**QUEEN'S
UNIVERSITY
BELFAST**

Singular diffusions, constant elasticity of variance processes and logarithmic rates of return

Liu, S., Melia, A., Song, X., & Tippet, M. (2020). Singular diffusions, constant elasticity of variance processes and logarithmic rates of return. *European Journal of Finance*, 26(9), 837-853.
<https://doi.org/10.1080/1351847X.2019.1709526>

Published in:
European Journal of Finance

Document Version:
Peer reviewed version

Queen's University Belfast - Research Portal:
[Link to publication record in Queen's University Belfast Research Portal](#)

Publisher rights

Copyright 2021 Informa UK Limited, trading as Taylor & Francis Group.
This manuscript is distributed under a Creative Commons Attribution-NonCommercial-NoDerivs License (<https://creativecommons.org/licenses/by-nc-nd/4.0/>), which permits distribution and reproduction for non-commercial purposes, provided the author and source are cited.

General rights

Copyright for the publications made accessible via the Queen's University Belfast Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The Research Portal is Queen's institutional repository that provides access to Queen's research output. Every effort has been made to ensure that content in the Research Portal does not infringe any person's rights, or applicable UK laws. If you discover content in the Research Portal that you believe breaches copyright or violates any law, please contact openaccess@qub.ac.uk.

Singular Diffusions, Constant Elasticity of Variance Processes and Logarithmic Rates of Return

Siqi Liu^a, Adrian Melia^{b,*}, Xiaojing Song^c and Mark Tippett^d

^a ICMA Centre, Henley Business School, University of Reading, Whiteknights, Reading, RG6 6BA, UK

^b Business School, University of Newcastle, Callaghan, NSW, 2308, Australia

^c Newcastle University Business School, 5 Barrack Road, Newcastle upon Tyne, NE1 7SE, UK

^d Business School, University of Sydney, Codrington Street, Sydney, NSW, 2008, Australia

The singular diffusion processes developed by William Feller occupy a central role in a number of disciplines including economics and finance. We identify a fundamental inconsistency between the probability densities stated in the Feller papers for these singular diffusion processes. Moreover, we apply the method of group-invariance to resolve this inconsistency. Since logarithmic returns are of considerable importance in economics and finance, we also illustrate a procedure for determining the conditional expected logarithmic rate of return for state variables which evolve in terms of the singular diffusion processes on which the Feller papers are based.

Keywords: Constant elasticity of variance (CEV), Kolmogorov equation, Group invariance, Logarithmic return, Singular diffusion process

JEL Classification: C15, C22, C46

*Corresponding author. Email: m.tippett@lboro.ac.uk. The authors acknowledge the helpful comments and suggestions of the referees and invaluable discussions with Huw Rhys, formerly of the University of Exeter, during the writing of this paper.

1. Introduction

The seminal papers of Feller (1951a, 1951b) use the Kolmogorov (that is, Fokker-Planck) equation to determine the distributional properties of a general class of singular diffusion processes (Karlin and Taylor 1981, 219-220).¹ Whilst early applications of these processes occurred mainly in biology and genetics (Gan and Waxman 2015), more recently they have been applied in a number of disciplines including economics and finance (Cox and Ross 1976; Jarrow and Rudd 1983; Cox, 1996). In this paper we identify a fundamental inconsistency between the general probability density which appears in Feller (1951b) for these singular diffusion processes and the probability density for a particular interpretation of the singular diffusion processes – namely, the continuous time branching process appearing in Feller (1951a)². We resolve this inconsistency by applying the method of group invariance under infinitesimal transformations (Nariboli 1977) to determine the correct form of the probability density for both the branching process and a second interpretation of the singular diffusion processes which has particular relevance to the issue of the volatility smile in the pricing of derivative securities (Chance, 2013). Both examples show that the general probability density summarised in Feller (1951b) is in error. Moreover, we illustrate how the Kolmogorov equation may be used to determine the conditional expected logarithmic rate of return for the singular diffusion processes on which the Feller (1951a, 1951b) papers are based. This result holds particular significance in economics and finance where the logarithmic rates of

¹ A diffusion process is said to be singular when there are zeros or singularities in the mean and/or variance of infinitesimal changes in the process which preclude the imposition of given boundary/initial conditions on the Kolmogorov equation for the process (Cox and Miller 1965, 225).

² See the chapter entitled “Continuous Time Branching Processes” by Schinazi (1999, 151-173) for a more detailed treatment of the analytical foundations of the branching process in continuous time with continuous state space.

return arising under the Feller singular diffusion processes are important in areas such as credit risk and bankruptcy (Nunes 2009), the term structure of interest rates (Cox, et al. 1985) and as previously noted, volatility smile in the valuation of derivative securities (Chance 2013)³.

We commence our analysis in section 2 by reviewing the distributional properties of the continuous time branching process as summarised in Feller (1951a, 235) – and show how they differ from the equivalent distributional properties for the branching process as developed from Feller (1951b). Sections 3 and 4 then apply the method of group invariance to determine the probability density for a second interpretation of the Feller (1951b) singular diffusion processes which is particularly relevant to the issue of the volatility smile in the valuation of derivative securities (Chance 2013). Again, distributional properties based on the method of group invariance differ from the distributional properties summarised in Feller (1951b). Moreover, the moments associated with the probability density summarised in Feller (1951b) for this second interpretation of the singular diffusion processes also differ from the moments based on the Kolmogorov equation for the stochastic differential equation on which the given singular diffusion process is based. Section 5 uses both the branching process and our second (volatility smile) example to illustrate how the Kolmogorov equation may be used to determine the conditional expected logarithmic rate of return for state variables that evolve in terms of the singular diffusion processes on which the Feller (1951b) paper is based. Our analysis shows in particular, that the expected logarithmic rate of return for the singular diffusion processes varies according to the probability of the state variable eventually being absorbed into the zero state (as, for

³ See Beckers (1980), Davydov and Linetsky (2001) and Baaquie et al. (2011) for a more exhaustive list of references to empirical applications of CEV processes in the finance literature.

example, when a firm enters bankruptcy). This in turn will mean that the expected logarithmic return will be non-stationary in time and will vary with the magnitude of the state variable at the beginning of the period over which the logarithmic return is determined.

2. Branching Process

Feller (1951b) considers a state variable, $x(t)$, that evolves in terms of the stochastic differential equation:

$$dx(t) = (c + bx(t))dt + \sqrt{ax(t)}dq(t) \quad (1)$$

where $a > 0$, b and $c < a$ are parameters and $dq(t)$ is a pure white noise process (Feller 1951b, 180). This, in turn will mean that the conditional probability density for $x(t)$ can be determined from the Kolmogorov equation corresponding to the stochastic differential equation (1); namely (Karlin and Taylor 1981 219-220):

$$\frac{\partial^2}{\partial x(t)^2} \left\{ \frac{1}{2} ax(t) p[x(0), x(t), t] \right\} - \frac{\partial}{\partial x(t)} \{ [c + bx(t)] p[x(0), x(t), t] \} = \frac{\partial p[x(0), x(t), t]}{\partial t} \quad (2)$$

Feller (1951b, 180) uses the Kolmogorov equation (2) to determine the Laplace transform of the conditional probability density of the state variable, $x(t)$. He then inverts the Laplace transform and thereby proposes that the conditional probability density, $p[x(0), x(t), t]$, of the state variable is given by:

$$p[x(0), x(t), t] = \frac{2b}{a(e^{bt} - 1)} \exp \left\{ -\frac{2b}{a} \frac{x(0)e^{bt} + x(t)}{e^{bt} - 1} \right\} \left\{ \frac{4b^2 e^{-bt} x(t)}{x(0)} \right\}^{\frac{(2c-a)}{2a}} I_{\frac{(a-2c)}{a}} \left\{ \frac{4b}{a} \frac{\sqrt{x(0)x(t)e^{bt}}}{e^{bt} - 1} \right\} \quad (3)$$

where $I_{\frac{(a-2c)}{a}}(\cdot)$ is the Modified Bessel Function of the first kind of order $\frac{(a-2c)}{a}$. A

simple interpretation of the above result is provided by the continuous time branching process, for which $a = \sigma^2$ is an intensity parameter defined on the white noise process $dq(t)$, $b = \mu$ is the expected infinitesimal proportionate change in the state variable on a per unit time basis and $c = 0$ (Feller 1951a, 235; Cox and Miller 1965, 215-216; Docherty et al. 2018, 428).⁴ Substituting these parameter values into equation (3) leads to the Feller (1951b, 180) expression for the conditional probability density of a state variable that evolves in terms of the continuous time branching process with parameters $a = \sigma^2$, $b = \mu$ and $c = 0$; namely:

$$p[x(0), x(t), t] = \frac{2\mu}{\sigma^2(e^{\mu t} - 1)} \exp\left\{-\frac{2\mu}{\sigma^2} \frac{x(0)e^{\mu t} + x(t)}{e^{\mu t} - 1}\right\} \sqrt{\frac{x(0)e^{\mu t}}{4\mu^2 x(t)}} I_1\left\{\frac{4\mu}{\sigma^2} \frac{\sqrt{x(0)x(t)e^{\mu t}}}{e^{\mu t} - 1}\right\} \quad (4a)$$

Moreover, the Modified Bessel Function of the first kind of order one possesses the following series representation (Kreyszig 1972, 134):

