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ON THE POINT SPECTRUM IN THE EKMAN BOUNDARY LAYER PROBLEM

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Abstract. New eigenvalue enclosures for the block operator problem arising in the study of stability of the Ekman boundary layer are proved. This solves an open problem in [19] on the existence of open sets of eigenvalues in domains of Fredholmness of the analyzed operator family.

1. Introduction

The stability of the Ekman boundary layer was studied in several works, ranging from experimental physics to a rigorous operator-theoretic approach, see in particular [13, 19, 22, 23, 24]. The system of coupled differential equations on $\mathbb{R}_+ := (0, \infty)$ which arises in this analysis has the form

$$A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \lambda B \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

where

$$\begin{aligned} A &= \begin{pmatrix} (-\partial^2 + \alpha^2)^2 + \mathrm{i}\alpha R V(-\partial^2 + \alpha^2) + \mathrm{i}\alpha R V'' & 2\partial \\ 2\partial + \mathrm{i}\alpha R U' & -\partial^2 + \alpha^2 + \mathrm{i}\alpha R V \end{pmatrix}, \\ B &= \begin{pmatrix} -\partial^2 + \alpha^2 & 0 \\ 0 & I \end{pmatrix} \end{aligned} \tag{1.1}$$

with formal boundary conditions

$$f_1(0) = f_1'(0) = f_2(0) = 0, \qquad f_1(\infty) = f_1'(\infty) = f_2(\infty) = 0.$$

Here $\alpha > 0$ is the wave number, $R \geq 0$ is the Reynolds number, λ is the spectral parameter and in the physical setting U, V are known real-valued smooth and exponentially decaying functions, see *e.g.* [19, 22, 24] for the derivations and further details. In particular in [22], the one-parametric class of functions

$$\begin{aligned} U_\epsilon(x) &= \cos(\epsilon) - e^{-x}\cos(x + \epsilon), \\ V_\epsilon(x) &= e^{-x}\sin(x + \epsilon), \qquad\qquad x \in \mathbb{R},\ \epsilon \in \mathbb{R} \end{aligned} \tag{1.2}$$

is considered.

The rigorous spectral analysis can be performed in the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+),$$

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disregarding the precise form of U, V and assuming merely that

$$U', V, V', V'' \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+), \tag{1.3}$$

see [19, 23]. From the operator theoretic point of view, the system (1.1) can be viewed as a spectral problem for the operator family

$$\begin{aligned} \mathcal{T}(\lambda) &= A - \lambda B, \\ \operatorname{Dom}(\mathcal{T}(\lambda)) &= \{(f_1, f_2) \in W^{4,2}(\mathbb{R}_+) \times W^{2,2}(\mathbb{R}_+) \, : \\ & \qquad f_1(0) = f_1'(0) = f_2(0) = 0\}, \qquad \lambda \in \mathbb{C}; \end{aligned} \tag{1.4}$$

recall that the spectrum of an operator family is defined as

$$\sigma(\mathcal{T}) = \{\lambda \in \mathbb{C} \, : \, 0 \in \sigma(\mathcal{T}(\lambda))\}$$

and analogously for various parts of the spectrum.

The following result is known on the spectrum of the operator family $\mathcal{T}$ in (1.4), precise references to individual claims are given below Theorem 1.1. Throughout the paper, we denote by $\|\cdot\|_{L^p}$ the standard L^p-norm over $\mathbb{R}_+$ for $p \in [1, \infty]$.

Theorem 1.1. *Let U and V satisfy* (1.3) *and let $\mathcal{T}(\lambda)$, $\lambda \in \mathbb{C}$, be as in* (1.4). *Then the following hold.*

i) *The essential spectrum of $\mathcal{T}$ reads as*

$$\sigma_{\mathrm{ess}}(\mathcal{T}) = \{\lambda \in \mathbb{C} \, : \, \exists \xi \in \mathbb{R}, \; p_\lambda(\xi) = 0\} \tag{1.5}$$

where

$$p_\lambda(\xi) = (\xi^2 + \alpha^2)(\xi^2 + \alpha^2 - \lambda)^2 + 4\xi^2, \qquad \xi \in \mathbb{R}. \tag{1.6}$$

ii) *The point spectrum of $\mathcal{T}$ satisfies*

$$\sigma_{\mathrm{p}}(T) \subset \{\lambda \in \mathbb{C} \, : \, \operatorname{Re} \lambda \geq \gamma, \, |\operatorname{Im} \lambda| \leq \eta\} \tag{1.7}$$

where

$$\begin{aligned} \gamma &= \alpha^2 - \frac{R}{2} \left(\|U'\|_{L^\infty} + \|V'\|_{L^\infty}\right) \left(\alpha + \frac{1}{\alpha}\right) - \alpha R \|V\|_{L^\infty}, \\ \eta &= 2 + 2\alpha R \|V\|_{L^\infty} + \frac{R}{2} \left(\|U'\|_{L^\infty} + \|V'\|_{L^\infty}\right) \left(\alpha + \frac{1}{\alpha}\right) + \frac{R}{\alpha} \|V''\|_{L^\infty}. \end{aligned}$$

iii) *Let*

$$\Omega = \mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{T}) \tag{1.8}$$

and let Ω_+ and Ω_- be the two connected components of Ω such that

$$\Omega = \Omega_- \cup \Omega_+, \qquad (-\infty, \alpha^2) \subset \Omega_-, \qquad (\alpha^2, \infty) \subset \Omega_+.$$

Then $\sigma_{\mathrm{p}}(\mathcal{T}) \cap \Omega_-$ is a discrete set.

The essential spectrum was calculated non-rigorously in [24]; in [19, Thm. 3.6] and [23, Thm. 3.1] this was proved using singular sequences and a relative compactness argument, respectively. To be more precise, the essential spectrum used in [19, 23] is

$$\sigma_{\mathrm{ess}}(\mathcal{T}) = \sigma_{\mathrm{e3}}(\mathcal{T}) = \{\lambda \in \mathbb{C} \, : \, \mathcal{T}(\lambda) \text{ is not Fredholm}\}.$$

Nonetheless, it readily follows from the results of [23] that (1.5) holds also for $\sigma_{\mathrm{e4}}(\mathcal{T})$, *i.e.* that $\mathcal{T}$ has index zero in its domain of Fredholmness, which implies

$$\sigma(\mathcal{T}) \setminus \sigma_{\mathrm{ess}}(\mathcal{T}) \subset \sigma_{\mathrm{p}}(\mathcal{T}).$$

The enclosure of the point spectrum in the semi-infinite strip (1.7) was proved in [19, Thm. 5.1]; notice that this strip always contains the essential spectrum.

Moreover, in [19, Thm. 6.2] it was established that $\sigma_{\mathrm{p}}(\mathcal{T}) \cap \Omega_-$ is discrete and, assuming that Ω_+ does not contain any open sets of eigenvalues either, the spectral exactness of approximations via domain truncation was shown in [19, Thm. 8.1]. However, the question on the structure of the point spectrum in Ω_+ was answered only partially in [19, Thm. 6.3] and it remained unsolved if $\sigma_{\mathrm{p}}(\mathcal{T}) \cap \Omega_+$ is discrete for arbitrary Reynolds numbers, see also [3, Open Problem 2013-13-ICMS].

Theorem 1.2 ([19, Thm. 6.3])**.** *Assume that $U \in C^1([0,\infty))$, $V \in C^2([0,\infty))$, $U', V, V'' \in L^1([0,\infty))$, and*

$$\lim_{x\to\infty} U'(x) = \lim_{x\to\infty} V(x) = \lim_{x\to\infty} V''(x) = 0.$$

Let $\mathfrak{R}_{OE}$ be the set of Reynolds numbers $\{R \in \mathbb{R} : R \geq 0\}$ such that the Ekman problem (1.4) *has a non-empty open set of eigenvalues.*

(1) *There exists $R_0 > 0$ such that $[0, R_0] \cap \mathfrak{R}_{OE} = \emptyset$.*
(2) *$\mathfrak{R}_{OE}$ has no accumulation points.*

Notice that due to the position of Ω_+ with respect to the essential spectrum, neither numerical range nor perturbation arguments based on Neumann series and relative boundedness apply. For the same reason, spectral exactness cannot be established using novel essential numerical range tools recently developed in [2].

In this paper we study location and structure of the point spectrum of $\mathcal{T}$ in Ω and, in particular, solve the open problem from [19] affirmatively. Our methodology is inspired by recent results on spectral bounds for non-self-adjoint Schrödinger type operators and benefits from an interesting interplay between operator theory, Fourier analysis and residue calculus.

Theorem 1.3. *Let U and V satisfy* (1.3) *and let $\mathcal{T}(\lambda)$, $\lambda \in \mathbb{C}$, be as in* (1.4)*. Then $\sigma_{\mathrm{p}}(\mathcal{T}) \cap \Omega$ is a discrete and bounded set.*

In fact, Theorem 1.3 is a simplified version of Theorem 3.4 below from which a new quantitative enclosure (3.14) for $\sigma_{\mathrm{p}}(\mathcal{T})$ can be obtained, see also Remark 3.5. Figure 1 illustrates the essential spectrum, our spectral enclosure (3.14), as well as the previously known result (1.7).

The strategy to prove Theorem 1.3 consists of two main steps. First in Section 2 we find a suitable formula for the inverse of the (unperturbed) operator family

$$\mathcal{L}(\lambda) = \begin{pmatrix} (-\partial^2+\alpha^2)(-\partial^2+\alpha^2-\lambda) & 2\partial \\ 2\partial & -\partial^2+\alpha^2-\lambda \end{pmatrix}, \quad \lambda \in \mathbb{C} \tag{1.9}$$

with the λ-independent domain

$$\begin{aligned} \operatorname{Dom}(\mathcal{L}(\lambda)) = \operatorname{Dom}(\mathcal{L}) = \big\{ (f_1, f_2) \in W^{4,2}(\mathbb{R}_+) \times W^{2,2}(\mathbb{R}_+) : \\ f_1(0) = f_1'(0) = f_2(0) = 0 \big\}; \end{aligned} \tag{1.10}$$

notice that $\mathcal{L}(\lambda)$, $\lambda \in \mathbb{C}$, are diagonally dominant operator matrices. In the second step, the original operator function $\mathcal{T}$ is expressed as

$$\mathcal{T}(\lambda) = \mathcal{L}(\lambda) + \mathrm{i}\alpha R \mathcal{V}$$

where

$$\mathcal{V} = \begin{pmatrix} V(-\partial^2+\alpha^2) + V'' & 0 \\ U' & V \end{pmatrix};$$

the perturbation is known to be relatively compact, see [23]. Our new results are then obtained by analyzing the Birman-Schwinger type operator

$$\mathcal{Q}(\lambda) = \mathcal{V}_1 \mathcal{L}(\lambda)^{-1} \mathcal{V}_2, \quad \lambda \in \Omega$$

where $\mathcal{V} = \mathcal{V}_2 \mathcal{V}_1$ is a suitable factorization of $\mathcal{V}$, see Section 3 for details. This is inspired by the pioneering work [1] on one-dimensional Schrödinger operators with complex potentials.

