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A Split-Combination Approach to Merging Knowledge Bases in Possibilistic Logic*

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Abstract

In this paper, we propose an adaptive approach to merging possibilistic knowledge bases that deploys multiple operators instead of a single operator in the merging process. The merging approach consists of two steps: one is called the splitting step and the other is called the combination step. The splitting step splits each knowledge base into two subbases and then in the second step, different classes of subbases are combined using different operators. Our approach is applied to knowledge bases which are self-consistent and the result of merging is also a consistent knowledge base. Two operators are proposed based on two different splitting methods. Both operators result in a possibilistic knowledge base which contains more information than that obtained by the *t-conorm* (such as the maximum) based merging methods. In the flat case, one of the operators provides a good alternative to syntax-based merging operators in classical logic.

Keywords: Knowledge representation, merging of knowledge bases, inconsistency handling, possibilistic logic.

1 Introduction

In many cases, we confront the problem of merging inconsistent information from different sources [9, 10, 12, 13, 21, 23, 24, 29, 36]. When merging different data sources, we often need to consider uncertainty. Possibilistic logic [15] provides a good framework to deal with fusion problems when information is pervaded with inconsistency and uncertainty where only partial or incomplete information is available [3, 5, 6, 7, 9, 10]. There are two different views for merging possibilistic knowledge bases. The first view considers inconsistency as unacceptable and the conflicting information between different sources should be resolved after merging [3, 7, 8]. In contrast, the second view claims that inconsistency is unavoidable and the resulting possibilistic knowledge base can be inconsistent after merging [9, 10, 35]. In this paper, we focus on the first view. A common deficiency of existing approaches which follow the first view is that they are usually defined by a

*This paper is a revised and extended version of [34]

single operator, so it is not possible to differentiate different classes of information, such as *free* and *conflict* information, during the process of merging.

In [3, 7, 8, 9], some merging operators were proposed. Among them, two merging operators, the *maximum* (or more generally, *t-conorm*) based merging operator and the *minimum* (or more generally, *t-norm*) based merging operator, are used to combine inconsistent and consistent sources of information respectively. Given two *possibilistic knowledge bases* $\mathcal{B}_1 = \{(\phi_i, \alpha_i), i = 1, \dots, n\}$ and $\mathcal{B}_2 = \{(\psi_j, \beta_j), j = 1, \dots, m\}$, where ϕ_i and ψ_j are classical propositional formulas, and α_i and β_j belonging to $[0,1]$ are necessity degrees of ϕ_i and ψ_j respectively, the syntactic results of merging \mathcal{B}_1 and \mathcal{B}_2 by the maximum based merging operator and the minimum based merging operator are $\mathcal{B}_{dm} = \{(\phi_i \vee \psi_j, \min(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{B}_1, (\psi_j, \beta_j) \in \mathcal{B}_2\}$ and $\mathcal{B}_{cm} = \mathcal{B}_1 \cup \mathcal{B}_2$ respectively. \mathcal{B}_{dm} is always consistent provided that \mathcal{B}_1 or \mathcal{B}_2 is consistent, whilst \mathcal{B}_{cm} is consistent only if the union of \mathcal{B}_1 and \mathcal{B}_2 is consistent. So the maximum based merging operator is more advisable than the minimum based merging operator for dealing with inconsistency. However, when the union of \mathcal{B}_1 and \mathcal{B}_2 is consistent, the minimum based merging operator results in a more *specific* possibilistic knowledge base. That is, the possibility distribution of the combination of \mathcal{B}_1 and \mathcal{B}_2 by the minimum based merging operator is more *specific* than that of the maximum based merging operator. Therefore, the maximum (or more general, t-conorm) combination mode is too *cautious* to be used for merging possibilistic knowledge bases that are consistent with each other.

In this paper, we propose two Split-Combination (S-C for short) operators that follow the first view on possibilistic merging. We divide the fusion process into two steps: the splitting step and the combination step. The splitting step splits each knowledge base into two subbases and then in the second step, different classes of subbases are combined using different operators.

We first introduce an Incremental Split-Combination (*I-S-C* for short) merging operator. Given two possibilistic knowledge bases \mathcal{B}_1 and \mathcal{B}_2 (where $\mathcal{B}_1 \cup \mathcal{B}_2$ is inconsistent but each of them is individually consistent), we first split each of them into two subbases such that $\mathcal{B}_1 = \mathcal{C}_1 \cup \mathcal{D}_1$ and $\mathcal{B}_2 = \mathcal{C}_2 \cup \mathcal{D}_2$ with regard to a value obtained an incremental algorithm. In the second step, we combine \mathcal{C}_1 and \mathcal{C}_2 using a t-conorm based merging operator, while combining \mathcal{D}_1 and \mathcal{D}_2 using a t-norm based merging operator. Finally, the union of the possibilistic bases obtained by the second step is taken as the result of the combination of \mathcal{B}_1 and \mathcal{B}_2 . We prove that the new merging operator reduces to the t-norm based merging operator when no conflict exists and that its resulting possibilistic knowledge base contains more important information than that of the t-conorm based merging operator. Furthermore, we adapt the set of postulates for merging propositional knowledge bases in [23] to possibilistic logic and discuss the logical properties of our merging operator.

The *I-S-C* merging operator is problematic to be applied to merge flat (or classical) knowledge bases, i.e., knowledge bases without any priorities between their elements, because the weight of a formula used to split the possibilistic bases is related to priority. Therefore we propose an alternative approach to split the knowledge bases which do not involve priority. The corresponding split-combination operator, called Free-formula based Split-Combination (*F-S-C* for short) merging operator, can then be applied to merging classical knowledge bases. We compare our *F-S-C* with propositional merging operators in the flat case and conclude that it is a good alternative to syntax-based merging operators in classical logic.

We compare our merging operators with existing possibilistic merging operators by considering the following criteria.

The first one is rationality. We generalize the set of rationality postulates in the propositional setting in [23] and discuss the logical properties of different possibilistic merging operators. The generalized postulates are not used to give a normative definition or characterization of possibilistic merging. The reason why we propose them is that we think they are helpful for users to choose among different possibilistic merging operators.

The second one is the inference power of the resulting possibilistic knowledge base of merging. Given two merging operators, we prefer the one leading to a merged base which can non-trivially infer more information.

The third one is the compatibility with merging operators in the classical setting. That is, we prefer possibilistic merging operators which are well-behaved in classical logic to those which are not.

The last one is computational complexity. This criterion has been adopted to evaluate a solution in many AI problems, such as belief revision and nonmonotonic reasoning. To implement a merging operator, computational efficiency is an important requirement. It is clear that computationally more efficient operators are preferred to more complex ones.

This paper is organized as follows. Section 2 provides some preliminary definitions in possibilistic logic. We then give a brief survey on existing merging methods in possibilistic logic in Section 3. In Section 4, we propose an Incremental Split Combination (*I-S-C* for short) merging operator. In Section 5, we discuss the semantic aspect of the *I-S-C* operator. Another split-combination operator, called the free-formula based split-combination method, is proposed in Section 6. Section 7 discusses related work. Finally, we conclude this paper in Section 8.

2 Background on Possibilistic Logic

In this section, we give a brief overview of possibilistic logic. More details on possibilistic logic can be found in [15, 9].

Throughout the paper, \mathcal{L} is a propositional language formed in the usual way from a finite set of propositional symbols \mathcal{P} . An interpretation is a truth assignment to the atoms in \mathcal{P} , i.e. a mapping from \mathcal{P} to $\{true, false\}$. We denote the set of classical interpretations by Ω , and the classical consequence relation by \models . Propositional symbols are denoted by p, q, r, \dots , and propositional formulas are denoted by Greek letters ϕ, ψ, χ, \dots . The satisfiability relation $\omega \models \phi$ is defined as usual between an interpretation ω and a formula ϕ . A *knowledge base* K is a finite set of propositional formulas and can be represented as the formula ϕ , which is the conjunction of the formulas in K . A *knowledge profile* is then defined as a multi-set E consisting of a finite number of knowledge bases, that is, $E = \{K_1, \dots, K_n\}$, where K_i may be the same as K_j for $i \neq j$. Two knowledge profiles E_1 and E_2 are equivalent, denoted $E_1 \equiv E_2$, iff there exists a bijection f between E_1 and E_2 such that for each $K \in E_1$, $f(K) \equiv K$. The union of multi-sets is denoted as \sqcup . The union of knowledge bases in E is defined as $\cup E = \cup_{i=1}^n K_i$.

2.1 Semantics of possibilistic logic

The semantics of possibilistic logic is based on the notion of a *possibility distribution* π which is a mapping from Ω to interval $[0,1]$. The unit interval is not necessary and can be replaced by any totally ordered scale. $\pi(\omega)$ represents the degree of compatibility of the interpretation ω with the available beliefs about the real world. $\pi(\omega) = 0$ means that it is impossible to be the real world, and $\pi(\omega) = 1$ means that nothing prevents ω from being the real world, while $0 < \pi(\omega) < 1$ means that it is only somewhat possible for ω to be the real world. When $\pi(\omega) > \pi(\omega')$, ω is preferred to ω' for being the real world. A possibility distribution is said to be normal if $\exists \omega \in \Omega$, such that $\pi(\omega) = 1$. Given two possibility distributions π and π' , π is said to be less specific (or less informative) than π' if $\forall \omega, \pi(\omega) \geq \pi'(\omega)$, and π is said to be strict less specific (or less informative) than π' if $\forall \omega, \pi(\omega) \geq \pi'(\omega)$ and $\exists \omega, \pi(\omega) > \pi'(\omega)$.

From a *possibility distribution* π , two measures defined on a set of propositional formulas can be determined. One is the possibility degree of formula ϕ , denoted as $\Pi_\pi(\phi) = \max\{\pi(\omega) : \omega \models \phi\}$.

