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A new hierarchical approach for the optimal design of a 5-dof hybrid
serial-parallel kinematic machine

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Abstract

Parallel kinematic machines are highly nonlinear and strongly coupled systems, which makes the driving-structural-dimensional parameters integrated design extremely difficult. To solve this problem, this paper proposes a new hierarchical optimal approach for a 5 degree-of-freedom hybrid serial-parallel kinematic machine. Considering application requirements in high speed machining, a global kinematic index, a global dynamic index, a global elastodynamic index as well as a global stiffness index are introduced and considered as objective functions. Taking the extension ratio and rotation angles as constraint conditions, optimal design is developed through a hierarchical optimization procedure. Results demonstrated that the proposed approach enables the robot to achieve good kinematic, dynamic and stiffness performance simultaneously and can be applied to other PKMs for machining.

Keywords: Parallel kinematic machine, optimal design, dimensional synthesis, stiffness, natural frequency

Notation

- \( a \) radius of the platform’s circumcircle
- \( a_i \) vector pointing from \( P \) to \( A_i \) in \( B_{c-xyz} \)
- \( b_k \) distance between \( B_4 \) and \( B_k \)
- \( b_k \) vector pointing from \( B_4 \) to \( B_k \)
- \( b_i \) distance from \( B_4 \) to \( B_i (B_3) \)
- \( b \) distance from \( B_4 \) to \( B_1 \)
- \( C^* \) coefficient matrix of \( \dot{U} \)
- \( C_p \) compliance matrix of the PKM
- \( D_1 \) external diameter of stretchable limb
- \( D_2 \) external diameter of oscillating limb
- \( D_3 \) external diameter of thick limb
- \( D_4 \) external diameter of thin limb
- \( D^{ai} \) transformation matrix of \( U_p \) with respect to \( \rho_{ai} \)
- \( D^{bi} \) transformation matrix of \( U_p \) with respect to \( \delta_{bi} \)
- \( f^* \) driving force of the system
- \( F \) synthesis index with weight factors
- \( F \) global external force vector
- \( F^* \) external force applied at the platform in the rigid dynamic model
- \( F_i \) global kinematic index
- \( f_i \) average of the first order natural frequency
- \( M_i \) the \( i \)th row of the mass matrix \( M^* \)
- \( \dot{M} \) mass matrix expressed in the joint space
- \( M_v \) mass matrix of the platform measured in \( B_{c-xyz} \)
- \( m_v \) direction of the spring axis
- \( p \) pitch of the lead-screw
- \( Q^* \) Coriolis force
- \( q_i \) length of \( \text{leg}_i \)
- \( q_i \) velocity of the \( i \)th prismatic joint
- \( q_{\text{min}} \) minimum length of \( \text{leg}_4 \)
- \( q_{\text{max}} \) maximum length of \( \text{leg}_4 \)
- \( q_{\text{min}} \) minimum length of the leg whose extension ratio is the largest among three \( \text{UPS} \) legs
- \( q_{\text{max}} \) maximum length of the leg whose extension ratio is the largest among three \( \text{UPS} \) legs
- \( R \) radius of the task workspace
- \( R \) rotation transformation matrix of \( P-uvw \) with respect to \( B_{c-xyz} \)
- \( r_{ai} \) vector pointing from \( B_4 \) to \( A_i \)
- \( r_j \) direction and distance to the line of action
- \( r_f \) position of the point \( P \)
- \( s_j \) unit eigenscrew to eigenstiffness \( \lambda_j \)
- \( t_i \) thickness of stretchable limb

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E-mail address: niuwentie@tju.edu.cn.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_2$</td>
<td>global dynamic index</td>
</tr>
<tr>
<td>$F_3$</td>
<td>global elastodynamic index</td>
</tr>
<tr>
<td>$F_4$</td>
<td>global stiffness index</td>
</tr>
<tr>
<td>$F_1$</td>
<td>average value of $F_1$</td>
</tr>
<tr>
<td>$F_2$</td>
<td>average value of $F_2$</td>
</tr>
<tr>
<td>$F_3$</td>
<td>average value of $F_3$</td>
</tr>
<tr>
<td>$F_{A_l}$</td>
<td>reaction force of joint $A_l$</td>
</tr>
<tr>
<td>$f_j$</td>
<td>external force vector of the leg $j$</td>
</tr>
<tr>
<td>$f_{M_{i}}$</td>
<td>reaction force applied at joint $M_i$ $(M=A, B)$ in $B_{i-xyz}$</td>
</tr>
<tr>
<td>$F_{M_{i}}$</td>
<td>reaction force applied at joint $M_i$ $(M=A, B)$ in $B_{i-xyz}$</td>
</tr>
<tr>
<td>$f_p$</td>
<td>external force including the platform’s gravity</td>
</tr>
<tr>
<td>$G^*$</td>
<td>gravity of the system</td>
</tr>
<tr>
<td>$h$</td>
<td>height of the task workspace</td>
</tr>
<tr>
<td>$H$</td>
<td>distance between the right plane of the task workspace and the global coordinate system</td>
</tr>
<tr>
<td>$h_j$</td>
<td>pitch of $s_j$</td>
</tr>
<tr>
<td>$l_i$</td>
<td>equivalent load moment of inertia applied to the servomotor</td>
</tr>
<tr>
<td>$l_{mean}$</td>
<td>average value throughout the whole task workspace</td>
</tr>
<tr>
<td>$I_M$</td>
<td>rotor inertia of the servomotor</td>
</tr>
<tr>
<td>$I_p$</td>
<td>inertia matrix of the platform measured in $B_{i-xyz}$</td>
</tr>
<tr>
<td>$J$</td>
<td>generalized Jacobian matrix</td>
</tr>
<tr>
<td>$J_a$</td>
<td>active Jacobian matrix</td>
</tr>
<tr>
<td>$J_l$</td>
<td>constraint Jacobian matrix</td>
</tr>
<tr>
<td>$J_{hu}$</td>
<td>homogeneous Jacobian matrix</td>
</tr>
<tr>
<td>$K$</td>
<td>stiffness matrix of the PKM</td>
</tr>
<tr>
<td>$k^*$</td>
<td>average stiffness of six spring constants at any configuration</td>
</tr>
<tr>
<td>$\bar{k}$</td>
<td>average of $k^*$ throughout the whole task workspace</td>
</tr>
<tr>
<td>$k_i$</td>
<td>stiffness matrix of the leg $i$</td>
</tr>
<tr>
<td>$k_j$</td>
<td>spring constant</td>
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<tr>
<td>$k_{M_{i}}$</td>
<td>linear stiffness of joint $M_i$ $(M=A, B)$ in x axis</td>
</tr>
<tr>
<td>$k_{M_{i}}$</td>
<td>linear stiffness of joint $M_i$ $(M=A, B)$ in y axis</td>
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<tr>
<td>$k_{M_{i}}$</td>
<td>linear stiffness of joint $M_i$ $(M=A, B)$ in z axis</td>
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<tr>
<td>$k_{M_{i}}$</td>
<td>angular stiffness of joint $M_i$ $(M=A, B)$ about x axis $(i=1~4)$</td>
</tr>
<tr>
<td>$k_{M_{i}}$</td>
<td>angular stiffness of joint $M_i$ $(M=A, B)$ about y axis</td>
</tr>
<tr>
<td>$t_2$</td>
<td>thickness of oscillating limb</td>
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<tr>
<td>$t_3$</td>
<td>thickness of thick limb</td>
</tr>
<tr>
<td>$t_4$</td>
<td>thickness of thin limb</td>
</tr>
<tr>
<td>$T_{A_i}$</td>
<td>reaction moment of joint $A_i$</td>
</tr>
<tr>
<td>$T_i$</td>
<td>nodal transformation matrix of $B_{i-xyz}$ with respect to $B_{i-xyz}$</td>
</tr>
<tr>
<td>$T_{M_{i}}$</td>
<td>reaction force applied at joint $M_i$ $(M=A, B)$ in $B_{i-xyz}$</td>
</tr>
<tr>
<td>$\mathbf{u}$</td>
<td>unit vector of $u$ axis</td>
</tr>
<tr>
<td>$\mathbf{v}$</td>
<td>unit vector of $v$ axis</td>
</tr>
<tr>
<td>$\mathbf{v}_p$</td>
<td>linear velocity of point $P$</td>
</tr>
<tr>
<td>$w_{4x}$</td>
<td>coordinate in the x axis of $\mathbf{w}_4$</td>
</tr>
<tr>
<td>$w_{4y}$</td>
<td>coordinate in the y axis of $\mathbf{w}_4$</td>
</tr>
<tr>
<td>$w_{4z}$</td>
<td>coordinate in the z axis of $\mathbf{w}_4$</td>
</tr>
<tr>
<td>$\mathbf{w}_i$</td>
<td>unit vector of $\mathbf{w}_i$</td>
</tr>
<tr>
<td>$\mathbf{w}_p$</td>
<td>angular velocity of the platform</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Euler angle about the x axis</td>
</tr>
<tr>
<td>$\psi_0$</td>
<td>allowable angle between the y axis and y axis</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Euler angle about the v axis</td>
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<tr>
<td>$\theta_0$</td>
<td>allowable angle between the z axis and z axis</td>
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<tr>
<td>$\mu$</td>
<td>extension ratio</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>allowable extension ratio</td>
</tr>
<tr>
<td>$\eta$</td>
<td>condition number of the homogeneous Jacobian matrix</td>
</tr>
<tr>
<td>$\mathbf{z}$</td>
<td>eigenstiffness of the stiffness matrix $K_p$</td>
</tr>
<tr>
<td>$\varsigma$</td>
<td>eigenscrew of the stiffness matrix $K_p$</td>
</tr>
<tr>
<td>$\delta_{A_p}$</td>
<td>linear displacement of the interface point $A_p$</td>
</tr>
<tr>
<td>$\rho_{A_i}$</td>
<td>angular displacement of the interface point $A_i$</td>
</tr>
<tr>
<td>$\varepsilon_{A_i}$</td>
<td>linear displacement of the interface point $A_i$</td>
</tr>
<tr>
<td>$\xi_{A_i}$</td>
<td>angular displacement of the interface point $A_i$</td>
</tr>
</tbody>
</table>
1. Introduction