⁴ The continuous time branching process is a member of the class of constant elasticity of variance (CEV) processes defined by the stochastic differential equation (Cox, 1975; Cox 1996):

$$dx(t) = \mu x(t)dt + \sigma[x(t)]^\beta dq(t)$$

where $0 < \beta < 1$ is called the exponent parameter. The properties of the branching process (for which $\beta = 1/2$) have been extensively documented in the literature – often in comparison with the more commonly applied Geometric Brownian Motion (for which $\beta = 1$) (Cox and Miller 1965, 235-237; Docherty et al. 2018). Given this, subsequent analysis will include a consideration of the CEV process with $\beta = 3/4$ – a kind of “half way house” between the branching process and the Geometric Brownian Motion – and which is of particular relevance to the issue of the volatility smile in the valuation of derivative securities (Chance 2013).

$$I_1 \left\{ \frac{4\mu}{\sigma^2} \frac{\sqrt{x(0)x(t)e^{\mu t}}}{e^{\mu t} - 1} \right\} = \frac{2\mu}{\sigma^2} \frac{\sqrt{x(0)x(t)e^{\mu t}}}{e^{\mu t} - 1} \sum_{j=0}^{\infty} \frac{\left\{ \frac{2\mu}{\sigma^2} \frac{\sqrt{x(0)x(t)e^{\mu t}}}{e^{\mu t} - 1} \right\}^{2j}}{j!(j+1)!}$$

Hence, substitution and cancelling of terms shows that the Feller (1951b, 180) expression for the conditional probability density of the state variable, $x(t)$, is given by:

$$p[x(0), x(t), t] = \frac{2\mu}{\sigma^4} \frac{x(0)e^{\mu t}}{(e^{\mu t} - 1)^2} \exp \left\{ -\frac{2\mu}{\sigma^2} \frac{x(0)e^{\mu t} + x(t)}{e^{\mu t} - 1} \right\} \sum_{j=0}^{\infty} \frac{\left\{ \frac{4\mu^2}{\sigma^4} \frac{x(0)x(t)e^{\mu t}}{(e^{\mu t} - 1)^2} \right\}^j}{j!(j+1)!} \quad (4b)$$

Now, here one can make the substitutions:

$$\xi = \frac{4\mu^2}{\sigma^4} \frac{x(0)x(t)e^{\mu t}}{(e^{\mu t} - 1)^2} \quad d\xi = \frac{4\mu^2}{\sigma^4} \frac{x(0)e^{\mu t}}{(e^{\mu t} - 1)^2} dx(t) \quad \gamma = \frac{2\mu}{\sigma^2} \frac{x(0)e^{\mu t}}{e^{\mu t} - 1}$$

and thereby show that the canonical representation of the probability density (4b) is given by:

$$p[x(0), x(t), t] = \frac{1}{2\mu} e^{-\gamma} \exp \left(-\frac{\xi}{\gamma} \right) \sum_{j=0}^{\infty} \frac{\xi^j}{j!(j+1)!} \quad (5)$$

It then follows from equation (5) that the mean of the continuous time branching process is given by:

$$E(\xi) = \frac{1}{2\mu} e^{-\gamma} \int_0^{\infty} \exp \left(-\frac{\xi}{\gamma} \right) \sum_{j=0}^{\infty} \frac{\xi^{j+1}}{j!(j+1)!} d\xi$$

where $E(\cdot)$ is the expectation operator. The further substitutions $y = \frac{\xi}{\gamma}$ and

$\gamma dy = d\xi$ will then show:

$$E(\xi) = \frac{1}{2\mu} e^{-\gamma} \sum_{j=0}^{\infty} \frac{\gamma}{j!(j+1)!} \int_0^{\infty} e^{-y} (\gamma y)^{j+1} dy = \frac{\gamma^2}{2\mu} e^{-\gamma} \sum_{j=0}^{\infty} \frac{\gamma^j}{j!} = \frac{\gamma^2}{2\mu}$$

or equivalently:

$$E(\xi) = \frac{4\mu^2}{\sigma^4} \frac{x(0)e^{\mu t}}{(e^{\mu t} - 1)^2} E[x(t)] = \frac{\gamma^2}{2\mu} = \frac{2\mu}{\sigma^4} \frac{[x(0)]^2 e^{2\mu t}}{(e^{\mu t} - 1)^2}$$

Cancelling terms in the above expression shows that the Feller (1951b, 180) formulation for the mean of the branching process is given by:

$$E[x(t)] = \frac{x(0)}{2\mu} e^{\mu t} \quad (6)$$

Here, however, it is important to note how Docherty et al. (2018) apply the method of group invariance under infinitesimal transformations to the Kolmogorov equation (Nariboli, 1977) to determine the following expression for the probability density of the continuous time branching process:

$$p[x(0), x(t), t] = \frac{4\mu^2}{\sigma^4} \frac{x(0)e^{\mu t}}{(e^{\mu t} - 1)^2} \exp\left\{-\frac{2\mu}{\sigma^2} \frac{x(0)e^{\mu t} + x(t)}{e^{\mu t} - 1}\right\} \sum_{j=0}^{\infty} \frac{\left\{\frac{4\mu^2}{\sigma^4} \frac{x(0)x(t)e^{\mu t}}{(e^{\mu t} - 1)^2}\right\}^j}{j!(j+1)!} \quad (7)$$

Feller (1951a, 235) too, obtains this expression for the probability density of the branching process – although his statement of the probability density “is given for the sake of completeness only” and is made without formal proof. However, the crucial

point which emerges from Feller (1951a, 235) is that the expression for the probability density (4) of the continuous time branching process with parameters $a = \sigma^2$, $b = \mu$ and $c = 0$ as summarised in Feller (1951b, 180) is in error by a factor of:

$$(4b^2)^{(2c-a)/2a} = \frac{1}{\sqrt{4b^2}} = \frac{1}{2\mu}$$

when compared with the correct probability density (7). From equation (6), this in turn will mean that the expected value of the state variable, $x(t)$, whose evolution is governed by the probability density (7) will be given by:

$$E[x(t)] = 2\mu \frac{x(0)}{2\mu} e^{\mu t} = x(0)e^{\mu t}$$

a result which is compatible with the statement for the mean of the continuous time branching process given by a number of other authors, including Cox and Miller (1965, 236) and Karlin and Taylor (1975, 416-417).

3. Group-Invariant Solutions

One can investigate this matter further by considering a state variable, $x(t)$, which evolves in terms of a CEV process with exponent parameter $\beta = \frac{3}{4}$. This in turn will mean that the sample paths of $x(t)$ are described by the following stochastic differential equation:

$$dx(t) = \mu x(t)dt + \sigma[x(t)]^{\frac{3}{4}} dq(t) \quad (8)$$

Here, as previously, μ is the expected proportionate change in the state variable on a per unit time basis and σ is an intensity parameter defined on the white noise process $dq(t)$. Taking the square root of the state variable:

$$y(t) = \sqrt{x(t)} \quad (9)$$

will then show:

$$\frac{\partial y}{\partial x(t)} = \frac{1}{2} [x(t)]^{-1/2} \quad \text{and} \quad \frac{\partial^2 y}{\partial x(t)^2} = -\frac{1}{4} [x(t)]^{-3/2}$$

One can then use the fact that $[dx(t)]^2 = \sigma^2 \{ [x(t)]^{3/4} \}^2 dt = \sigma^2 [x(t)]^{3/2} dt$ in conjunction with Itô's formula (Karlin and Taylor 1981, 372-373) to show that infinitesimal variations in the transformed state variable, $y(t)$, will evolve in terms of the following stochastic differential equation:

$$\begin{aligned} dy(t) &= \frac{\partial y}{\partial x(t)} dx(t) + \frac{1}{2} \frac{\partial^2 y}{\partial x(t)^2} [dx(t)]^2 = \\ &= \frac{1}{2} [x(t)]^{-1/2} \{ \mu x(t) dt + \sigma [x(t)]^{3/4} dq(t) \} - \frac{1}{8} [x(t)]^{-3/2} \sigma^2 [x(t)]^{3/2} dt \end{aligned}$$

or equivalently:

$$dy(t) = \left\{ \frac{\mu}{2} y(t) - \frac{\sigma^2}{8} \right\} dt + \frac{\sigma}{2} \sqrt{y(t)} dq(t) \quad (10)$$

This in turn will mean that $a = \frac{\sigma^2}{4}$, $b = \frac{\mu}{2}$ and $c = -\frac{\sigma^2}{8}$ in equation (1). Now, the differential equation (10) taken in conjunction with equation (2) shows that the Kolmogorov equation for the conditional probability density, $p[y(0), y(t), t]$, of the state variable, $y(t)$, is given by:

$$\frac{\partial^2}{\partial y(t)^2} \left\{ \frac{\sigma^2}{8} y(t) p[y(0), y(t), t] \right\} - \frac{\partial}{\partial y(t)} \left\{ \left[-\frac{\sigma^2}{8} + \frac{\mu}{2} y(t) \right] p[y(0), y(t), t] \right\} = \frac{\partial p[y(0), y(t), t]}{\partial t}$$

or equivalently:

$$\frac{1}{8} \sigma^2 y(t) \frac{\partial^2 p}{\partial y(t)^2} + \left[\frac{3\sigma^2}{8} - \frac{\mu}{2} y(t) \right] \frac{\partial p}{\partial y(t)} - \frac{\mu}{2} p[y(0), y(t), t] - \frac{\partial p[y(0), y(t), t]}{\partial t} = 0 \quad (11)$$

Moreover, at $t=0$ the conditional probability density will take the form of a Dirac delta function under which the entire probability mass is concentrated at $y(0)$.

