges as compared to the rest of the problem domain. This freedom of assigning approximation orders independently to each block provides an efficient and accurate way for modelling composite laminates. The computation framework can either accept the user-defined ply-level homogenised elastic material properties or calculates these directly from the underlying representative volume element consisting of matrix and fibre using the computational homogenisation. The model developed for the computational homogenisation has the flexibility of unified imposition of representative volume element boundary conditions, which allows convenient switching between linear displacement, uniform traction and periodic boundary conditions. The computational framework has additional flexibility of high-performance computing and makes use of state-of-the-art computational libraries including Portable, Extensible Toolkit for Scientific Computation (PETSc) and the Mesh-Oriented datABase (MOAB). Symmetric cross-ply, angle-ply and quasi-isotropic laminates subjected to uniaxial loading are used as test cases to demonstrate the correct implementation and computational efficiency of the developed computational framework.

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Keywords: Multi-scale modelling, Computational homogenisation, Hierarchic finite elements, Composites laminates, Free-edge effect, interlaminar stresses

1. Introduction

Fibre reinforced polymer (FRP) composite materials offer exceptional chemical and mechanical properties and are therefore used in a variety of industrial applications [1–3]. The freedom of choice of the fibre volume fraction and their orientation in each ply within composite laminates provide enormous flexibility in the design of composite structures. On the other hand, the mismatch in material properties across different plies resulting in very high interlaminar stresses, especially on approaching the free edges. This often leads to initiation of delamination and subsequent failure of composite laminates due to their very low interlaminar strengths. The classical analytical model cannot accurately predict these stresses ([4, 5]). Therefore, numerical methods are an ideal choice for the accurate calculation of interlaminar stresses. Due to the heterogeneous nature of composite materials, their macro-level material properties are dependent on their underlying micro-level structures. Therefore, multi-scale modelling consisting of computational homogenisation (CH) of the micro-level constituents, coupled with the macro-level numerical analysis, provides a consistent computational framework for the numerical simulation of these structures.

A variety of analytical and numerical methods, dating back to 1967, have been proposed in the literature for calculating interlaminar stresses in composite laminates. Analytical methods ([5–12]) can efficiently provide an approximation to interlaminar stresses within a composite laminate. On the other hand, numerical methods provide a very accurate framework for the simulation of composite laminates. In literature, numerous numerical methods have been proposed for the analysis of composite laminates subjected to a variety of loading scenarios. A numerical procedure based on finite-difference was proposed in [13]. Two-dimensional four-nodes quadrilateral elements, based on the formulation of constant strain, were developed in [14]. These elements were subsequently used for the simulation of symmetric cross-ply $([0/90]_s)$, angle-ply $([\pm 45]_s)$ and quasi-isotropic $([\pm 45/0/90]_s)$ laminates. Three-dimensional eight-nodes hexahedral elements were developed in [15] and were used for the simulation of symmetric cross-ply $([90/0]_s)$ and angle-ply $([\pm 45]_s)$ laminates. A multi-particle model of multilayered material (M4) was developed in [16]. An elasticity-based finite element formulation was developed in [17] for T300/934 $([\pm 25/90]_s)$ laminates. For composite laminates subjected to uniaxial tension, bending, twisting and thermal loading, a mixed-field multi-term extended Kantorovich method (MMEKM) was developed in [4]. The effect of layer shifting on intralaminar and interlaminar stresses in woven composites was studied in [18] using finite elements. In [19], a numerical approach was introduced to investigate generic laminated composites with arbitrary geometries.

The model used a higher-order displacement-based beam theory. Interlaminar stresses in both conventional and wrap-around laminates were investigated using finite element analysis in [20]. In [21], commercial finite element software, ANSYS, was used to study hybrid and non-hybrid composite laminates consisting of both carbon and glass fibres. A coupled stress and energy criteria, using finite fracture mechanics, was used in [22]. The scale-boundary finite element technique was used as a numerical tool, which due to its semi-analytical nature, reduces the numerical effort significantly as compared to standard finite element analysis.

Due to the heterogeneous nature of composite laminates, multi-scale modelling is the arguably most appropriate computational framework for their simulation. The macro-level homogenised or effective properties of these materials are calculated using CH [3, 23]. These homogenised properties are then used for the analysis of the macro-level structures. As compared to analytical homogenisation procedures, numerical techniques provide a better prediction of their homogenised mechanical properties. In [24], a multi-scale model was used for the modelling of composite laminates. A macro-level model was used to capture the laminate stacking sequence and the global stress field for a given loading. A micro-level model was used for the prediction of stresses at the fibre and matrix level. The model was used to examine the $[90/90]$ interface of a $[25_n / -25_n / 90_n]_s$ carbon/epoxy laminate. Another multi-scale model was developed in [25], for the investigation of the effect of heterogeneity in the micro-structure for a $[\pm 45/0/90]_s$ laminate subjected to uniaxial tension. The 0° and 90° layers were modelled at the micro level including random fibres within the polymer matrix. On the other hand, the $\pm 45^\circ$ layers were modelled using homogenised or effective properties. In [26], a global-local approach was used. The available software packages were used for macro scale analysis and Carrera Unified Formulation (CUF) was used for the in-depth analysis of critical domains.

With the conventional finite element analysis often used in the literature, dense meshes are required for the determination of accurate interlaminar stresses in composite laminates in the vicinity of free edges due to singularity of stresses. This paper presents a hierarchical finite element-based computational framework for the multi-scale modelling of composite laminates. The hierarchic finite elements allow changing the order of approximation locally or globally without changing the underlying finite element mesh, leading to very accurate results for relatively coarse meshes [27–30]. In hierarchic finite elements, high order shape functions are calculated from low order shape functions recursively and therefore maintain the continuity in the shape functions across the element boundaries in the case of localised p -refinement. In our previous publications ([23, 31–33]), the approximation order was increased globally within the problem domain for improving the accuracy. In this paper, the flexibility of the hierarchical finite elements is fully exploited. The macro-level problem domain of composite laminates is divided into several blocks during the pre-processing stage and approximation orders are assigned to each block independently. As compared to the rest of the problem domain, higher