Parallel kinematic machines (PKMs) have been applied in the locomotive and aeronautical industries due to high stiffness, high speed and high accuracy. However, the obvious disadvantage of PKMs is their limited workspace due to distributions of legs. In order to increase workspace and dexterity of PKMs, hybrid serial-parallel kinematic machines (HSPKMs) are proposed by connecting serial kinematic machines (SKMs) serially to PKMs. Therefore, HSPKMs will present good performance of PKMs and large workspace of SKMs theoretically, which makes them an alternative solution to high speed machining (HSM). This has been fully exemplified by the commercial success of Exechon [1,2] and the Tricept robots [3,4]. However, most existing HSPKMs are still in the laboratory stage. Inspired by the Tricept and Exechon, a novel HSPKM named Trimule [5] was proposed by Tianjin University as shown in Fig. 1. The HSPKM consists of a 3 degree-of-freedom (DOF) PKM for positioning and a 2-DOF SKM for machining, which can be used in milling and drilling for large scale components.

![Fig. 1. A CAD model of the Trimule robot.](image)
parameters (parameters of legs’ cross-sections, for example, radius, thickness, width and height). Due to the complicated structural features of this kind of PKMs, most work related to the early optimal design is single objective optimization based on a simple and intuitive kinematic performance. Kinematic indices are mainly related to dexterity, workspace, and motion/force transmission measures [6-10].

However, optimal design based on kinematic indices cannot guarantee good dynamic and stiffness performance especially when PKMs are working at a high speed and high acceleration, which will affect machining efficiency and accuracy in practice. Therefore, combining kinematic indices and dynamic/stiffness indices, two-objective optimization is conducted. Dynamic indices are usually based on the system’s inertial matrix. In order to minimize the required driving force and maximize the reachable workspace, optimal design of both symmetric and unsymmetrical hexapod PKMs was conducted [11]. Based on the torque and power indices, dynamic optimization of the Delta robot was conducted [12]. Different from pick-and-place parallel manipulators, PKMs usually work at a relatively low speed. Therefore, dynamic optimization of PKMs is very scarce. In order to achieve good position accuracy, stiffness needs to be considered. In order to maximum system stiffness and workspace, optimization for a 3-DOF parallel manipulator was conducted [13]. It is noted that only actuator’s stiffness is considered in most literatures, while joints and limbs are assumed to be rigid [14].

Even stiffness indices are considered, elastodynamic indices should be considered for PKMs may own large inertia, which is also conducive to vibration reduction and vibration suppression. Considering the conditional number of the Jacobian matrix and the first order natural frequency simultaneously, Wu et al [15] presented optimal design of a 2-DOF PKM with redundancy. Taking workspace and the first order resonance frequency as objective functions, optimization of the Gantry-Tau PKM was conducted [16]. Actually, due to the complex structures, there is no too much work about optimization based on elastodynamic indices of PKMs for HSM.

It is noted that optimization with two indices are not enough according to application requirements of PKMs. Thus, more and more work is focused on multi-objective optimization to achieve a good and comprehensive performance system. Taking workspace volume, dexterity, static efficiency and stiffness as objective functions, Russo et al proposed a multi-objective optimization method for a 3-UPR PKM for a robotic leg [17]. Considering motion transmission, velocity transmission and acceleration transmission, the multi-objective optimization of a 3-PUU PKM was carried out by the improved non-dominated sorting genetic algorithm [18]. Combining transmission, stiffness and dynamic indices, architecture optimization of a parallel manipulator for pick-and-place applications was conducted [19].