Now, suppose one seeks a solution of the Kolmogorov equation using the method of group invariance based on the self-similar solution (Nariboli 1977, 159-161):

$$p[y(0), y(t), t] = D[y(0), y(t), t] \nu(\xi) \quad (12a)$$

where:

$$\xi = \frac{16\mu^2}{\sigma^4} \frac{y(0)y(t) \exp\left(\frac{\mu}{2}t\right)}{\left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\}^2} \quad (12b)$$

is the similarity variable and:

$$D[y(0), y(t), t] = \frac{64\mu^3}{\sigma^6} \frac{[y(0)]^2 e^{\mu t}}{\left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\}^3} \exp \left\{ -\frac{4\mu}{\sigma^2} \frac{y(0) \exp\left(\frac{\mu}{2}t\right) + y(t)}{\exp\left(\frac{\mu}{2}t\right) - 1} \right\} \quad (12c)$$

Then straightforward but arduous differentiation and algebraic manipulation will show that these transformations reduce the Kolmogorov equation (11) to the following ordinary differential equation:

$$\xi v''(\xi) + 3v'(\xi) - v(\xi) = 0 \quad (13)$$

which is stated purely in terms of the similarity variable, ξ . One can then determine solutions of the above differential equation using the power-series method:

$$v(\xi) = \sum_{j=0}^{\infty} a_j \xi^{j+r} \quad (14a)$$

where a_j is the coefficient associated with the j^{th} term in the series expansion and r represents the root(s) of the indicial equation (Kreyszig 1972, 115).⁵ Substituting equation (14a) into equation (13) shows that when $r = 0$, the relationship between the coefficients defining the power series expansion for $v(\xi)$ are given by:

$$a_j = \frac{a_{j-1}}{j(j+2)} \quad (14b)$$

This in turn will mean:

$$v(\xi) = \sum_{j=0}^{\infty} a_j \xi^j = 2a_0 \sum_{j=0}^{\infty} \frac{\xi^j}{j!(j+2)!} \quad (14c)$$

⁵ The indicial equation of the differential equation (13) is given by $r(r+2) = 0$ and thus, the roots are $r = 0$ and $r = -2$. It is the larger of these roots which is compatible with the requirement at $t = 0$ that the probability density takes the form of a Dirac delta function with the entire probability mass concentrated at $y(0)$.

where $a_0 = \frac{1}{2}$ is determined so as satisfy the initial condition of a unit probability

mass under the probability density at $t=0$. It then follows that the self-similar solution of the Kolmogorov equation (11) is given by:

$$p[y(0), y(t), t] = D[y(0), y(t), t]v(\xi) =$$

$$\frac{64\mu^3}{\sigma^6} \frac{[y(0)]^2 e^{\mu t}}{\left\{\exp\left(\frac{\mu}{2}t\right) - 1\right\}^3} \exp\left\{-\frac{4\mu}{\sigma^2} \frac{y(0)\exp\left(\frac{\mu}{2}t\right) + y(t)}{\exp\left(\frac{\mu}{2}t\right) - 1}\right\} \sum_{j=0}^{\infty} \left[\frac{16\mu^2}{\sigma^4} \frac{y(0)y(t)\exp\left(\frac{\mu}{2}t\right)}{\left\{\exp\left(\frac{\mu}{2}t\right) - 1\right\}^2} \right]^j \frac{1}{j!(j+2)!}$$

(15)

One may contrast the self-similar solution of the Kolmogorov equation summarised here with the Feller (1951b, 180) expression for the probability density which, on substituting the parameters $a = \frac{\sigma^2}{4}$, $b = \frac{\mu}{2}$ and $c = -\frac{\sigma^2}{8}$ into equation (3), turns out to be:

$$p[y(0), y(t), t] =$$

$$\frac{4\mu}{\sigma^2 \left\{\exp\left(\frac{\mu}{2}t\right) - 1\right\}} \exp\left\{-\frac{4\mu}{\sigma^2} \frac{y(0)\exp\left(\frac{\mu}{2}t\right) + y(t)}{\exp\left(\frac{\mu}{2}t\right) - 1}\right\} \left[\frac{y(0)\exp\left(\frac{\mu}{2}t\right)}{\mu^2 y(t)} \right] I_2 \left[\frac{8\mu}{\sigma^2} \sqrt{\frac{y(0)y(t)\exp\left(\frac{\mu}{2}t\right)}{\exp\left(\frac{\mu}{2}t\right) - 1}} \right]$$

(16)

Moreover, the Modified Bessel Function of the first kind of order two possesses the following series representation (Kreyszig 1972, 134):

$$I_2 \left\{ \frac{8\mu}{\sigma^2} \frac{\sqrt{y(0)y(t)\exp\left(\frac{\mu}{2}t\right)}}{\exp\left(\frac{\mu}{2}t\right)-1} \right\} = \left[\frac{4\mu}{\sigma^2} \frac{\sqrt{y(0)y(t)\exp\left(\frac{\mu}{2}t\right)}}{\exp\left(\frac{\mu}{2}t\right)-1} \right]^2 \sum_{j=0}^{\infty} \left[\frac{\frac{4\mu}{\sigma^2} \sqrt{y(0)y(t)\exp\left(\frac{\mu}{2}t\right)}}{\exp\left(\frac{\mu}{2}t\right)-1} \right]^{2j} \frac{1}{j!(j+2)!}$$

Substituting this expression into equation (16) shows that the Feller (1951b, 180) expression for the probability density may be re-stated as:

$$p[y(0), y(t), t] = \frac{64\mu}{\sigma^6} \frac{[y(0)]^2 e^{\mu t}}{\left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\}^3} \exp \left\{ - \frac{4\mu}{\sigma^2} \frac{y(0)\exp\left(\frac{\mu}{2}t\right) + y(t)}{\exp\left(\frac{\mu}{2}t\right) - 1} \right\} \sum_{j=0}^{\infty} \left[\frac{\frac{16\mu^2}{\sigma^4} \frac{y(0)y(t)\exp\left(\frac{\mu}{2}t\right)}{\left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\}^2}}{j!(j+2)!} \right]^j \quad (17)$$

Substituting the parameters $a = \frac{\sigma^2}{4}$, $b = \frac{\mu}{2}$ and $c = -\frac{\sigma^2}{8}$ shows how on this occasion the self-similar probability density (15) differs from the Feller (1951b, 180) probability density in equation (17) by a factor of:

$$(4b^2)^{(2c-a)/2a} = \left(4 \frac{\mu^2}{4} \right)^{-1} = \frac{1}{\mu^2}$$

This again demonstrates that the probability density (17) determined from Feller (1951b, 180) is in error.⁶

One may cast further light on this matter by transforming the probability density (15) by the similarity variable (12b) in which case we have:

$$\xi = \frac{16\mu^2}{\sigma^4} \frac{y(0)y(t)\exp\left(\frac{\mu}{2}t\right)}{\left\{\exp\left(\frac{\mu}{2}t\right)-1\right\}^2} \quad \text{and} \quad d\xi = \frac{16\mu^2}{\sigma^4} \frac{y(0)\exp\left(\frac{\mu}{2}t\right)}{\left\{\exp\left(\frac{\mu}{2}t\right)-1\right\}^2} dy(t)$$

Substitution will then show that the canonical representation of the probability density (15) is given by:⁷

$$p[y(0), y(t), t] = \gamma e^{-\gamma} \exp\left(-\frac{\xi}{\gamma}\right) \sum_{j=0}^{\infty} \frac{\xi^j}{j!(j+2)!} \quad (18)$$

where:

⁶ An anonymous referee notes that, “in an age long before electronic typesetting, it is likely the error [in Feller (1951b, 180)] crept in at the galley proof stage of the publication process and Feller, or an assistant, failed to spot it when checking the proofs. However the error occurred, [the authors, along with the anonymous referee] find it hard to contemplate that Feller was not fully aware of the correct formula.”

⁷ When the method of group invariance is applied to the general case of a state variable governed by the stochastic differential equation (1), it leads to the following canonical representation of the conditional probability density for the state variable:

$$p[x(0), x(t), t] = \gamma^{-\frac{2c}{a}} e^{-\gamma} \exp\left(-\frac{\xi}{\gamma}\right) \sum_{j=0}^{\infty} \frac{\xi^j}{j! \Gamma(j + \frac{(a-2c)}{a} + 1)}$$

where $\xi = \frac{4b^2}{a^2} \frac{x(0)x(t)e^{bt}}{(e^{bt}-1)^2}$ is the similarity variable and $\gamma = \frac{2b}{a} \frac{x(0)e^{bt}}{e^{bt}-1}$ is the conditioning variable. Note

that setting $a = \frac{\sigma^2}{4}$, $b = \frac{\mu}{2}$ and $c = -\frac{\sigma^2}{8}$ in the general expression for $p[x(0), x(t), t]$ given here, leads to the probability density (18) summarised in the text.

$$\gamma = \frac{4\mu}{\sigma^2} \frac{y(0) \exp\left(\frac{\mu}{2}t\right)}{\exp\left(\frac{\mu}{2}t\right) - 1} \quad (19)$$

is the conditioning variable, and:

$$\frac{\xi}{\gamma} = \frac{\frac{16\mu^2}{\sigma^4} \frac{y(0)y(t) \exp\left(\frac{\mu}{2}t\right)}{\left\{\exp\left(\frac{\mu}{2}t\right) - 1\right\}^2}}{\frac{4\mu}{\sigma^2} \frac{y(0) \exp\left(\frac{\mu}{2}t\right)}{\exp\left(\frac{\mu}{2}t\right) - 1}} = \frac{4\mu}{\sigma^2} \frac{y(t)}{\exp\left(\frac{\mu}{2}t\right) - 1} \quad (20)$$