approximation orders are used near the free edges. This provides an efficient and accurate way of capturing a sharp increase of interlaminar stresses near the free edges. A detailed study is also conducted to compare the computational cost in terms of degrees of freedom and simulation time for increasing the approximation order globally and locally. The computational framework allows to change the approximation order for each block and angle for each ply within laminate from the command line and therefore provide a very convenient way to change these without changing the code or input files. Moreover, as compared to [33], the computational framework in the current paper has additional flexibility of either accepting the ply-level homogenised elastic material properties or calculates these directly from the CH of the underlying representative volume element (RVE) consisting of matrix and fibres. The formulation for the CH has the flexibility of unified imposition of the RVE boundary conditions, which allows convenient switching between linear displacement, uniform traction and periodic boundary conditions [23, 31, 32, 34]. As compared to the local p -refinement on the macro-level structure, global p -refinement is used in the case of the CH. Tetrahedral elements are used for the discretisation of both micro- and macro-level structures. Symmetric cross-ply ($[90/0]_s$ and $[0/90]_s$), angle-ply ($([\pm 45]_s)$) and quasi-isotropic ($[45/-45/0/90]_s$ and $[90/0/-45/45]_s$) laminates, subjected to uniaxial loading are considered as test cases to demonstrate the correct implementation and computational efficiency of the developed computational framework. On a micro level, the matrix is modelled as linear elastic and isotropic material while fibres are modelled as linear elastic and transversely-isotropic material. On a macro level, each ply within the composite laminates is modelled as a linear elastic, homogenised and orthotropic material. The computational model is implemented in an open-source finite element software library, Mesh-Oriented Finite Element Method (MoFEM) [30], which make use of the state-of-the-art computational libraries including Portable, Extensible Toolkit for Scientific Computation (PETSc) [35] and the Mesh-Oriented datABase (MOAB) [36].

The rest of the paper is organised as follows. The computational framework is given in Section 2. A brief introduction of the implementational aspects of the hierarchic basis functions is given in Section 2.1. Formulation for the CH is given in Section 2.2. A detailed description of the theoretical background and mathematical formulation of composite laminates are given in Section 2.3. Four numerical examples are presented in Section 3 to demonstrate the correct implementation and efficiency of the computational framework. The first (Section 3.1), second (Section 3.2) and third (Section 3.3) numerical examples are symmetric cross-ply, angle-ply and quasi-isotropic laminates respectively. The fourth numerical example about the multi-scale modelling is presented in Section 3.4. Finally, concluding remarks are given in Section 4.

2. Computational framework

The developed multi-scale computational framework consists of the calculation of the effective or homogenised mechanical properties using the CH. It also provides a detailed formulation for the simulation of composite laminates at the macro-level. Both micro- and macro-level structures are discretised with hierarchic finite elements. A detailed description of these constituents of the computational framework are provided in the following.

2.1. Hierarchic finite element formulation

Both micro- and macro-level structures are discretised with tetrahedral elements. Therefore, a brief description of the construction of hierarchic shape functions for these elements is provided, further details of which are given in [27–30, 33, 37, 38]. These elements allow changing the order of approximation both globally and locally without changing the underlying finite element mesh. Due to their hierarchic nature, these elements preserve the continuity of the shape functions across the element boundaries. For the first approximation order, i.e. $p = 1$, one linear shape function is associated with each of the four vertices of the tetrahedral element. With increasing the approximation order, the linear shape functions associated with the vertices, remain unchanged and additional high-order shape functions are added to its edges, faces and interior. These high-order shape functions add additional degrees of freedom to the element.

Consider a one-dimensional reference interval $I = [-1, 1]$ for the construction of hierarchic shape functions $\{\phi_l : l = 0, 1, \dots\}$. The shape functions associated with its vertices are the standard linear finite element shape functions. The high order shape functions associated with increasing the order of approximation are constructed by combining these linear shape functions and some arbitrary polynomials $\{\psi_l : l = 0, 1, \dots\}$ of degree l . There is no restriction on the choice of polynomials but Legendre polynomials are often used. Legendre polynomials of high order are obtained recursively from their low order counterparts. The equation for Legendre polynomial L_l of degree l is given as:

$$\begin{aligned} L_0(s) &= 1, & L_1(s) &= s, \\ L_{l+1}(s) &= \frac{2l+1}{l+1}sL_l(s) - \frac{l}{l+1}L_{l-1}(s), & l &= 1, 2, \dots \end{aligned} \tag{1}$$

For a line, triangle and tetrahedron, the linear shape functions associated with their vertices

are given as:

$$\begin{aligned}
N_o &= 1 - \xi, & N_i &= \xi, & \text{Edge} \\
N_o &= 1 - \xi - \eta, & N_i &= \xi, & N_j &= \eta, & \text{Triangle} \\
N_o &= 1 - \xi - \eta - \zeta, & N_i &= \xi, & N_j &= \eta, & N_k &= \zeta, & \text{Tetrahedron}
\end{aligned} \tag{2}$$

Where ξ , η and ζ are the reference coordinates of the corresponding line, triangle and tetrahedron elements. With known linear shape functions (Equation (2)) and Legendre polynomials (Equation (1)), the high order shape functions for edges $e = [o, i]$, faces $f = [o, i, j]$ and interior of the tetrahedron $t = [o, i, j, k]$ are given as:

$$\begin{aligned}
\phi_l^e &= \beta_{oi} \psi_l(\hat{\zeta}_{oi}), & 0 \leq l \leq p-2 & & \text{Edge} \\
\phi_{lm}^f &= \beta_{oij} \psi_l(\hat{\zeta}_{oi}) \psi_m(\hat{\zeta}_{oj}), & 0 \leq l, m, \quad l+m \leq p-3 & & \text{Face} \\
\phi_{lmn}^t &= \beta_{oijk} \psi_l(\hat{\zeta}_{oi}) \psi_m(\hat{\zeta}_{oj}) \psi_n(\hat{\zeta}_{ok}), & 0 \leq l, m, n \quad l+m+n \leq p-4 & & \text{Tetrahedron Interior}
\end{aligned} \tag{3}$$

where $\beta_{oi} = N_o N_i$, $\beta_{oij} = N_o N_i N_j$, $\beta_{oijk} = N_o N_i N_j N_k$, $\hat{\zeta}_{oi} = N_i - N_o$, $\hat{\zeta}_{oj} = N_j - N_o$ and $\hat{\zeta}_{ok} = N_k - N_o$. Moreover, p is the user-defined order of approximation for the finite element analysis. For the first order of approximation, i.e. $p = 1$, there will be only linear shape functions N_0 and N_1 associated with its vertices. For $p = 2$, in addition to the linear shape functions N_0 and N_1 , we will have one high order shape function, i.e. ϕ_0^{oi} . For a three-dimensional mechanical problem, each shape function is used for the approximation of three degrees of freedom, i.e. displacement in x -, y - and z -direction. The number of degrees of freedom associated with high-order shape functions for each edge (DOF_e), face (DOF_f) and interior (DOF_t) of the tetrahedral element, depend on the order of approximation (p) and are given as:

$$\begin{aligned}
\text{DOF}_v &= 3, \quad \text{DOF}_e = 3 \times (p-1), \quad \text{DOF}_f = 3 \times \frac{(p-2)(p-1)}{2}, \\
\text{DOF}_t &= 3 \times \frac{(p-3)(p-2)(p-1)}{6}
\end{aligned} \tag{4}$$

For standard finite elements, numerical integration is well documented but literature is very limited in the case of hierarchic finite elements. The number of integration points and their corresponding position are selected accordance to Grundmann and Moller integration rule [37], i.e.

$$n_g = \frac{n!}{l!(n-1)!}, \quad n = d + l + 1, \tag{5}$$

where l is the degree of polynomial and $d = 1, 2$, or 3 is the dimensionality of problem.