The other is the necessity degree of formula ϕ , and is defined as $N_\pi(\phi) = 1 - \Pi_\pi(\neg\phi)$. The possibility degree of ϕ evaluates to what extent ϕ is consistent with knowledge expressed by π and the necessity degree of ϕ evaluates to what extent ϕ is entailed by the available knowledge. $N_\pi(\phi) = 1$ means that ϕ is a totally certain piece of knowledge, while $N_\pi(\phi) = 0$ expresses the complete lack of knowledge of priority about ϕ , but does not mean that ϕ is or should be false. We have $N_\pi(\text{true}) = 1$ and $N_\pi(\phi \wedge \psi) = \min(N_\pi(\phi), N_\pi(\psi))$ for all ϕ and ψ .

2.2 Possibilistic knowledge bases

At the syntactic level, a formula, called a *possibilistic formula*, is represented by a pair (ϕ, α) where ϕ is a propositional formula and $\alpha \in [0, 1]$, which means that the necessity degree of ϕ is at least equal to α , i.e. $N(\phi) \geq \alpha$. Then uncertain pieces of information can be represented by a *possibilistic knowledge base* which is a finite set of *possibilistic formulas* of the form $\mathcal{B} = \{(\phi_i, \alpha_i) : i = 1, \dots, n\}$. A *possibilistic knowledge profile* \mathcal{E} is a multi-set of possibilistic knowledge bases. In this paper, we only consider possibilistic knowledge bases where every formula ϕ is a classical propositional formula. The weights attached to formulas are denoted by $\alpha, \beta, \gamma, \dots$. The classical base associated with \mathcal{B} is denoted as \mathcal{B}^* , namely $\mathcal{B}^* = \{\phi_i | (\phi_i, \alpha_i) \in \mathcal{B}\}$. A *possibilistic base* \mathcal{B} is consistent if and only if its classical base \mathcal{B}^* is consistent.

Definition 1 Let \mathcal{B} be a possibilistic knowledge base, and $\alpha \in [0, 1]$. The α -cut (resp. strict α -cut) of \mathcal{B} is $\mathcal{B}_{\geq \alpha} = \{\phi \in \mathcal{B}^* | (\phi, \beta) \in \mathcal{B} \text{ and } \beta \geq \alpha\}$ (resp. $\mathcal{B}_{> \alpha} = \{\phi \in \mathcal{B}^* | (\phi, \beta) \in \mathcal{B} \text{ and } \beta > \alpha\}$).

Given a possibilistic base \mathcal{B} , we can associate with it a semantics *w.r.t* possibility distributions.

Definition 2 Let \mathcal{B} be a possibilistic knowledge base and $(\phi, \alpha) \in \mathcal{B}$. A possibility distribution π is said to be compatible with (ϕ, α) if $N_\pi(\phi) \geq \alpha$ and it is compatible with \mathcal{B} if for each $(\phi_i, \alpha_i) \in \mathcal{B}$, we have $N_\pi(\phi_i) \geq \alpha_i$

Generally, there are several possibility distributions compatible with a possibilistic knowledge base \mathcal{B} . However, a unique possibility distribution, denoted by $\pi_{\mathcal{B}}$ can be obtained by the principle of minimum specificity. That is, among the possibility distributions compatible with \mathcal{B} , we choose one which is the least specific, i.e. there is no possibility distribution π' such that π' is more specific than π . This possibility distribution can be computed as follows [15]. For all $\omega \in \Omega$,

$$\pi_{\mathcal{B}}(\omega) = \begin{cases} 1 & \text{if } \forall (\phi_i, \alpha_i) \in \mathcal{B}, \omega \models \phi_i, \\ 1 - \max\{\alpha_i | \omega \not\models \phi_i\} & \text{otherwise.} \end{cases} \quad (1)$$

It is clear that a possibilistic knowledge base \mathcal{B} is consistent iff its associated possibility $\pi_{\mathcal{B}}$ is normal.

Let us look at an example.

Example 1 Let $\mathcal{B} = \{(p, 0.9), (q, 0.6), (\neg q \vee r, 0.5), (\neg r, 0.5), (r, 0.3)\}$ be a possibilistic knowledge base. By Equation 1, the least specific possibility distribution associated with \mathcal{B} is defined by $\pi_{\mathcal{B}}(pqr) = 0.5$, $\pi_{\mathcal{B}}(pq\neg r) = 0.5$, $\pi_{\mathcal{B}}(p\neg q r) = 0.4$, $\pi_{\mathcal{B}}(p\neg q\neg r) = 0.4$, $\pi_{\mathcal{B}}(\neg pqr) = 0.1$, $\pi_{\mathcal{B}}(\neg pq\neg r) = 0.1$, $\pi_{\mathcal{B}}(\neg p\neg q r) = 0.1$, $\pi_{\mathcal{B}}(\neg p\neg q\neg r) = 0.1$.

Two possibilistic knowledge bases \mathcal{B}_1 and \mathcal{B}_2 are said to be equivalent, denoted by $\mathcal{B}_1 \equiv_s \mathcal{B}_2$ iff $\pi_{\mathcal{B}_1} = \pi_{\mathcal{B}_2}$, that is, their associated possibility distributions are the same. The equivalence of two possibilistic knowledge bases can also be defined as $\mathcal{B}_1 \equiv_s \mathcal{B}_2$ iff $\forall \alpha \in [0, 1]$, $(\mathcal{B}_1)_{\geq \alpha} \equiv (\mathcal{B}_2)_{\geq \alpha}$ [10]. Moreover, two possibilistic knowledge profiles \mathcal{E}_1 and \mathcal{E}_2 are equivalent, denoted $\mathcal{E}_1 \equiv_s \mathcal{E}_2$ iff there exists a bijection f between \mathcal{E}_1 and \mathcal{E}_2 such that for each $\mathcal{B} \in \mathcal{E}_1$, $f(\mathcal{B}) \equiv_s \mathcal{B}$.

2.3 Possibilistic inference

Given a possibilistic knowledge base \mathcal{B} , we can define its level of inconsistency as follows.

Definition 3 *Let \mathcal{B} be a possibilistic knowledge base. The inconsistency degree of \mathcal{B} is:*

$$Inc(\mathcal{B}) = \max\{\alpha_i : \mathcal{B}_{\geq\alpha_i} \text{ is inconsistent}\}.$$

That is, the inconsistency degree of \mathcal{B} is the largest weight α_i such that the α_i -cut of \mathcal{B} is inconsistent. It can be equivalently defined by the possibility distribution associated with \mathcal{B} as $Inc(\mathcal{B}) = 1 - \max_{\omega} \pi_{\mathcal{B}}(\omega)$. When \mathcal{B} is consistent, we have $Inc(\mathcal{B}) = 0$. So if the possibility distribution $\pi_{\mathcal{B}}$ is subnormal, i.e. $\forall \omega, \pi_{\mathcal{B}}(\omega) < 1$, \mathcal{B} is not consistent.

The possibilistic consequence relation is defined as follows.

Definition 4 [10] *Let \mathcal{B} be a possibilistic base. A formula ϕ is said to be a consequence of \mathcal{B} , denoted by $\mathcal{B} \vdash_{\pi} (\phi, \alpha)$, iff*

- (i) $\mathcal{B}_{\geq\alpha}$ is consistent,
- (ii) $\mathcal{B}_{\geq\alpha} \models \phi$,
- (iii) $\forall \beta > \alpha, \mathcal{B}_{\geq\beta} \not\models \phi$.

It is required that weights of possibilistic formulas which are consequences of \mathcal{B} be greater than the inconsistency degree of \mathcal{B} . This is because for any possibilistic formula (ϕ, α) , if $\alpha \leq Inc(\mathcal{B})$, then $\mathcal{B}_{\geq\alpha} \vdash \phi$. That is, (ϕ, α) can be inferred from \mathcal{B} trivially.

Subsumption can be defined as follows:

Definition 5 *Let (ϕ, α) be a possibilistic formula in \mathcal{B} . (ϕ, α) is said to be subsumed by \mathcal{B} if $(\mathcal{B} \setminus \{(\phi, \alpha)\})_{\geq\alpha} \models \phi$.*

Subsumed formulas can be viewed as redundant by the following proposition.

Proposition 1 [9] *Let (ϕ, α) be a subsumed formula in \mathcal{B} . Then \mathcal{B} and $\mathcal{B} \setminus \{(\phi, \alpha)\}$ are equivalent.*

By Proposition 1, possibilistic formulas with a null degree, i.e. having the form $(\phi_i, 0)$, are subsumed in any possibilistic knowledge base. So they are not included in the knowledge base explicitly.

It has been shown in [28] that the possibilistic inference has a computational complexity similar to that of classical logic, that is, it needs $\lceil \log_2 n \rceil$ satisfiability checks, where n is the number of certainty levels used in \mathcal{B} .

Although possibilistic inference is inconsistency tolerant, it suffers from the ‘‘drowning problem’’ [2]. That is, given an inconsistent possibilistic knowledge base \mathcal{B} , formulas whose certainty degrees are not larger than $Inc(\mathcal{B})$ are completely useless for nontrivial deductions. For instance, let $\mathcal{B} = \{(p, 0.9), (\neg p, 0.8), (r, 0.6), (q, 0.7)\}$, it is clear that \mathcal{B} is equivalent to $\mathcal{B} = \{(p, 0.9), (\neg p, 0.8)\}$ because $Inc(\mathcal{B}) = 0.8$. So $(q, 0.7)$ and $(r, 0.6)$ are not used in the possibilistic inference. Since possibilistic inference has the drowning problem, inconsistency is not desirable and should be avoided if possible.

3 Merging Approaches in Possibilistic Operator

Many approaches have been proposed to merge prioritized knowledge bases in possibilistic logic [3, 7, 8, 9, 10]. There are two different views for merging possibilistic knowledge bases. The first view considers inconsistency as unacceptable and conflicting information between different sources should be resolved after merging [3, 7, 8]. In contrast, the second view claims that inconsistency is unavoidable and the resulting possibilistic knowledge base can be inconsistent after merging [9, 10].