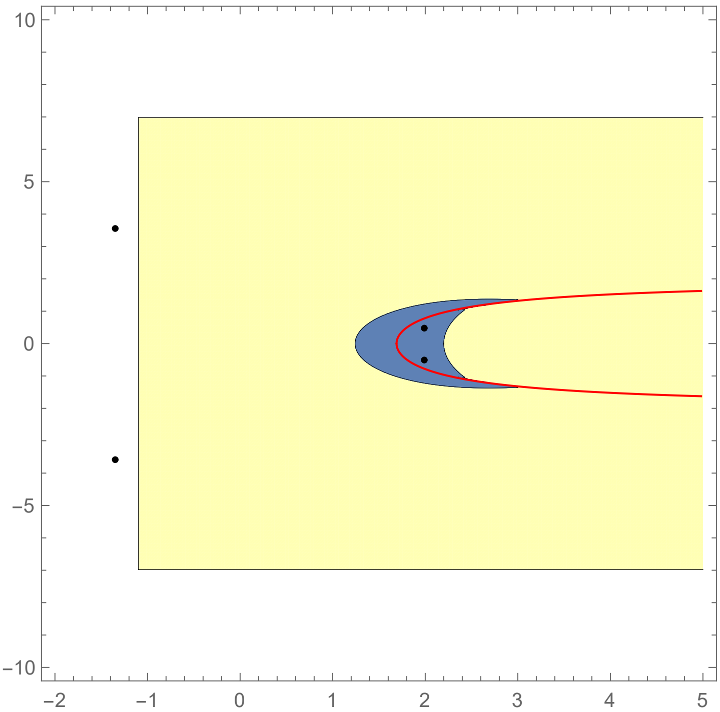

FIGURE 1. An illustration of the essential spectrum of $\mathcal{T}$ (in red) and the enclosures of the point spectrum of $\mathcal{T}$ ((1.7) in yellow, (3.14) in blue) for $\alpha = 1.3$, $R = 1.12$ and $U = U_0$, $V = V_0$ from (1.2) and the decomposition with $W_1(x) = e^{-x/2}$, see (3.1). The blue dots are the finite set B_α, see (1.11). In Remark 3.5 ii) and iii) some details on the performed steps in the numerical calculation are described.

It is essential to find the Green's function of $\mathcal{L}(\lambda)$, which is done by means of the distributional Fourier transform. Although it has a more complicated form, it shares the same qualitative properties as the Green's function of the one-dimensional Laplace operator, see *e.g.* (2.20); in particular, it has a singularity at $\lambda = \alpha^2$, *i.e.* at the tip of the essential spectrum, see Propositions 2.7 and 2.8 for the precise formulas. Overall, our analysis resembles the original one-dimensional Schrödinger operator case in [1] eventually. Nonetheless, the complex nature of the problem results in several obstacles which usually do not occur simultaneously. Not only does the Ekman boundary layer give rise to a spectral problem for a linear operator family having an operator matrix structure with higher order differential entries, but also boundary conditions at 0 need to be included, the essential spectrum is not a subset of $\mathbb{R}$ and, even though the unperturbed problem is normal, it is not always self-adjoint (see *e.g.* [4, 7, 11] on non-self-adjoint matrix differential problems).

The crucial role for both the analysis of the Green's function of $\mathcal{L}(\lambda)$ and the subsequent Birman-Schwinger type argument is played by the zeros μ_j, $j = 1, \dots, 6$ of the polynomial p_λ in (1.6), see Lemma 2.1. Although $\{\mu_j\}$ can be expressed via Cardano's formula, these explicit but complicated expressions are not used; needed properties of $\{\mu_j\}$ and their asymptotic behavior are instead proved independently, see Section 2.1, using also the previously established results in [19].

Some arguments in [19] require special considerations when λ is in the finite set

$$B_\alpha = \{\lambda \in \mathbb{C} : p_\lambda \text{ has multiple roots}\}. \tag{1.11}$$

This set occurs naturally also here and it is special at least in that the formulas for the Green's function of $\mathcal{L}(\lambda)$ seem to have singularities; these are, however, only apparent, as can easily be seen from the formulas in Propositions 2.7 and 2.8. Nonetheless, these effects prevent one to give simple and explicit enclosures for the point spectrum in terms of $\{\mu_j\}$ and the L^1-norms of the coefficients (*e.g.* as for one-dimensional Schrödinger operators), see Remark 3.5.

The Birman-Schwinger type argument used to prove Theorem 1.3 relies on the asymptotic behavior of the L^∞-norms of the Green's function of $\mathcal{L}(\lambda)$, see Lemma 3.3, which in turn requires the knowledge of the asymptotic behaviour of the zeros $\{\mu_j\}$ for large λ. Interestingly, due to a peculiar asymptotic behavior of the Green's function, the norm of the Birman-Schwinger operator is merely bounded in λ and thus the standard approach to show the invertibility of $I + \mathrm{i}\alpha R\mathcal{Q}(\lambda)$ does not immediately yield the result. Instead, one can estimate the spectral radius of $\mathcal{Q}(\lambda)$ which decays as $\lambda \to \infty$, see Lemma 3.3 and Theorem 3.4 below for details.

Using known techniques, our result can be extended in several directions. In particular, the integrability conditions on the coefficients U, V can be relaxed to L^p conditions with $p \in [1, \infty)$ and the estimate of the Birman-Schwinger operator can be modified accordingly by employing Stein's interpolation theorem, see *e.g.* [6, 8, 9, 10, 12, 16, 17, 21]. The L^∞ assumption is as usual only technical. Moreover, one can obtain bounds on (possibly) embedded eigenvalues by establishing a limiting-absorption type principle *e.g.* as in [14, 15].

2. Inverse of the operator family $\mathcal{L}$

For the explicit description of the inverse of $\mathcal{L}(\lambda)$, it is convenient to first reflect and extend the problem to the whole real line. Nonetheless, we keep the boundary conditions and so the extended operator remains disconnected at 0. In more detail, we consider the Hilbert space

$$\mathcal{H}_0 = L^2_{\mathrm{even}}(\mathbb{R}) \oplus L^2_{\mathrm{odd}}(\mathbb{R})$$

and define an operator family $\mathcal{L}_0(\lambda)$, $\lambda \in \mathbb{C}$, in $\mathcal{H}_0$ as

$$\begin{aligned} \mathcal{L}_0(\lambda) &= \begin{pmatrix} (-\partial_0^2 + \alpha^2)(-\partial_0^2 + \alpha^2 - \lambda) & 2\partial_0 \\ 2\partial_0 & -\partial_0^2 + \alpha^2 - \lambda \end{pmatrix}, \\ \mathrm{Dom}(\mathcal{L}_0(\lambda)) &= \mathrm{Dom}(\mathcal{L}_0) \\ &= \Big\{ (f_1, f_2) \in (W^{4,2}(\mathbb{R} \setminus \{0\}) \times W^{2,2}(\mathbb{R} \setminus \{0\})) \cap \mathcal{H}_0 \ : \\ &\qquad\qquad f_1(0) = f_1'(0) = f_2(0) = 0 \Big\}; \end{aligned} \tag{2.1}$$

here ∂_0 denotes the distributional derivative on $\mathbb{R} \setminus \{0\}$. Note that, by the Sobolev embedding theorem, f_1 is continuous on $\mathbb{R} \setminus \{0\}$ and $f_1(0-)$ and $f_1(0+)$ exist; the same follows for f_1', f_1'', f_1''', f_2 and f_2'. The imposed boundary conditions are understood accordingly, *e.g.* as $f_1(0-) = 0 = f_1(0+)$.

A suitable formula for the inverse of the family $\mathcal{L}_0$ is derived via the distributional Fourier transform. We adopt here the conventions of [25, 26], see in particular [25, §9]. Namely, the Fourier transform is defined on the Schwartz space $\mathcal{S}(\mathbb{R})$ by

$$\mathcal{F}[\phi(x)](\xi) = \int_{\mathbb{R}} e^{\mathrm{i}\xi x} \phi(x) \, \mathrm{d}x, \qquad \phi \in \mathcal{S}(\mathbb{R}), \tag{2.2}$$

and extended in the standard way to the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions. For all $\phi \in \mathcal{S}(\mathbb{R})$ and $f \in \mathcal{S}'(\mathbb{R})$, we have the following well-known identities

$$\begin{aligned} \mathcal{F}^{-1}[\phi(\xi)](x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mathrm{i}\xi x}\phi(\xi)\,\mathrm{d}\xi, \\ \mathcal{F}[f'(x)](\xi) &= -\mathrm{i}\xi\mathcal{F}[f(x)](\xi), & (2.3) \\ \mathcal{F}[f * \phi] &= \mathcal{F}[f]\mathcal{F}[\phi] & (2.4) \end{aligned}$$

Moreover, it is well-known that

$$\begin{aligned} \mathcal{F}[\delta(x)](\xi) &= 1, & (2.5) \\ \mathcal{F}[e^{-\nu|x|}](\xi) &= \frac{2\nu}{\xi^2+\nu^2}, \qquad \operatorname{Re}\nu > 0. & (2.6) \end{aligned}$$

In our analysis, a naturally appearing object is the matrix funcion (a formal symbol of $\mathcal{L}(\lambda)$)

$$\mathcal{M}_\lambda(\xi) = \begin{pmatrix} (\xi^2+\alpha^2)(\xi^2+\alpha^2-\lambda) & -2\mathrm{i}\xi \\ -2\mathrm{i}\xi & \xi^2+\alpha^2-\lambda \end{pmatrix}, \quad \lambda \in \mathbb{C}, \quad \xi \in \mathbb{R}.$$

The inverse of $\mathcal{M}_\lambda$ reads as

$$\mathcal{M}_\lambda^{-1}(\xi) = \frac{1}{p_\lambda(\xi)} \begin{pmatrix} \xi^2+\alpha^2-\lambda & 2\mathrm{i}\xi \\ 2\mathrm{i}\xi & (\xi^2+\alpha^2)(\xi^2+\alpha^2-\lambda) \end{pmatrix} \tag{2.7}$$

where p_λ is as in (1.6). Note that $\mathcal{M}_\lambda^{-1}$ is bounded if and only if $\lambda \in \Omega$, as expected.