Only tetrahedral elements are used in this paper, automatically generated using Trelis. Tetrahedral elements are normally used for meshing complicated geometries with automatic (or semi-automatic) mesh generation algorithms [39]. Both tetrahedral and hexahedral meshes are used in [40] for the calculation of homogenised mechanical properties of 2×2 Twill woven composites. For the same number of elements, tetrahedral elements required a significantly lower number of degrees of freedom, leading to computational efficiency. In [41], tetrahedral elements were used for the woven composites due to their complicated geometries.

2.2. Computational homogenisation

The working principle of multi-scale CH is shown in Fig. 1. The macro-level structure is a composite laminate consisting of four plies. Each ply is modelled as homogenous, linear elastic and orthotropic material. These homogenised material properties, e.g. at a macro-level integration point $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ are calculated from the CH of the underlying RVE, consisting of matrix and fibres. This transition from macro to micro consists of construction of a micro-level boundary value problem using the macro-level strain $\bar{\boldsymbol{\varepsilon}} = \begin{bmatrix} \bar{\varepsilon}_{11} & \bar{\varepsilon}_{22} & \bar{\varepsilon}_{33} & 2\bar{\varepsilon}_{12} & 2\bar{\varepsilon}_{23} & 2\bar{\varepsilon}_{31} \end{bmatrix}^T$. The solution of the RVE boundary value problem is followed by the calculation of the homogenised stress $\bar{\boldsymbol{\sigma}} = \begin{bmatrix} \bar{\sigma}_{11} & \bar{\sigma}_{22} & \bar{\sigma}_{33} & \bar{\sigma}_{12} & \bar{\sigma}_{23} & \bar{\sigma}_{31} \end{bmatrix}^T$ and stiffness matrix $\bar{\mathbf{C}}$. This step is also referred to as a transition from micro to macro. An integration point associated with the RVE is represented by $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$. At this micro-level integration point, the displacement field is written as

$$\mathbf{u}_\mu(\mathbf{y}) = \bar{\boldsymbol{\varepsilon}}(\mathbf{x})\mathbf{y} + \tilde{\mathbf{u}}_\mu(\mathbf{y}), \quad (6)$$

where $\bar{\boldsymbol{\varepsilon}}\mathbf{y}$ and $\tilde{\mathbf{u}}_\mu$ are linear and fluctuating displacement terms. Strain at point $\mathbf{y}$ is written as

$$\boldsymbol{\varepsilon}_\mu(\mathbf{y}) = \nabla^s \mathbf{u}_\mu = \bar{\boldsymbol{\varepsilon}}(\mathbf{x}) + \tilde{\boldsymbol{\varepsilon}}(\mathbf{y}), \quad (7)$$

consisting of macro-level strain $\bar{\boldsymbol{\varepsilon}}(\mathbf{x})$ and strain fluctuation $\tilde{\boldsymbol{\varepsilon}}(\mathbf{y})$.

In this paper, the following three types of RVE boundary conditions are considered [32]:

1. Linear boundary displacement: In this case, it is assumed that the displacement fluctuation $\tilde{\mathbf{u}}_\mu(\mathbf{y})$ is zero over the RVE boundary leading to fully prescribed displacement on the RVE boundary.
2. Periodic boundary conditions: In this case displacement ($\tilde{\mathbf{u}}_\mu(\mathbf{y}^+)$) and traction ($\mathbf{t}(\mathbf{y}^+)$)

Figure 1: Multi-scale computational homogenisation.

are assumed to be periodic and anti-periodic respectively, i.e.

$$\left. \begin{aligned} \tilde{\mathbf{u}}_\mu(\mathbf{y}^+) &= \tilde{\mathbf{u}}_\mu(\mathbf{y}^-) \\ \mathbf{t}(\mathbf{y}^+) &= -\mathbf{t}(\mathbf{y}^-) \end{aligned} \right\} \quad \forall \text{ pairs } \left\{ \mathbf{y}^+, \mathbf{y}^- \right\}, \quad (8)$$

here, $\mathbf{y}^+ \in \partial\Omega_\mu^+$ and $\mathbf{y}^- \in \partial\Omega_\mu^-$ are two opposite points on the RVE boundary.

3. Uniform traction boundary conditions: In this case the traction on the RVE boundary is prescribed using the macro-level stress $\bar{\boldsymbol{\sigma}}$, i.e.

$$\mathbf{t} = \bar{\boldsymbol{\sigma}} \cdot \mathbf{n}, \quad (9)$$

where $\mathbf{n}$ represents normal to the RVE boundary.

These three types of RVE boundary conditions generate different RVE behaviour. The linear displacement boundary conditions generate the stiffest RVE response. On the other hand, the traction boundary condition leads to a least kinematically constrained response. The response of the periodic boundary conditions lies between the displacement and traction boundary conditions. In this paper, the RVE boundary conditions are implemented in a unified manner [23, 31, 32, 34]. The discretised system of equations for the RVE is written as:

$$\begin{bmatrix} \mathbf{K} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{D}\bar{\boldsymbol{\epsilon}} \end{Bmatrix}, \quad (10)$$

here $\mathbf{K}$ is the stiffness matrix and $\mathbf{u}$ is the a vector of unknown displacements. Moreover, $\boldsymbol{\lambda}$ is

an unknown vector of Lagrange multipliers and Matrices $\mathbf{C}$ and $\mathbf{D}$ are calculated over the RVE boundary and are given as

$$\mathbf{C} = \int_{\partial\Omega_\mu} \mathbf{H}\mathbf{N}^T \mathbf{N} d\partial\Omega_\mu, \quad \mathbf{D} = \int_{\partial\Omega_\mu} \mathbf{H}\mathbf{N}^T \mathbf{X} d\partial\Omega_\mu. \quad (11)$$