Let \mathcal{B}_1 and \mathcal{B}_2 be two possibilistic bases, π_1 and π_2 be their associated possibility distributions. Semantically, a two place function \oplus from $[0,1] \times [0,1]$ to $[0,1]$, is applied to aggregate the two possibility distributions π_1 and π_2 into a new one π_{\oplus} , i.e. $\pi_{\oplus}(\omega) = \pi_1(\omega) \oplus \pi_2(\omega)$. Generally, the operator \oplus is very weakly constrained, i.e. the only requirements for it are the following properties [8, 9]:

1. $1 \oplus 1 = 1$, and
2. if $a \geq c, b \geq d$ then $a \oplus b \geq c \oplus d$, where $a, b, c, d \in [0, 1]$ (monotonicity).

The first property states that if two sources agree that an interpretation ω is fully possible, then the result of merging should confirm it. The second property is the monotonicity condition, that is, a degree resulting from a combination cannot decrease if the degrees to be combined increase.

In the case of n sources $\mathcal{B}_1, \dots, \mathcal{B}_n$, the semantic combination of their possibility distributions π_1, \dots, π_n can be performed easily when \oplus is associative. That is, we have $\pi_{\oplus}(\omega) = (\dots((\pi_1(\omega) \oplus \pi_2(\omega)) \oplus \pi_3(\omega)) \oplus \dots) \oplus \pi_n(\omega)$. When the operator is not associative, it need to be generalized as a unary operator defined on vector (π_1, \dots, π_n) of possibility distributions such that:

1. $\oplus(1, \dots) = 1$, and
2. if $\forall i = 1, n, a_i \geq b_i$ then $\oplus(a_1, \dots, a_n) \geq \oplus(b_1, \dots, b_n)$, where $a_i, b_i \in [0, 1]$.

Two basic operators are the maximum and the minimum. The merging methods based on maximum and minimum operator have no reinforcement effect. That is, given an interpretation ω , if expert 1 assigns possibility $\pi_1(\omega) < 1$ and expert 2 assigns possibility $\pi_2(\omega) < 1$ to ω , then overall $\pi_{dm}(\omega) = \pi_2(\omega)$ (or $\pi_{cm}(\omega) = \pi_1(\omega)$) if $\pi_1(\omega) < \pi_2(\omega)$, regardless of the value of $\pi_1(\omega)$ (or $\pi_2(\omega)$). To obtain a reinforcement effect, we can use a triangular norm operator other than the minimum for conjunctive combination, and a triangular conorm operator other than the maximum for disjunctive combination.

Definition 6 [19] *A triangular norm (t-norm) tn is a two place real-valued function $tn : [0, 1] \times [0, 1] \rightarrow [0, 1]$ which satisfies the following conditions:*

1. $tn(0,0)=0$, and $tn(a,1)=tn(1,a)=a$, for every a (boundary condition);
2. $tn(a,b) \leq tn(c,d)$ whenever $a \leq c$ and $b \leq d$ (monotonicity);
3. $tn(a,b)=tn(b,a)$ (symmetry);
4. $tn(a,tn(b,c))=tn(tn(a,b),c)$ (associativity).

A triangular conorm (t-conorm) ct is a two place real-valued function whose domain is the unit square $[0,1] \times [0,1]$ and which satisfies the conditions 2-4 given in Definition 6 plus the following revised boundary conditions:

- 1'. $ct(1,1)=1, ct(a,0)=ct(0,a)=a$.

Any t-conorm ct can be generated from a t-norm through the duality transformation:

$$ct(a, b) = 1 - tn(1 - a, 1 - b)$$

and conversely.

It is easy to check that the maximum operator is a t-conorm and the minimum operator is a t-norm. Other frequently used t-norms are the product operator ab and the *Lukasiewicz t-norm* ($\max(0, a+b-1)$). The duality relation yields the following t-conorms respectively: the *probabilistic sum* ($a + b - ab$), and the *bounded sum* ($\min(1, a + b)$).

Given two possibilistic bases \mathcal{B}_1 and \mathcal{B}_2 with possibility distributions π_1 and π_2 respectively, the semantic results of their combination by a t-norm tn and a t-conorm ct are

$$\forall \omega, \pi_{tn}(\omega) = tn(\pi_1(\omega), \pi_2(\omega)), \quad (2)$$

$$\forall \omega, \pi_{ct}(\omega) = ct(\pi_1(\omega), \pi_2(\omega)). \quad (3)$$

When the original possibility distributions are normal, the merging methods based on t-conorms preserve normalization. However, the merging methods based on t-norms may result in subnormal results, i.e. $\forall \omega, \pi_{tn}(\omega) < 1$ (or equivalently, the possibilistic knowledge base associated to π_{tn} is inconsistent). In that case, we may think of renormalizing π_{tn} (that is, if we follow the first views on possibilistic merging). Let π be a possibility distribution which is subnormal, π_N be the possibility distribution renormalized from π . Then π_N should satisfy the following conditions:

1. $\exists \omega, \pi_N(\omega) = 1$,
2. if π is normal then $\pi_N = \pi$,
3. $\forall \omega, \omega', \pi(\omega) < \pi(\omega')$ if and only if $\pi_N(\omega) < \pi_N(\omega')$.

For example, let $h(\pi_{tn}) = \max_{\omega} \{\pi_{tn}(\omega)\}$, the following equation provides a normalization rule.

$$\pi_{N,tn}(\omega) = \begin{cases} 1 & \text{if } \pi_{tn}(\omega) = h(\pi_{tn}), \\ \pi_{tn}(\omega) & \text{otherwise.} \end{cases} \quad (4)$$

The normalization rule defined by Equation 4 resolves inconsistency because the inconsistency degree of any possibilistic knowledge base associated to $\pi_{N,tn}$ is zero. Other normalization rules can be found in [3].

The syntactical counterpart of the fusion of π_1 and π_2 is to obtain a possibilistic knowledge base whose possibility distribution is π_{\oplus} . In [8], it has been shown that this knowledge base have the following form:

$$\begin{aligned} \mathcal{B}_{\oplus} = & \{(\phi_i, 1 - (1 - a_i) \oplus 1) : (\phi_i, a_i) \in \mathcal{B}_1\} \cup \{(\psi_j, 1 - 1 \oplus (1 - b_j)) : (\psi_j, b_j) \\ & \in \mathcal{B}_2\} \cup \{(\phi_i \vee \psi_j, 1 - (1 - a_i) \oplus (1 - b_j)) : (\phi_i, a_i) \in \mathcal{B}_1 \text{ and } (\psi_j, b_j) \in \mathcal{B}_2\}. \end{aligned} \quad (5)$$

That is, we have $\pi_{\mathcal{B}_{\oplus}}(\omega) = \pi_{\oplus}(\omega) = \pi_1(\omega) \oplus \pi_2(\omega)$, where $\pi_{\mathcal{B}_{\oplus}}$ is the possibility distribution associated with \mathcal{B}_{\oplus} . It is clear that when \oplus is associative, the syntactic computation of the resulting base is easily generalized to n sources. The syntactic generalization for a non-associative operator can be carried out as follows.

Proposition 2 [9] Let $\mathcal{E} = \{\mathcal{B}_1, \dots, \mathcal{B}_n\}$ be a set of n possibilistic knowledge bases and (π_1, \dots, π_n) be their associated possibility distributions. Let $\pi_{\mathcal{B}_{\oplus}}$ be the result of combining (π_1, \dots, π_n) with \oplus . The possibilistic knowledge base associated to $\pi_{\mathcal{B}_{\oplus}}$ is:

$$\mathcal{B}_{\oplus} = \{(D_j, 1 - \oplus(x_1, \dots, x_n)) : j = 1, \dots, n\}, \quad (6)$$

where D_j are disjunctions of size j between formulas taken from different \mathcal{B}_i 's ($i = 1, \dots, n$) and x_i is either equal to $1 - \alpha_i$ if ϕ_i belongs to D_j and 1 if it does not.

By Equation 6, the possibilistic knowledge bases which are the syntactic counterparts of the semantic based merging using the maximum and the minimum are

$$\mathcal{B}_{dm} = \{(\phi_i \vee \psi_j, \min(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{B}_1, \text{ and } (\psi_j, \beta_j) \in \mathcal{B}_2\}, \quad (7)$$

$$\mathcal{B}_{cm} = \mathcal{B}_1 \cup \mathcal{B}_2, \quad (8)$$

respectively. \mathcal{B}_{dm} and \mathcal{B}_{cm} are referred to as the results of *disjunctive* and *conjunctive* combination respectively. More generally, the syntactic results associated with π_{tn} and π_{ct} are the following knowledge bases respectively [3]:

$$\mathcal{B}_{tn} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \{(\phi_i \vee \psi_j, ct(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{B}_1 \text{ and } (\psi_j, \beta_j) \in \mathcal{B}_2\}, \quad (9)$$

$$\mathcal{B}_{ct} = \{(\phi_i \vee \psi_j, tn(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{B}_1 \text{ and } (\psi_j, \beta_j) \in \mathcal{B}_2\}. \quad (10)$$

By Equation 9, the possibilistic knowledge base \mathcal{B}_{tn} may be inconsistent. Let $\pi_{N,tn}$ be the possibility distribution obtained by Equation 4, then the possibilistic knowledge base associated with it has the following form:

$$\mathcal{B}_{N,tn} = \{(\phi_i, \alpha_i) : (\phi_i, \alpha_i) \in \mathcal{B}_{tn} \text{ and } \alpha_i > Inc(\mathcal{B}_{tn})\}. \quad (11)$$

$\mathcal{B}_{N,tn}$ restores consistency of \mathcal{B}_{tn} by dropping formulas whose weights are less than or equal to the inconsistency degree of \mathcal{B}_{tn} . We call the merging operator obtained by Equation 11 a renormalization based merging operator. It is clear that $\mathcal{B}_{N,tn}$ may drop too much information from \mathcal{B}_{tn} if $Inc(\mathcal{B}_{tn})$ is large, for example, 0.8.