Now, the probability density (18) possesses compact support over the semi-infinite real line in which case one can determine its moment generating function through the following integral:

$$E(e^{\lambda\xi}) = \gamma e^{-\gamma} \int_0^\infty e^{\lambda\xi} \exp\left(-\frac{\xi}{\gamma}\right) \sum_{j=0}^\infty \frac{\xi^j}{j!(j+2)!} d\xi = \gamma e^{-\gamma} \sum_{j=0}^\infty \frac{1}{j!(j+2)!} \int_0^\infty \exp\left(\lambda\xi - \frac{\xi}{\gamma}\right) \xi^j d\xi \quad (21)$$

where λ is an arbitrary real parameter. One can then make the substitutions

$$q = \left(\frac{1}{\gamma} - \lambda\right)\xi = \left(\frac{1-\gamma\lambda}{\gamma}\right)\xi \text{ and } \frac{\gamma}{1-\gamma\lambda} dq = d\xi \text{ in which case we have:}$$

$$E(e^{\lambda\xi}) = \frac{\gamma^2}{1-\gamma\lambda} e^{-\gamma} \sum_{j=0}^\infty \frac{1}{j!(j+2)!} \int_0^\infty e^{-q} \left\{\left(\frac{\gamma}{1-\gamma\lambda}\right)q\right\}^j dq$$

or equivalently:

$$E(e^{\lambda\xi}) = \frac{\gamma^2}{1-\gamma\lambda} e^{-\gamma} \sum_{j=0}^{\infty} \frac{\left(\frac{\gamma}{1-\gamma\lambda}\right)^j}{j!(j+2)!} \int_0^{\infty} e^{-q} q^j dq = (1-\gamma\lambda) e^{-\gamma} \sum_{j=0}^{\infty} \frac{\left(\frac{\gamma}{1-\gamma\lambda}\right)^{j+2}}{(j+2)!}$$

But here one can use a Taylor expansion to show:

$$\exp\left(\frac{\gamma}{1-\gamma\lambda}\right) = 1 + \frac{\gamma}{1-\gamma\lambda} + \sum_{j=0}^{\infty} \frac{\left(\frac{\gamma}{1-\gamma\lambda}\right)^{j+2}}{(j+2)!}$$

in which case it follows:

$$E(e^{\lambda\xi}) = (1-\gamma\lambda) e^{-\gamma} \sum_{j=0}^{\infty} \frac{\left(\frac{\gamma}{1-\gamma\lambda}\right)^{j+2}}{(j+2)!} = (1-\gamma\lambda) e^{-\gamma} \left\{ \exp\left(\frac{\gamma}{1-\gamma\lambda}\right) - 1 - \frac{\gamma}{1-\gamma\lambda} \right\}$$

Simple algebraic manipulation will then show that the moment generating function possesses the following canonical representation:

$$E(e^{\lambda\xi}) = (1-\gamma\lambda) \exp\left(\frac{\gamma^2\lambda}{1-\gamma\lambda}\right) - (1+\gamma-\gamma\lambda) e^{-\gamma} \quad (22)$$

Now, letting $\lambda=0$ in the above expression shows that the total area under the probability density (15) amounts to:

$$\gamma e^{-\gamma} \int_0^{\infty} \exp\left(-\frac{\xi}{\gamma}\right) \sum_{j=0}^{\infty} \frac{\xi^j}{j!(j+2)!} d\xi = 1 - (1+\gamma) e^{-\gamma} \quad (23)$$

Moreover, from equation (18) it will be observed that $\lim_{t \rightarrow 0} e^{-\gamma} = \lim_{t \rightarrow 0} \gamma e^{-\gamma} = 0$ in

which case it follows:

$$\lim_{t \rightarrow 0} p[y(0), y(t), t] = \lim_{t \rightarrow 0} \gamma e^{-\gamma} \exp\left(-\frac{\xi}{\gamma}\right) \sum_{j=0}^{\infty} \frac{\xi^j}{j!(j+2)!} = 1 \quad (24)$$

This in turn will mean that at $t=0$ the conditional probability density takes the form of a Dirac delta function under which the entire probability mass is concentrated at $y(0)$. Beyond this point, however, equation (23) shows that integrating through the probability density results in a less than unit probability mass. The “missing” probability mass captures the likelihood of the state variable being absorbed into the zero state (Feller 1951a, 236; Cox and Ross 1976, 161). This will mean that the probability of the state variable being absorbed into the zero state (that is, the probability of extinction) before time $t > 0$ is given by:⁸

$$1 - \gamma e^{-\gamma} \int_0^{\infty} \exp\left(-\frac{\xi}{\gamma}\right) \sum_{j=0}^{\infty} \frac{\xi^j}{j!(j+2)!} d\xi =$$

$$e^{-\gamma} (1 + \gamma) = \left(1 + \frac{4\mu}{\sigma^2} \frac{y(0)}{1 - \exp\left(-\frac{\mu}{2}t\right)} \right) \exp\left\{ -\frac{4\mu}{\sigma^2} \frac{y(0)}{1 - \exp\left(-\frac{\mu}{2}t\right)} \right\}$$

(25a)

⁸ For the general case of a state variable governed by the stochastic differential equation (1), the probability of absorption at zero is given by:

$$1 - \int_0^{\infty} p[x(0), x(t), t] dx(t) = 1 - \gamma^{-\frac{2c}{a}} e^{-\gamma} \int_0^{\infty} \exp\left(-\frac{\xi}{\gamma}\right) \sum_{j=0}^{\infty} \frac{\xi^j}{j! \Gamma(j + \frac{(a-2c)}{a} + 1)} d\xi = \frac{\Gamma(n+1, \gamma)}{n!}$$

where $\xi = \frac{4b^2}{a^2} \frac{x(0)x(t)e^{bt}}{(e^{bt}-1)^2}$ is the similarity variable, $\gamma = \frac{2b}{a} \frac{x(0)e^{bt}}{e^{bt}-1}$ is the conditioning variable and:

$$\Gamma(n+1, \gamma) = n! e^{-\gamma} \sum_{j=0}^n \frac{\gamma^j}{j!}$$

is the upper incomplete gamma function (Jameson, 2016).

Moreover, the probability of the state variable being absorbed at zero at some unspecified future time (that is, the probability of ultimate extinction) is given by:

$$\lim_{t \rightarrow \infty} 1 - \gamma e^{-\gamma} \int_0^{\infty} \exp\left(-\frac{\xi}{\gamma}\right) \sum_{j=0}^{\infty} \frac{\xi^j}{j!(j+2)!} d\xi = \left(1 + \frac{4\mu}{\sigma^2} y(0)\right) \exp\left\{-\frac{4\mu}{\sigma^2} y(0)\right\} \quad (25b)$$

Differentiating through the expression $e^{-\gamma}(I + \gamma)$ shows that a stationary point occurs when $-\gamma e^{-\gamma} = 0$ or equivalently, when $\gamma = 0$. Moreover, evaluating the second derivative at this point shows $-(I - \gamma)e^{-\gamma} = -I$. This in turn will mean there is a maximum at the point $\gamma = 0$ or that the probability of extinction cannot exceed unity.

4. Conditional Moments

One can, of course, use the moment generating function (22) to determine the moments of the probability density (15). However, given the apparent inconsistency in the distributional properties which appear in Feller (1951a) and Feller (1951b), it is instructive to approach the determination of the moments in a slightly different manner. Here it will be recalled that the $(2n)^{th}$ moment about the origin for the probability density (18) is defined as follows:

$$E(\xi^{2n}) = \gamma e^{-\gamma} \int_0^{\infty} \exp\left(-\frac{\xi}{\gamma}\right) \sum_{j=0}^{\infty} \frac{\xi^{j+2n}}{j!(j+2)!} d\xi = \gamma e^{-\gamma} \sum_{j=0}^{\infty} \frac{1}{j!(j+2)!} \int_0^{\infty} \exp\left(-\frac{\xi}{\gamma}\right) \xi^{j+2n} d\xi$$

The substitutions $q = \frac{\xi}{\gamma}$ and $\gamma dq = d\xi$ will then show:

$$E(\xi^{2n}) = \gamma^2 e^{-\gamma} \sum_{j=0}^{\infty} \frac{1}{j!(j+2)!} \int_0^{\infty} e^{-q} (\gamma q)^{j+2n} dq = \gamma^2 e^{-\gamma} \sum_{j=0}^{\infty} \frac{\gamma^{j+2n}}{j!(j+2)!} (j+2n)!$$

or equivalently:

$$E(\xi^{2n}) = \gamma^{2n} e^{-\gamma} \sum_{j=0}^{\infty} \frac{(j+3)(j+4) \dots (j+2n) \gamma^{j+2}}{j!} = \gamma^{2n} e^{-\gamma} \frac{d^{2(n-1)}}{d\gamma^{2(n-1)}} \left(\sum_{j=0}^{\infty} \frac{\gamma^{j+2n}}{j!} \right)$$

This in turn will mean that the $(2n)^{th}$ moment about the origin can be expressed as:

$$E(\xi^{2n}) = \gamma^{2n} e^{-\gamma} \frac{d^{2(n-1)}}{d\gamma^{2(n-1)}} (\gamma^{2n} e^{\gamma}) \quad (26)$$

Thus, letting $n=1$ in the above result shows $E(\xi^2) = \gamma^4 e^{-\gamma} e^{\gamma} = \gamma^4$. From equation (12b) and equation (19) this will mean $E\{[y(t)]^2\} = [y(0)]^2 e^{\mu t}$. But here it will be recalled that $y(t) = \sqrt{x(t)}$ in which case substitution shows:

$$E[x(t)] = x(0) e^{\mu t} \quad (27)$$

is the mean of the probability density (15). This indicates that the state variable, $x(t)$, grows at an expected infinitesimal proportionate rate of μ per unit time (Cox and Miller 1965, 215-216).