A detailed description of matrices $\mathbf{N}$, $\mathbf{X}$ and $\mathbf{H}$ are given in [23, 31, 34] and is not repeated here. Finally, homogenised stress is calculated from the resulted Lagrange multipliers $\boldsymbol{\lambda}$ using

$$\bar{\boldsymbol{\sigma}} = \frac{1}{V} \mathbf{D}^T \boldsymbol{\lambda}, \quad (12)$$

where V is the volume of the RVE. Similarly, the homogenised stiffness matrix $\mathbf{C}$, which relates the macro-level stress $\bar{\boldsymbol{\sigma}}$ and strain $\bar{\boldsymbol{\varepsilon}}$, is calculated by solving six RVE problems subjected to six unit strains leading to

$$\bar{\mathbf{C}} = \begin{bmatrix} \bar{\boldsymbol{\sigma}}^1 & \bar{\boldsymbol{\sigma}}^2 & \bar{\boldsymbol{\sigma}}^3 & \bar{\boldsymbol{\sigma}}^4 & \bar{\boldsymbol{\sigma}}^5 & \bar{\boldsymbol{\sigma}}^6 \end{bmatrix}, \quad (13)$$

where for example:

$$\begin{aligned} \bar{\boldsymbol{\sigma}}^1 : & \text{ for } \bar{\boldsymbol{\varepsilon}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \bar{\boldsymbol{\sigma}}^4 : & \text{ for } \bar{\boldsymbol{\varepsilon}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T. \end{aligned} \quad (14)$$

Finally, a detailed formulation for the calculation of mechanical properties including Young's moduli, shear moduli and Poisson's ratios, from the calculated stiffness matrix $\bar{\mathbf{C}}$, is given in [32]. In the proposed computational framework, periodic geometries and meshes are prerequisites to impose the periodic boundary condition on the RVE. On the other hand, for the linear displacement and uniform traction boundary conditions, there is no requirement of the periodic geometries and meshes and therefore will allow random packing of fibres. A similar analysis using the same computational framework is provided by the authors in [32]. In the future, the computational framework will be extended to implement the periodic boundary conditions for nonperiodic geometries and meshes.

2.3. Composite laminates

Consider a square symmetric $([\pm\theta]_s)$ laminate of length $2l$, consisting of four plies, each of thickness h as shown in Fig. 2. Local coordinates for each ply are represented by 123, where 1, 2 and 3 are in-plane and out-of-plane directions respectively. For the laminate, the global coordinate system is represented with xyz , where x , y and z are the in-plane and out-of-plane directions respectively. The in-plane angle between the local and global coordinate systems is represented by θ . The laminate is subjected to unidirectional strain, ϵ_0 , along the y -axis. Each ply is modelled as a linear elastic, homogenous and orthotropic material. The computation

framework has additional flexibility of either accepting the ply-level homogenised elastic material properties or calculates these directly from the micro-level elastic properties of matrix and fibre using the CH.

Figure 2: Geometry, boundary conditions and coordinate systems for composite laminate

The global stiffness matrix ($\mathbf{C}_{xyz}$) for the laminate is calculated from the local stiffness matrix ($\mathbf{C}_{123}$) of a ply using:

$$\mathbf{C}_{xyz} = \mathbf{T}^{-1} \mathbf{C}_{123} \mathbf{T}^{-T}, \quad (15)$$

here $\mathbf{C}_{123} = \mathbf{S}_{123}^{-1}$, where $\mathbf{S}_{123}$ is the ply-level compliance matrix and is given as:

$$\mathbf{S}_{123} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} & -\frac{\nu_{31}}{E_{33}} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} & -\frac{\nu_{32}}{E_{33}} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_{11}} & -\frac{\nu_{23}}{E_{22}} & \frac{1}{E_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{23}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{31}} \end{bmatrix}, \quad (16)$$

where E_{11} , E_{22} and E_{33} are the ply-level Young's moduli in the fibre, transverse and through-thickness directions respectively. Similarly, ν_{ij} and G_{ij} are ply-level Poisson's ratios and shear

moduli respectively. $\mathbf{T}$ is a transformation matrix and is given as:

$$\mathbf{T} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & 0 & 0 & 0 & \sin 2\theta \\ \sin^2 \theta & \cos^2 \theta & 0 & 0 & 0 & -\sin 2\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \theta & -\sin \theta & 0 \\ 0 & 0 & 0 & \sin \theta & \cos \theta & 0 \\ -\frac{1}{2} \sin 2\theta & \frac{1}{2} \sin 2\theta & 0 & 0 & 0 & \cos 2\theta \end{bmatrix}. \quad (17)$$

3. Numerical examples

Four numerical examples are presented in the following to demonstrate the correct implementation and performance of the developed computational framework. The first and second numerical examples are symmetric cross-ply laminate ($[90/0]_s$ and $[0/90]_s$) and angle-ply laminate ($[\pm 45]_s$) respectively. Quasi-isotropic symmetric laminates ($[45/-45/0/90]_s$ and $[90/0/-45/45]_s$) are provided as a third numerical example. Multi-scale modelling of symmetric cross-ply laminates is presented as a fourth numerical example. Although, only symmetric laminates are provided as numerical examples but the computational framework can be equally used for non-symmetric laminates. To fully exploit the flexibility offered by the hierarchical finite elements, composite laminates in the following numerical examples are divided into three blocks during the pre-processing stage. These blocks are selected manually during the pre-processing stage to create relatively fine meshes and use higher approximation orders near free edges. Ideally, these regions should be selected using an adaptive procedure, which will be pursued in future studies.

3.1. Cross-ply laminate

The geometry, boundary conditions and coordinate system for the cross-ply laminate are shown in Fig. 3. A strain of $\epsilon_0 = 1 \times 10^{-6}$ is applied to its y -faces, while the remaining faces are traction free. Due to symmetry about the x -, y - and z -axis, only one-eighth of the laminate is modelled, as shown in the same figure. Symmetric boundary conditions are applied to faces at $x = 0$, $y = 0$ and $z = 0$. Displacement of $u_y = l\epsilon_0$ is applied to the y -face at $y = l$. The face at $x = l$ is a free-edge. In this paper, the relative dimensions of the laminate are chosen as $h = 0.25$ in and $l = 8h$. The ply-level material properties used in this case are given in Table 1, where 1, 2 and 3 represent the fibre, transverse and through-the-thickness directions respectively. All geometrical parameters and material properties are in compatible units. The same problem with the same geometrical parameters and material properties was also used as a test case in [13–15]. The mesh for this problem is shown in Fig. 4(a) consisting of 9,428 tetrahedral

elements and 2,034 nodes. The arrangement of blocks for assigning arbitrary order is shown in Fig. 4(b). Two types of cross laminates, i.e. $[90/0]_s$ and $[0/90]_s$ are considered.