Example 2 Let $\mathcal{B}_1 = \{(p, 0.9), (q, 0.7)\}$ and $\mathcal{B}_2 = \{(\neg p, 0.8), (r, 0.6), (p \vee q, 0.5)\}$. Suppose the operator is the maximum, then by Equation 7, we have $\mathcal{B}_{dm} = \{(p \vee r, 0.6), (p \vee q, 0.5), (\neg p \vee q, 0.7), (q \vee r, 0.6)\}$. It is clear that the the maximum based merging method is very cautious, that is, all the formulas are weakened as disjunctions. By contrast, if we choose the minimum, then by Equation 8, we have $\mathcal{B}_{cm} = \{(p, 0.9), (\neg p, 0.8), (q, 0.7), (r, 0.6), (p \vee q, 0.5)\}$. \mathcal{B}_{cm} is inconsistent. Suppose we apply the normalization rule (Equation 4) to the possibility distribution associated to \mathcal{B}_{cm} , then by Equation 11 the possibilistic knowledge base associated with the normalized possibility distribution is $\mathcal{B}_{N,cm} = \{(p, 0.9)\}$. $(q, 0.7)$ and $(r, 0.6)$ are not involved in conflict between \mathcal{B}_1 and \mathcal{B}_2 . However, they are deleted after merging.

Example 2 illustrates that, the merging methods based on t-conorms are too cautious when most of the formulas are not involved in conflict while the renormalization based merging method may result in a knowledge base which deletes too much important information.

4 Incremental Split-Combination Merging Approach

In this section, we introduce an Incremental Split-Combination (*I-S-C*) approach to merging possibilistic knowledge bases. We follow the first view on possibilistic merging, that is, the resulting knowledge base should be consistent. We further assume that the original possibilistic knowledge bases are individually consistent. According to the analysis in Section 3, the t-conorm (for example, the maximum) based merging methods can be used to weaken conflicting information, while the t-norm (for example, the minimum) based merging methods are more advisable to be used to exploit the symbolic complementarities between sources, i.e. all the symbolic information is recovered. In this section and Section 6, we propose two split-combination operators for merging individually consistent possibilistic knowledge bases by utilizing both t-norm and t-conorm based merging operators.

The general idea of the *S-C* approach can be described as follows. Given two possibilistic knowledge bases \mathcal{B}_1 and \mathcal{B}_2 , in the first step, we split them into $\mathcal{B}_1 = \langle \mathcal{C}_1, \mathcal{D}_1 \rangle$ and $\mathcal{B}_2 = \langle \mathcal{C}_2, \mathcal{D}_2 \rangle$ with regard to a splitting method such that \mathcal{C}_i ($i = 1, 2$) contain information which would be weakened and \mathcal{D}_i ($i = 1, 2$) contain formulas which are “safe” to be kept. In the second step, we combine \mathcal{C}_1 and \mathcal{C}_2 by a t-conorm operator (the result is a possibilistic knowledge base \mathcal{C}) and combine \mathcal{D}_1 and \mathcal{D}_2 by a t-norm operator (the result is a possibilistic knowledge base \mathcal{D}). The final result of the *S-C* combination method, denoted by \mathcal{B}_{S-C} , is $\mathcal{C} \cup \mathcal{D}$. Different S-C methods can be developed by incorporating different ways of splitting the knowledge bases, while retaining the general S-C approach.

4.1 Incremental *S-C* Operator

Our first splitting method is to split possibilistic knowledge bases using necessity degrees in the possibilistic knowledge bases. Let \mathcal{B}_1 and \mathcal{B}_2 be two possibilistic knowledge bases, and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{(\phi_i, \alpha_i) : i = 1, \dots, n\}$. Let $\alpha_{max} = \max(\alpha_i : i = 1, \dots, n)$. Suppose the weights of the formulas in \mathcal{B} are rearranged in such a way that $\alpha_1 = \alpha_{max} > \alpha_2 > \dots > \alpha_{n_1}$. Let $\alpha_{n_1+1} = 0$. Suppose \mathcal{B}_1 and \mathcal{B}_2 are split *w.r.t* some α_m ($1 \leq m \leq n_1 + 1$) into $\mathcal{B}_1 = \langle \mathcal{C}_1, \mathcal{D}_1 \rangle$ and $\mathcal{B}_2 = \langle \mathcal{C}_2, \mathcal{D}_2 \rangle$, where $\mathcal{C}_i = \{(\phi_i, \alpha_i) : (\phi_i, \alpha_i) \in \mathcal{B}_i, \alpha_i \leq \alpha_m\}$ and $\mathcal{D}_i = \mathcal{B}_i \setminus \mathcal{C}_i$, for $i = 1, 2$. Suppose \mathcal{D}_1 and \mathcal{D}_2 are combined¹ by a t-norm operator tn , and \mathcal{C}_1 and \mathcal{C}_2 are combined by a t-conorm ct . By Equation 9 and 10, we have $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \{(\phi_i \vee \psi_j, ct(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{D}_1 \text{ and } (\psi_j, \beta_j) \in \mathcal{D}_2\}$ and $\mathcal{C} = \{(\phi_i \vee \psi_j, tn(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{C}_1 \text{ and } (\psi_j, \beta_j) \in \mathcal{C}_2\}$. The final result of the S-C method is $\mathcal{B}_{S-C} = \mathcal{C} \cup \mathcal{D}$. The main problem is how to select the splitting point. Let $Inc(\mathcal{B}) = \alpha_k$. When $\alpha_m < \alpha_k$, by Definition 3, \mathcal{D} is inconsistent. So $\mathcal{B}_{S-C} = \mathcal{C} \cup \mathcal{D}$ is inconsistent. Therefore, we cannot use α_m such that $\alpha_m < \alpha_k$ as the splitting point. Suppose α_k is the splitting point, then both \mathcal{C} and \mathcal{D} are consistent. However, \mathcal{B}_{S-C} may still be inconsistent. Let us look at an example.

Example 3 Let $\mathcal{B}_1 = \{(p, 0.9), (-r, 0.7)\}$ and $\mathcal{B}_2 = \{(r, 0.8), (-p, 0.6)\}$. It is easy to check that $Inc(\mathcal{B}_1 \cup \mathcal{B}_2) = 0.7$. Suppose we select 0.7 as the splitting point, then \mathcal{B}_1 is split into $\mathcal{D}_1 = \{(p, 0.9)\}$ and $\mathcal{C}_1 = \{(-r, 0.7)\}$, and \mathcal{B}_2 is split into $\mathcal{D}_2 = \{(r, 0.8)\}$ and $\mathcal{C}_2 = \{(-p, 0.6)\}$. Suppose \mathcal{D}_1 and \mathcal{D}_2 are combined by the minimum and \mathcal{C}_1 and \mathcal{C}_2 are combined by the maximum. Then $\mathcal{D} = \{(p, 0.9), (r, 0.8)\}$ and $\mathcal{C} = \{(-p \vee -r, 0.6)\}$. So $\mathcal{B}_{S-C} = \mathcal{C} \cup \mathcal{D} = \{(p, 0.9), (r, 0.8), (-p \vee -r, 0.6)\}$. It is clear that \mathcal{B}_{S-C} is inconsistent.

There is no guarantee that the final merged knowledge base is consistent, when selecting an α_i such that $\alpha = \alpha_k$ as the splitting point. We need to find the point incrementally. To do so, we give

¹When we say that two possibilistic knowledge bases are combined by a t-norm (or t-conorm) operator, we mean they are syntactically combined using Equation 9 (or Equation 10).

the following algorithm to find the value step by step and use it to split both \mathcal{B}_1 and \mathcal{B}_2 .

Algorithm 1

Input: two possibilistic knowledge bases $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, \dots, n\}$ and $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, \dots, m\}$, a t-conorm ct and a t-norm tn .

Output: a splitting point γ .

Step 1 Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{(\varphi_i, \gamma_i) : i = 1, \dots, n+m\}$. Rearrange the weights of formulas in \mathcal{B} such that $\gamma_1 > \gamma_2 > \dots > \gamma_{n_1}$. Let $\gamma_{n_1+1} = 0$.

Step 2 Compute $Inc(\mathcal{B})$. Assume $Inc(\mathcal{B}) = \gamma_k$. Let $l = k$.

Step 3 Split \mathcal{B}_1 and \mathcal{B}_2 with regard to γ_l such that $\mathcal{B}_1 = \langle \mathcal{C}_1, \mathcal{D}_1 \rangle$ and $\mathcal{B}_2 = \langle \mathcal{C}_2, \mathcal{D}_2 \rangle$, where $\mathcal{C}_1 = \{(\phi_i, \alpha_i) : (\phi_i, \alpha_i) \in \mathcal{B}_1, \alpha_i \leq \gamma_l\}$ and $\mathcal{D}_1 = \mathcal{B}_1 \setminus \mathcal{C}_1$, and $\mathcal{C}_2 = \{(\psi_j, \beta_j) : (\psi_j, \beta_j) \in \mathcal{B}_2, \beta_j \leq \gamma_l\}$ and $\mathcal{D}_2 = \mathcal{B}_2 \setminus \mathcal{C}_2$.