Although there are many work focused on optimization design of PKMs, most design variables in the published literatures are mainly about dimensional parameters. Optimization of structural parameters and matching design of driving parameters to the whole system are seldom discussed. Actually, structural and driving parameters have an important influence on the system’s stiffness and dynamic performance, which should be considered in the optimal design [20-23]. Since a PKM is a highly nonlinear and strongly coupled system with multi-inputs and multi-outputs, it will be a challenging but meaningful work to deal with the integrated design of dimensional, structural and driving parameters.

The Trimule robot is a new type of HPKMs and has potential to be applied in HSM. Thus, this paper will deal with the integrated design of driving-dimensional-structural parameters of this robot. Considering application requirements, a kinematic index, two dynamic indices as well as a stiffness index are introduced and taken as objective functions. Since a lot of design variables are involved, it would be a huge workload to optimize all with one-time optimization. Therefore, a hierarchical approach is proposed. Besides, the proposed approach can make each design stage concise. Considering the coupling relationships between indices and design variables, dimensional synthesis, structural parameters optimization and matching design of driving parameters are conducted step by step. The reminder of this paper is organized as follows. Inverse kinematics of the Trimule robot is described in Section 2, followed by the definition of design variables and constraint conditions in Section 3. A performance evaluation system is built in Section 4, based on which a hierarchical optimal design method is introduced in Section 5. Finally, main conclusions are drawn in Section 6.

2. Inverse kinematics of the robot

As shown in Fig. 1, the Trimule mainly consists of one base, one platform, four legs, one A axis and one C axis. The four legs are denoted as leg1, leg2, leg3, and leg4, respectively. leg1 and leg2 connect to the base by a universal joint (U joint) while connecting to the platform through a spherical joint (S joint). Therefore, they have an identical UPS architecture, where P stands for a linear actuator. leg3 is slightly different from the above three legs in that it is connected to the platform fixedly. Kinematically, leg4 can be regarded as a UP architecture, where P represents a prismatic joint. Structure of the A axis and C axis is relatively simple, which will not be analyzed in the followings.
Denote $A_i$, $B_i$ and $C_i$ ($i=1$–4) as joint centers and end points of leg$_i$, respectively. Specifically, $A_4$ is the connection point on the interface of the platform and leg$_4$. The schematic diagram of the PKM module and several coordinate systems are depicted in Fig. 2. For the convenience of derivation, the platform and base are set to be an isosceles right triangle and an equilateral triangle, respectively. The platform body-fixed coordinate system is the position angle of joint platform and base are set to be an isosceles right triangle and an equilateral triangle, respectively. The platform body-fixed coordinate system is shown in Fig. 2.

As demonstrated before, the PKM module has 3 DOFs, which can translate in the $xyz$ module and several coordinate systems are depicted in Fig. 2. For the convenience of derivation, the global coordinate system is attached to the U joint’s center, where the x axis points from $B_1$ to $B_2$ and the y axis is vertically up to $B_1B_2$. Then the z axis can be determined by the right-hand rule. The platform body-fixed coordinate system $Puvw$ is set at the midpoint of $A_1A_2$, with the u axis parallel to $A_3A_2$ and the w axis vertical to the platform. Accordingly, the v axis can be decided by the right-hand rule. The limb body-fixed coordinate system $B_4xyz$ is set at the point $B_4$, where the z axis is along the axis of the kth limb, the $y_k$ axis is along the rotation axis vertical to the limb of the kth universal joint and the $x_k$ axis can be determined by the right-hand rule. To be specific, $B_1-x_1y_1z_1$ in leg$_1$ is shown in Fig. 2.

As demonstrated before, the PKM module has 3 DOFs, which can translate in the $xyz$ axis and rotate about the x and y axes. Therefore, the rotation transformation matrix of the moving coordinate system $P-uvw$ with respect to the global coordinate system $B_4-xyz$ can be defined as

$$R = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ \sin \psi \sin \theta & \cos \psi & -\sin \psi \cos \theta \\ -\cos \psi \sin \theta & \sin \psi & \cos \psi \cos \theta \end{bmatrix}, \quad (1)$$

where $\psi$ and $\theta$ are Euler angles about the x and y axes.

Position of the point $P r_p$ can be expressed in the global coordinate system $B_4-xyz$ as

$$r_p = b_k + q_k w_k - a_k, \quad (k=1 \sim 3), \quad (2)$$

$$r_p = q_k w_k, \quad (3)$$

where $b_k$ is the vector pointing from $B_4$ to $B_k$; $q_k$ and $w_k$ denote the length and unit vector of leg$_k$, respectively; $a_k$ is the vector pointing from $P$ to $A_k$ expressed in the global coordinate system $B_4-xyz$, $w_k = R a_k$. And there exist

$$a_k = a \begin{bmatrix} \cos \gamma_k & \sin \gamma_k & 0 \end{bmatrix}^T, \quad b_k = \begin{bmatrix} b_k \cos \gamma_k & b_k \sin \gamma_k & 0 \end{bmatrix}^T, \quad (4)$$

where $a$ is the radius of the platform’s circumcircle; $b_k$ is the distance between $B_k$ and $B_4$. To be specific, $b_1=b_2=b_3=b_4$. $\gamma_k$ is the position angle of joint $A_k$ ($B_k$) and $\gamma_1=\frac{\pi}{2}, \gamma_2=0, \gamma_3=\pi$.

Taking norm operation of Eq. (3), the following can be derived

$$q_k = |r_p|, \quad w_k = r_p / q_k. \quad (5)$$

From Eq. (1), there exists

$$w_k = \begin{bmatrix} \sin \theta \\ -\sin \psi \cos \theta \\ \cos \psi \cos \theta \end{bmatrix}, \quad (6)$$

Combing Eqs. (5) and (6), the two Euler angles can be derived

$$\theta = \arcsin (w_{kz}), \quad \psi = \arctan(-w_{ky} / w_{kx}), \quad (7)$$

where $w_{kx}$, $w_{ky}$, and $w_{kz}$ are three coordinate components of the unit vector $w_k$ in the x, y and z axes, respectively.
Then, the inverse kinematic solutions, namely the length and unit vector of the leg, can be derived
\[ q_i = \left[ r_p + a_i - b_i \right], w_i = \left( r_p + a_i - b_i \right) / q_i. \] (8)

Taking derivation of Eqs. (2) and (3) yields
\[ v_p = \dot{q}_i w_i + q_i (o_w \times w)_i - o_p \times a_i, \] (9)
\[ v_p = \dot{q}_i w_i + q_i (o_p \times w)_i, \] (10)
where \( v_p \) and \( w_p \) are the linear velocity of the point \( P \) and the angular velocity of the platform, respectively; \( \dot{q}_i \) and \( o_w \) denote the velocity of the \( i \)th prismatic joint and the angular velocity of the leg, respectively.