Table 1: Ply-level elastic material properties ([13–15])

E_{11} psi	$E_{22} = E_{33}$ psi	$G_{12} = G_{13} = G_{23}$ psi	$\nu_{12} = \nu_{13} = \nu_{23}$
20×10^6	2.1×10^6	0.85×10^6	0.21

This problem is initially solved with global p -refinement with four global approximation orders ranging from $p = 1$ to $p = 4$. The distribution of the out-of-plane normal stress, σ_{zz} , versus x/l at $y = 0$ and $z = 0$ is shown in Fig. 5(b). Results are compared with the reference results from [14]. These reference numerical results were obtained using the four-nodes quadrilateral elements, based on the formulation of constant strain. For very low global approximation order, i.e. $p = 1$, stresses are not smooth, especially in the vicinity of the free-edge. This is due to the use of only linear shape functions which lead to constant strain and stress within each element. The use of higher approximation order, i.e. $p = 2$ smoothened the stress distribution and results match exactly with the reference results. For this case, there is no further improvement in σ_{zz} by further increasing the global approximation order, i.e. using $p = 3, 4$. The distribution of out-of-plane normal (σ_{zz}) and shear (σ_{zx}) stresses versus x/l at $y = 0$ and $z = h$ are shown in Fig. 5(c) and Fig. 5(d) respectively. Similar to σ_{zz} at $z = 0$, results obtained with $p = 1$ are not smooth and do not match well with the reference results. With increasing the order of approximation, both σ_{zz} and σ_{zx} approaching the reference results. These results are not the same as reference results near the free-edge, which is due to the use of very coarse mesh in this case. Very good agreement with the reference results was achieved in our previous publication [33] with relatively fine mesh near the free-edge. Through-thickness distribution of out-of-plane stress σ_{zz} versus z/h near the free-edge for both $[90/0]_s$ and $[0/90]_s$ is shown in Fig. 6. These stresses are provided at two locations along the length of the laminate, i.e. at $0.977l$ and $0.999l$. For $0.977l$, stresses are shown in Fig. 6(a) and for $0.999l$, stresses are shown in Fig. 6(b). Reference results from the literature ([14]) are also provided in these figures for comparison. These stresses are not very smooth in the case of lower global approximation order, i.e. $p = 1$. With increasing global approximation order, i.e. $p = 2, 3, 4$, the calculated stresses are very smooth and in a good agreement with the reference results. As mentioned previously, further improvement in these stresses can be achieved by combining p -refinement with h -refinement.

To fully exploit the flexibility offered by the hierarchical finite elements, composite laminate is divided into three blocks during the pre-processing stage as shown in Fig. 4 (a). These blocks are named as Block-1, Block-2 and Block-3. Relative dimensions of these blocks are also shown in the same figure. Block-3 is a region with relative radius $R/l = 0.2$ centred at $x = l$ and Block-2 lies between radii $R/l = 0.2$ and $R/l = 0.4$. The computational framework allows assigning approximation orders independently to each block. Due to a sharp increase in the interlaminar

stresses, higher approximation orders are used near the free-edge as compared to the rest of the problem domain. In this case, only a $[0/90]_s$ laminate is considered. The distribution of σ_{zz} versus x/l at $y = 0$ and $z = 0$ for a variety of approximation orders within these blocks are shown in Fig. 7(b)-(e). The first, second and third number in the legend, e.g. "Order-112" in Fig. 7(b) are local approximation orders in Block-1, Block-2 and Block-3 respectively. In Fig. 7(b), results are obtained while using $p = 1$ in Block-1 and Block-2, and $p = 2, 3$ are used in Block-3. The distribution of σ_{zz} is compared with reference results from [14]. As expected, $p = 1$ in Block-1 and Block-2 leads to σ_{zz} in these block, which are not very smooth. Higher approximation orders ($p = 2, 3$) in Block-3 provides a very smooth and accurate distribution of σ_{zz} in the vicinity of the free-edge. This combination of low approximation orders away from the free-edge and higher-order in the vicinity of the free-edge leads to very accurate results for relatively coarse meshes. Different combination of approximation orders are used in Fig. 7(c)-7(e) including 122, 123, 223, 224, 334 and 335. In this case, convergence for σ_{zz} is achieved for 223 and there is no further improvement in the results by increasing the approximation orders.

Results for σ_{zz} and σ_{zx} at $y = 0$ and $z = h$ for a variety of approximation orders are given in Fig. 8 and Fig. 9 respectively. In these cases, relatively low approximation orders, i.e. $p = 1, 2, 3$ are used in Block-1 and Block-2 but in Block-3 the approximation order is increased to fifth order, i.e. $p = 5$. The discrepancy in the results near the free-edge can be improved by refining the mesh locally, as explained previously. The effect of changing the local approximation order on the through-thickness distribution of σ_{zz} at two locations along the length of the laminate, i.e. $x/l = 0.977$ and $x/l = 0.999$ are shown in Fig. 10. These stresses are calculated within Block-3, where relatively higher approximation orders are used as compared to Block-1 and Block-2. In Fig. 10(b), first approximation order, i.e. $p = 1$ is used in Block-1 and Block-2 while the approximation order in Block-3 is increased from 2 to 5. Similarly, approximation orders are kept constant in Block 1 and Block-2 in Fig. 10(c) to Fig. 10(e) and only approximation order is changed in Block-3. The use of a relatively higher approximation order in Block-3 (i.e. $p \geq 3$) coupled with a very low approximation order (i.e. $p = 1$) provide a very smooth and accurate result for relatively coarse meshes.

For the different combination of approximation orders in Block-1, Block-2 and Block-3, degrees of freedom and associated simulation time are given in 2. The simulations are run on MacBook Pro with 2.6 GHz 6-Core Intel Core i7 processor and with 16 GB 2400 MHz DDR4 memory. The first two columns provided the effect of the global approximation orders, i.e. same approximation orders are used in all three blocks. For the approximation order 111, the degrees of freedoms are 2,670 and the simulation time is 1.6s. The approximation order 222 increase the degrees of freedom to 18,255 and simulation time to 5.09. Finally, with approximation order 555, the degrees of freedom are 258,708 and simulation time is 723.51s. There is a rapid increase in the degrees of freedom and corresponding simulation time with increasing the approximation

order globally. The effect of local approximation order on the number of degrees of freedom and simulation time are given in columns 3 to 8. In these cases, approximation orders are kept constant in Block-1 and Block-2, while approximation order is increased up to 5 in Block-3. It is clear that combining relatively lower approximation order in Block-1 and Block-2 and higher approximation order in Block-3 provide a reduction in computational cost in term of degrees of freedom and simulation time.