Step 4 Combine \mathcal{C}_1 and \mathcal{C}_2 by ct and combine \mathcal{D}_1 and \mathcal{D}_2 by tn , as shown by Equation 10 and Equation 9, the results are respectively

$$\mathcal{C} = \{(\phi_i \vee \psi_j, tn(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{C}_1 \text{ and } (\psi_j, \beta_j) \in \mathcal{C}_2\}, \quad (12)$$

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \{(\phi_i \vee \psi_j, ct(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{D}_1 \text{ and } (\psi_j, \beta_j) \in \mathcal{D}_2\}. \quad (13)$$

Step 5 Let $\mathcal{B}_{S-C} = \mathcal{C} \cup \mathcal{D}$.

Step 6 If \mathcal{B}_{S-C} is inconsistent, let $l = l - 1$ and go to Step 3.

Step 7 Return γ_l .

In Algorithm 1, we first rearrange all the weights of formulas in the union of \mathcal{B}_1 and \mathcal{B}_2 in a decreasing order. We then search the weights incrementally until we find the minimal weight such that the resulting possibilistic knowledge base of the split-combination approach is consistent.

We now define the *I-S-C* merging operator based on Algorithm 1.

Definition 7 Let $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, \dots, n\}$ and $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, \dots, m\}$ be two possibilistic knowledge bases, ct be a t-conorm and tn be a t-norm. Let γ be the splitting point obtained by Algorithm 1. Suppose \mathcal{B}_i ($i = 1, 2$) are split into $\mathcal{B}_i = \langle \mathcal{C}_i, \mathcal{D}_i \rangle$ w.r.t γ , and \mathcal{C} and \mathcal{D} are obtained by Equation 12 and 13 respectively. The resulting possibilistic knowledge base of the *I-S-C* merging operator², denoted \mathcal{B}_{I-S-C} , is defined as $\mathcal{B}_{I-S-C} = \mathcal{C} \cup \mathcal{D}$.

Let us look at an example.

Example 4 Given two possibilistic bases $\mathcal{B}_1 = \{(\neg\phi \vee \varphi, 0.8), (\neg\phi \vee \psi, 0.6), (\phi, 0.5)\}$ and $\mathcal{B}_2 = \{(\phi \vee \delta, 0.9), (\neg\psi \vee \delta, 0.7), (\delta \vee \varphi, 0.5), (\neg\phi \vee \neg\psi, 0.4), (\psi, 0.3)\}$. Let $ct = \max$ and $tn = \min$. Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{(\phi \vee \delta, 0.9), (\neg\phi \vee \varphi, 0.8), (\neg\psi \vee \delta, 0.7), (\neg\phi \vee \psi, 0.6), (\phi, 0.5), (\delta \vee \varphi, 0.5), (\neg\phi \vee \neg\psi, 0.4), (\psi, 0.3)\}$. The weights of formulas in \mathcal{B} are rearranged as $\gamma_1 = 0.9 > \gamma_2 = 0.8 > \gamma_3 = 0.7 > \gamma_4 = 0.6 > \gamma_5 = 0.5 > \gamma_6 = 0.4 > \gamma_7 = 0.3$. Let $\gamma_8 = 0$. The inconsistency degree of $\mathcal{B}_1 \cup \mathcal{B}_2$ is $Inc(\mathcal{B}_1 \cup \mathcal{B}_2) = 0.4$. Let $l = 6$. \mathcal{B}_1 and \mathcal{B}_2 are split with regard to $\gamma_6 = 0.4$ into

$$\mathcal{C}_1 = \emptyset, \quad \mathcal{D}_1 = \{(\neg\phi \vee \varphi, 0.8), (\neg\phi \vee \psi, 0.6), (\phi, 0.5)\}$$

$$\mathcal{C}_2 = \{(\neg\phi \vee \neg\psi, 0.4), (\psi, 0.3)\}, \quad \mathcal{D}_2 = \{(\phi \vee \delta, 0.9), (\neg\psi \vee \delta, 0.7), (\delta \vee \varphi, 0.5)\}.$$

²The merging operator is based on a t-conorm ct and a t-norm tn . For notional simplicity, we omit ct and tn and simply call *I-S-C* operator in the following.

Combining \mathcal{C}_1 and \mathcal{C}_2 using the maximum and combining \mathcal{D}_1 and \mathcal{D}_2 using the minimum respectively give

$$\mathcal{C} = \emptyset, \mathcal{D} = \{(\phi \vee \delta, 0.9), (\neg\phi \vee \varphi, 0.8), (\neg\psi \vee \delta, 0.7), (\neg\phi \vee \psi, 0.6), (\delta \vee \varphi, 0.5), (\phi, 0.5)\}.$$

So $\mathcal{B}_{S-C} = \{(\phi \vee \delta, 0.9), (\neg\phi \vee \varphi, 0.8), (\neg\psi \vee \delta, 0.7), (\neg\phi \vee \psi, 0.6), (\delta \vee \varphi, 0.5), (\phi, 0.5)\}$. Since \mathcal{B}_{S-C} is consistent, so $\gamma = 0.4$ is the splitting point and $\mathcal{B}_{I-S-C} = \{(\phi \vee \delta, 0.9), (\neg\phi \vee \varphi, 0.8), (\neg\psi \vee \delta, 0.7), (\neg\phi \vee \psi, 0.6), (\delta \vee \varphi, 0.5), (\phi, 0.5)\}$.

4.2 Properties

4.2.1 Upper bound of the splitting point

Algorithm 1 stops in finite steps because there are finite certainty levels in $\mathcal{B}_1 \cup \mathcal{B}_2$. In this section, we introduce an *upper free degree* and we show that it is the upper bound of the splitting point obtained by Algorithm 1. We first introduce some definitions in [4, 6].

Definition 8 Let K be a classical knowledge base. A subbase K' of K is said to be *minimally inconsistent* if and only if it satisfies the following two requirements: (1) $K' \models \text{false}$ and (2) $\forall \phi \in K', K' - \{\phi\} \not\models \text{false}$.

Definition 9 Let K be a classical knowledge base. A formula ϕ is said to be *free* in K if it does not belong to any minimally inconsistent subbase of K and is said to be *in conflict* otherwise. $\text{Free}(K)$ denotes the set of free formulas in K .

Definition 10 Let \mathcal{B} be a possibilistic knowledge base. A possibilistic formula (ϕ, α) is said to be *free* in \mathcal{B} if ϕ is free in \mathcal{B}^* and it is *in conflict* otherwise.

We now define the upper free degree of a possibilistic knowledge base.

Definition 11 The *upper free degree* of a possibilistic base $\mathcal{B} = \{(\phi_i, \alpha_i) : i = 1, \dots, n\}$ is defined as:

$$\text{Free}_{\text{upp}}(\mathcal{B}) = \min\{\alpha \in \{\alpha_1, \dots, \alpha_n\} : \mathcal{B}_{>\alpha} \text{ does not contain any conflict formulas in } \mathcal{B}^*\}. \quad (14)$$

$\text{Free}_{\text{upp}}(\mathcal{B}) = 0$ when \mathcal{B} is consistent. $\mathcal{B}_{>\text{Free}_{\text{upp}}(\mathcal{B})}$ contains some free formulas of \mathcal{B} , but not all of them.

Definition 12 (*upper-free-degree-based splitting*) Given a possibilistic base \mathcal{B} , the *splitting* of \mathcal{B} with regard to $\text{Free}_{\text{upp}}(\mathcal{B})$ is defined as a pair $\langle \mathcal{C}, \mathcal{D} \rangle$ such that $\mathcal{B} = \mathcal{C} \cup \mathcal{D}$, where

$$\mathcal{C} = \{(\phi, \alpha) \in \mathcal{B} \mid \alpha \leq \text{Free}_{\text{upp}}(\mathcal{B})\} \text{ and } \mathcal{D} = \{(\phi, \alpha) \in \mathcal{B} \mid \alpha > \text{Free}_{\text{upp}}(\mathcal{B})\}.$$

It is clear that $\text{Free}_{\text{upp}}(\mathcal{B}) > \text{Inc}(\mathcal{B})$, for each possibilistic knowledge base \mathcal{B} . By Definition 12, \mathcal{C} is inconsistent if $\text{Free}_{\text{upp}}(\mathcal{B}) > 0$ and \mathcal{D} is always consistent.

Let us look at an example to illustrate how to split a possibilistic base *w.r.t* the upper free degree.

Example 5 Given a possibilistic knowledge base $\mathcal{B} = \{(\neg\psi \vee \delta, 0.9), (\phi \vee \delta, 0.7), (\neg\phi \vee \neg\varphi, 0.6), (\neg\psi \vee \varphi, 0.5), (\phi, 0.4), (\neg\phi \vee \psi, 0.3)\}$, by Definition 12, the upper free degree of \mathcal{B} is 0.6. \mathcal{B} is then split into $\langle \mathcal{C}, \mathcal{D} \rangle$ such that $\mathcal{C} = \{(\neg\phi \vee \neg\varphi, 0.6), (\neg\psi \vee \varphi, 0.5), (\phi, 0.4), (\neg\phi \vee \psi, 0.3)\}$, and $\mathcal{D} = \{(\neg\psi \vee \delta, 0.9), (\phi \vee \delta, 0.7)\}$.

Suppose the splitting method is the upper-free-degree-based splitting method, the corresponding $S-C$ method can be defined as follows.

Definition 13 Let $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, \dots, n\}$ and $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, \dots, m\}$ be two possibilistic knowledge bases. Suppose \mathcal{B}_i are split into $\mathcal{B}_i = \langle \mathcal{C}_i, \mathcal{D}_i \rangle$ w.r.t $Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, and \mathcal{C} and \mathcal{D} are obtained by Equation 12 and 13 respectively. The resulting possibilistic knowledge base of the upper-free-degree based $S-C$ ($U-S-C$ for short) merging operator, denoted $\mathcal{B}_{Upper-S-C}$, is defined as $\mathcal{B}_{Upper-S-C} = \mathcal{C} \cup \mathcal{D}$.