Taking inner dot of Eq. (9) by \( w_i \) leads to
\[ \dot{q}_i = w_i^T v_p + (a_i \times w_i)^T o_p. \] (11)

Taking inner dot of Eq. (10) by \( u \) and \( v \) (\( u \) and \( v \) are unit vectors of \( u \) and \( v \) axes) respectively yields
\[ u^T v_p + q_i (u \times w_i)^T o_p = 0, \quad v^T v_p + q_i (v \times w_i)^T o_p = 0. \] (12)

According to the principle of linear superposition, the angular velocity of the platform can be expressed as
\[ o_p = \psi^T x + \dot{\theta} v. \] (13)

Taking inner dot of Eq. (13) by \( n = v \times x \) yields
\[ n^T o_p = 0. \] (14)

Eqs. (11)–(12) and (14) can be written in a matrix form as follows
\[ \begin{bmatrix} J^T \\ \hline v_p \\ \hline o_p \end{bmatrix} = \begin{bmatrix} \dot{q}_i \end{bmatrix}, \] (15)
where \( \dot{q} = [\dot{q}_1, \dot{q}_2, \dot{q}_3, 0, 0, 0]^T \); \( \dot{J}_a \) and \( \dot{J}_c \) are the generalized Jacobian matrix of the PKM module, \( J_a \) and \( J_c \) are defined as the active Jacobian matrix and the constraint Jacobian matrix respectively, which can be expressed as
\[ J_a = \begin{bmatrix} w_1 \\ \hline a_2 \times w_2 \\ \hline a_3 \times w_3 \end{bmatrix}, \quad J_c = \begin{bmatrix} u \\ \hline v \\ \hline a_i (u \times w_i) \\ \hline q_i (v \times w_i) \\ \hline n \end{bmatrix}. \] (16)

**3. Question description and constraint conditions**

**3.1. Design variables**

Design variables involved in this paper are driving parameters (parameters of selected servomotors), structural parameters (external diameters and thickness of legs 1–4) and dimensional parameters (size of the platform and base, length of legs 1–4 and the distance between the task workspace and the origin of the global coordinate system). To make it clear, relationship between the task workspace and dimensional parameters is depicted in Fig. 3. The task workspace is referred to a cylinder whose radius is \( R \) and height is \( h \). Herein, \( H \) represents the distance between the right plane of the task workspace and the global coordinate system. \( b_i \) and \( b_4 \) are distance from \( B_4 \) to \( B_2 \) (\( B_1 \)) and \( B_1 \), respectively. Since the base is an equilateral triangle, there exists \( b_4 = \sqrt{3} b_1 \).

As can be seen from Fig. 3, the minimum length and maximum length of \( \text{leg}_4 \) can be expressed as
\[ q_{4_{\text{min}}} = H, \quad q_{4_{\text{max}}} = \sqrt{R^2 + (H + h)^2}. \] (17)

Similarly, the minimum length and maximum length of \( \text{leg}_2-\text{leg}_3 \) can be expressed in certain equations related to \( a, b, c, R, h \) and \( H \). For the convenience of serialization design, take normalization with \( b_i \) as
\[ \lambda_{a/b} = a / b_i, \quad \lambda_{R/b} = R / b_i, \quad \lambda_{H/b} = H / b_i, \] (18)
where \( \lambda_{a/b} \) is defined as the platform/base ratio; \( \lambda_{R/b} \) is defined as the workspace/footprint ratio.

Define \( \lambda_{h/R} = h / R \) as the height/radius ratio of the task workspace, which has an allowable range of 0.4–0.5 when referring to design of Tricept. Therefore, optimization of this PKM can be described as **how to determine design parameters to make the PKM perform best with the given \( \lambda_{h/R} = h / R \) and corresponding constraint conditions.**
3.2. Constraint conditions

In this paper, the following two constraint conditions are considered.

In order to assure the lateral stiffness of each UPS leg, the extension ratio is defined as

$$\mu = \frac{q_{\text{max}} - q_{\text{min}}}{q_{\text{min}}} \leq \mu_0,$$

(19)

where $\mu_0$ is the allowable extension ratio, which is usually set to be 0.7–0.8. $q_{\text{max}}$ and $q_{\text{min}}$ are the maximum and minimum length of the leg whose extension ratio is the largest among the three UPS legs.

Rotation angle of the universal joint should be limited as

$$\left| \theta \right|_{\text{max}} \leq \theta_{\text{io}} \leq \left| \psi \right|_{\text{max}},$$

(20)

where $\theta_{\text{io}}$ and $\psi_{\text{io}}$ represent the two allowable rotation angles of the universal joint in leg $i$, respectively. To be specific, $\psi_{\text{io}}$ denotes the allowable angle between the $y_i$ axis and $y$ axis while $\theta_{\text{io}}$ represents the allowable angle of leg $i$ about the $y_i$ axis.

4. Performance evaluation system

In order to evaluate the performance of the system comprehensively, a performance evaluation system is proposed, which includes the kinematic index, dynamic index, elastodynamic index and stiffness index.

4.1. Global kinematic index

The condition number of a Jacobian matrix is a commonly used index to evaluate motion/force transfer characteristics, which can be described as the degree close to singularity. It is noted that the homogenous Jacobian matrix should be derived first if one PKM owns both translation and rotation mobility [28,29].

Making separation of Eq. (15) results in

$$J_{pa} v_p = \dot{q}_a,$$

(21)

where $J_{pa} = J_{vw} - J_{wv} J_{uv} J_{vw}$, $J_{vw} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}$, $J_{wv} = \begin{bmatrix} a_1 \times w_1 & a_2 \times w_2 & a_3 \times w_3 \end{bmatrix}$, $J_{uv} = \begin{bmatrix} u \ v \ 0 \end{bmatrix}$, $J_{wv} = \begin{bmatrix} q_1 (u \times w_1) & q_2 (v \times w_1) \end{bmatrix}$, $\dot{q}_a = \begin{bmatrix} \dot{q}_1 \ \dot{q}_2 \ \dot{q}_3 \end{bmatrix}^T$.

As can be found from Eq. (21), units of $v_p$ and $\dot{q}_a$ are the same. Therefore, $J_{pa}$ is a homogeneous Jacobian matrix. The condition number of the homogeneous Jacobian matrix $J_{pa}$ can be expressed as

$$\eta = \text{cond}(J_{pa}).$$

(22)

Taking the Trimule with parameters in [30,31] as an example, distributions of the condition number over a working plane are depicted in Fig. 4. As shown, distributions of the condition number are strongly position-dependent and demonstrate plane symmetry.
In order to evaluate the kinematic index throughout the whole task workspace, a global kinematic index can be expressed as

$$ F_1 = \frac{\int V \eta dV}{\int V dV}, $$

where $V$ is the volume of the task workspace. Obviously, the smaller the $F_1$, the better the PKM’s kinematic performance.

### 4.2. Global dynamic index

For PKMs applied to HSM, the acceleration ability is required to improve the PKM’s working efficiency. The acceleration ability is referred to the driving torque required to make the platform realize a unit velocity/acceleration with a unit load. Therefore, a rigid dynamic model is required.