Similarly, letting $n=2$ in equation (26) shows:

$$E(\xi^4) = \gamma^4 e^{-\gamma} \frac{d^2}{d\gamma^2} (\gamma^4 e^{\gamma}) = (12\gamma^6 + 8\gamma^7 + \gamma^8)$$

However, using equation (12b) and equation (19) will show that the above expression reduces to:

$$E\{[y(t)]^4\} = \frac{3\sigma^4}{4\mu^2} [y(0)]^2 e^{\mu t} \left\{ \exp\left(\frac{\mu}{2} t\right) - 1 \right\}^2 + \frac{2\sigma^2}{\mu} [y(0)]^3 \exp\left(\frac{3\mu}{2} t\right) \left\{ \exp\left(\frac{\mu}{2} t\right) - 1 \right\} + [y(0)]^4 e^{2\mu t}$$

Finally, substituting $y(t) = \sqrt{x(t)}$ will then show:

$$E\{[x(t)]^2\} = \frac{3\sigma^4}{4\mu^2} x(0)e^{\mu t} \left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\}^2 + \frac{2\sigma^2}{\mu} [x(0)]^{\frac{3}{2}} \exp\left(\frac{3\mu}{2}t\right) \left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\} + [x(0)]^2 e^{2\mu t}$$

This in turn will mean that the conditional variance of the state variable, $x(t)$, turns out to be:

$$\begin{aligned} \text{Var}[x(t)] &= E\{[x(t)]^2\} - \{E[x(t)]\}^2 = \\ &= \frac{2\sigma^2}{\mu} [x(0)e^{\mu t}]^{\frac{3}{2}} \left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\} + \frac{3\sigma^4}{4\mu^2} x(0)e^{\mu t} \left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\}^2 \end{aligned} \quad (28)$$

Higher order conditional moments may be determined in a similar manner. The validity of these results, however, hinges on the correct specification of the probability density, $p[y(0), y(t), t]$, as summarised by equation (18). Hence, given the inconsistency between Feller (1951a) and Feller (1951b) we now approach the determination of the mean and variance of the state variable, $x(t)$, using an alternative procedure which avoids the direct use of the probability density (18).

Here, it will be recalled that mild regularity conditions imply that there is a one to one correspondence between the probability density and the stochastic differential equation (8) which describes the evolution of the state variable, $x(t)$ (Cox and Miller 1965, 213-215; Karlin and Taylor 1981, 219-220). This in turn will mean that if the state variable's probability density is incorrectly specified, moments determined using the Kolmogorov equation will differ from moments determined directly from the

probability density, $p[x(0), x(t), t]$, as summarised by equation (15). We thus define the n^{th} conditional moment about the origin for the state variable, $x(t)$, as follows:

$$M(t) \equiv E\{[x(t)]^n\} = \int_0^\infty [x(t)]^n p[x(0), x(t), t] dx(t) \quad (29)$$

where, as previously, $p[x(0), x(t), t]$ is the conditional probability density for $x(t)$. In the appendix we use the Kolmogorov equation to demonstrate that $M(t)$ will satisfy the following differential equation:

$$M'(t) = n\mu M(t) + \frac{1}{2} n(n-1)\sigma^2 E\{[x(t)]^{n+2\beta-2}\} \quad (30)$$

Substituting $n=1$ leads to the differential equation $M'(t) = \mu M(t)$ – from which it follows the conditional mean amounts to:⁹

$$E[x(t)] = x(0)e^{t\mu} \quad (31)$$

Likewise, setting $n = 3/2$ and using the exponent parameter $\beta = 3/4$ associated with equation (8) shows:

$$M'(t) = \frac{3}{2}\mu M(t) + \frac{3}{8}\sigma^2 E[x(t)] = \frac{3}{2}\mu M(t) + \frac{3}{8}\sigma^2 x(0)e^{t\mu}$$

Standard techniques (Boyce and DiPrima 2005, 32) will then show that under the initial condition $M(0) = [x(0)]^{3/2}$, the unique solution of the above differential equation is given by:

⁹ Note how the exponent parameter, $\beta = 3/4$, associated with equation (8) is redundant in this calculation. This shows that the functional form of the first moment ($n = 1$) of all CEV processes is given by equation (31) irrespective of the value assumed by the exponent parameter, β .

$$M(t) = E\{[x(t)]^{\frac{3}{2}}\} = \frac{3\sigma^2}{4\mu} x(0)e^{\mu t} \left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\} + [x(0)]^{\frac{3}{2}} \exp\left(\frac{3\mu}{2}t\right) \quad (32)$$

Finally, letting $n = 2$ and again using the exponent parameter $\beta = \frac{3}{4}$ associated with equation (8) leads to the differential equation:

$$M'(t) = 2\mu M(t) + \sigma^2 E\{[x(t)]^{\frac{3}{2}}\} = \\ 2\mu M(t) + \frac{3\sigma^4}{4\mu} x(0)e^{\mu t} \left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\} + \sigma^2 [x(0)]^{\frac{3}{2}} \exp\left(\frac{3\mu}{2}t\right)$$

Again, one can use standard techniques in conjunction with equation (32) and thereby show that under the initial condition $M(0) = [x(0)]^2$ the unique solution of the above differential equation is given by:

$$M(t) = E\{[x(t)]^2\} = \frac{3\sigma^4}{4\mu^2} x(0)e^{\mu t} \left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\}^2 + \\ \frac{2\sigma^2}{\mu} [x(0)]^{\frac{3}{2}} \exp\left(\frac{3\mu}{2}t\right) \left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\} + [x(0)]^2 e^{2\mu t} \quad (33)$$

It then follows:

$$Var[x(t)] = E\{[x(t)]^2\} - \{E[x(t)]\}^2 = \\ \frac{2\sigma^2}{\mu} [x(0)e^{\mu t}]^{\frac{3}{2}} \left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\} + \frac{3\sigma^4}{4\mu^2} x(0)e^{\mu t} \left\{ \exp\left(\frac{\mu}{2}t\right) - 1 \right\}^2 \quad (34)$$

will be the conditional variance of the state variable, $x(t)$, at time t . The important point here is that the expressions for the conditional mean (as given by equation (27)

and equation (31)) and conditional variance (as given by equation (28) and equation (34)) are the same irrespective of whether one uses the probability density (15) directly or the moments are determined using the Kolmogorov equation (11) based on the stochastic differential equation (10). We would emphasise that this will *not* be the case if instead of the probability density (15), one uses the probability density (17) based on Feller (1951b, 180) to determine the moments.

5. Logarithmic Rates of Return

We have previously observed how logarithmic rates of return arise in economics and finance, including credit risk and bankruptcy (Nunes 2009), the term structure of interest rates (Cox et al. 1985) and the volatility smile in the valuation of derivative securities (Chance 2013). Given this, we now illustrate how conditional expected logarithmic rates of return may be determined for the singular diffusion processes on which the Feller (1951a, 1951b) papers are based.

We begin with the state variable, $y(t)$, that evolves in terms of the stochastic differential equation (10) and for which the conditional expected logarithmic return is defined as follows:

$$M(t) \equiv E \left\{ \log \left[\frac{y(t)}{y(0)} \right] \right\} = \int_0^\infty \log \left[\frac{y(t)}{y(0)} \right] p[y(0), y(t), t] dy(t) \quad (35)$$

where $p[y(0), y(t), t]$ is the conditional probability density defined by equation (15).

Whilst the above integral will have to be evaluated numerically (that is, it cannot be expressed in terms of elementary functions), it also possesses an alternative representation which allows one to develop many of its important properties. One can demonstrate this by differentiating through equation (35) and then substituting the

Kolmogorov equation, (11), corresponding to the stochastic differential equation (10).

Having done this, one can then follow procedures similar to those implemented in the first part of the Appendix for the derivation of equation (30) – in which case it follows that $M(t)$ satisfies the following differential equation:

$$M'(t) = \frac{\mu}{2} - \frac{1}{4} \sigma^2 E \left\{ \frac{1}{y(t)} \right\} \quad (36)$$

Integrating through the above expression under the initial condition $M(0) = 0$ will then show:

$$M(t) = E \left\{ \log \left[\frac{y(t)}{y(0)} \right] \right\} = \frac{\mu}{2} t - \frac{1}{4} \sigma^2 \int_0^t E \left\{ \frac{1}{y(s)} \right\} ds \quad (37)$$

where $\frac{1}{4} \sigma^2 \int_0^t E \left\{ \frac{1}{y(s)} \right\} ds > 0$ given that the state variable, $y(t)$, is strictly non-negative.