Example 6 (Continue Example 4) The upper free degree of $\mathcal{B}_1 \cup \mathcal{B}_2$ is 0.6. Therefore we split \mathcal{B}_1 and \mathcal{B}_2 into $\langle \mathcal{C}_1, \mathcal{D}_1 \rangle$ and $\langle \mathcal{C}_2, \mathcal{D}_2 \rangle$ such that

$$\mathcal{C}_1 = \{(\neg\phi \vee \psi, 0.6), (\phi, 0.5)\}, \quad \mathcal{D}_1 = \{(\neg\phi \vee \varphi, 0.8)\},$$

$$\mathcal{C}_2 = \{(\delta \vee \varphi, 0.5), (\neg\phi \vee \neg\psi, 0.4), (\psi, 0.3)\}, \quad \mathcal{D}_2 = \{(\phi \vee \delta, 0.9), (\neg\psi \vee \delta, 0.7)\}.$$

Combining \mathcal{C}_1 and \mathcal{C}_2 , and combining \mathcal{D}_1 and \mathcal{D}_2 using the maximum and the minimum respectively give

$$\mathcal{C} = \{(\phi \vee \delta \vee \varphi, 0.5), (\neg\phi \vee \psi \vee \delta \vee \varphi, 0.5), (\phi \vee \psi, 0.3), (\neg\phi \vee \psi, 0.3)\},$$

$$\mathcal{D} = \{(\phi \vee \delta, 0.9), (\neg\phi \vee \varphi, 0.8), (\neg\psi \vee \delta, 0.7)\}.$$

So we have

$$\begin{aligned} \mathcal{B}_{Upper-S-C} = & \{(\phi \vee \delta, 0.9), (\neg\phi \vee \varphi, 0.8), (\neg\psi \vee \delta, 0.7), (\neg\phi \vee \psi \vee \delta \vee \varphi, 0.5), \\ & (\phi \vee \delta \vee \varphi, 0.5), (\phi \vee \psi, 0.3), (\neg\phi \vee \psi, 0.3)\}. \end{aligned}$$

Given two possibilistic bases \mathcal{B}_1 and \mathcal{B}_2 , if $\mathcal{B}_1 \cup \mathcal{B}_2$ is consistent, by Definition 12, we have $Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2) = 0$. When we split \mathcal{B}_1 and \mathcal{B}_2 using $Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, we obtain $\mathcal{C}_1 = \emptyset$, $\mathcal{D}_1 = \mathcal{B}_1$ and $\mathcal{C}_2 = \emptyset$, $\mathcal{D}_2 = \mathcal{B}_2$, which results in $\mathcal{B}_{Upper-S-C} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \{(\phi_i \vee \psi_j, ct(\alpha_i, \beta_j)) | (\phi_i, \alpha_i) \in \mathcal{B}_1 \text{ and } (\psi_j, \beta_j) \in \mathcal{B}_2\}$. Therefore, the $U-S-C$ operator is equivalent to the t-norm based merging operator when sources are consistent. Next we give some properties of $U-S-C$ operator when $\mathcal{B}_1 \cup \mathcal{B}_2$ is in inconsistent.

Proposition 3 The resulting possibilistic base $\mathcal{B}_{Upper-S-C}$ of $U-S-C$ operator is consistent.

Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{(\varphi_i, \alpha_i) : i = 1, \dots, n\}$. Suppose the weights of the formulas in \mathcal{B} are rearranged in the way that $\alpha_1 = 1 > \alpha_2 > \dots > \alpha_{n_1}$. Let $\alpha_{n_1+1} = 0$. Since $Inc(\mathcal{B}) < Free_{upp}(\mathcal{B})$, if $Inc(\mathcal{B}) = \alpha_k$ and $Free_{upp}(\mathcal{B}) = \alpha_m$, we have $m < k$, then by Proposition 3, Algorithm 1 terminates after at most $k-m+1$ iterations.

Proposition 4 Given two possibilistic knowledge bases \mathcal{B}_1 and \mathcal{B}_2 , suppose $\mathcal{B}_{Upper-S-C}$ and \mathcal{B}_{I-S-C} are the merging results of \mathcal{B}_1 and \mathcal{B}_2 using $U-S-C$ operator and $I-S-C$ operator respectively, then we have

$$\mathcal{B}_{I-S-C} \vdash_{\pi} (\phi, \alpha), \quad \text{for all } (\phi, \alpha) \in \mathcal{B}_{Upper-S-C},$$

but not vice versa.

Proposition 4 shows that the upper-free-degree of $\mathcal{B}_1 \cup \mathcal{B}_2$ is the upper bound of the splitting point of Algorithm 1, that is, Algorithm 1 must terminate when the splitting point reaches the upper-free-degree of $\mathcal{B}_1 \cup \mathcal{B}_2$, if it has not terminated beforehand.

Proposition 5 *Given two possibilistic bases \mathcal{B}_1 and \mathcal{B}_2 , let $\mathcal{B}_{Uppper-S-C}$ be the possibilistic base obtained by the $U-S-C$ operator (which is based on a t -norm tn and a t -conorm ct) and \mathcal{B}_{ct} be the resulting possibilistic base of merging using ct , then*

$$\mathcal{B}_{Uppper-S-C} \vdash_{\pi} (\phi, \alpha), \quad \text{for all } (\phi, \alpha) \in \mathcal{B}_{ct} \quad (15)$$

The converse of Proposition 5 is false. Let us look at a counter-example.

Example 7 *(Continue Example 6) By Example 6, we have*

$$\begin{aligned} \mathcal{B}_{Uppper-S-C} = & \{(\phi \vee \delta, 0.9), (\neg\phi \vee \varphi, 0.8), (\neg\psi \vee \delta, 0.7), (\neg\phi \vee \psi \vee \delta \vee \varphi, 0.5), \\ & (\phi \vee \delta \vee \varphi, 0.5), (\phi \vee \psi, 0.3), (\neg\phi \vee \psi, 0.3)\}. \end{aligned}$$

If we combine \mathcal{B}_1 and \mathcal{B}_2 by the maximum operator, the result is

$$\begin{aligned} \mathcal{B}_{dm} = & \{(\neg\phi \vee \neg\psi \vee \delta \vee \varphi, 0.7), (\phi \vee \neg\psi \vee \delta, 0.5), (\neg\phi \vee \psi \vee \delta \vee \varphi, 0.5), (\phi \vee \delta, 0.5), \\ & (\neg\phi \vee \delta \vee \varphi, 0.5), (\phi \vee \delta \vee \varphi, 0.5), (\neg\phi \vee \neg\psi \vee \varphi, 0.4), (\neg\phi \vee \psi, 0.3), \\ & (\neg\phi \vee \psi \vee \varphi, 0.3), (\phi \vee \psi, 0.3)\} \end{aligned}$$

It is easy to check that all the possibilistic formulas in \mathcal{B}_{dm} can be inferred from $\mathcal{B}_{Uppper-S-C}$. However, $\mathcal{B}_{Uppper-S-C}$ contains $(\phi \vee \delta, 0.9), (\neg\phi \vee \varphi, 0.8), (\neg\psi \vee \delta, 0.7)$, which are not involved in conflict in $\mathcal{B}_1 \cup \mathcal{B}_2$ and cannot be inferred from \mathcal{B}_{dm} .

Proposition 5 and Example 7 show that the resulting possibilistic knowledge base of the $U-S-C$ method contains more important formulas than that of the combination method based on the t -conorm.

Clearly we have the following corollary.

Corollary 1 *Given two possibilistic bases \mathcal{B}_1 and \mathcal{B}_2 , let \mathcal{B}_{I-S-C} be the possibilistic base obtained by the $I-S-C$ operator (which is based on a t -norm tn and a t -conorm ct) and \mathcal{B}_{ct} be the resulting possibilistic base of merging using ct , then*

$$\mathcal{B}_{I-S-C} \vdash_{\pi} (\phi, \alpha), \quad \text{for all } (\phi, \alpha) \in \mathcal{B}_{ct}, \quad (16)$$

but not vice versa.

We have shown that $I-S-C$ operator maintains more original information than the merging operator based on the t -conorm based operator. Next, we compare our $I-S-C$ operator with the renormalization based merging operator (see Equation 11).

We have the following proposition.

Proposition 6 *Let \mathcal{B}_1 and \mathcal{B}_2 be two possibilistic knowledge bases. Suppose tn is the minimum and ct is an arbitrary t -conorm in Algorithm 1. Let γ be the splitting point obtained by Algorithm 1. Suppose $\gamma = Inc(\mathcal{B}_1 \cup \mathcal{B}_2)$, then $\mathcal{B}_{N,min} \subseteq \mathcal{B}_{I-S-C}$, but not vice versa.*

The proof of Proposition 6 is trivial. We do not provide it here. Proposition 6 states that, when the t -norm is the minimum, the resulting knowledge base of our merging operator contains more information than that of the renormalization based merging operator. However, this conclusion does not hold for other t -norms generally. Let us look at an example.