Based on the principle of virtual work, the rigid dynamic model of the system can be derived

$$ M^* \ddot{U}^* + C^* \dot{U}^* + G^* + Q^* - J^* F^* = f^*, $$

where $\dot{U}^*$ and $\ddot{U}^*$ are the velocity and acceleration of the platform, respectively. $M^*$ is the mass matrix, $C^*$ is the coefficient matrix of $\dot{U}^*$, $G^*$ and $Q^*$ are the gravity and Coriolis force of the system. $F^*$ is the external force applied at the platform. $f^*$ is the driving force of the system. For content limitation, the detailed derivation will not be addressed in this paper.

In general, effects of $C^* \dot{U}$ and $Q^*$ on the driving force are relatively small, while that of the gravity can be compensated by the feedforward control. Therefore, the driving force is mainly caused by the inertia force when the external force is zero. Accordingly, the driving torque of leg$k$ can be expressed

$$ \tau_{a,k} = \frac{p}{2\pi} M_k \ddot{U}^*, $$

where $M_k$ is the $k$th row of the mass matrix $M^*$, $p$ is the pitch of the lead-screw.

Based on the principle of singular value decomposition, the maximum of $2\pi \tau_{a,k} / p$ required to realize the unit acceleration of the platform ($\|\dot{U}\| = 1$) can be expressed as

$$ \tau_{a,k,\max} = \sqrt{M_k^* M_k^\top}. $$

Based on mass parameters in [30], distributions of the maximum driving force among the three actuators $\max_{k=1,3} \tau_{a,k,\max}$ over a working plane are calculated and shown in Fig. 5. As shown, the maximum driving force among the three actuators is strongly position-dependent and symmetric to the plane of $x=0$. Moreover, variations of the maximum driving force among the three actuators with respect to $y$ are not monotonous. To be specific, the maximum driving force among the three actuators decreases first then increases with the increment of $y$.

In order to enable the PKM to achieve good dynamic performance throughout the whole task workspace, the maximum value of $\max_{k=1,3} \tau_{a,k,\max}$ throughout the whole workspace is defined as a global dynamic index

$$ F_2 = \max_{x=\alpha} \max_{k=1,3} \tau_{a,k,\max}. $$
4.3. Global elastodynamic index

PKMs applied to HSM need to have good inherent behaviors to decrease vibration in high-speed machining to guarantee machining accuracy. Therefore, an elastodynamic model of this PKM should be proposed first. A stiffness model is presented in our previous work based on the substructure synthesis technique [31], which is proved with good accuracy when compared to finite element analysis. Inspired by the previous work [31], an elastodynamic model of this PKM is built in this paper.

4.3.1. Dynamic model of leg subsystems

Structural diagrams of the UPS leg and UP leg are shown in Fig. 6. In order to decrease mass of the PKM, limbs are made of carbon fiber. Considering tensile-compressional, torsional and bending deformation, limbs are modeled as Euler-Bernoulli beams. Accordingly, universal joints, spherical joints as well as the fixed joint are simplified into 6 dof springs with equivalent stiffness and mass at their geometrical centers.

![Diagram of UPS leg and UP leg](image)

Fig. 6. Structural diagrams of the UPS leg and UP leg.

Based on the above description, the UPS leg and UP leg can be simplified into two spatial beam systems constrained by two groups of springs, as shown in Fig. 7. Herein, $k_{d_t1}$, $k_{d_t2}$, $k_{d_t3}$, $k_{d_t4}$ and $k_{d_t5}$ are three linear stiffness and three angular stiffness coefficients of joint $A_i$; $k_{B11}$, $k_{B12}$, $k_{B13}$, $k_{B14}$, $k_{B15}$ and $k_{B16}$ are three linear stiffness and three angular stiffness coefficients of joint $B_i$ ($i=1\sim4$).

![Force diagrams of UPS leg and UP leg](image)

Fig. 7. Force diagrams of the UPS leg and UP leg.

For the convenience of derivation, the UPS leg and the UP leg are both divided into $n$ elements, with $A_i$ being the 1st node and $C_i$ being the $(n+1)$th node respectively. The dynamic equation of leg$_i$ subsystem can be expressed as $B_{r,x,y,z}$, where
\[
m \ddot{u}_i + k_u u_i = f_i,
\]

where \(m_i\) and \(k_i\) are the mass and stiffness matrices of the leg; \(u_i\) and \(f_i\) are the coordinate vector and external force vector of the leg, respectively.

Eq. (28) can be expressed in the global coordinate system \(B_x\)-xyz as

\[
M_i \ddot{U}_i + K_i U_i = F_i,
\]

where \(M_i = T m_i T_i^T\), \(K_i = T k_i T_i^T\), \(U_i = T u_i\), \(F_i = T f_i\). \(T_i\) is the nodal transformation matrix of \(B_x\)-xyz with respect to the global coordinate system \(B_1\)-xyz.

4.3.2. Dynamic model of the platform subsystem

Comparing to rigidity of legs 1−4, that of the platform is relatively large. Thus, the platform is treated as a rigid body. The force diagram of the platform is shown in Fig. 8. Herein, \(f_p\) and \(\rho_i\) are the external force and moment including the platform’s gravity; \(F_{Ai}\) and \(T_{Ai}\) are reaction force and moment of joint \(A_i\), \(r_{Ai}\) is the vector pointing from \(B_1\) to point \(A_i\).

![Fig. 8. Force diagram of the platform.](image)

Assuming that \(\epsilon_p\) and \(\xi_p\) stand for linear displacement and angular displacement of the platform respectively, the dynamic equations of the platform subsystem can be expressed in \(B_x\)-xyz as

\[
- \sum_{i=1}^{4} F_{Ai} \dot{r}_{Ai} = m_p \ddot{\epsilon}_p - \sum_{i=1}^{4} r_{Ai} \times F_{Ai} + \tau_p - \sum_{i=1}^{4} T_{Ai} = I_p \ddot{\xi}_p,
\]

where \(m_p\) and \(I_p\) are the mass and inertia matrices of the platform measured in \(B_x\)-xyz.

4.3.3. Deformation compatibility conditions

Displacement relationship between the platform and leg is demonstrated in Fig. 9, where \(A_{ip}\) and \(A_{ii}\) are interface points on the platform and leg, \(\delta_{ii}\) and \(\rho_{ii}\) are linear and angular displacements of the interface point \(A_{ip}\) while \(\epsilon_{ii}\) and \(\xi_{ii}\) are linear and angular displacements of the interface point \(A_{ii}\). \(k_{ii}\) and \(k_{iiij}\) are the linear stiffness matrix and angular stiffness matrix of joint \(A_i\).

![Fig. 9. Displacement relationship between the platform and leg.](image)

Linear and angular displacements of the interface point \(A_{ip}\) can be expressed as

\[
\delta_{ii} = R_i^T D^i U_p, \quad \rho_{ii} = R_i^T D^\omega U_p,
\]

where \(R_i\) is the transformation matrix of \(B_x\)-xyz, \(U_p = [\epsilon_p, \xi_p]^T\); \(D^i\) and \(D^\omega\) are transformation matrices of \(U_p\) with respect to \(\delta_{ii}\) and \(\rho_{ii}\).