2.1. **Zeros of p_λ.** In order to find $\mathcal{L}_0(\lambda)^{-1}$, we need certain properties of the zeros of p_λ. Recall that B_α is defined as the finite set where roots of the polynomial p_λ are multiple, see (1.11).

Lemma 2.1. *Let p_λ, $\lambda \in \mathbb{C}$, be as in* (1.6) *and let Ω and B_α be as in* (1.8) *and* (1.11)*, respectively. Then p_λ can be factorized as*

$$p_\lambda(\xi) = (\xi^2-\mu_1(\lambda)^2)(\xi^2-\mu_2(\lambda)^2)(\xi^2-\mu_3(\lambda)^2), \quad \xi \in \mathbb{R}. \tag{2.8}$$

If $\lambda \in \Omega$, there are three roots of p_λ with negative imaginary parts given by

$$\mu_j(\lambda) \equiv \mu_j = \sqrt{\mu_j(\lambda)^2}, \quad j = 1, 2, 3$$

where the complex square root is understood as a mapping

$$\mathbb{C} \setminus [0,\infty) \to \{z \in \mathbb{C} : \operatorname{Im} z < 0\};$$

the remaining roots are given by $-\mu_j$, $j = 1, 2, 3$. Moreover, if $\lambda \in \Omega \setminus B_\alpha$, we can decompose

$$\frac{1}{p_\lambda(\xi)} = \frac{c_1}{\xi^2-\mu_1^2} + \frac{c_2}{\xi^2-\mu_2^2} + \frac{c_3}{\xi^2-\mu_3^2} \tag{2.9}$$

where

$$\begin{aligned} c_1(\lambda) \equiv c_1 &= \frac{1}{(\mu_1^2-\mu_2^2)(\mu_1^2-\mu_3^2)}, \\ c_2(\lambda) \equiv c_2 &= \frac{1}{(\mu_2^2-\mu_1^2)(\mu_2^2-\mu_3^2)}, \\ c_3(\lambda) \equiv c_3 &= \frac{1}{(\mu_3^2-\mu_1^2)(\mu_3^2-\mu_2^2)}. \end{aligned}$$

Proof. As p_λ is cubic in ξ^2, it can clearly be factorized as in (2.8). If $\lambda \in \Omega$, then p_λ has no real roots, thus $\mu_j^2 \notin [0,\infty)$ and the zeros in the lower half plane are obtained by the claimed branch of the complex square root. The decomposition (2.9) immediately follows from the partial fraction expansion (recall that if $\lambda \notin B_\alpha$ and $i \neq j$ then $\mu_j^2 - \mu_i^2 \neq 0$). ☐

For later use, we derive asymptotic formulas for the roots of p_λ as $\lambda \to \infty$.

Lemma 2.2. *Let μ_j, $j = 1, 2, 3$, be as in Lemma 2.1. Then*

$$\begin{aligned}
&\mu_1^2 = -\alpha^2 + \frac{4\alpha^2}{\lambda^2} + \mathcal{O}(\lambda^{-4}), \\
&\mu_2^2 = \lambda - \alpha^2 - 2\mathrm{i} + \mathcal{O}(\lambda^{-1}), \qquad \mu_3^2 = \lambda - \alpha^2 + 2\mathrm{i} + \mathcal{O}(\lambda^{-1}), \qquad \lambda \to \infty.
\end{aligned} \tag{2.10}$$

Hence, as $\lambda \to \infty$,

$$\begin{aligned}
|\mu_1| &= \alpha + \mathcal{O}(\lambda^{-2}), \qquad |\mu_2| = |\lambda|^{\frac{1}{2}} + \mathcal{O}(\lambda^{-\frac{1}{2}}), \quad |\mu_3| = |\lambda|^{\frac{1}{2}} + \mathcal{O}(\lambda^{-\frac{1}{2}}), \\
|c_1| &= \frac{1}{|\lambda|^2} + \mathcal{O}(\lambda^{-4}), \quad |c_2| = \frac{1}{4|\lambda|} + \mathcal{O}(\lambda^{-2}), \quad |c_3| = \frac{1}{4|\lambda|} + \mathcal{O}(\lambda^{-2}).
\end{aligned} \tag{2.11}$$

Proof. Clearly, we have $p_\lambda(\xi) = P_\lambda(\xi^2)$ with

$$P_\lambda(z) = (z + \alpha^2)(z + \alpha^2 - \lambda)^2 + 4z, \qquad z \in \mathbb{C}.$$

In what follows, all asymptotic relations shall be understood for $\lambda \to \infty$.

Relying on Rouché's theorem, see *e.g.* [5, Thm. V.3.8], we prove that P_λ has a simple root

$$z_\lambda = u_\lambda + \mathcal{O}(\lambda^{-4}), \qquad u_\lambda = -\alpha^2 + \frac{4\alpha^2}{\lambda^2}.$$

Consider the polynomials

$$q_\lambda(z) = P_\lambda(u_\lambda + z), \qquad r_\lambda(z) = z(u_\lambda + z - \lambda)^2$$

and let $C > 16\alpha^2$. For $|z| = C|\lambda|^{-4}$, one can easily compute and estimate

$$\begin{aligned}
|q_\lambda(z) - r_\lambda(z)| &= |16\alpha^2\lambda^{-2} + \mathcal{O}(1)z^2 + \mathcal{O}(\lambda)z + \mathcal{O}(\lambda^{-3})| \\
&\le 16\alpha^2|\lambda|^{-2} + \mathcal{O}(\lambda^{-3}).
\end{aligned} \tag{2.12}$$

On the other hand, an easy calculation yields the estimate

$$|r_\lambda(z)| = |\lambda^2 z + z^3 + \mathcal{O}(\lambda)z^2 + \mathcal{O}(\lambda)z| \ge C|\lambda|^{-2} + \mathcal{O}(\lambda^{-3}). \tag{2.13}$$

Since $C > 16\alpha^2$, it follows from (2.12) and (2.13) that

$$|q_\lambda(z) - r_\lambda(z)| < |r_\lambda(z)|, \qquad |z| = C|\lambda|^{-4}$$

if $|\lambda|$ is sufficiently large. By Rouché's Theorem, we conclude that q_λ has exactly one root in the ball $B_{C|\lambda|^{-4}}(0)$. Equivalently, if $|\lambda|$ is sufficiently large, then P_λ has exactly one root z_λ in $B_{C|\lambda|^{-4}}(u_\lambda)$, which then satisfies

$$|z_\lambda - u_\lambda| \le C|\lambda|^{-4}.$$

Setting $\mu_1(\lambda)^2 := z_\lambda$, this proves the first claim in (2.10); the remaining two are proven analogously.

The asymptotic expansions in (2.11) follow in a straightforward way from (2.10) and the relation $(1 + x)^\beta = 1 + \mathcal{O}(x)$ as $x \to 0$. □

The following lemma on the roots of p_λ is analogous to a part of [19, Thm. 6.1], where only $\lambda \in \Omega_+$ was considered; some modifications of the proof therein allow us to extend the claim to the whole Ω.

Lemma 2.3. *Let μ_j, $j = 1, 2, 3$ be as in Lemma 2.1 and Ω as in* (1.8). *Then*

$$(\alpha^2 - \lambda)(\mu_1 + \mu_2 + \mu_3) - \mu_1\mu_2\mu_3 \neq 0, \quad \lambda \in \Omega.$$

Proof. We proceed by contradiction, *i.e.* we assume that

$$(\alpha^2 - \lambda)(\mu_1 + \mu_2 + \mu_3) = \mu_1\mu_2\mu_3. \tag{2.14}$$

By expanding the polynomial $p_\lambda(\xi)$ in ξ^2 and applying Vieta's theorem, we get

$$\mu_1^2 + \mu_2^2 + \mu_3^2 = 2\lambda - 3\alpha^2, \tag{2.15}$$

$$\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_3^2\mu_1^2 = \lambda^2 - 4\lambda\alpha^2 + 3\alpha^4 + 4, \tag{2.16}$$

$$\mu_1^2\mu_2^2\mu_3^2 = -\alpha^2(\alpha^2 - \lambda)^2. \tag{2.17}$$

Notice that $\lambda \neq \alpha^2$, since otherwise p_λ would have the real root $\xi = 0$, which is impossible as $\lambda \in \Omega$. In view of this, we square (2.14) and use the trinomial identity

$$(\mu_1 + \mu_2 + \mu_3)^2 = \mu_1^2 + \mu_2^2 + \mu_3^2 + 2(\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1)$$

together with (2.15), (2.17) and obtain

$$\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 = \alpha^2 - \lambda. \tag{2.18}$$

Next, using the square of (2.18), the trinomial identity and (2.16), as well as (2.14), (2.17) and $\lambda \neq \alpha^2$, we get

$$\begin{aligned}(\alpha^2 - \lambda)^2 &= \lambda^2 - 4\lambda\alpha^2 + 3\alpha^4 + 4 + 2\mu_1\mu_2\mu_3(\mu_1 + \mu_2 + \mu_3)\\ &= \lambda^2 - 4\lambda\alpha^2 + 3\alpha^4 + 4 - 2\alpha^2(\alpha^2 - \lambda)\\ &= (\alpha^2 - \lambda)^2 + 4.\end{aligned}$$

However, this is a contradiction. □

2.2. **Inverse of $\mathcal{L}_0(\lambda)$.** We start with finding the inverse Fourier transform of $\mathcal{M}_\lambda^{-1}$.