Example 8 Let $\mathcal{B}_1 = \{(\psi, 0.7), (\phi, 0.7)\}$ and $\mathcal{B}_2 = \{(\neg\phi, 0.6), (\psi, 0.4)\}$. The inconsistency degree of $\mathcal{B}_1 \cup \mathcal{B}_2$ is 0.6. So \mathcal{B}_1 and \mathcal{B}_2 are split w.r.t 0.6 into $\langle \mathcal{C}_1, \mathcal{D}_1 \rangle$ and $\langle \mathcal{C}_2, \mathcal{D}_2 \rangle$ respectively, where $\mathcal{C}_1 = \emptyset$, $\mathcal{D}_1 = \mathcal{B}_1$, $\mathcal{C}_2 = \mathcal{B}_2$ and $\mathcal{D}_2 = \emptyset$. Combining \mathcal{C}_1 and \mathcal{C}_2 using the probabilistic sum and combining \mathcal{D}_1 and \mathcal{D}_2 using the product operator we get $\mathcal{C} = \emptyset$ and $\mathcal{D} = \mathcal{B}_1$ respectively. So $\mathcal{B}_{S-C} = \mathcal{B}_1$. It is clear that \mathcal{B}_{S-C} is consistent. So $\mathcal{B}_{I-S-C} = \mathcal{B}_1$. In contrast, suppose we combine \mathcal{B}_1 and \mathcal{B}_2 using the product operator, then the resulting possibilistic knowledge base is $\{(\neg\phi \vee \psi, 0.88), (\psi, 0.82), (\phi \vee \psi, 0.82), (\psi, 0.7), (\phi, 0.7), (\neg\phi, 0.6), (\psi, 0.4)\}$, which is equivalent to $\{(\neg\phi \vee \psi, 0.88), (\psi, 0.82), (\phi, 0.7), (\neg\phi, 0.6)\}$. So we have $\mathcal{B}_{N,tn} \equiv_s \{(\neg\phi \vee \psi, 0.88), (\psi, 0.82), (\phi, 0.7)\}$, where tn is the product operator. It is clear that every formula in \mathcal{B}_{I-S-C} can be inferred from $\mathcal{B}_{N,tn}$, while the converse is false.

When the original knowledge bases are consistent, the $I-S-C$ operator is equivalent to the t -norm based operator. That is, we have the following proposition.

Proposition 7 Let \mathcal{B}_1 and \mathcal{B}_2 be two possibilistic knowledge bases. If $\mathcal{B}_1 \cup \mathcal{B}_2$ is consistent, then $\mathcal{B}_{tn} = \mathcal{B}_{I-S-C}$.

The proof of Proposition 7 is trivial.

4.2.2 Logical properties

In this section, we discuss the logical properties of possibilistic merging operators by generalizing postulates or logical properties for propositional merging operators. The generalized postulates are not used to give a normative definition or characterization of possibilistic merging. In fact, as we will see, none of possibilistic merging operators satisfy all of the generalized postulates. The reason why we propose them is that we think they are helpful for users to choose among different possibilistic merging operators.

We first introduce postulates for characterizing a propositional merging operator proposed in [23].

Definition 14 [23] Let Δ be an operator which assigns to each knowledge profile E a knowledge base $\Delta(E)$. Let E_1 and E_2 be two knowledge profiles, K and K' be two knowledge bases. Δ is a merging operator iff it satisfies the following postulates:

- (A1) $\Delta(E)$ is consistent
- (A2) If E is consistent, then $\Delta(E) \equiv \bigwedge E$, where $\bigwedge E = \bigwedge_{K_i \in E} K_i$
- (A3) If $E_1 \equiv E_2$, then $\Delta(E_1) \equiv \Delta(E_2)$
- (A4) If $K \wedge K'$ is not consistent, then $\Delta(\{K\} \sqcup \{K'\}) \not\equiv K$
- (A5) $\Delta(E_1) \wedge \Delta(E_2) \models \Delta(E_1 \sqcup E_2)$
- (A6) If $\Delta(E_1) \wedge \Delta(E_2)$ is consistent, then $\Delta(E_1 \sqcup E_2) \models \Delta(E_1) \wedge \Delta(E_2)$.

(A1) requires that the resulting knowledge base of a merging operator be consistent. (A2) says that the resulting knowledge base should be the conjunction of the original knowledge bases. (A3) is the principle of irrelevance of syntax. (A4) is the fairness postulate, which means that if two knowledge bases are in conflict, merging operators must not give preference to any one of them. (A5) and (A6) together state that if there are two subgroups whose merged results are consistent,

then the result of merging of the global group is the conjunction of merged results of subgroups.

(A5) can be equivalently expressed as

(A5) If $\Delta(E_1) \wedge \Delta(E_2)$ is consistent, $\Delta(E_1) \wedge \Delta(E_2) \models \Delta(E_1 \sqcup E_2)$.

So (A5) and (A6) can be merged as the following postulate.

(A7) If $\Delta(E_1) \wedge \Delta(E_2)$ is consistent, $\Delta(E_1) \wedge \Delta(E_2) \equiv \Delta(E_1 \sqcup E_2)$.

We now propose some postulates for a possibilistic merging operator based on postulates in Definition 14. In the following, \mathcal{E} , \mathcal{E}_1 and \mathcal{E}_2 denote possibilistic knowledge profiles, and \mathcal{B}_1 and \mathcal{B}_2 denote possibilistic knowledge bases.

(P1) $\Delta(\mathcal{E})$ is consistent

(P2) Let $\mathcal{E} = \{\mathcal{B}_1, \dots, \mathcal{B}_n\}$. If $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$ is consistent, then $(\Delta(\mathcal{E}))^* \equiv (\mathcal{B}_1 \cup \dots \cup \mathcal{B}_n)^*$ and $\forall \phi$, if $\exists i$ such that $\mathcal{B}_i \vdash_{\pi} (\phi, \alpha)$ then $\exists \beta$ such that $\beta \geq \alpha$ and $\Delta(\mathcal{E}) \vdash_{\pi} (\phi, \beta)$.

(P3) If $\mathcal{E}_1 \equiv_s \mathcal{E}_2$, then $\Delta(\mathcal{E}_1) \equiv_s \Delta(\mathcal{E}_2)$

(P4) $\Delta(\{\Delta(\mathcal{E}_1)\}, \{\Delta(\mathcal{E}_2)\}) \vdash_{\pi} (\phi, \alpha)$ for all $(\phi, \alpha) \in \Delta(\mathcal{E}_1 \sqcup \mathcal{E}_2)$

(P5) $\Delta(\mathcal{E}_1 \sqcup \mathcal{E}_2) \vdash_{\pi} (\phi, \alpha)$ for all $(\phi, \alpha) \in \Delta(\{\Delta(\mathcal{E}_1)\}, \{\Delta(\mathcal{E}_2)\})$.

(P6) If $\Delta(\mathcal{E}_1) \cup \Delta(\mathcal{E}_2)$ is consistent, then $\Delta(\mathcal{E}_1)^* \wedge \Delta(\mathcal{E}_2)^* \equiv (\Delta(\mathcal{E}_1 \sqcup \mathcal{E}_2))^*$ and if $\Delta(\mathcal{E}_1) \cup \Delta(\mathcal{E}_2) \vdash_{\pi} (\phi, \alpha)$, then $\exists \beta$ such that $\beta \geq \alpha$ and $\Delta(\mathcal{E}_1 \sqcup \mathcal{E}_2) \vdash_{\pi} (\phi, \beta)$.

(P1) is a mandatory condition because we presume the first view on possibilistic merging. (P2) is adapted from a postulate in [10]. It is stronger than (A2) in that when there is no conflict among original knowledge bases, it requires not only all the original information be restored, but also the weights of the formulas in original knowledge bases should not be decreased after merging. (P3) is the principle of irrelevance of syntax, that is, it is a generalizations of (A3). Note we do not consider the generalization of (A4) because it is a controversial postulate (see [26] for a discussion) and not intuitive in the context of possibilistic merging because weights are attached to formulas. (P4) and (P5) together are the associativity condition, that is, the fusion of n bases can be decomposed in several steps. (P6) is a generalization of (A7). We do not use $\Delta(\mathcal{E}_1) \cup \Delta(\mathcal{E}_2) \equiv_s \Delta(\mathcal{E}_1 \sqcup \mathcal{E}_2)$ because it is equivalent to assuming that we combine $\Delta(\mathcal{E}_1)$ and $\Delta(\mathcal{E}_2)$ using the minimum operator, which violates (P2).

Before checking the logical properties of our I - S - C operator, we need to generalize it to n possibilistic knowledge bases. It is easy to check that I - S - C operator is not *associative*, that is, the order of combination will influence the result of merging. For example, given three possibilistic knowledge bases $\mathcal{B}_1 = \{(\phi, 0.6)\}$, $\mathcal{B}_2 = \{(\neg\phi, 0.7)\}$, and $\mathcal{B}_3 = \{(\phi, 0.7)\}$. Let tn be the product operator and $ct = \max$. Suppose we combine \mathcal{B}_1 and \mathcal{B}_2 first, the result of combination is $\mathcal{B}_4 = \{(\neg\phi, 0.7)\}$. We then combine \mathcal{B}_4 and \mathcal{B}_3 , the result of combination is $\mathcal{B} = \{(\top, 0.7)\}$. If we combine \mathcal{B}_1 and \mathcal{B}_3 first, the result of combination is $\mathcal{B}_5 \equiv_s \{(\phi, 0.88)\}$. By combining \mathcal{B}_5 and \mathcal{B}_2 we get $\mathcal{B}' = \{(\phi, 0.88)\}$. It is clear $\mathcal{B} \not\equiv_s \mathcal{B}'$.

In the following, we generalize Algorithm 1 to compute a splitting point for n possibilistic knowledge base.

Algorithm 2

Input: a set of possibilistic knowledge bases $\{\mathcal{B}_1, \dots, \mathcal{B}_n\}$, a t-conorm ct and a t-norm tn .

Output: a splitting point γ .

Step 1 Let $\mathcal{B} = \cup_{i=1}^n \mathcal{B}_i = \{(\varphi_i, \gamma_i) : i = 1, \dots, n + m\}$. Rearrange the weights of formulas in \mathcal{B} such that $\gamma_1 > \gamma_2 > \dots > \gamma_{n+1}$. Let $\gamma_{n+1} = 0$.

Step 2 Compute $Inc(\mathcal{B})$. Assume $Inc(\mathcal{B}) = \gamma_k$. Let $l = k$.

Step 3 Split \mathcal{B}_i with regard to γ_l such that $\mathcal{B}_i = \langle \mathcal{C}_i, \mathcal{D}_i \rangle$, where $\mathcal{C}_i = \{(\phi_{ij}, \alpha_{ij}) : (\phi_{ij}, \alpha_{ij}) \in \mathcal{B}_i, \alpha_{ij} \leq \gamma_l\}$ and $\mathcal{D}_i = \mathcal{B}_i \setminus \mathcal{C}_i$.