Therefore, reactions applied at joint \(A_i\) can be expressed in the coordinate system \(B_x\)-xyz as

\[
f_{Ai} = -k_{ii} \dot{\delta}_{ii}, \quad \tau_{Ai} = -k_{iiij} (\xi_{ii} - \rho_{ii}).
\]

Eq. (32) can be expressed in the global coordinate system \(B_1\)-xyz as

\[
F_{Ai} = R_i f_{Ai}, \quad T_{Ai} = R_i \tau_{Ai}.
\]

Similarly, reactions applied at joint \(B_i\) can be expressed as

\[
f_{Bi} = -k_{ii} \dot{\epsilon}_{ii}, \quad \tau_{Bi} = -k_{ii} \dot{\xi}_{ii},
\]

where \(k_{ii}\) and \(k_{iiij}\) are the linear stiffness matrix and angular stiffness matrix of the universal joint in
leg: $\xi_{Bi}$ and $\zeta_{Bi}$ are linear and angular displacements of joint $B_i$.

Eq. (34) can be expressed in the global coordinate system $B_{x'y'z'}$ as

$$F_{Bi} = R_i f_{Bi}', T_{Bi} = R_i \tau_{Bi}'.$$  \hfill (35)

### 4.3.4. Dynamic Equation of the PKM Module

Substituting Eqs. (33) and (35) into (29) and (30), the dynamic equation of the PKM module can be obtained

$$MU + KU = F.$$ \hfill (36)

where $M$ and $K$ are the mass matrix and stiffness matrix of the PKM; $U$ and $F$ are the global coordinate vector and external force vector of the PKM.

### 4.3.5. Definition of the elastodynamic index

Supposing that the external force $F$ is equal to zero, natural frequencies of the system can be derived by solving Eq. (36). Distributions of the first two order natural frequencies over a working plane are shown in Fig. 10. Obviously, the first two order natural frequencies are strongly position-dependent and symmetric to the plane of $x=0$.

![Fig. 10. Distributions of lower natural frequencies over the working plane $z=0.7$ m.](image)

As shown in Fig. 10, the first two order natural frequencies are very close and distributions of which are the same over the task workspace. Therefore, a global elastodynamic index based on the first order natural frequency is expressed

$$F_3 = \sqrt{\int_V \left( f_1 - \bar{f}_1 \right)^2 dV},$$ \hfill (37)

where $\bar{f}_1$ is the average of the first order natural frequency $f_1$ throughout the whole task workspace. $F_3$ stands for the fluctuation of the first order natural frequency over the whole task workspace. The smaller the $F_3$, the more uniform the dynamic performance, which is helpful to the design of controller.

### 4.4. Global Stiffness index

It can be concluded that stiffness of a PKM is relatively high if its lower natural frequencies are high. However, it cannot make sure that distributions of the stiffness are uniform. Uniform stiffness distributions are helpful to the selection of machining area, error compensation and accuracy control.

There is a typical feature in the PKM that legs are coupled. Therefore, the system’s stiffness matrix $K$ in Eq. (36) is off-diagonal, making it difficult to propose an individual stiffness index. Meanwhile, units in the stiffness matrix $K$ are not unified. Based on the above discussions, decomposition based on the screw theory will be addressed as follows.

In order to derive stiffness of the platform, the compliance matrix of the PKM must be obtained by conducting inverse operation of the stiffness matrix $K$. Extracting the lower right corner 6×6 block matrix from the PKM’s compliance matrix as $C_P$, stiffness of the platform can be expressed by

$$T_T P P \text{diag}(f_1, f_2) T_T P P = K R R C R R.$$ \hfill (38)

The eigenscrew problem of $K_P$ can be described in ray coordinates as $K_P \lambda \varsigma = \lambda \varsigma,$ \hfill (39)

where the eigenvalue $\lambda$ and the screw $\varsigma$ are defined as the eigenstiffness and eigenscrew of the stiffness matrix $K_P$; $A$ denotes the conversion between two types of coordinates, which can be expressed

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$ \hfill (40)

where $0$ and $I$ denote a zero matrix and an identity matrix in $3 \times 3$, respectively.

Then the eigenscrew decomposition of the stiffness matrix can be expressed as
$$K_p = \sum_{j=1}^{6} k_j s_j^T,$$

where $s_j$ is the unit eigenscrew corresponding to eigenstiffness $\lambda_j$; $k_j$ is the spring constant. And there exist

$$s_j = \begin{pmatrix} n_j \\ r_j \times n_j + h_j n_j \end{pmatrix},$$

$$k_j = \frac{\lambda_j}{2h_j}, h_j = \frac{1}{2} s_j^T \Delta s_j,$$

where $n_j$ is a unit vector representing the direction of the spring axis; $r_j$ identifies the direction and distance to the line of action; $h_j$ is the pitch of $s_j$ ($j=1\sim6$).

Therefore, the PKM at any configuration can be regarded as a rigid body supported by six screw springs $n_j$ with spring constant $k_j$. Distributions of the decomposed stiffness over a working plane are shown in Fig. 11. Obviously, the six spring constants are position-dependent. Furthermore, distributions of the 1st spring constant $k_1$ are symmetric to those of the 6th spring constant $k_6$ about the plane of $x=0$. This phenomenon can be found in distributions of the 2nd spring constant $k_2$ and the 5th spring constant $k_3$ as well as the 3rd spring constant $k_3$ and the 4th spring constant $k_4$. Therefore, the first three spring constants can be used to evaluate stiffness of the system in a certain situation.

![Fig. 11. Distributions of decomposed stiffness over the working plane $z=0.7$ m.](image)

Based on the above discussions, a global stiffness index is proposed

$$F_4 = \left( \frac{\int_{V} (k^* - \bar{k}^*)^2 dV}{\int_{V} k^* dV} \right)^{1/2},$$

where $k^* = \frac{1}{6} \sum_{j=1}^{6} k_j$ is the average stiffness of the six spring constants at any configuration;

$$\bar{k}^* = \frac{\int_{V} k^* dV}{\int_{V} dV}$$

is the average of $k^*$ throughout the whole task workspace. $F_4$ demonstrates the fluctuation of the system’s stiffness. Obviously, if the global stiffness index becomes smaller, distributions of the PKM’s stiffness throughout are more uniform, which can make machining accuracy easy to control.