In the second part of the Appendix we demonstrate how this latter integral may be evaluated in terms of a convolution of the upper incomplete gamma function (which, as previously noted, captures the probability of the state variable, $y(t)$, being absorbed into the zero state) and the exponential integral. Applying this result in conjunction with the probability of ultimate extinction as given by equation (25b), will then show:

$$\begin{aligned} E \left\{ \Gamma(2, \gamma) \int_{y(0)}^{y(t)} \frac{\exp \left(\frac{-4\mu}{\sigma^2} w \right)}{w} dw \right\} = \\ E \left\{ \left(1 + \frac{4\mu}{\sigma^2} y(t) \right) \int_{y(0)}^{y(t)} \frac{\exp \left[\frac{-4\mu}{\sigma^2} (y(t) - w) \right]}{w} dw \right\} = -\frac{1}{4} \sigma^2 \int_0^t E \left\{ \frac{1}{y(s)} \right\} ds \end{aligned} \quad (38)$$

where from equation (19) we have:

$$\gamma = \lim_{t \rightarrow \infty} \frac{4\mu}{\sigma^2} \frac{y(0) \exp\left(\frac{\mu}{2}t\right)}{\exp\left(\frac{\mu}{2}t\right) - 1} = \frac{4\mu}{\sigma^2} y(0)$$

and $\Gamma(2, \gamma) = \int_{\gamma}^{\infty} p e^{-p} dp = 1! e^{-\gamma} \sum_{j=0}^{\infty} \frac{\gamma^j}{j!}$ is the upper incomplete gamma function (Jameson 2016).

Using this result in conjunction with equation (38) will then show that the conditional expected logarithmic return for the state variable, $y(t)$, is given by:

$$E\left\{\log\left[\frac{y(t)}{y(0)}\right]\right\} = \frac{\mu}{2}t + E\left\{\left(1 + \frac{4\mu}{\sigma^2} y(t)\right) \int_{y(0)}^{y(t)} \frac{\exp\left[-\frac{4\mu}{\sigma^2}(y(t) - w)\right]}{w} dw\right\} \quad (39)$$

Moreover, since from equation (9) we have $y(t) = \sqrt{x(t)}$, the above result will mean that the conditional expected logarithmic return for the state variable, $x(t)$, which evolves in terms of the stochastic differential equation (8), amounts to:

$$E\left\{\log\left[\frac{x(t)}{x(0)}\right]\right\} = \mu t + 2E\left\{\left(1 + \frac{4\mu}{\sigma^2} \sqrt{x(t)}\right) \int_{\sqrt{x(0)}}^{\sqrt{x(t)}} \frac{\exp\left[-\frac{4\mu}{\sigma^2}(\sqrt{x(t)} - w)\right]}{w} dw\right\} \quad (40)$$

This result shows that the expected conditional logarithmic return for the state variable, $x(t)$, will be equal to the expected proportionate change in the state variable, μt , over the period from time zero until time t , plus a downward adjustment which

hinges on the probability of the state variable reaching the absorbing state of zero at some future point in time. This latter point follows from the fact that the probability of ultimate absorption into the zero state as summarised by equation (25b), appears in the expression for the downward adjustment which is made to the expected proportionate change, μt , to determine the conditional expected logarithmic return for the state variable in both equations (39) and (40). This also has the important implication that larger opening values of the state variable, $x(0)$, will lead to larger expected logarithmic returns – reflecting the declining probability of absorption into the zero state as the state variable grows in magnitude.¹⁰

A significant advantage of closed form expressions for the conditional expected logarithmic return like this is that they allow one to investigate the dependence and sensitivity of the logarithmic return on the parameters of the underlying stochastic process responsible for generating the sample paths of the state variable, $x(t)$. As an illustration of this point, we contrast the abnormal returns which accrue on the state variable under equation (40) with those which arise under the more conventionally applied Geometric Brownian Motion (GBM). Now, the difference between the expected conditional logarithmic return for the CEV process (with exponent

¹⁰ One can also apply a Taylor expansion to $\log[x(t)]$ about the point $x(0)e^{\mu t}$. If one then takes expectations across the series expansion and substitutes the expressions for the mean and variance as given by equation (31) and equation (34) respectively, it follows:

$$E\left\{\log\left[\frac{x(t)}{x(0)}\right]\right\} = \mu t - \frac{\sigma^2}{\mu\sqrt{x(0)}} \left\{1 - \exp\left(-\frac{\mu}{2}t\right)\right\} - \frac{3\sigma^4}{8\mu^2 x(0)} \left\{1 - \exp\left(-\frac{\mu}{2}t\right)\right\}^2 + \dots$$

Note how the above result shows that larger opening values of the state variable $x(0)$, will lead to larger expected logarithmic returns – reflecting the decreasing probability of absorption into the zero state under the Feller (1951b) singular diffusion processes as the state variable grows in magnitude.

parameter $\beta = \frac{3}{4}$) and the expected conditional logarithmic return under the GBM amounts to:¹¹

$$\frac{1}{2}\delta^2 t + 2E\left\{\left(1 + \frac{4\mu}{\sigma^2}\sqrt{x(t)}\right)\int_{\sqrt{x(0)}}^{\sqrt{x(t)}} \frac{\exp\left[-\frac{4\mu}{\sigma^2}(\sqrt{x(t)} - w)\right]}{w} dw\right\} \quad (41)$$

where δ^2 is the infinitesimal variance of the logarithmic return for the GBM. Here one can replicate the previously applied procedures in the Appendix involving the Komolgorov equation to derive the following limit condition for the conditional variance of the logarithmic return under the CEV process:

$$\lim_{x(0) \rightarrow \infty} Var\left\{\log\left[\frac{x(t)}{x(0)}\right]\right\} \rightarrow -4E\left\{\left(1 + \frac{4\mu}{\sigma^2}\sqrt{x(t)}\right)\int_{\sqrt{x(0)}}^{\sqrt{x(t)}} \frac{\exp\left[-\frac{4\mu}{\sigma^2}(\sqrt{x(t)} - w)\right]}{w} dw\right\} \quad (42)$$

This result shows that as the opening value of the state variable, $x(0)$, grows in magnitude, the conditional variance of the logarithmic return for the CEV process converges in absolute terms towards *twice* the downward adjustment made to the

¹¹ The conditional expected logarithmic return for the GBM is given by:

$$E\left\{\log\left[\frac{x(t)}{x(0)}\right]\right\} = \left(\mu - \frac{1}{2}\delta^2\right)t$$

where μ is the infinitesimal mean and δ^2 is the infinitesimal variance associated with infinitesimal increments in the GBM. Subtracting the conditional expected logarithmic return for the GBM from equation (40) leads directly to equation (41).

conditional expected logarithmic return in equation (40).¹² Moreover, one can combine equation (40) with equation (42) to show that when the state variable evolves in terms of the given CEV process, its conditional expected logarithmic return will have the following asymptotic representation:

$$\lim_{x(0) \rightarrow \infty} E \left\{ \log \left[\frac{x(t)}{x(0)} \right] \right\} = \mu t - \frac{1}{2} \text{Var} \left\{ \log \left[\frac{x(t)}{x(0)} \right] \right\} \quad (43)$$

When taken in conjunction with equation (41), this result has the important implication that as the opening value, $x(0)$, of the state variable grows in magnitude, then the abnormal return computed using the conditional expected logarithmic return formula for the GBM will converge towards the abnormal return computed using the conditional expected logarithmic return formula for the CEV process. In contrast, when the opening values of the state variable are relatively low, large differences may

¹² The result stated here generalises to CEV processes with other exponent parameter values. Thus, if one sets $n = 0$ in the second part of the Appendix so that the state variable evolves in terms of a CEV process with exponent parameter $\beta = 1/2$ (that is, the continuous time branching process), then the expected conditional logarithmic return is given by (Docherty et al. 2018, 430-432):

$$E \left\{ \log \left[\frac{x(t)}{x(0)} \right] \right\} = \mu t + E \left\{ \int_{x(0)}^{x(t)} \frac{\exp \left[\frac{-2\mu}{\sigma^2} (x(t) - w) \right]}{w} dw \right\}$$

The limiting condition for the variance of the conditional logarithmic return for the CEV process with exponent parameter $\beta = 1/2$ then turns out to be (Melia et al. 2019, 851-853):

$$\lim_{x(0) \rightarrow \infty} \text{Var} \left\{ \log \left[\frac{x(t)}{x(0)} \right] \right\} \rightarrow -2E \left\{ \int_{x(0)}^{x(t)} \frac{\exp \left[\frac{-2\mu}{\sigma^2} (x(t) - w) \right]}{w} dw \right\}$$

Similar, though generally more complicated expressions for the asymptotic variance of the conditional logarithmic return of other CEV processes may be obtained by setting $n = 2, 3, \dots$.

arise between the abnormal return computed under the GBM and those calculated under the CEV process.

6. Conclusions

In this paper we identify a fundamental inconsistency between the probability density which appears in Feller (1951a) and the probability density which appears in Feller (1951b). Moreover, we apply the method of group-invariance to two interpretations of the Constant Elasticity of Variance (CEV) stochastic processes in order to determine the nature of this inconsistency. This shows that the probability density summarised in Feller (1951b) is in error. We also use the Kolmogorov equation to determine the expected logarithmic return for a state variable that evolves in terms of the singular diffusion processes on which the Feller (1951b) paper is based. Here our analysis shows that the expected conditional logarithmic return varies according to the probability of the state variable eventually being absorbed into the zero state. This in turn will mean that the expected conditional logarithmic return will be non-stationary in time and will vary with the magnitude of the state variable at the beginning of the period over which the logarithmic return is being determined. Lower values of the state variable will imply lower conditional expected logarithmic returns and vice versa.