Lemma 2.4. *Let Ω and B_α be as in (1.8) and (1.11), respectively, let $\mathcal{M}_\lambda^{-1}$ be as in (2.7) and let $\mathcal{F}$ be the Fourier transform as in (2.2). Then*

$$\mathcal{F}^{-1}[\mathcal{M}_\lambda^{-1}(\xi)](x) = \mathcal{G}_\lambda(x) = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{pmatrix}(x), \quad x \in \mathbb{R} \tag{2.19}$$

depends analytically on $\lambda \in \Omega$ (for fixed $x \in \mathbb{R}$). Moreover, for all $\lambda \in \Omega$, the convolution with $\mathcal{G}_\lambda$ is a bounded operator on $\mathcal{H}_0$. If $\lambda \in \Omega \setminus B_\alpha$, then

$$\begin{aligned}\mathcal{G}_{11}(x) &= -\frac{\mathrm{i}}{2}\sum_{j=1}^{3}\frac{c_j}{\mu_j}(\mu_j^2 + \alpha^2 - \lambda)e^{-\mathrm{i}\mu_j|x|},\\ \mathcal{G}_{12}(x) = \mathcal{G}_{21}(x) &= \operatorname{sgn} x\sum_{j=1}^{3}c_j e^{-\mathrm{i}\mu_j|x|},\\ \mathcal{G}_{22}(x) &= -\frac{\mathrm{i}}{2}\sum_{j=1}^{3}\frac{c_j}{\mu_j}(\mu_j^2 + \alpha^2 - \lambda)(\mu_j^2 + \alpha^2)e^{-\mathrm{i}\mu_j|x|}, \quad x \in \mathbb{R}\end{aligned} \tag{2.20}$$

with μ_j, c_j, $j = 1, 2, 3$ as in Lemma 2.1.

Proof. Since $\mathrm{Re}(\mathrm{i}\mu_j) > 0$, see Lemma 2.1, employing formula (2.6) gives

$$-\frac{\mathrm{i}}{2\mu_j}\mathrm{e}^{-\mathrm{i}\mu_j|x|} = \mathcal{F}^{-1}\Big[\frac{1}{\xi^2 - \mu_j^2}\Big], \qquad j = 1, 2, 3.$$

Taking into account the decomposition (2.9), we find that, for $\lambda \in \Omega \setminus B_\alpha$,

$$\tau(x) := \mathcal{F}^{-1}[p_\lambda^{-1}(\xi)](x) = -\frac{\mathrm{i}}{2}\left(\frac{c_1}{\mu_1}e^{-\mathrm{i}\mu_1|x|} + \frac{c_2}{\mu_2}e^{-\mathrm{i}\mu_2|x|} + \frac{c_3}{\mu_3}e^{-\mathrm{i}\mu_3|x|}\right).$$

To get (2.20), we calculate distributional derivatives of τ, namely,

$$\tau'(x) = -\frac{1}{2}\left(c_1 e^{-\mathrm{i}\mu_1|x|} + c_2 e^{-\mathrm{i}\mu_2|x|} + c_3 e^{-\mathrm{i}\mu_3|x|}\right)\operatorname{sgn} x,$$
$$\tau''(x) = \frac{\mathrm{i}}{2}\left(c_1\mu_1 e^{-\mathrm{i}\mu_1|x|} + c_2\mu_2 e^{-\mathrm{i}\mu_2|x|} + c_3\mu_3 e^{-\mathrm{i}\mu_3|x|}\right) - (c_1+c_2+c_3)\delta.$$

It is obvious from (2.9) that $c_1 + c_2 + c_3 = 0$, thus

$$\tau''(x) = \frac{\mathrm{i}}{2}\left(c_1\mu_1 e^{-\mathrm{i}\mu_1|x|} + c_2\mu_2 e^{-\mathrm{i}\mu_2|x|} + c_3\mu_3 e^{-\mathrm{i}\mu_3|x|}\right). \tag{2.21}$$

Analogously, using the relation $c_1\mu_1^2 + c_2\mu_2^2 + c_3\mu_3^2 = 0$, we find

$$\tau'''(x) = \frac{1}{2}\left(c_1\mu_1^2 e^{-\mathrm{i}\mu_1|x|} + c_2\mu_2^2 e^{-\mathrm{i}\mu_2|x|} + c_3\mu_3^2 e^{-\mathrm{i}\mu_3|x|}\right)\operatorname{sgn} x,$$
$$\tau^{(4)}(x) = -\frac{\mathrm{i}}{2}\left(c_1\mu_1^3 e^{-\mathrm{i}\mu_1|x|} + c_2\mu_2^3 e^{-\mathrm{i}\mu_2|x|} + c_3\mu_3^3 e^{-\mathrm{i}\mu_3|x|}\right).$$

The formulas in (2.20) are now obtained in a straightforward way by applying the rule (2.3) successively.

The holomorphicity of $\mathcal{G}_\lambda$ on Ω can be verified in a straightforward way from its definition in (2.19) by applying the dominated convergence theorem. It remains to show the boundedness of the convolution operator $\mathcal{G}_\lambda * \cdot$ on $\mathcal{H}_0$. Let therefore $G = (g_1, g_2) \in \mathcal{H}_0$. As $\mathcal{G}_{11}$ and $\mathcal{G}_{22}$ are even and $\mathcal{G}_{12} = \mathcal{G}_{21}$ is odd, it follows that the first and second components of $\mathcal{G}_\lambda * G$ are even and odd, respectively. By Young's convolution inequality,

$$\begin{aligned}\|\mathcal{G}_\lambda * G\|_{\mathcal{H}_0}^2 &\lesssim \sum_{i,j=1}^{2} \|\mathcal{G}_{ij} * g_j\|_{L^2(\mathbb{R})}^2 \le \sum_{i,j=1}^{2} \|\mathcal{G}_{ij}\|_{L^1(\mathbb{R})}^2 \|g_j\|_{L^2(\mathbb{R})}^2 \\ &\lesssim \sum_{i,j=1}^{2} \|\mathcal{G}_{ij}\|_{L^1(\mathbb{R})}^2 \|G\|_{\mathcal{H}_0}^2.\end{aligned}$$

The proof is concluded by checking the integrability of $\mathcal{G}_\lambda$, which follows from (2.19), (2.7) and the properties of the Fourier transform (we remark that for $\lambda \in \Omega \setminus B_\alpha$, this is immediate from (2.20) and the choice of μ_j, $j = 1, 2, 3$, see Lemma 2.1). $\square$

To find the inverse of $\mathcal{L}_0(\lambda)$ we further need the two following lemmas.

Lemma 2.5. *Let Ω be as in* (1.8)*, let $\mathcal{M}_\lambda^{-1}$ be as in* (2.7) *and let*

$$\mathcal{A}(\lambda) = \int_{\mathbb{R}} \begin{pmatrix} -\mathrm{i}\xi & 0 \\ 0 & 1 \end{pmatrix} \mathcal{M}_\lambda^{-1} \begin{pmatrix} \mathrm{i}\xi & 0 \\ 0 & 1 \end{pmatrix} \mathrm{d}\xi, \quad \lambda \in \Omega. \tag{2.22}$$

Then $\det \mathcal{A}(\lambda) \neq 0$*, $\lambda \in \Omega$.*

Proof. We first find a suitable formula for $\det \mathcal{A}(\lambda)$ when $\lambda \in \Omega \setminus B_\alpha$. It can be easily verified from the definition (2.22) that $\det \mathcal{A}(\lambda)$ is a continuous function of $\lambda \in \Omega$ and so one can obtain a formula for $\det \mathcal{A}(\lambda)$, $\lambda \in \Omega \cap B_\alpha$, by passing to the limit in λ; see (2.31) for the final formula.

The determinant of $\mathcal{A}(\lambda)$ can be written as

$$\det \mathcal{A}(\lambda) = I_1(\lambda) I_2(\lambda) + I_3(\lambda)^2, \qquad I_i(\lambda) = \int_{\mathbb{R}} p_\lambda^{-1}(\xi) f_i(\xi) \mathrm{d}\xi$$

where the functions f_i, $i = 1, 2, 3$, are given by

$$f_1(\xi) = \xi^2(\xi^2 + \alpha^2 - \lambda), \; f_2(\xi) = (\xi^2 + \alpha^2)(\xi^2 + \alpha^2 - \lambda), \; f_3(\xi) = 2\xi^2.$$

We determine $I_1(\lambda)$ by a standard residue calculation. Let $r > 0$ and Γ_r be a negatively oriented, simple parametrization of the boundary of the semi-disc

$$B_r(0) \cap \{z \in \mathbb{C} : \operatorname{Im} z < 0\}.$$

Since f_1 is entire, $\operatorname{Re}\mu_j < 0$, $j = 1, 2, 3$, and the remaining roots of p_λ lie in the upper half plane, the residue theorem yields, for sufficiently large r,

$$\begin{aligned} -2\pi\mathrm{i}\sum_{j=1}^{3}\operatorname{res}(p_\lambda^{-1}f_1;\mu_j) &= \int_{\Gamma_r} p_\lambda^{-1}(z)f_1(z)\mathrm{d}z \\ &= \int_{-r}^{r} p_\lambda^{-1}(\xi)f_1(\xi)\mathrm{d}\xi + \int_{-\pi}^{0} \mathrm{i}r\mathrm{e}^{\mathrm{i}\xi}p_\lambda^{-1}(r\mathrm{e}^{\mathrm{i}\xi})f_1(r\mathrm{e}^{\mathrm{i}\xi})\mathrm{d}\xi \end{aligned} \tag{2.23}$$

where the last integral along the semi-circle tends to 0 as $r \to \infty$; this is a consequence of the dominated convergence theorem and the fact that the integrand is continuous, uniformly bounded in r and converges pointwise to 0. From (2.23), we thus obtain

$$I_1(\lambda) = \lim_{r\to\infty}\int_{-r}^{r} p_\lambda^{-1}(\xi)f_1(\xi)\mathrm{d}\xi = -2\pi\mathrm{i}\sum_{j=1}^{3}\operatorname{res}(p_\lambda^{-1}f_1;\mu_j). \tag{2.24}$$

It remains to calculate the residues. Since we assumed that $\lambda \notin B_\alpha$, we use the decomposition (2.9) and as μ_j are simple roots of p_λ in this case, we derive

$$\operatorname{res}(p_\lambda^{-1}f_1;\mu_j) = \lim_{z\to\mu_j}(z-\mu_j)p_\lambda^{-1}(z)f_1(z) = \frac{c_jf_1(\mu_j)}{2\mu_j}$$

and further, by (2.24),

$$I_1(\lambda) = -\pi\mathrm{i}\sum_{j=1}^{3} c_j\mu_j(\mu_j^2+\alpha^2-\lambda). \tag{2.25}$$

In the same way, one calculates the integrals

$$I_2(\lambda) = -\pi\mathrm{i}\sum_{j=1}^{3}\frac{c_j}{\mu_j}(\mu_j^2+\alpha^2)(\mu_j^2+\alpha^2-\lambda), \qquad I_3(\lambda) = -2\pi\mathrm{i}\sum_{j=1}^{3}c_j\mu_j.$$