5. **Optimal design**

The radius $R$ and height $h$ of the task workspace are set to be 300 mm and 150 mm, respectively. The design parameters include driving parameters (mainly refer to the inertia of a motor rotor), dimensional parameters ($a$, $b_1$, $b_2$, $H$, $l_1$, $l_2$, $l_3$, and $l_4$) and structural parameters (external diameters: $D_1$, $D_2$, $D_3$, and $D_4$; thickness: $t_1$, $t_2$, $t_3$, and $t_4$). Since there are many design variables, it would be a huge workload to identify all variables after one time optimization. Therefore, a hierarchical optimization design process is depicted in Fig. 12.
The hierarchical procedure can be described as follows. Firstly, driving motors are selected initially according to engineering experience. Secondly, structural parameters are discrete in a reasonable range and a group of structural parameters are selected initially to provide inertial parameters. Thirdly, dimensional synthesis can be conducted based on constraint conditions and objective functions. Then, based on the derived dimensional parameters from Step 3, structural parameters optimization can be conducted. If the derived structural parameters are not consistent with the initial ones in Step 2, another group of structural parameters should be chosen and dimensional synthesis in Step 3 should be repeated until the terminating condition \( TC_1 \) is satisfied. When the terminating condition \( TC_1 \) is satisfied, the selected motors in Step 1 should be judged whether they are matched with the optimal structural and dimensional parameters. If so, one group of optimal design parameters can be obtained. Otherwise, another motor should be chosen and the procedure should be repeated.

The initial structural parameters are listed in Table 1. Herein, \( D_1, D_2, D_3, D_4, t_1, t_2, t_3 \) and \( t_4 \) are external diameters and thickness of the stretchable limb, oscillating limb, thick limb and thin limb. The power of the initial selected servomotor is 1000 W, and inertia of the motor rotor is \( 2.45 \times 10^{-4} \text{ kg} \cdot \text{m}^2 \).

**Table 1** Initial structural parameters of the PKM (unit: mm).

<table>
<thead>
<tr>
<th>( D_1 )</th>
<th>( t_1 )</th>
<th>( D_2 )</th>
<th>( t_2 )</th>
<th>( D_3 )</th>
<th>( t_3 )</th>
<th>( D_4 )</th>
<th>( t_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>6</td>
<td>62</td>
<td>8</td>
<td>144</td>
<td>7</td>
<td>92</td>
<td>7</td>
</tr>
</tbody>
</table>

Based on the proposed procedure, the following analysis can be conducted.

### 5.1. Dimensional synthesis

In order to achieve good kinematic and dynamic behaviors, dimensional synthesis of the PKM can be described as a nonlinear multi-objective programming problem with constraints

\[
\chi \to \min , \\
F - (\chi + 1)F^* \leq 0, F^* = (\min F_1 \min F_2 \min F_3)\top , \\
F_1 = \frac{\eta dV}{\int \eta dV}, \quad F_2 = \max \max_{x \in [0,1]} \tau_{x,1,\ldots,3} \max_{i,j} \tau_{i,j,k}, \quad F_3 = \sqrt{\int_{V} \int_{V} \left( f_i - f_j \right)^2 dV}.
\]

s.t.

\[
\mu = \frac{q_{\text{max}} - q_{\text{min}}}{q_{\text{min}}} \leq \mu_0, \quad |\theta|_{\text{max}} \leq \theta_{\text{r}}, \quad |\psi|_{\text{max}} \leq \psi_{\text{r}}.
\]
In order to analyze effects of $\lambda_{w/b}$, $\lambda_{H/b}$ and $\lambda_{R/b}$ on the global kinematic index $F_1$, dynamic index $F_2$ and elastodynamic index $F_3$, variations of $F_1$, $F_2$ and $F_3$ with respect to $\lambda_{w/b}$, $\lambda_{H/b}$ and $\lambda_{R/b}$ are depicted in Fig. 13. It is noted that values of $\lambda_{w/b}$, $\lambda_{H/b}$ and $\lambda_{R/b}$ are referred to those in [32]. As can be observed clearly, variations of $F_1$, $F_2$ and $F_3$ are not absolutely the same. To be specific, the global kinematic index $F_1$ increases monotonously with the increment of $\lambda_{H/b}$ and $\lambda_{R/b}$. It seems that effects of $\lambda_{w/b}$ on the global kinematic index $F_1$ are very ‘weak’ even though the global kinematic index $F_1$ decreases monotonously with the increment of $\lambda_{w/b}$. The global dynamic index $F_2$ decreases first then increases with the increment of $\lambda_{H/b}$ while increasing monotonously with that of $\lambda_{w/b}$ and $\lambda_{R/b}$. However, effects of $\lambda_{w/b}$ on the global dynamic index $F_2$ become weak with the increment of $\lambda_{H/b}$. Variations of the global elastodynamic index $F_3$ are much different from $F_1$ and $F_2$ in that it increases first then decreases with the increment of $\lambda_{H/b}$ while decreasing first then increasing with that of $\lambda_{R/b}$ and $\lambda_{w/b}$. Furthermore, effects of $\lambda_{w/b}$ on the global index $F_3$ are not very large when the global index $F_3$ increases with the increment of $\lambda_{w/b}$. Thus, by combing the discussion above, the optimal parameter $\lambda_{w/b}$ is chosen to be 0.4 to guarantee design size of the platform.

![Fig. 13. Variations of $F_1$, $F_2$ and $F_3$ with respect to $\lambda_{w/b}$, $\lambda_{H/b}$ and $\lambda_{R/b}$](image)

According to Fig. 13, it is known that the three indices are conflicting, which proves that optimization based on an individual index is not enough. In order to solve this problem, the following synthesis index with weight factors is adopted

$$F = \sqrt{F_2'^2 + (\eta_1 F_1')^2 + (\eta_2 F_3')^2},$$

(45)

where $\eta_1 = \bar{F}_2' \cdot F_1$, $\eta_2 = \bar{F}_3' \cdot F_1$, $\bar{F}_1'$ and $\bar{F}_3'$ denote the average values of $F_1$, $F_2'$ and $F_3'$, respectively. And $F_2'$ and $F_3'$ are dimensionless quantities, which can be expressed as

$$F_2' = \frac{\bar{F}_2'}{\bar{F}_2}, \quad F_3' = \frac{\bar{F}_3'}{\bar{F}_3},$$

(46)

where $\bar{F}_2$ and $\bar{F}_3$ denote the minimum values of $F_2$ and $F_3$, respectively.

Variations of the extension ratio $\mu$, the synthesis index $F$ and maximum rotation angles of universal joints with respect to $\lambda_{H/b}$ and $\lambda_{R/b}$ are addressed in Fig. 14.

![Fig. 14. Variations of $\mu$, $F$ and rotation angles with respect to $\lambda_{H/b}$ and $\lambda_{R/b}$](image)
As shown, the extension ratio $\mu$ as well as the maximum rotation angles of universal joints increases monotonously with the increment of $\lambda_{R/h}$ while decreasing with that of $\lambda_{H/b}$. Relatively speaking, variations of the synthesis index $F$ with respect to $\lambda_{H/b}$ are complicated. Specifically, the synthesis index $F$ decreases first then increases then decreases then increases with the increment of $\lambda_{H/b}$. In general, the synthesis index $F$ increases with the increment of $\lambda_{R/h}$. Supposing the allowable extension ratio $\mu_0$ is 0.8, the range of the optimal parameter $\hat{\lambda}_{H/b}$ is [2.56, 3]. And the synthesis index $F$ increases with the increment of $\lambda_{R/h}$ when $\lambda_{H/b} \in [2.56, 3]$. Observing variations of the synthesis index $F$ with respect to $\lambda_{R/h}$, we can get the optimal parameter $\hat{\lambda}_{R/h}$ ($\lambda_{R/h}^* = 1.3$). Moreover, the synthesis index $F$ decreases first then increases with the increment of $\lambda_{H/b}$ when $\lambda_{H/b} \in [2.56, 3]$. Therefore, making the synthesis index $F$ be the minimize value, we can get the optimal parameter $\hat{\lambda}_{H/b}$ ($\lambda_{H/b}^* = 2.60$). Accordingly, the allowable rotation angles of universal joints can be derived as shown in Fig. 14, which is useful to the mechanical design.