Appendix: Conditional Moments of the CEV Processes

Consider the stochastic differential equation:

$$dx(t) = \mu x(t)dt + \sigma[x(t)]^\beta dq(t)$$

where μ is the expected proportionate infinitesimal change in the state variable, $x(t)$, on a per unit time basis, σ is an intensity parameter defined on the white noise process $dq(t)$ and $0 < \beta < 1$ is the exponent parameter. The n^{th} moment about the origin for the state variable, $x(t)$, is then defined as:

$$M(t) \equiv E\{[x(t)]^n\} = \int_0^\infty [x(t)]^n p[x(0), x(t), t] dx(t)$$

Differentiating under the integral sign will then show (Flanders, 1973):

$$M'(t) = \int_0^\infty [x(t)]^n \frac{\partial p[x(0), x(t), t]}{\partial t} dx(t)$$

Substituting the Kolmogorov equation for the state variable, $x(t)$, will then mean (Cox and Miller 1965, 213-215):

$$M'(t) = \int_0^\infty [x(t)]^n \frac{1}{2} \frac{\partial^2}{\partial x(t)^2} \{ \sigma^2 [x(t)]^{2\beta} p[x(0), x(t), t] \} dx(t) - \int_0^\infty [x(t)]^n \frac{\partial}{\partial x(t)} \{ \mu x(t) p[x(0), x(t), t] \} dx(t)$$

Evaluating the second term for $M'(t)$ shows:

$$\int_0^\infty [x(t)]^n \frac{\partial}{\partial x(t)} \{ \mu x(t) p[x(0), x(t), t] \} dx(t) = \mu [x(t)]^{n+1} p[x(0), x(t), t] \Big|_0^\infty - n\mu \int_0^\infty [x(t)]^n p[x(0), x(t), t] dx(t)$$

or, equivalently:

$$\int_0^\infty [x(t)]^n \frac{\partial}{\partial x(t)} \{ \mu x(t) p[x(0), x(t), t] \} dx(t) = -n\mu \int_0^\infty [x(t)]^n p[x(0), x(t), t] dx(t) = -n\mu M(t)$$

Repeated integration by parts applied to the first term for $M'(t)$ also shows:

$$\int_0^\infty [x(t)]^n \frac{1}{2} \frac{\partial^2}{\partial x(t)^2} \{ \sigma^2 [x(t)]^{2\beta} p[x(0), x(t), t] \} dx(t) = -\frac{1}{2} n \int_0^\infty [x(t)]^{n-1} \frac{\partial}{\partial x(t)} \{ \sigma^2 [x(t)]^{2\beta} p[x(0), x(t), t] \} dx(t) = \frac{1}{2} n(n-1) \sigma^2 E\{[x(t)]^{n+2\beta-2}\}$$

Bringing these results together will then mean:

$$M'(t) = \int_0^\infty [x(t)]^n \frac{1}{2} \frac{\partial^2}{\partial x(t)^2} \{ \sigma^2 [x(t)]^{2\beta} p[x(0), x(t), t] \} dx(t) - \int_0^\infty [x(t)]^n \frac{\partial}{\partial x(t)} \{ \mu x(t) p[x(0), x(t), t] \} dx(t) = n\mu M(t) + \frac{1}{2} n(n-1) \sigma^2 E\{[x(t)]^{n+2\beta-2}\}$$

Appendix: Conditional Expected Logarithmic Rates of Return for CEV Processes

Let $x(t)$ evolve in terms of a CEV process with exponent parameter $\beta = (2n+1)/(2n+2)$; that is:

$$dx(t) = \mu x(t) dt + \sigma [x(t)]^{(2n+1)/(2n+2)} dq(t)$$

for integral n . Make the substitution $y(t) = [x(t)]^{1/(n+1)}$ in which case Itô's formula will mean $y(t)$ evolves in terms of the stochastic differential equation (Karlin and Taylor 1981, 372-373):

$$dy(t) = \left\{ \frac{\mu}{(n+1)} y(t) - \frac{n\sigma^2}{2(n+1)^2} \right\} dt + \frac{\sigma}{(n+1)} \sqrt{y(t)} dz(t)$$

One can then use the Kolmogorov equation in conjunction with the above stochastic differential equation and follow the procedures summarised in the first part of the Appendix to show that the conditional expected logarithmic return for the state variable $y(t)$, is given by:

$$E\left\{\log\left[\frac{y(t)}{y(0)}\right]\right\} = \int_0^t \log\left[\frac{y(s)}{y(0)}\right] p[y(0), y(s), s] dy(s) = \frac{\mu t}{(n+1)} - \frac{\sigma^2}{2(n+1)} \int_0^t E\left\{\frac{1}{y(s)}\right\} ds$$

Now, define the conditional expectation:

$$M(t) \equiv E\left[\frac{\Gamma\left\{n+1, \frac{2\mu(n+1)}{\sigma^2} y(t)\right\}}{n!} \int_{y(0)}^{y(t)} \frac{\exp\left\{\frac{2\mu(n+1)}{\sigma^2} w\right\}}{w} dw\right] = \int_0^t \int_{y(0)}^{y(t)} \sum_{j=0}^n \frac{\left\{\frac{2\mu(n+1)}{\sigma^2} y(t)\right\}^j}{j!} \frac{\exp\left\{-\frac{2\mu(n+1)}{\sigma^2} (y(t) - w)\right\}}{w} p[y(0), y(t), t] dw dy(t)$$

where:

$$\Gamma\left\{n+1, \frac{2\mu(n+1)}{\sigma^2} y(t)\right\} = \int_{\frac{2\mu(n+1)}{\sigma^2} y(t)}^{\infty} p^n e^{-p} dp = n! \exp\left\{-\frac{2\mu(n+1)}{\sigma^2} y(t)\right\} \sum_{j=0}^n \frac{\left\{\frac{2\mu(n+1)}{\sigma^2} y(t)\right\}^j}{j!}$$

is the upper incomplete gamma function (Jameson 2016). On substituting the Kolmogorov equation it follows (Flanders 1973):

$$\begin{aligned}
M'(t) &= \int_0^\infty \int_{y(0)}^{y(t)} \sum_{j=0}^n \frac{\left\{ \frac{2\mu(n+1)}{\sigma^2} y(t) \right\}^j}{j!} \frac{\exp\left\{ -\frac{2\mu(n+1)}{\sigma^2} (y(t) - w) \right\}}{w} dw \frac{\partial p[y(0), y(t), t]}{\partial t} dy(t) = \\
&\int_0^\infty \int_{y(0)}^{y(t)} \sum_{j=0}^n \frac{\left\{ \frac{2\mu(n+1)}{\sigma^2} y(t) \right\}^j}{j!} \frac{\exp\left\{ -\frac{2\mu(n+1)}{\sigma^2} (y(t) - w) \right\}}{w} dw \left\langle \frac{\partial^2}{\partial y(t)^2} \left\{ \frac{\sigma^2}{2(n+1)^2} y(t) p[y(0), y(t), t] \right\} - \frac{\partial}{\partial x(t)} \left\{ \left[-\frac{n\sigma^2}{2(n+1)^2} + \frac{\mu}{(n+1)} y(t) \right] p[y(0), y(t), t] \right\} \right\rangle dy(t)
\end{aligned}$$

Applying integration by parts to the right hand side of the above equation will enable one to retrieve the appropriate expression for the integral

$\sigma^2 \int_0^t E \left\{ \frac{1}{y(s)} \right\} ds$. One can illustrate the procedures involved by considering the case of $n = 1$ as given by the stochastic differential equation (10)

of the text. It then follows:

$$\begin{aligned}
M'(t) &= \int_0^\infty \int_{y(0)}^{y(t)} \left(1 + \frac{4\mu}{\sigma^2} y(t) \right) \frac{\exp\left[\frac{-4\mu}{\sigma^2} (y(t) - w) \right]}{w} \frac{\partial p[y(0), y(t), t]}{\partial t} dw dy(t) = \\
&\int_0^\infty \int_{y(0)}^{y(t)} \left(1 + \frac{4\mu}{\sigma^2} y(t) \right) \frac{\exp\left[\frac{-4\mu}{\sigma^2} (y(t) - w) \right]}{w} \left\langle \frac{\partial^2}{\partial y(t)^2} \left\{ \frac{1}{8} \sigma^2 y(t) p[y(0), y(t), t] \right\} - \frac{\partial}{\partial y(t)} \left\{ \left[\frac{\mu}{2} y(t) - \frac{\sigma^2}{8} \right] p[y(0), y(t), t] \right\} \right\rangle dw dy(t)
\end{aligned}$$

Now, apply integration by parts to the second term on the right hand side of the above expression:

$$\begin{aligned}
& \int_0^\infty \left(1 + \frac{4\mu}{\sigma^2} y(t)\right) \exp\left[\frac{-4\mu}{\sigma^2} y(t)\right] \int_{y(0)}^{y(t)} \frac{\exp\left[\frac{4\mu}{\sigma^2} w\right]}{w} dw \frac{\partial}{\partial y(t)} \left\{ \left[\frac{\mu}{2} y(t) - \frac{\sigma^2}{8}\right] p[y(0), y(t), t] \right\} dy(t) = \\
& \left\{ \int_0^\infty \frac{16\mu^2}{\sigma^4} y(t) \exp\left[\frac{-4\mu}{\sigma^2} y(t)\right] \int_{y(0)}^{y(t)} \frac{\exp\left[\frac{4\mu}{\sigma^2} w\right]}{w} dw \left(\frac{\mu}{2} y(t) - \frac{\sigma^2}{8}\right) p[y(0), y(t), t] - \left(\frac{4\mu}{\sigma^2} + \frac{1}{y(t)}\right) \left(\frac{\mu}{2} y(t) - \frac{\sigma^2}{8}\right) p[y(0), y(t), t] \right\} dy(t)
\end{aligned}$$