Combining this with (2.25), we conclude that

$$\frac{\det\mathcal{A}(\lambda)}{-\pi^2} = \sum_{i,j=1}^{3} c_ic_j\left(\frac{\mu_i}{\mu_j}(\mu_i^2+\alpha^2-\lambda)(\mu_j^2+\alpha^2-\lambda)(\mu_j^2+\alpha^2)+4\mu_i\mu_j\right). \tag{2.26}$$

Recall that as $\lambda \in \Omega$, all roots of p_λ are non-zero and so the calculations above are justified. Moreover, $p_\lambda(\mu_j) = 0$ then implies that $\mu_j^2+\alpha^2-\lambda \neq 0$ and

$$(\mu_j^2+\alpha^2-\lambda)(\mu_j^2+\alpha^2) = -\frac{4\mu_j^2}{\mu_j^2+\alpha^2-\lambda}.$$

Using this identity, (2.26) simplifies to

$$\frac{\det\mathcal{A}(\lambda)}{-\pi^2} = 4\sum_{i,j=1}^{3} c_ic_j\mu_i\mu_j\frac{\mu_j^2-\mu_i^2}{\mu_j^2+\alpha^2-\lambda}.$$

Introducing the notations

$$\begin{aligned} &C \equiv C_1C_2C_3 \equiv (\mu_2^2-\mu_3^2)(\mu_3^2-\mu_1^2)(\mu_1^2-\mu_2^2) \neq 0, \\ &D \equiv \prod_{j=1}^{3}(\mu_j^2+\alpha^2-\lambda), \qquad D_j \equiv \prod_{k=1,\ k\neq j}^{3}(\mu_k^2+\alpha^2-\lambda), \end{aligned} \tag{2.27}$$

and using that $c_jC = -C_j$, we infer

$$\frac{\det\mathcal{A}(\lambda)}{-4\pi^2} = \sum_{i,j=1,i\neq j}^{3}\frac{C_iC_jD_j}{C^2D}\mu_i\mu_j\left(\mu_j^2-\mu_i^2\right). \tag{2.28}$$

Expanding the right hand side of (2.28) and using (2.27), we further obtain that

$$\frac{\det \mathcal{A}(\lambda)}{-4\pi^2} = \frac{1}{CD}\left(\mu_1\mu_2(D_1 - D_2) + \mu_2\mu_3(D_2 - D_3) + \mu_3\mu_1(D_3 - D_1)\right).$$

By inserting the formulas for D_j, we get

$$\begin{aligned}\frac{\det \mathcal{A}(\lambda)}{-4\pi^2} = \frac{1}{CD}\Big(&\mu_1\mu_2(\mu_3^2 + \alpha^2 - \lambda)(\mu_2^2 - \mu_1^2)\\ &+ \mu_2\mu_3(\mu_1^2 + \alpha^2 - \lambda)(\mu_3^2 - \mu_2^2)\\ &+ \mu_3\mu_1(\mu_2^2 + \alpha^2 - \lambda)(\mu_1^2 - \mu_3^2)\Big).\end{aligned} \tag{2.29}$$

Using (2.29), by elementary manipulations (more precisely, expanding and simplifying the right-hand side of (2.30)), we see that

$$\frac{\det \mathcal{A}(\lambda)}{-4\pi^2} = \frac{(\mu_1 - \mu_2)(\mu_2 - \mu_3)(\mu_3 - \mu_1)}{CD}((\alpha^2 - \lambda)(\mu_1 + \mu_2 + \mu_3) - \mu_1\mu_2\mu_3). \tag{2.30}$$

Hence, we finally obtain for all $\lambda \in \Omega$ that

$$\det \mathcal{A}(\lambda) = -4\pi^2 \frac{(\alpha^2 - \lambda)(\mu_1 + \mu_2 + \mu_3) - \mu_1\mu_2\mu_3}{(\mu_1 + \mu_2)(\mu_2 + \mu_3)(\mu_3 + \mu_1)\prod_{j=1}^{3}(\mu_j^2 + \alpha^2 - \lambda)}; \tag{2.31}$$

notice that we include also $\lambda \in \Omega \cap B_\alpha$ since the formula has no longer (apparent) singularities for multiple roots. Finally, we conclude by Lemma 2.3 that $\det A(\lambda) \neq 0$ for $\lambda \in \Omega$. □

Lemma 2.6. *Let Ω and $\mathcal{G}_\lambda$ be as in* (1.8) *and* (2.19)*, respectively. Then*

$$\mathcal{G}_{11}(0) = \frac{\mathrm{i}}{2}\frac{\mu_1\mu_2\mu_3 - (\alpha^2 - \lambda)(\mu_1 + \mu_2 + \mu_3)}{\mu_1\mu_2\mu_3(\mu_1 + \mu_2)(\mu_2 + \mu_3)(\mu_3 + \mu_1)} \neq 0, \quad \lambda \in \Omega. \tag{2.32}$$

Proof. We show the first equality in (2.32), the claim then follows by Lemma 2.3. Let $\lambda \in \Omega \setminus B_\alpha$, then we get from (2.20)

$$\begin{aligned}\mathcal{G}_{11}(0) &= -\frac{\mathrm{i}}{2}\sum_{j=1}^{3}\frac{c_j}{\mu_j}(\mu_j^2 + \alpha^2 - \lambda)\\ &= \frac{\mathrm{i}}{2C\mu_1\mu_2\mu_3}\Big(\mu_2\mu_3(\mu_2^2 - \mu_3^2)(\mu_1^2 + \alpha^2 - \lambda)\\ &\qquad + \mu_3\mu_1(\mu_3^2 - \mu_1^2)(\mu_2^2 + \alpha^2 - \lambda)\\ &\qquad + \mu_1\mu_2(\mu_1^2 - \mu_2^2)(\mu_3^2 + \alpha^2 - \lambda)\Big)\end{aligned}$$

where C as in (2.27). The formula in (2.32) for $\lambda \in \Omega \setminus B_\alpha$ follows by manipulations analogous to (2.29) and (2.30); moreover, it can be extended to $\lambda \in B_\alpha \cap \Omega$ by continuity of both left and right hand side. □

The inverse of $\mathcal{L}_0(\lambda)$ is found via Fourier transform and by employing Lemmas 2.5 and 2.6; notice that the second term in (2.33) is a rank-one operator.

Proposition 2.7. *Let Ω be as in* (1.8) *and let the family $\mathcal{L}_0$ be as in* (2.1)*. Then, with $\mathcal{G}_\lambda$ as in* (2.19)*, we have for all $\lambda \in \Omega$*

$$\mathcal{L}_0(\lambda)^{-1}G = \int_{\mathbb{R}} \mathcal{G}_\lambda(\cdot - y)G(y)\,\mathrm{d}y + \int_{\mathbb{R}} \mathcal{K}_\lambda(\cdot, y)G(y)\,\mathrm{d}y, \qquad G \in \mathcal{H}_0 \tag{2.33}$$

where (with $x, y \in \mathbb{R}$)

$$\mathcal{K}_\lambda(x, y) = \frac{1}{\mathcal{G}_{11}(0)}\begin{pmatrix} -\mathcal{G}_{11}(x)\mathcal{G}_{11}(y) & \mathcal{G}_{11}(x)\mathcal{G}_{12}(y) \\ -\mathcal{G}_{12}(x)\mathcal{G}_{11}(y) & \mathcal{G}_{12}(x)\mathcal{G}_{12}(y)\end{pmatrix}. \tag{2.34}$$

Proof. We find the formula for $\mathcal{L}_0(\lambda)^{-1}$ by taking the Fourier transform of the resolvent equation

$$\mathcal{L}_0(\lambda)F = G, \qquad F \in \operatorname{Dom}(\mathcal{L}_0),\ G \in \mathcal{H}_0 \cap \mathcal{S}(\mathbb{R})^2. \tag{2.35}$$

To this end, we rewrite (2.35) as an equation in tempered distributions. Using the boundary conditions which $F = (f_1, f_2)$ satisfies, see (2.1), we derive the following identities between its distributional derivatives on the whole real line and on $\mathbb{R}\setminus\{0\}$

$$\begin{aligned} f_1' &= \partial_0 f_1, & f_1'' &= \partial_0^2 f_1, \\ f_1''' &= \partial_0^3 f_1 + [f_1'']_0\,\delta, & f_1^{(4)} &= \partial_0^4 f_1 + [f_1''']_0\,\delta + [f_1'']_0\,\delta', \\ f_2' &= \partial_0 f_2, & f_2'' &= \partial_0^2 f_2 + [f_2']_0\,\delta; \end{aligned}$$

here $[g]_0 = g(0+) - g(0-)$ is the jump of a piece-wise continuous function g at 0 and δ is the Dirac δ at 0. Hence, denoting by $\mathcal{L}_{\mathbb{R}}(\lambda)$ the operator $\mathcal{L}_0(\lambda)$ with standard distributional derivatives on whole $\mathbb{R}$ instead of ∂_0, (2.35) reads

$$\mathcal{L}_{\mathbb{R}}(\lambda)F = G + \begin{pmatrix} [f_1''']_0\,\delta + [f_1'']_0\,\delta' \\ -[f_2']_0\,\delta \end{pmatrix}. \tag{2.36}$$

Applying the Fourier transform $\mathcal{F}$ on (2.36) and using the standard rules for $\mathcal{F}$, see (2.3) and (2.5), we obtain

$$\mathcal{M}_\lambda \mathcal{F}[F] = \mathcal{F}[G] + \begin{pmatrix} [f_1''']_0 - \mathrm{i}\xi[f_1'']_0 \\ -[f_2']_0 \end{pmatrix}.$$

Hence, inverting the matrix $\mathcal{M}_\lambda$ for $\lambda \in \Omega$, we get

$$\mathcal{F}[F] = \mathcal{M}_\lambda^{-1}\mathcal{F}[G] + \mathcal{M}_\lambda^{-1}\begin{pmatrix} [f_1''']_0 - \mathrm{i}\xi[f_1'']_0 \\ -[f_2']_0 \end{pmatrix}. \tag{2.37}$$

Applying the inverse Fourier transform to (2.37) and using the convolution theorem (2.4), it follows that

$$F = (\mathcal{F}^{-1}\mathcal{M}_\lambda^{-1}) * G + \mathcal{F}^{-1}\left[\mathcal{M}_\lambda^{-1}\begin{pmatrix} [f_1''']_0 - \mathrm{i}\xi[f_1'']_0 \\ -[f_2']_0 \end{pmatrix}\right]. \tag{2.38}$$

Recalling Lemma 2.4 and $\mathcal{M}_\lambda^{-1} = \mathcal{F}[\mathcal{G}_\lambda]$, we thus obtain the first term in (2.41).