5.2. Structural parameters optimization

Based on the optimal parameters $\lambda_{R/h}^*$, $\lambda_{R/b}^*$ and $\lambda_{H/b}^*$, optimization of structural parameters are conducted. In this section, the global stiffness index $F_4$ is chosen as the corresponding objection function.

Variations of the global stiffness index $F_4$ with respect to structural parameters are shown in Fig. 15.
In general, the global stiffness index $F_4$ decreases with the increment of $D_1$, $D_2$, $t_1$ and $t_4$ while increasing with that of $D_3$, $D_4$, $t_2$ and $t_3$. Relatively speaking, the thickness of the oscillating limb $t_2$ has a weak impact on the system’s rigidity. In order to make the global stiffness index $F_4$ to be the minimum, the optimal structural parameters $D_1$, $t_1$, $D_2$, $t_2$, $D_3$, $t_3$, $D_4$ and $t_4$ can be chosen as 42 mm, 8 mm, 58 mm, 6 mm, 140 mm, 6 mm, 96 mm and 8 mm, respectively.

![Variations of the global stiffness index $F_4$ with respect to structural parameters.](image)

**Fig. 15.** Variations of the global stiffness index $F_4$ with respect to structural parameters.

It is noted that the optimal structural parameters are not consistent with the initial ones in Table 1. Therefore, dimensional synthesis needs to be conducted with another group of structural parameters. Finally, considering the design margin, the optimal dimensional and structural parameters are obtained as listed in Tables 2 and 3.

<p>| Table 2 Optimal dimensional parameters (unit: mm). |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|</p>
<table>
<thead>
<tr>
<th>$a$</th>
<th>$b_x$</th>
<th>$b_y$</th>
<th>$H$</th>
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<th>$l_2$</th>
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<td>267</td>
<td>573</td>
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</table>

<p>| Table 3 Optimal structural parameters (unit: mm). |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|</p>
<table>
<thead>
<tr>
<th>$D_1$</th>
<th>$d_1$</th>
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<td>46</td>
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</table>

### 5.3. Matching design of driving parameters

Based the optimal design parameters listed in Tables 2 and 3, driving parameters of the selected servomotor needs to be checked whether they are matched to the optimal system. In this section, the principle of inertia matching is selected. If the principle of inertia matching is satisfied, the whole optimization process is done. Otherwise, another servomotor needs to be selected and the optimization process needs to be repeated.

Transform Eq. (24) in the joint space and denote $\mathbf{\dot{M}}$ as the mass matrix expressed in the joint space. The equivalent load moment of inertia applied to the servomotor can be expressed as

$$I_k = \frac{D^2}{4\pi^2} \mathbf{\dot{M}}(k,k) - I_{\text{rot}}, k = 1 \sim 3,$$

where $\mathbf{\dot{M}}(k,k)$ is the $k$th diagonal element of $\mathbf{\dot{M}}$; $I_{\text{rot}}$ is the rotor inertia of the servomotor.

Similar to the kinematic performance, dynamic performance and stiffness performance discussed in Section 4, the equivalent load moment of inertia $I_k$ is position-depend. Therefore, the average value throughout the whole task workspace $I_{\text{mean}}$ is taken into calculation. According to the principle of
inertia matching, there exists

\[ 1 \leq \frac{I_{\text{mean}}}{I_M} \leq 3. \]  

(48)

The average equivalent load inertia of the leg 1 \( I_{\text{mean}} \) is \( 3.02 \times 10^{-4} \) kg m\(^2\), while that of leg 2 (leg 3) \( I_{\text{mean}} \) is \( 2.87 \times 10^{-4} \) kg m\(^2\). According to Eq. (46), there are

\[ \frac{I_{\text{mean}}}{I_M} = 1.23 \leq 3, \quad \frac{I_{\text{mean}}}{I_M} = 1.17 \leq 3. \]

The above results demonstrate that the initial selected servomotor is suitable. Therefore, we can get an optimal robot based on the initial selected servomotors and optimal dimensional and structural parameters as listed in Tables 2 and 3.

5.4. Performance comparison

Based on the optimal design variables in this paper and parameters in \([30,31]\), performance comparison of the two Trimule robots is listed in Table 4. To make it reasonable, performance comparison is conducted in their whole task workspace. As shown in Table 4, the Trimule robot in \([30,31]\) presents better kinematic performance throughout its whole task workspace. On the contrary, the optimal one in this paper has better dynamic behaviors and stiffness throughout the whole task workspace even though its kinematic performance is not good as the one in \([30,31]\). Especially, there is a relatively large difference in the dynamic index (acceleration ability) \( F_2 \) and stiffness index \( F_4 \), which demonstrates that the optimal Trimule in this paper can achieve better working efficiency and rigidity. This would be helpful in HSM, which may result in high performance machining. This can verify that the proposed hierarchical design method is useful for this kind of PKMs.

| Table 4 Performance comparison throughout the whole task workspace |
|------------------|----------------|----------------|----------------|
|                  | \( F_1 \)     | \( F_2 \)     | \( F_3 \)     | \( F_4 \)     |
| Trimule in \([30,31]\) | 3.38         | 242.04        | 4.49          | \( 1.42 \times 10^6 \) |
| Optimal Trimule   | 4.12         | 102.09        | 3.88          | \( 7.56 \times 10^5 \) |

6. Conclusions

An optimal design method for the Trimule robot is proposed based on a comprehensive performance index system. The main contributions can be drawn as follows.

1. Considering kinematic, dynamic and elastodynamic performance as well as stiffness, a performance evaluation system for PKMs in HSM is built. Results show that performance of the system is position-dependent and symmetric due to its symmetry in structural features.

2. Considering there are many design variables, a hierarchical optimization design process is proposed, which is helpful to obtain the optimal driving, dimensional and structural parameters. Meanwhile, the optimal system has good kinematic, dynamic and stiffness performance throughout the whole task workspace.

3. Results show that variations of the proposed performance indices with respect to the design variables are conflicting, which demonstrates that optimization based on a single and two indices cannot make the robot own good kinematic performance, rigid dynamic performance and elastodynamic performance simultaneously. A synthesis index with weighted factors is proposed to make compromises on different performance indices.

4. Although only the rotor inertia of the servomotor is considered in this paper for demonstrating the proposed approach, the required maximum torque and speed in a typical task can also be considered when necessary.

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