One can also apply integration by parts to the first term in the above expression in which case we have:

$$\begin{aligned}
& \frac{1}{8} \sigma^2 \int_0^\infty \left(1 + \frac{4\mu}{\sigma^2} y(t)\right) \exp\left[\frac{-4\mu}{\sigma^2} y(t)\right] \int_{y(0)}^{y(t)} \frac{\exp\left[\frac{4\mu}{\sigma^2} w\right]}{w} dw \frac{\partial^2}{\partial y(t)^2} \{y(t) p[y(0), y(t), t]\} dy(t) \\
& = \frac{1}{8} \sigma^2 \int_0^\infty \left\{ \frac{16\mu^2}{\sigma^4} y(t) \exp\left[\frac{-4\mu}{\sigma^2} y(t)\right] \int_{y(0)}^{y(t)} \frac{\exp\left[\frac{4\mu}{\sigma^2} w\right]}{w} dw - \left(\frac{4\mu}{\sigma^2} + \frac{1}{y(t)}\right) \right\} \frac{\partial}{\partial y(t)} \{y(t) p[y(0), y(t), t]\} dy(t)
\end{aligned}$$

Applying integration by parts to this latter expression will then show:

$$\begin{aligned} & \frac{1}{8} \sigma^2 \int_0^\infty \left(1 + \frac{4\mu}{\sigma^2} y(t) \right) \exp \left[\frac{-4\mu}{\sigma^2} y(t) \right] \int_{y(0)}^{y(t)} \frac{\exp \left[\frac{4\mu}{\sigma^2} w \right]}{w} dw \frac{\partial^2}{\partial y(t)^2} \{ y(t) p[y(0), y(t), t] \} dy(t) = \\ & \int_0^\infty \left\{ \left\langle \frac{8\mu^3}{\sigma^4} y(t) \exp \left[\frac{-4\mu}{\sigma^2} y(t) \right] - \frac{2\mu^2}{\sigma^2} \exp \left[\frac{-4\mu}{\sigma^2} y(t) \right] \right\rangle \int_{y(0)}^{y(t)} \frac{\exp \left[\frac{4\mu}{\sigma^2} w \right]}{w} dw - \left(\frac{2\mu^2}{\sigma^2} + \frac{\sigma^2}{8y^2(t)} \right) \right\} y(t) p[y(0), y(t), t] dy(t) \end{aligned}$$

Bringing these results together will then mean:

$$\begin{aligned} M'(t) &= \int_0^\infty \int_{y(0)}^{y(t)} \left(1 + \frac{4\mu}{\sigma^2} y(t) \right) \frac{\exp \left[\frac{-4\mu}{\sigma^2} (y(t) - w) \right]}{w} \frac{\partial p[y(0), y(t), t]}{\partial t} dw dy(t) = \\ & \int_0^\infty \left\{ \left\langle \frac{8\mu^3}{\sigma^4} y(t) \exp \left[\frac{-4\mu}{\sigma^2} y(t) \right] - \frac{2\mu^2}{\sigma^2} \exp \left[\frac{-4\mu}{\sigma^2} y(t) \right] \right\rangle \int_{y(0)}^{y(t)} \frac{\exp \left[\frac{4\mu}{\sigma^2} w \right]}{w} dw - \left(\frac{2\mu^2}{\sigma^2} + \frac{\sigma^2}{8y^2(t)} \right) \right\} y(t) p[y(0), y(t), t] dy(t) - \\ & \left\{ \int_0^\infty \frac{16\mu^2}{\sigma^4} y(t) \exp \left[\frac{-4\mu}{\sigma^2} y(t) \right] \int_{y(0)}^{y(t)} \frac{\exp \left[\frac{4\mu}{\sigma^2} w \right]}{w} dw \left(\frac{\mu}{2} y(t) - \frac{\sigma^2}{8} \right) p[y(0), y(t), t] - \left(\frac{2\mu^2}{\sigma^2} y(t) - \frac{\sigma^2}{8y(t)} \right) p[y(0), y(t), t] \right\} dy(t) \end{aligned}$$

or equivalently:

$$M'(t) = \int_0^\infty \int_{y(0)}^{y(t)} \left(1 + \frac{4\mu}{\sigma^2} y(t)\right) \frac{\exp\left[\frac{-4\mu}{\sigma^2} (y(t) - w)\right]}{w} \frac{\partial p[y(0), y(t), t]}{\partial t} dw dy(t) = -\frac{1}{4} \sigma^2 \int_0^\infty \frac{p[y(0), y(t), t]}{y(t)} = -\frac{1}{4} \sigma^2 E\left\{\frac{1}{y(t)}\right\}$$

Integrating through the above expression using the initial condition $M(0) = 0$ will then show:

$$M(t) \equiv E\left\{\left(1 + \frac{4\mu}{\sigma^2} y(t)\right) \int_{y(0)}^{y(t)} \frac{\exp\left[\frac{-4\mu}{\sigma^2} (y(t) - w)\right]}{w} dw\right\} = -\frac{1}{4} \sigma^2 \int_0^t E\left\{\frac{1}{y(s)}\right\} ds$$

Moreover, using this result in conjunction with $n = 1$ will then mean that the conditional expected logarithmic return for the state variable $y(t)$, may be expressed as:

$$E\left\{\log\left[\frac{y(t)}{y(0)}\right]\right\} = \frac{\mu}{2} t - \frac{1}{4} \sigma^2 \int_0^t E\left\{\frac{1}{y(s)}\right\} ds = \frac{\mu}{2} t + E\left\{\left(1 + \frac{4\mu}{\sigma^2} y(t)\right) \int_{y(0)}^{y(t)} \frac{\exp\left[\frac{-4\mu}{\sigma^2} (y(t) - w)\right]}{w} dw\right\}$$

References

- Baaquie, B., Pan, T. and Bhanap, J. 2011. Empirical analysis and calibration of the CEV process for pricing equity default swaps. *Quantitative Finance*, 11 (12): 1815-1823.
- Beckers, S. 1980. The Constant Elasticity of Variance model and its implications for option pricing. *Journal of Finance*, 35 (3): 661-673.
- Boyce, W. and DiPrima, R. 2005. *Elementary differential equations and boundary value problems*. New York: Wiley.
- Chance, D. 2013. *Introduction to derivatives and risk management*, Mason, Ohio: South-Western.
- Cox, D. and Miller, H. 1965. *Theory of stochastic processes*, London: Chapman and Hall.
- Cox, J. 1975. Notes on option pricing I: Constant elasticity of diffusions. Unpublished Draft, Stanford University.
- Cox, J. 1996. The constant elasticity of variance option pricing model. *Journal of Portfolio Management*, Special Issue, 15-17.
- Cox, J. and Ross, S. 1976. The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3 (1-2): 145-166.
- Cox, J., Ingersoll, J. and Ross, S. 1985. A theory of the term structure of interest rates. *Econometrica*, 53(2): 385-407.
- Davidson, I., Song, X. and Tippet, M. 2015. "Time Varying Costs of Capital and the Expected Present Value of Future Cash Flows." *European Journal of Finance* 21 (2): 129-146.
- Davydov, D. and Linetsky, V. 2001. Pricing and hedging path-dependent options under the CEV process. *Management Science*, 47 (7): 949-965.
- Docherty, P., Dong, Y., Song, X. and Tippet, M. 2018. The Feller diffusion, filter rules and abnormal stock returns. *European Journal of Finance*, 24 (5): 426-438.
- Feller, W. 1951a. Diffusion processes in genetics. *Proceedings of the Second Berkley Symposium on Mathematical Statistics and Probability*. 227-246.
- Feller, W. 1951b. Two singular diffusion processes. *Annals of Mathematics*, 54 (1), 173-182.
- Gan, X., and Waxman, D. 2015. Singular solution of the Feller diffusion equation via a spectral decomposition. *Physical Review E*, 91 (1): 012123-1-10.

Flanders, H. 1973. Differentiation under the integral sign. *American Mathematical Monthly*, 80 (6): 615-627.

Jameson, G. 2016. The incomplete gamma functions. *The Mathematical Gazette*, 100 (548): 298-306.

Jarrow, R. and Rudd, A. 1983. *Option pricing*. Homewood, Illinois: Irwin.

Karlin, S. and Taylor, H. 1975. *A first course in stochastic processes*. London: Academic Press.

Karlin, S., and Taylor, H. 1981. *A second course in stochastic processes*. London: Academic Press.

Kreyszig, E. 1972. *Advanced Engineering Mathematics*. New York: Wiley.

Melia, A., Song, X. and Tippett, M. 2019. Subtle is the Lord, but malicious He is not: the calculation of abnormal stock returns in applied research. *European Journal of Finance*, 25(9); 835-855.

Nariboli, G. 1977. Group-invariant solutions of the Fokker-Planck equation. *Stochastic Processes and their Applications*, 5(2): 157-171.

Nunes, J. 2009. Pricing American options under the constant elasticity of variance model and subject to bankruptcy. *Journal of Financial and Quantitative Analysis*, 44 (5): 1231-1263.

Schinazi, R. 1999. *Classical and Spatial Stochastic Processes*. New York: Springer.