Table 2: Effect of global and local approximation orders on the number of degrees of freedom and simulation time for cross-ply laminate

Order (DOFs)	Time (s)	Order (DOFs)	Time (s)	Order (DOFs)	Time (s)	Order (DOFs)	Time (s)
111 (2,670)	1.6	112 (5,463)	2.06	223 (25,179)	7.6	334 (71,310)	43.6
222 (18,255)	5.09	113 (12,387)	3.77	224 (38,112)	15.69	335 (92,130)	78.389
333 (58,377)	30.39	114 (25,320)	9.21	225 (58,932)	37.77		
444 (134,655)	154.02	115 (46,140)	27.06				
555 (258,708)	723.51						

The effect of mesh refinement on the distribution of stresses is presented for $[0/90]_s$ laminate. Three meshes with increasing mesh density, i.e. coarse, medium and fine are shown in Figs. 11(a), 11 (b) and 11(c) respectively. The number of tetrahedral elements in the coarse, medium and fine meshes are 31814, 12269 and 3873 respectively while the number of nodes are 6024, 2485 and 890 respectively. Mesh are only refined near the free edge. Distribution of σ_{zz} versus x/l at $y = 0$ and $z = 0$ for first approximation order is shown in Fig. 12(a) while for the second approximation order it is shown in Fig. 12(b). As expected, at constant approximation order, the accuracy of σ_{zz} improve with increasing the mesh refinement. Results for σ_{zz} versus x/l at $y = 0$ and $z = h$ are shown in Figs. 12(c) and 12(d) for the first and second approximation orders respectively. Moreover, σ_{zx} versus x/l at $y = 0$ and $z = h$ are shown in Figs. 12(e) and 12(f) respectively for the first and second approximation orders. The effect of mesh density on the through-thickness distribution of the out-of-plane stress σ_{zz} versus z/h near the free-edge are shown in Figs. 13(a-d).

3.2. Angle-ply laminate

The geometry, boundary conditions and coordinate system for a symmetric angle-ply laminate ($[\pm 45]_s$) are shown in Fig. 14. The laminate is subjected to unidirectional strain ϵ_0 at its two y -faces. Angle-ply laminate is symmetric only about the z -axis and therefore half of the problem domain is modelled as shown in the same figure. This leads to a problem domain of size four times as compared to the one used for the cross-ply laminate. Symmetric boundary conditions, i.e. $u_z = 0$ are applied to the face at $z = 0$. Displacement $u_y = 0$ is applied to the face at $y = 0$ and displacement $u_y = 2l\epsilon_0$ is applied to the face at $y = 2l$. Faces with $x = 0$ and $x = 2l$ are free-edges. Similar geometric parameters and material properties are used as in the case of the

Figure 3: Geometry, boundary conditions and coordinate system for the cross-ply laminate

Figure 4: Mesh and blocks for defining the local approximation order for the cross-ply laminate

cross-ply laminate (Section 3.1). The mesh for this problem is shown in Fig. 15(a), consisting of 15,687 tetrahedral elements and 3,320 nodes. The arrangement of blocks for assigning arbitrary approximation orders is shown in Fig. 15(b). There are two free-edges in this case but Block-2 and Block-3 with higher approximation orders are used only at $x = 2l$. This arrangement of blocks is chosen for the computational efficiency as stresses are extracted along the line, starting at the centre of the laminate toward the free-edge at $x = 2l$. Similarly, through-the-thickness stresses are extracted at the vicinity of the same free-edge. As shown in Fig. 15(b), Block-3 is centred at the free-edge at $x = 2l$ and having relative radius $R/l = 0.3$. Block-2 is adjacent to Block-3 and is lying between $R/l = 0.3$ and $R/l = 0.6$.

In this numerical example four sets of approximation orders are used, i.e. 122, 123, 222 and 223. Similar to the previous numerical example, in this case, the first, second and third digit refers to the approximation orders used in Block-1, Block-2 and Block-3 respectively. First and second orders of approximation, i.e. $p = 1, 2$ are used in Block-1. For all four sets of

(a) Calculate stress along these lines

(b) $y = 0, z = 0$

(c) $y = 0, z = h$

(d) $y = 0, z = h$

Figure 5: Effect of the global approximation order on the distribution of σ_{zz} and σ_{xz} for the cross-ply laminate

approximation orders, $p = 2$ are used in Block-2, while second and third approximation order, i.e. $p = 2, 3$ are used in Block-3. The distribution of in-plane stresses, i.e. σ_{yy} and σ_{xy} versus x/l at $y = l$ and $z = h$ (along the green line shown in Fig. 16(a)) are shown in Fig. 16. Reference results from literature ([14]) are also provided in these figures for comparison. These current results with four sets of approximation orders provide good agreement with reference results. Increasing the approximation order near the free-edge improve the stresses. For the same four sets of approximation orders, the distribution of out-of-plane normal (σ_{zz}) and shear (σ_{xz}, σ_{yz}) stresses are shown in Fig. 17. In these cases, there is a very good agreement for σ_{zz} and σ_{yz} with the reference results for all four sets of approximation orders. There is a slight variation for σ_{yz} , especially in a region in the vicinity of the free-edge. The use of relatively low approximation orders in Block-1 and Block-2 which constitute the majority of the problem domain and higher approximation orders in Block-1, provide a good balance of accuracy and computational cost.

The through-thickness distribution of σ_{yz} at three different locations in the vicinity of the free-

(a) Calculate stress along these lines

Figure 6: Effect of the global approximation order on the through-thickness distribution of σ_{zz} near the free-edge for the cross-ply laminate

edge is shown in Fig. 18. Results were compared with reference numerical results from the literature ([13]), which were obtained using numerical procedure based on finite-difference. At $x = 0.89l$, the through-thickness stresses are given in 18(a). As expected, the stresses σ_{yz} are increasing on approaching the interface, i.e. at $z/h = 1$, where, our results start to deviate from the reference results. The four sets of approximation order give very similar results away from the interface but there is slight variation at the interface. These stresses are calculated in Block-3, where relatively higher approximation orders, i.e. $p = 2, 3$ are used. Similarly, the through-thickness distribution of σ_{yz} at $x = 0.96l$ and $x = 1.0l$ are shown in Fig. 18(b) and Fig. 18(c) respectively. In these cases, results obtained with the approximation orders 122 and 222 are very similar, which is due to the use of $p = 2$ in Block-3. Similarly, due to the use of $p = 3$, results obtained with approximation orders 123 and 223 are very similar. Moreover, as compared to $p = 2$ in Block-3, results obtained with $p = 3$ are in close agreement with the reference results.

Figure 7: Effect of the local approximation order on the distribution of σ_{zz} at $y = 0$ and $z = 0$ for the cross-ply laminate

3.3. Quasi-Isotropic laminates

In this numerical example, two types of eight-layer quasi-isotropic laminates, i.e. $[45/-45/0/90]_s$ and $[90/0/-45/45]_s$ are considered. In this case, four plies are modelled as the laminate is only symmetric about the z -axis. In this example, the thickness is $8h$ and width

Figure 8: Effect of the local approximation order on the distribution of σ_{zz} at $y = 0$ and $z = h$ for the cross-ply laminate

is $2l = 32h$. Therefore the physical size of this problem is eight times that of the angle-ply laminate. The geometry, mesh and block arrangement for assigning the approximation orders are given in Fig. 19. Similar to the angle-ply laminate, Block-2 and Block-3 are used only in the vicinity of one of the free-edges where stresses will be extracted and will be compared with the reference results. Here, Block-3 is centred the right free-edge and with relative radius $R/l = 0.1$ and Block-2 is adjacent to Block-3 and lies between $R/l = 0.1$ and $R/l = 0.3$. The

Figure 9: Effect of the local approximation order on the distribution of σ_{zx} at $y = 0$ and $z = h$ for the cross-ply laminate

remaining domain is considered as Block-1. A relatively coarse mesh was used for the problem discretisation consisting of 20,885 tetrahedral elements and 4,439 nodes. Given the relative size and the level of discretisation (20,885 tetrahedral elements and 4,439 nodes) of the quasi-isotropic laminate as compared to the angle-ply laminate (15,687 elements and 3,320 nodes),

Figure 10: Effect of the local approximation order on the through-thickness distribution of σ_{zz} near the free-edge for the cross-ply laminate

it is expected that the results, in this case, will be less accurate.