We proceed by finding the formula for $\mathcal{K}_\lambda$. The function F in (2.38) must be an element of $\operatorname{Dom}(\mathcal{L}_0)$, in particular, it must satisfy the boundary conditions at 0. In the Fourier space, these can be expressed as

$$\begin{aligned} f_j(0) &= 2\pi \int_{\mathbb{R}} \mathcal{F}[f_j](\xi)\,\mathrm{d}\xi = 0, \quad j = 1, 2, \\ f_1'(0) &= -2\pi\mathrm{i}\int_{\mathbb{R}} \xi\mathcal{F}[f_1](\xi)\,\mathrm{d}\xi = 0. \end{aligned}$$

Combining the relations above with (2.37), we obtain three equations. Namely, (with $\mathcal{A}(\lambda)$ is as in Lemma 2.5)

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \int_{\mathbb{R}} \begin{pmatrix} -\mathrm{i}\xi & 0 \\ 0 & 1 \end{pmatrix} \mathcal{M}_\lambda^{-1}(\xi)\left(\mathcal{F}[G](\xi) + \begin{pmatrix} [f_1''']_0 \\ 0 \end{pmatrix}\right)\mathrm{d}\xi - \mathcal{A}(\lambda)\begin{pmatrix} [f_1'']_0 \\ [f_2']_0 \end{pmatrix} \tag{2.39}$$

and (with $\mathcal{M}_\lambda^{-1} = \mathcal{F}[\mathcal{G}_\lambda]$ and $G = (g_1, g_2)$)

$$\begin{aligned} 0 = &\int_{\mathbb{R}} (\mathcal{F}[\mathcal{G}_{11}](\xi)\mathcal{F}[g_1](\xi) + \mathcal{F}[\mathcal{G}_{12}](\xi)\mathcal{F}[g_2](\xi))\,\mathrm{d}\xi \\ &+ \int_{\mathbb{R}} ((\mathcal{M}_\lambda^{-1})_{11}(\xi)([f_1''']_0 - \mathrm{i}\xi[f_1'']_0) - (\mathcal{M}_\lambda^{-1})_{12}(\xi)[f_2']_0)\,\mathrm{d}\xi. \end{aligned} \tag{2.40}$$

Since $G \in \mathcal{H}_0 \cap \mathcal{S}(\mathbb{R})^2$, considering the defining formula (2.2), we have $\mathcal{F}[G] \in \mathcal{H}_0$ and both components of the integrand in (2.39) are odd functions. Thus the integral vanishes and equation (2.39) simplifies to

$$\mathcal{A}(\lambda)\begin{pmatrix}[f_1'']_0\\ [f_2']_0\end{pmatrix} = 0,$$

which in turn by Lemma 2.5 implies $[f_1'']_0 = [f_2']_0 = 0$. Consequently, using that $\mathcal{G}_{11} = \mathcal{F}^{-1}[(\mathcal{M}_\lambda)_{11}^{-1}]$ and that $\mathcal{G}_{11}(0) \neq 0$ by Lemma 2.6, equation (2.40) reads

$$[f_1''']_0 = -\frac{\langle \mathcal{F}[g_1], \overline{\mathcal{F}[\mathcal{G}_{11}]}\rangle_{L^2(\mathbb{R})} + \langle \mathcal{F}[g_2], \overline{\mathcal{F}[\mathcal{G}_{12}]}\rangle_{L^2(\mathbb{R})}}{2\pi\mathcal{G}_{11}(0)}.$$

Using in addition that $\overline{\mathcal{F}[f]} = \mathcal{F}[\overline{f}]$ if f is even and $\overline{\mathcal{F}[f]} = -\mathcal{F}[\overline{f}]$ if f is odd, as well as the identity

$$\langle \mathcal{F}[f], \mathcal{F}[g]\rangle_{L^2(\mathbb{R})} = 2\pi\langle f, g\rangle_{L^2(\mathbb{R})}, \quad f, g \in L^2(\mathbb{R}),$$

we obtain the kernel $\mathcal{K}_\lambda$ in (2.34). Combining this with (2.38), we see that F is equal to the right-hand side of (2.33), which in particular implies that $\mathcal{L}_0(\lambda)$ is injective and its left inverse is given by (2.33). Moreover, considering the definition of $\mathcal{G}_\lambda$ in (2.19), using the properties of the Fourier transform, Fubini's theorem and the dominated convergence theorem, one can verify in a straightforward way that $F = \mathcal{L}_0(\lambda)^{-1}G$ defined by (2.33) with arbitrary $G \in \mathcal{H}_0$ satisfies $F \in \mathrm{Dom}(\mathcal{L}_0)$ and $\mathcal{L}_0(\lambda)F = G$, *i.e.* that the operator in (2.33) is also a right inverse of $\mathcal{L}_0(\lambda)$. We conclude that $\mathcal{L}_0(\lambda)$ is bijective and its inverse is given by the formula in (2.33). $\square$

2.3. **Inverse of $\mathcal{L}(\lambda)$.** The inverse of $\mathcal{L}(\lambda)$ can be found by rewriting the formula for $\mathcal{L}_0(\lambda)^{-1}$ in (2.33).

Proposition 2.8. *Let Ω be as in* (1.8)*, let the family $\mathcal{L}$ be as in* (1.9) *and* (1.10)*, let $\mathcal{G}_\lambda$ be as in* (2.19) *and let $\mathcal{K}_\lambda$ be as in* (2.34)*. Then*

$$\mathcal{L}(\lambda)^{-1}\Psi = \int_{\mathbb{R}_+} \mathcal{L}_\lambda(\cdot, y)\Psi(y)\,\mathrm{d}y, \quad \Psi \in \mathcal{H},\ \lambda \in \Omega \tag{2.41}$$

where (with $x, y \in \mathbb{R}_+$)

$$\begin{aligned}\mathcal{L}_\lambda(x,y) &= \mathcal{G}_\lambda^+(x,y) + 2\mathcal{K}_\lambda(x,y),\\ \mathcal{G}_\lambda^+(x,y) &= \begin{pmatrix}\mathcal{G}_{11}(x-y) + \mathcal{G}_{11}(x+y) & \mathcal{G}_{12}(x-y) - \mathcal{G}_{12}(x+y)\\ \mathcal{G}_{12}(x-y) + \mathcal{G}_{12}(x+y) & \mathcal{G}_{22}(x-y) - \mathcal{G}_{22}(x+y)\end{pmatrix}.\end{aligned} \tag{2.42}$$

Proof. We aim to solve the equation

$$\mathcal{L}(\lambda)\Psi = \Phi, \qquad \Phi \in \mathcal{H},\ \Psi \in \mathrm{Dom}(\mathcal{L}). \tag{2.43}$$

To this end, we extend $\Phi, \Psi \in \mathcal{H}$ uniquely to $\Phi_0, \Psi_0 \in \mathcal{H}_0$ so that $\Phi_0 \restriction \mathbb{R}_+ = \Phi$ and $\Psi_0 \restriction \mathbb{R}_+ = \Psi$. Then it is not difficult to see that

$$\Psi_0 \in \mathrm{Dom}(\mathcal{L}_0), \qquad (\mathcal{L}_0(\lambda)\Psi_0) \restriction \mathbb{R}_+ = \mathcal{L}(\lambda)\Psi.$$

Moreover, solving (2.43) (uniquely) is equivalent to solving $\mathcal{L}_0(\lambda)\Psi_0 = \Phi_0$ (uniquely) and restricting Ψ_0 to $\mathbb{R}_+$, *i.e.*

$$\Psi = (\mathcal{L}_0(\lambda)^{-1}\Phi_0) \restriction \mathbb{R}_+. \tag{2.44}$$

Using that the components of Φ_0 are even and odd, we can further simplify the terms in (2.44), namely, for $x \in \mathbb{R}_+$,

$$\begin{aligned}(\mathcal{G}_\lambda * \Phi_0)(x) &= \int_{\mathbb{R}_+} \mathcal{G}_\lambda(x-y)\Phi(y)\,\mathrm{d}y + \int_{\mathbb{R}_+} \mathcal{G}_\lambda(x+y)\Phi_0(-y)\,\mathrm{d}y\\ &= \int_{\mathbb{R}_+} \mathcal{G}_\lambda^+(x,y)\Phi(y)\,\mathrm{d}y\end{aligned}$$

and similarly

$$\int_{\mathbb{R}} \mathcal{K}_\lambda(x,y)\Phi_0(y)\,\mathrm{d}y = \int_{\mathbb{R}_+} \mathcal{K}_\lambda(x,y)\Phi(y)\,\mathrm{d}y + \int_{\mathbb{R}_+} \mathcal{K}_\lambda(x,-y)\Phi_0(-y)\,\mathrm{d}y$$
$$= 2\int_{\mathbb{R}_+} \mathcal{K}_\lambda(x,y)\Phi(y)\,\mathrm{d}y$$

where we have used in addition that $\mathcal{G}_{11}$ and $\mathcal{G}_{12}$ are even and odd. □

Remark 2.9. Notice that the entries of the kernel $\mathcal{G}_\lambda^+$ can be simplified. In detail, for $\lambda \in \Omega \setminus B_\alpha$, we have (with $x, y \in \mathbb{R}_+$)

$$\mathcal{G}_{11}^+(x,y) = -\mathrm{i}\sum_{j=1}^{3} \frac{c_j}{\mu_j}(\mu_j^2+\alpha^2-\lambda) \begin{cases} e^{-\mathrm{i}\mu_j x}\cos(\mu_j y), & y < x, \\ e^{-\mathrm{i}\mu_j y}\cos(\mu_j x), & y > x, \end{cases}$$
$$\mathcal{G}_{12}^+(x,y) = 2\mathrm{i}\sum_{j=1}^{3} c_j \begin{cases} e^{-\mathrm{i}\mu_j x}\sin(\mu_j y), & y < x, \\ \mathrm{i}e^{-\mathrm{i}\mu_j y}\cos(\mu_j x), & y > x, \end{cases}$$
$$\mathcal{G}_{21}^+(x,y) = -2\mathrm{i}\sum_{j=1}^{3} c_j \begin{cases} \mathrm{i}e^{-\mathrm{i}\mu_j x}\cos(\mu_j y), & y < x, \\ e^{-\mathrm{i}\mu_j y}\sin(\mu_j x), & y > x, \end{cases}$$
$$\mathcal{G}_{22}^+(x,y) = \sum_{j=1}^{3} \frac{c_j}{\mu_j}(\mu_j^2+\alpha^2-\lambda)(\mu_j^2+\alpha^2) \begin{cases} e^{-\mathrm{i}\mu_j x}\sin(\mu_j y), & y < x, \\ e^{-\mathrm{i}\mu_j y}\sin(\mu_j x), & y > x. \end{cases}$$