Results for the in-plane and out-of-plane stresses for $[45/-45/0/90]_s$ are shown in Fig. 20. These stresses are calculated along the lines starting at the centre of the laminates and ranges towards the right free-edge. Here, at the centre of the laminate $x/l = 0$ and at right free-edge $x/l = 1$. There are five lines, i.e. at $z = 0, h, 2h, 3h$ and $4h$ which are shown in Fig. 20(a). Results for σ_{yy} at $z = 0$, σ_{xz} at $z = h$ and σ_{yz} at $z = 3h$ versus x/l are shown in Fig. 20(b).

Figure 11: Coarse, medium and fine mesh for the mesh sensitivity study for the cross-ply laminate

These stresses are compared against the reference results from the literature ([14]). Similar to the angle-ply laminate, four set of approximation orders, i.e. 122, 123, 222 and 223 are used in this case. There are fluctuation in the stresses from $x/l = 0$ to $x/l = 0.7$ for the cases 122 and 123, which is due to the use of $p = 1$ in Block-1. In the same region stresses are very smooth and accurate for the set of approximation orders 222 and 223. Due to the use of $p = 2$ in Block-2 and $p = 2, 3$ in Block-3 stress in the region from $x/l = 0.7$ to $x/l = 1$ are very smooth and accurate. Similarly, results for σ_{zz} at $z = 0$ and $z = 3h$ versus x/l are shown in Fig. 20(c) and Fig. 20(d) respectively. With increasing approximation order, these stresses match exactly with the reference results.

For the second type of quasi-isotropic laminate, i.e. $[90/0/-45/45]_s$, the out-of-plane normal stresses σ_{zz} versus x/l at $z = 0, h, 2h$ and $3h$ are given in Fig. 21. Similar to $[45/-45/0/90]_s$, four sets of approximation orders, i.e. 122, 123, 222 and 223 are used. Reference results from the literature ([14]) are provided in the same figures for comparison. Similar to the previous case, for the sets of approximation orders 122 and 123, stresses in Block-1 region are not smooth. For $[90/0/-45/45]_s$, the through-thickness distribution of σ_{zz} and σ_{yz} versus z/h at two location along the length of the laminate, i.e. $x = 0.96l$ and $x = 0.999l$ are shown in Fig. 22(b) and Fig. 22(c) respectively. These stresses are compared with the reference results

Figure 12: Effect of mesh density and approximation order on the distribution of σ_{zz} versus σ_{zx} near the free-edge for the cross-ply laminate

from the literature ([14]). Although, there is some discrepancy between our results and reference results, which, as explained earlier, can be improved by refining the mesh near the free-edge. As compared to the stresses shown in Figures 20 and 21, the through-thickness stresses are calculated within the Block-3. These stresses provide smooth and consistent results due to the

Figure 13: Effect of mesh density and approximation order on the through-thickness distribution of σ_{zz} near the free-edge for the cross-ply laminate

Figure 14: Geometry, boundary conditions and coordinate system for the angle-ply laminate

use of high approximation order $p = 2, 3$ within Block-3 as compared to Block-1 and Block-2.

Figure 15: Mesh and blocks for defining local approximation orders for the angle-ply laminate

Figure 16: Effect of the local approximation order on the distribution of in-plane stresses for the angle-ply laminate

3.4. Multi-scale modelling of cross-ply laminate

The multi-scale modelling of cross-ply laminates is considered as the final numerical example. In this numerical example, the ply-level elastic mechanical properties are calculated directly

Figure 17: Effect of the local approximation order on the distribution of out-of-plane stresses for the angle-ply laminate

from the CH of the underlying RVE. The composite used in this example is carbon fibre reinforced epoxy matrix. The matrix is considered as isotropic and the fibre is considered as transversely-isotropic. Material properties for fibres and matrix are taken from [42–44], and are given in Table 3, where E , G and ν are Young’s modulus, shear modulus and Poisson’s ratio respectively. Index 1 refers to the fibre direction and 2 and 3 refer to transverse directions. The RVE considered is hexagonal packing arrangement of fibres as shown in Fig. 23, consisting of 2, 368 tetrahedral elements and 606 nodes. As compared to the square packing arrangement, hexagonal packing arrangement of fibres in the RVE ensure transverse-isotropy in each ply. The calculated nine homogenised or effective elastic material properties of the resultant orthotropic composite ply are shown in Fig. 24 and Fig. 25. These calculated homogenised mechanical properties are compared with the experimental results from [42] and numerical results from [43, 44].

Figure 18: Effect of the local approximation order on the through-thickness distribution of σ_{yz} for the angle-ply laminate

Figure 19: Mesh and blocks for defining local approximation orders for the quasi-isotropic laminate

Results for the homogenised Young's modulus $\bar{E}_{11}$ versus the degrees of freedom for displacement, periodic and traction boundary conditions are shown in Fig. 24(a). In Fig. 24(a), P_1 , P_2 and P_3 refers to first, second and third order of approximation. Reference experimental and

Figure 20: Effect of the local approximation order on the distribution of both in-plane and out-of-plane stresses for the quasi-isotropic laminate $[45/-45/0/90]_s$

numerical results are provided as straight horizontal line in these figures. In the case of $\bar{E}_{11}$, results for the displacement and periodic boundary conditions are very similar and are in very good agreement with numerical results from [43, 44]. Prediction for $\bar{E}_{11}$ is slightly higher as compared to the experimental results from [42]. Results of $\bar{E}_{11}$ for traction boundary condition is very low as compared to the displacement, periodic and reference results. This discrepancy can be explained by the deformed RVEs shown in Fig. 26. Deformed RVEs, in the case of displacement, periodic and traction boundary conditions subjected to strain ε_{11} are shown in Fig. 26(a), 26(b) and 26(c) respectively. In the case of displacement and periodic boundary conditions, uniform matrix and fibre deformation was achieved. On the other hand, in the case of traction boundary conditions, only the matrix is deformed due to its less kinematically constraint nature [32]. Results for the homogenised Young's moduli $\bar{E}_{22}$ and $\bar{E}_{33}$ are shown in Fig. 24(b) and Fig. 24(c) respectively. Due to the transverse-isotropic nature of each composite ply, these value are exactly the same. As expected, results for the periodic boundary conditions lie between the displacement and traction boundary conditions. Results of the periodic boundary condition are in a very good agreement with the reference numerical results from the