3. Birman-Schwinger operator

Let $\lambda \in \Omega$, thus $\mathcal{L}(\lambda)^{-1}$ exists and is given by (2.41) in Proposition 2.8. To employ a Birman-Schwinger type argument, we factorize the perturbation $\mathcal{V} = \mathcal{V}_2\mathcal{V}_1$ with

$$\mathcal{V}_2 = \begin{pmatrix} W_1 & 0 \\ 0 & W_1 \end{pmatrix}, \qquad \mathcal{V}_1 = \begin{pmatrix} W_2(-\partial^2+\alpha^2)+W_3 & 0 \\ W_4 & W_2 \end{pmatrix} \tag{3.1}$$

where

$$W_1 = \max\{|V|^{\frac12}, |V''|^{\frac12}, |U'|^{\frac12}\} \tag{3.2}$$

and

$$W_2 = \frac{V}{W_1}, \quad W_3 = \frac{V''}{W_1}, \quad W_4 = \frac{U'}{W_1}.$$

It follows from the assumptions on U and V, see (1.3), that

$$W_j \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+), \quad j = 1, \ldots, 4,$$

and

$$\begin{aligned} &\|W_1\|_{L^2}^2 = \|\max\{|V|, |V''|, |U'|\}\|_{L^1}, && \|W_2\|_{L^2}^2 \le \|V\|_{L^1}, \\ &\|W_3\|_{L^2}^2 \le \|V''\|_{L^1}, && \|W_4\|_{L^2}^2 \le \|U'\|_{L^1}. \end{aligned} \tag{3.3}$$

The choice of W_1 in (3.2) guarantees that $W_j \in L^2(\mathbb{R}_+)$, $j = 1, \ldots, 4$, which is essential in the next steps. Nonetheless, in particular situations, like in the physical setting with $U = U_\epsilon$ and $V = V_\epsilon$ as in (1.2), it can be more convenient to choose a different and simpler W_1, *e.g.* $W_1(x) = e^{-x/2}$.

We next analyze the Birman-Schwinger type operator

$$\mathcal{Q}(\lambda) = \mathcal{V}_1\mathcal{L}(\lambda)^{-1}\mathcal{V}_2, \qquad \lambda \in \Omega. \tag{3.4}$$

To express the integral kernel of $\mathcal{Q}(\lambda)$, we first derive terms produced by the differential operator $(-\partial^2+\alpha^2)$ in $\mathcal{V}_1$.

Lemma 3.1. *Let $\mathcal{G}_\lambda$ and $\mathcal{G}_\lambda^+$ be as in* (2.19) *and* (2.42)*, respectively. Then*

$$\mathcal{G}_{22} = (-\partial^2 + \alpha^2)\mathcal{G}_{11}, \qquad r_\lambda := (-\partial^2 + \alpha^2)\mathcal{G}_{12} \tag{3.5}$$

are analytic in λ on Ω (for every $x \in \mathbb{R}$). Moreover, for $x \in \mathbb{R}$ and $\lambda \in \Omega \setminus B_\alpha$,

$$r_\lambda(x) = \operatorname{sgn}(x) \sum_{j=1}^{3} c_j(\mu_j^2 + \alpha^2)\mathrm{e}^{-\mathrm{i}\mu_j|x|}. \tag{3.6}$$

Proof. From the definition of $\mathcal{G}_\lambda$ in (2.19) and the properties of the Fourier transform, it can be derived easily that

$$(-\partial^2 + \alpha^2)\mathcal{G}_{11} = \mathcal{G}_{22}, \qquad (-\partial^2 + \alpha^2)\mathcal{G}_{12} = \mathcal{F}^{-1}\left[\frac{2\mathrm{i}\xi(\xi^2 + \alpha^2)}{p_\lambda(\xi)}\right]$$

which in turn yields the claimed analyticity in λ by the dominated convergence theorem. Moreover, using $c_1 + c_2 + c_3 = 0$ for $\lambda \in \Omega \setminus B_\alpha$, one obtains (3.6) similarly to (2.21). □

Lemma 3.2. *Let Ω be as in* (1.8)*, let $\mathcal{V}_1$ and $\mathcal{V}_2$ be as in* (3.1) *with* $\mathrm{Dom}(\mathcal{V}_1) = \mathrm{Dom}(\mathcal{L})$ *and* $\mathrm{Dom}(\mathcal{V}_2) = \mathcal{H}$ *and let the families $\mathcal{L}^{-1}$ and $\mathcal{L}_\lambda$ be as in* (2.41) *and* (2.42)*, respectively. Then*

$$\mathcal{Q}(\lambda) = \mathcal{V}_1\mathcal{L}(\lambda)^{-1}\mathcal{V}_2, \quad \lambda \in \Omega$$

is a holomorphic family of Hilbert-Schmidt integral operators on $\mathcal{H}$ with kernel

$$\begin{aligned}\mathcal{Q}_\lambda(x,y) = &\begin{pmatrix} W_3(x) & 0 \\ W_4(x) & W_2(x)\end{pmatrix}\mathcal{L}_\lambda(x,y)W_1(y) \\ &+ W_2(x)\begin{pmatrix} q_{11}(x,y) & q_{12}(x,y) \\ 0 & 0\end{pmatrix}W_1(y)\end{aligned} \tag{3.7}$$

where

$$\begin{aligned} q_{11}(x,y) &= \mathcal{G}_{22}(x-y) + \mathcal{G}_{22}(x+y) - \frac{2}{\mathcal{G}_{11}(0)}\mathcal{G}_{22}(x)\mathcal{G}_{11}(y), \\ q_{12}(x,y) &= r_\lambda(x-y) - r_\lambda(x+y) + \frac{2}{\mathcal{G}_{11}(0)}\mathcal{G}_{22}(x)\mathcal{G}_{12}(y), \quad x,y \in \mathbb{R}_+ \end{aligned} \tag{3.8}$$

with r_λ and $\mathcal{G}_\lambda$ as in (3.5) *and* (2.19)*, respectively.*

Proof. Let $\lambda \in \Omega$. We first note that $\mathcal{V}_1\mathcal{L}(\lambda)^{-1}\mathcal{V}_2$ is everywhere defined in $\mathcal{H}$. The formula (3.7) for its integral kernel follows by composing $\mathcal{V}_1\mathcal{L}(\lambda)^{-1}\mathcal{V}_2$ and using Lemma 3.1 together with the dominated convergence theorem (in order to interchange $-\partial^2$ and the integral).

To show that $\mathcal{V}_1\mathcal{L}(\lambda)^{-1}\mathcal{V}_2$ is a Hilbert-Schmidt operator, it suffices to consider its kernel in (3.7), notice that $\mathcal{G}_{ij}$, $i,j = 1,2$, q_{11} and q_{12} are bounded functions and that $W_j \in L^2(\mathbb{R}_+)$, $j = 1,\dots,4$, see (3.3). Moreover, going back to the definitions of the functions $\mathcal{G}_\lambda^+$, q_{11}, q_{12} and r_λ, see (2.42), (3.8) and (3.5), as well as $\mathcal{G}_\lambda$ originally given via the inverse Fourier transform in (2.19), using the dominated convergence theorem one can verify in a straightforward way that all these functions are holomorphic in $\lambda \in \Omega$ and that $\mathcal{V}_1\mathcal{L}(\lambda)^{-1}\mathcal{V}_2$, $\lambda \in \Omega$, is indeed a bounded analytic family in Ω. □

In the next lemma we show that the spectral radius $r(\mathcal{Q}(\lambda))$ decays as $\lambda \to \infty$ in Ω. Notice that the method we use does not yield a decay of the norm of $\mathcal{Q}(\lambda)$, for which we merely obtain $\|\mathcal{Q}(\lambda)\| = \mathcal{O}(1)$ as $\lambda \to \infty$ in Ω.

Lemma 3.3. *Let $\mathcal{Q}(\lambda)$ be the integral operator with the kernel in* (3.7)*. Then*

$$r(\mathcal{Q}(\lambda)) = \mathcal{O}(|\lambda|^{-\frac{1}{2}}), \qquad \lambda \to \infty \text{ in } \Omega. \tag{3.9}$$

Proof. Let $\mathcal{G}_\lambda$, $\mathcal{L}_\lambda$ and q_{11}, q_{12} be as in (2.19), (2.42) and (3.8), respectively. By estimating the Hilbert-Schmidt norms of $\mathcal{Q}(\lambda)_{ij}$, $i,j=1,2$, one obtains

$$\begin{aligned}
\|\mathcal{Q}(\lambda)_{11}\| &\leq (\|W_3\|_{L^2}\|(\mathcal{L}_\lambda)_{11}\|_{L^\infty} + \|W_2\|_{L^2}\|q_{11}\|_{L^\infty})\,\|W_1\|_{L^2},\\
\|\mathcal{Q}(\lambda)_{12}\| &\leq (\|W_3\|_{L^2}\|(\mathcal{L}_\lambda)_{12}\|_{L^\infty} + \|W_2\|_{L^2}\|q_{12}\|_{L^\infty})\,\|W_1\|_{L^2},\\
\|\mathcal{Q}(\lambda)_{21}\| &\leq (\|W_4\|_{L^2}\|(\mathcal{L}_\lambda)_{11}\|_{L^\infty} + \|W_2\|_{L^2}\|(\mathcal{L}_\lambda)_{21}\|_{L^\infty})\,\|W_1\|_{L^2},\\
\|\mathcal{Q}(\lambda)_{22}\| &\leq (\|W_4\|_{L^2}\|(\mathcal{L}_\lambda)_{12}\|_{L^\infty} + \|W_2\|_{L^2}\|(\mathcal{L}_\lambda)_{22}\|_{L^\infty})\,\|W_1\|_{L^2}.
\end{aligned} \tag{3.10}$$