Figure 21: Effect of the local approximation order on the distribution of out-of-plane stresses for the quasi-isotropic laminate $[90/0/-45/45]_s$

literature ([43, 44]). Experimental results from literature ([42]) are slightly higher. Deformed RVEs, in the case of displacement, periodic and traction boundary conditions subjected to ε_{33} are shown in Fig. 27(a), 27(b) and 27(c) respectively. Results for shear moduli $\bar{G}_{12}$ and $\bar{G}_{13}$ are shown in Fig. 24(d) and Fig. 24(e) respectively, which are very similar. Reference numerical results are in a good agreement with the results obtained with periodic boundary conditions. Results for $\bar{G}_{23}$ are shown in Fig. 24(f). Similar to $\bar{G}_{12}$ and $\bar{G}_{13}$, results obtained with periodic boundary condition are in a good agreement with the reference numerical results. Deformed RVEs subjected to ε_{12} in the case of displacement, periodic and traction boundary conditions are shown in Fig. 28. Similarly, results for $\bar{\nu}_{12}$, $\bar{\nu}_{13}$ and $\bar{\nu}_{23}$ are shown in Fig. 25(a), Fig. 25(b) and Fig. 25(c) respectively.

The homogenised material properties obtained with the three types of RVE boundary conditions are subsequently used for the simulation of symmetric cross-ply laminate. The homogenised

Figure 22: Effect of the local approximation order on the through-thickness distribution of σ_{zz} and σ_{yz} for the quasi-isotropic laminate $[90/0/-45/45]_s$

material properties obtained with the third approximation order, i.e. $p = 3$ are used. The geometry and boundary conditions of the symmetric cross-ply laminate and its block arrangement for assigning arbitrary approximation order are the same as given in Section 3.1 and shown in Fig. 29(a). In this case, only one set of approximation order, i.e. 223 is used. Both $[90/0]_s$ and $[0/90]_s$ laminates are analysed. Results for the stress σ_{zz} versus x/l at $y = 0$ and $z = 0$ are shown in Fig. 29(b). Similarly, results for σ_{zz} and σ_{zx} versus x/l at $y = 0$ and $z = h$ are shown in Fig. 29(c) and Fig. 29(d) respectively. No reference results are provided in these figures for comparison, as the computational framework was validated individually at both micro and macro level. The calculated stresses obtained with the material properties from the displacement boundary condition are higher as compared to stresses obtained with the material properties calculated with the rest of the two boundary conditions. Stresses obtained using material properties calculated with periodic boundary conditions are very similar but slightly lower to ones obtained with displacement boundary conditions.

Table 3: Elastic material properties of carbon fibre and epoxy matrix([42–44])

	E_{11} (GPa)	$E_{22} = E_{33}$ (GPa)	$G_{12} = G_{13}$ (GPa)	G_{23} (GPa)	$\nu_{12} = \nu_{13} = \nu_{23}$
carbon fibre	255	15	15	7	0.2
epoxy matrix	4.2	4.2	1.567	1.567	0.34

Figure 23: Representative volume element

4. Concluding remarks

A three-dimensional computational framework based on hierarchical finite elements was presented for the multi-scale modelling of composite laminates. These hierarchical finite elements provide a simple and accurate framework for the calculation of free edge effect. The freedom of choosing arbitrary approximation order within each user-defined block, provide a good balance between accuracy and computational cost. The approximation order for each block and angle for each ply within laminate is provided from the command line and therefore provide a very convenient way to change these without changing the code or input files. The computation framework has additional flexibility of either accepting the ply-level homogenised elastic material properties or calculates these directly from the micro-level elastic properties of matrix and fibre using computational homogenisation. The model developed for the computational homogenisation has the flexibility of the unified imposition of representative volume element boundary conditions, which allows convenient switching between linear displacement, uniform traction and periodic boundary conditions. The correct implementation of the computational framework was demonstrated with four numerical examples. In the case of symmetric cross-ply laminates, both $[90/0]_s$ and $[0/90]_s$ were considered. Relatively coarse tetrahedral meshes were used for the problem discretisation. Initially, global p -refinement, i.e. $p = 1, 2, 3, 4$ were used and in-plane, out-of-plane and through-thickness stresses were compared with reference results from the literature. It was found that with increasing the approximation order, the computational framework was able to capture the complex profiles of these stresses, especially

Figure 24: Effect of the RVE boundary conditions and approximation orders on the homogenised Young's and shear moduli

near the free-edge. Next, the same problem domain was divided into three blocks during the pre-processing stage and the approximation order of each block was changed independently. As compared to the rest of the problem domain, higher approximation orders were used in the vicinity of the free-edge. As compared to the global p -refinement, this local increase in the approximation order provided accurate results at low computational cost.

Figure 25: Effect of the RVE boundary conditions and approximation orders on the homogenised Poisson's ratios

Figure 26: Deformed RVEs in the case of different boundary conditions subjected to applied strain $\varepsilon_{11} = 1$

In the case of symmetric angle-ply and quasi-isotropic laminates, three blocks and four sets of approximation orders (122, 123, 222, 223) were used. Block-2 and Block-3 with higher

Figure 27: Deformed RVEs in the case of different boundary conditions subjected to applied strain $\varepsilon_{33} = 1$

Figure 28: Deformed RVEs in the case of different boundary conditions subjected to applied strain $\varepsilon_{12} = 1$

approximation orders were used in one of the two free edges. Results of in-plane, out-of-plane and through-thickness stresses at different location within these laminates were compared with reference results from the literature and found to be good agreement. For the fourth numerical example, an RVE consisting of hexagonal packing arrangement of fibre was considered which ensures transverse-isotropy in each composite ply. Three types of RVE boundary conditions, i.e. displacement, periodic and traction, and three global approximation orders, i.e. $p = 1, 2, 3$ were used for the computational homogenisation. The calculated homogenised or effective properties were compared with experimental and numerical results from the literature. It was found that properties calculated with periodic boundary condition were in very good agreement compared to the properties obtained with displacement and traction boundary conditions. The calculated material properties were used subsequently for the simulation of cross-ply laminates. The computational framework will be extended in the future to include adaptive analysis to

Figure 29: Effect of the RVE boundary conditions on the distribution of σ_{zz} and σ_{zx} for the cross-ply laminate using approximation order of 223

automatically convert elements from low approximation orders to higher approximation orders. Moreover, the unified imposition of RVE boundary conditions were previously used by the same authors for the calculation of elastic material properties of 2D- and 3D woven composites. Therefore, it will be a straightforward exercise to extend the multi-scale modelling presented in this paper to woven laminates.

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