Moreover, from (2.20), (3.6) and Lemma 2.2, it follows readily that

$$\begin{aligned}
&\|\mathcal{G}_{11}\|_{L^\infty} = \mathcal{O}(|\lambda|^{-1}), && |\mathcal{G}_{11}(0)|^{-1} = \mathcal{O}(|\lambda|),\\
&\|\mathcal{G}_{12}\|_{L^\infty} = \mathcal{O}(|\lambda|^{-1}), && \|q_{11}\|_{L^\infty} = \mathcal{O}(|\lambda|^{-\frac{1}{2}}),\\
&\|\mathcal{G}_{22}\|_{L^\infty} = \mathcal{O}(|\lambda|^{-\frac{1}{2}}), && \|q_{12}\|_{L^\infty} = \mathcal{O}(1), \qquad \lambda\to\infty \text{ in } \Omega;
\end{aligned} \tag{3.11}$$

recall that B_α is a finite set, so (2.20) and (3.6) can be used for λ with sufficiently large modulus. From (3.10) and (3.11), we derive

$$\begin{aligned}
&\|\mathcal{Q}(\lambda)_{11}\| = \mathcal{O}(|\lambda|^{-\frac{1}{2}}), && \|\mathcal{Q}(\lambda)_{12}\| = \mathcal{O}(1),\\
&\|\mathcal{Q}(\lambda)_{21}\| = \mathcal{O}(|\lambda|^{-1}), && \|\mathcal{Q}(\lambda)_{22}\| = \mathcal{O}(|\lambda|^{-\frac{1}{2}}), \qquad \lambda\to\infty \text{ in } \Omega.
\end{aligned} \tag{3.12}$$

Next, we employ a simple similarity transform of $\mathcal{Q}(\lambda)$ and obtain

$$\widetilde{\mathcal{Q}}(\lambda) = \operatorname{diag}(I, |\lambda|^{\frac{1}{2}} I)\mathcal{Q}(\lambda)\operatorname{diag}(I, |\lambda|^{-\frac{1}{2}} I) = \begin{pmatrix} \mathcal{Q}(\lambda)_{11} & |\lambda|^{-\frac{1}{2}}\mathcal{Q}(\lambda)_{12} \\ |\lambda|^{\frac{1}{2}}\mathcal{Q}(\lambda)_{21} & \mathcal{Q}(\lambda)_{22} \end{pmatrix}.$$

Since similarity transforms leave the spectrum and thus the spectral radius invariant, we have

$$r(\mathcal{Q}(\lambda)) = r(\widetilde{\mathcal{Q}}(\lambda)) \leq \|\widetilde{\mathcal{Q}}(\lambda)\| \tag{3.13}$$

and the asymptotic relation in (3.9) follows from (3.13) and (3.12). ☐

The following is our main result.

Theorem 3.4. *Let U and V satisfy* (1.3)*, let $\mathcal{T}(\lambda)$, $\lambda\in\mathbb{C}$, be as in* (1.4) *and let $\mathcal{Q}(\lambda)$ be as in* (3.4)*. Then*

$$\sigma_{\mathrm{p}}(\mathcal{T})\cap\Omega \subset \{\lambda\in\Omega \,:\, \alpha R\, r(\mathcal{Q}(\lambda)) \geq 1\}. \tag{3.14}$$

Moreover, the asymptotic relation

$$r(\mathcal{Q}(\lambda)) = \mathcal{O}(|\lambda|^{-\frac{1}{2}}), \qquad \lambda\to\infty \text{ in } \Omega \tag{3.15}$$

implies that $\sigma_{\mathrm{p}}(\mathcal{T})\cap\Omega$ is a bounded and discrete set.

Proof. We first show that if $\lambda\in\Omega$ and

$$\alpha R\, r(\mathcal{Q}(\lambda)) < 1, \tag{3.16}$$

then $\lambda\in\rho(\mathcal{T})$; our proof relies on a Birman-Schwinger argument, *cf.* [18] for its discussion in full generality.

If (3.16) holds, then $I + \mathrm{i}\alpha R\mathcal{Q}(\lambda)$ is invertible and the inverse is bounded on $\mathcal{H}$. Moreover, one can easily show that $\mathcal{V}_1$ is relatively bounded with respect to $\mathcal{L}(\lambda)$. The latter implies that $\mathcal{V}_1\mathcal{L}(\lambda)^{-1}$ is bounded on $\mathcal{H}$, and thus

$$\mathcal{R}(\lambda) := \mathcal{L}(\lambda)^{-1} - \mathrm{i}\alpha R\mathcal{L}(\lambda)^{-1}\mathcal{V}_2(I + \mathrm{i}\alpha R\mathcal{Q}(\lambda))^{-1}\mathcal{V}_1\mathcal{L}(\lambda)^{-1} \tag{3.17}$$

is a bounded operator on $\mathcal{H}$; notice that $0\in\rho(\mathcal{L}(\lambda))$ by Lemma 2.4. Using (3.17), it is straightforward to show that

$$\operatorname{Ran}(\mathcal{R}(\lambda)) \subset \operatorname{Dom}(\mathcal{T}), \qquad \mathcal{T}(\lambda)\mathcal{R}(\lambda) = \mathcal{I}, \qquad \mathcal{R}(\lambda)\mathcal{T}(\lambda) \subset \mathcal{I},$$

i.e. that $\mathcal{R}(\lambda) = \mathcal{T}(\lambda)^{-1}$ and thus $\lambda\in\rho(\mathcal{T})$. From this we conclude (3.14).

The asymptotic relation (3.15) is showed in Lemma 3.3 and implies that the set $\{\lambda \in \Omega : \alpha R\, r(\mathcal{Q}(\lambda)) \geq 1\}$ is indeed bounded. It remains to show that $\sigma_{\mathrm{p}}(\mathcal{T}) \cap \Omega$ is discrete. This however, is a consequence of [20, Thm. VII.1.9]; indeed, since $\mathcal{Q}(\lambda)$ is a holomorphic family of compact operators on Ω and since there exist points $\lambda_\pm \in \Omega_\pm$ such that $I + \mathrm{i}\alpha R\mathcal{Q}(\lambda_\pm)$ are boundedly invertible, we conclude that $I + \mathrm{i}\alpha R\mathcal{Q}(\lambda_\pm)$ is not boundedly invertible only for λ in a discrete subset of Ω. Regarding (3.17), this proves the claim. □

Remark 3.5. i) Notice that (3.15) resembles an analogous asymptotic bound for one-dimensional Schrödinger operators $A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ in $L^2(\mathbb{R})$ with $V \in L^1(\mathbb{R})$, see (3.19) below. Indeed, in this case, it is well known that the corresponding Birman-Schwinger operator has the kernel

$$\mathcal{P}_\lambda(x, y) = V_1(x) \frac{e^{-k|x-y|}}{2k} V_2(y), \tag{3.18}$$

where $\lambda = -k^2$, $\operatorname{Re} k > 0$, $V_2 = |V|^{\frac{1}{2}}$ and V_1 is defined by $V_1 V_2 = V$, see [1] for details. Estimating the Hilbert-Schmidt norm of the integral operator $\mathcal{P}(\lambda)$ with kernel $\mathcal{P}_\lambda$ in (3.18) yields

$$\|\mathcal{P}(\lambda)\| \leq \|V_1\|_{L^2} \|V_2\|_{L^2} \frac{1}{2|\lambda|^{\frac{1}{2}}} = \frac{\|V\|_{L^1}}{2|\lambda|^{\frac{1}{2}}}. \tag{3.19}$$

The estimate (3.19) immediately gives the enclosure

$$\sigma_{\mathrm{p}}(A) \setminus [0, \infty) \subset \left\{ \lambda \in \mathbb{C} \setminus [0, \infty) : |\lambda| \leq \frac{1}{4} \|V\|_{L^1}^2 \right\} \tag{3.20}$$

for the point spectrum of A, which is known to be optimal (in a suitable sense).

ii) The spectral enclosure (3.14) can be made more explicit to resemble (3.20) by estimating $r(\mathcal{Q}(\lambda))$ by $\|\widetilde{\mathcal{Q}}(\lambda)\|$ and employing (3.10). For λ outside of B_α, one can further estimate the norms $\|\mathcal{G}_{ij}\|_{L^\infty}$, $i, j = 1, 2$, using (2.20) and $|e^{-\mathrm{i}\mu_j |x|}| \leq 1$.

Such an estimate results in a formula analogous to (3.19), exhibiting an explicit dependence on the L^2-norms of W_j, $j = 1, \dots, 4$, and the zeros $\{\mu_j\}$ of p_λ. However, Cardano's formula for the latter provide only a limited insight. More importantly, in such an analogue of (3.19), artificial singularities for $\lambda \in B_\alpha$ are created, implying that a neighborhood of B_α would automatically be included in an eigenvalue enclosure. This drawback is illustrated in Figure 1, where the parameters are selected such that two points of B_α are not included in the enclosure (1.7).

iii) Our illustration of (3.14) in a particular case in Figure 1 avoids the steps described in ii). Instead, using the decomposition (3.1) with $W_1(x) = e^{-x/2}$, we compute directly the Hilbert-Schmidt norms of $\mathcal{Q}(\lambda)_{ij}$, $i, j = 1, 2$, for λ on a square grid (with edges of length 0.03) in the box $[0, 3] \times \mathrm{i}[-2.7, 2.7] \subset \mathbb{C}$ calling "NIntegrate" in Mathematica. In this computation, we employ the formulas for the explicit integral kernels $\{\mathcal{G}_{ij}\}$ in (2.42), (2.20), (2.34). The spectral radius of $\mathcal{Q}(\lambda)$ is then estimated using Gelfand's formula and inequality $\|\mathcal{Q}^k\| \leq \|\mathcal{Q}_{\mathrm{HS}}^k\|$, $k \in \mathbb{N}$, with $\mathcal{Q}_{\mathrm{HS}} \in \mathbb{R}^{2\times 2}$ being a matrix with elements $\|\mathcal{Q}(\lambda)_{ij}\|_{\mathrm{HS}}$, $i, j = 1, 2$; for our calculation the upper bound with $k = 20$ is chosen. The result is interpolated by "ListInterpolation" on the box and "RegionPlot" is called to produce the blue set in Figure 1. The numerical integration seems to be less stable near the essential spectrum, resulting in numerical artifacts visible in Figure 1.

iv) Numerical computation of the enclosure (3.14) complies with numerically found eigenvalues in [19, Table 1] for the special case (1.2) with various values of ϵ, $\alpha = 0.5$ and corresponding critical R, see [19, Sec. 10.1] for more details.

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