Step 4 Combine \mathcal{C}_i ($i = 1, \dots, n$) by ct (the result is \mathcal{C}) and combine \mathcal{D}_i ($i = 1, \dots, n$) by tn (the result is \mathcal{D}).

Step 5 Let $\mathcal{B}_{S-C} = \mathcal{C} \cup \mathcal{D}$.

Step 6 If \mathcal{B}_{S-C} is inconsistent, let $l = l - 1$ and go to Step 3.

Step 7 Return γ_l .

Definition 15 Let $\mathcal{E} = \{\mathcal{B}_1, \dots, \mathcal{B}_n\}$ be a set of n possibilistic knowledge bases. Let ct be a t -conorm and tn be a t -norm. Let γ be the splitting point obtained by Algorithm 2. Suppose \mathcal{B}_i are split into $\mathcal{B}_i = \langle \mathcal{C}_i, \mathcal{D}_i \rangle$ w.r.t γ , and \mathcal{C} and \mathcal{D} are obtained by merging \mathcal{C}_i ($i = 1, \dots, n$) by ct and merging \mathcal{D}_i ($i = 1, \dots, n$) by tn respectively. The resulting possibilistic knowledge base of the generalized incremental $S-C$ ($G-I-S-C$ for short) merging operator, denoted as $\mathcal{B}_{G-I-S-C}$, is defined as $\mathcal{B}_{G-I-S-C} = \mathcal{C} \cup \mathcal{D}$.

Example 9 Let $\mathcal{B}_1 = \{(\phi, 0.7), (\psi, 0.5), (\varphi, 0.4)\}$, $\mathcal{B}_2 = \{(\phi \rightarrow \neg\psi, 0.6), (\delta, 0.5)\}$, and $\mathcal{B}_3 = \{(\phi, 0.8), (\varphi, 0.7)\}$ be three possibilistic knowledge bases. The union of \mathcal{B}_i is $\mathcal{B} = \{(\phi, 0.8), (\varphi, 0.7), (\phi, 0.7), (\phi \rightarrow \neg\psi, 0.6), (\psi, 0.5), (\delta, 0.5), (\varphi, 0.4)\}$ and its inconsistency degree is 0.5. The weights of formulas in \mathcal{B} are rearranged as $\gamma_1 = 0.8 > \gamma_2 = 0.7 > \gamma_3 = 0.6 > \gamma_4 = 0.5 > \gamma_5 = 0.4$. Let $\gamma_6 = 0$. So $Inc(\mathcal{B}) = \gamma_4$. Let $l = 4$. \mathcal{B}_1 , \mathcal{B}_2 , and \mathcal{B}_3 are split w.r.t $\gamma_4 = 0.5$ as $\mathcal{D}_1 = \{(\phi, 0.7)\}$ and $\mathcal{C}_1 = \{(\psi, 0.5), (\varphi, 0.4)\}$, $\mathcal{D}_2 = \{(\phi \rightarrow \neg\psi, 0.6)\}$ and $\mathcal{C}_2 = \{(\delta, 0.5)\}$, and $\mathcal{D}_3 = \{(\phi, 0.8), (\varphi, 0.7)\}$ and $\mathcal{C}_3 = \emptyset$ respectively. Combining \mathcal{C}_i by the bounded sum and combining \mathcal{D}_i by the Lukasiewicz t -norm, we get $\mathcal{D} \equiv_s \{(\phi, 1), (\neg\phi \vee \neg\psi \vee \varphi, 1), (\varphi, 0.7), (\phi \rightarrow \neg\psi, 0.6)\}$ and $\mathcal{C} = \emptyset$. Since $\mathcal{D} \cup \mathcal{C}$ is consistent, we have $\gamma = 0.5$. So the final result of merging is $\mathcal{B}_{G-I-S-C} = \mathcal{D} \equiv_s \{(\phi, 1), (\neg\phi \vee \neg\psi \vee \varphi, 1), (\varphi, 0.7), (\phi \rightarrow \neg\psi, 0.6)\}$. In $\mathcal{B}_{G-I-S-C}$, the weight of ϕ is increased to 1 because there is a reinforcement between $(\phi, 0.7)$ and $(\phi, 0.8)$.

We have compared the resulting possibilistic knowledge base of our merging operator with those of the merging operators based on t -conorm and renormalization. We now compare their logical properties. To make the notation consistent with those used in (P1)-(P5), we use $\Delta_{G-I-S-C}$, Δ_{ct} and $\Delta_{N,tn}$ to denote $G-I-S-C$, the t -conorm based, and renormalization based merging operators respectively.

Proposition 8 The $G-I-S-C$ operator $\Delta_{G-I-S-C}$ satisfies (P1)-(P3). It does not satisfy (P4)-(P6) in general.

Proposition 9 The t -conorm based merging operator Δ_{ct} satisfies (P1), (P3)-(P5). It doesn't not satisfy (P2) and (P6) in general. The operator $\Delta_{N,tn}$ satisfies (P1)-(P3). It does not satisfy (P4)-(P6) in general.

Proposition 8 and Proposition 9 show that the $G-I-S-C$ operator and the renormalization based operator satisfy (P1)-(P3). However, they both do not satisfy (P4)-(P6) in general. In contrast, the t -conorm based operator satisfies (P4) and (P5), but it does not satisfy (P2) and (P6) in general.

4.2.3 Computational complexity

We analyze the computational complexity of the $I-S-C$ operator by the following proposition.

Proposition 10 *Generating a consistent possibilistic knowledge base by I - S - C operator is in $F\Delta_2^p(\mathcal{O}(n))$, where $\Delta_2^p(\mathcal{O}(n))$ denotes the set of decision problems decidable in polynomial time with no more than $\mathcal{O}(n)$ calls to an NP oracle, n is the number of different valuations involved in \mathcal{B} [22], and “ F ” in $F\Delta_2^p(\mathcal{O}(n))$ stands for function and is intended to turn a complexity class for decision problem into one for search problem, i.e., problems that have answers.*

We also have the following results for computational complexity of a t-conorm and renormalization based operators.

Proposition 11 *Generating a consistent possibilistic knowledge base by a t-conorm based merging operator is in FP and generating a consistent possibilistic knowledge base by a renormalization based merging operator is in $F\Delta_2^p$.*

By Proposition 10 and Proposition 11, the computational complexity of I - S - C operator and that of the renormalization based merging operator lie in the same level of the boolean hierarchy, and the computational complexity of t-conorm based merging operator is tractable.

5 Semantic Aspects of the I - S - C Operator

In this section, we provide a semantic analysis of the I - S - C operator. We first give a definition on splitting a possibility distribution *w.r.t* a weight.

Definition 16 *Suppose $\mathcal{B} = \{(\phi_i, \alpha_i) : i = 1, \dots, n\}$ is a possibilistic knowledge base and $\pi_{\mathcal{B}}$ is the possibility distribution associated with it. For weight α_k of formula ϕ_k , we can split $\pi_{\mathcal{B}}$ *w.r.t* α_k as $\langle \pi_1, \pi_2 \rangle$, where*

$$\pi_1(\omega) = \begin{cases} 1 & \text{if } \omega \models \phi_i, \forall (\phi_i, \alpha_i) \in \mathcal{B} \text{ where } \alpha_i \leq \alpha_k, \\ 1 - \max\{\alpha_i | \omega \not\models \phi_i, (\phi_i, \alpha_i) \in \mathcal{B} \text{ and } \alpha_i \leq \alpha_k\} & \text{otherwise.} \end{cases} \quad (17)$$

and

$$\pi_2(\omega) = \begin{cases} 1 & \text{if } \omega \models \phi_i, \forall (\phi_i, \alpha_i) \in \mathcal{B} \text{ where } \alpha_i > \alpha_k, \\ 1 - \max\{\alpha_i | \omega \not\models \phi_i, (\phi_i, \alpha_i) \in \mathcal{B} \text{ and } \alpha_i > \alpha_k\} & \text{otherwise.} \end{cases} \quad (18)$$

Clearly, we have the following propositions.

Proposition 12 *Suppose $\mathcal{B} = \{(\phi_i, \alpha_i) : i = 1, \dots, n\}$ is a possibilistic knowledge base and $\pi_{\mathcal{B}}$ is the possibility distribution associated with it. (ϕ_i, α_i) is a formula of \mathcal{B} . Suppose \mathcal{B} is split *w.r.t* α_i as $\langle \mathcal{C}, \mathcal{D} \rangle$, $\pi_{\mathcal{B}}$ is split *w.r.t* α_i into π_1 and π_2 . We then have $\pi_1 = \pi_{\mathcal{C}}$ and $\pi_2 = \pi_{\mathcal{D}}$, where $\pi_{\mathcal{C}}$ and $\pi_{\mathcal{D}}$ are possibility distributions of \mathcal{C} and \mathcal{D} respectively.*

Proposition 13 *Let \mathcal{B}_1 and \mathcal{B}_2 be two equivalent possibilistic knowledge base, and $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ are possibility distributions associated with them. Let $(\phi_k, \alpha_k) \in \mathcal{B}_1$. Suppose π_{11} and π_{12} are possibility distributions obtained by splitting $\pi_{\mathcal{B}_1}$ *w.r.t* α_k using Equation 17 and Equation 18 respectively. Suppose π_{21} and π_{22} are possibility distributions obtained by splitting $\pi_{\mathcal{B}_2}$ *w.r.t* α_k using Equation 17 and Equation 18 respectively. We then have $\pi_{11}(\omega) = \pi_{21}(\omega)$ and $\pi_{12}(\omega) = \pi_{22}(\omega)$ for all ω .*

Proposition 13 shows that the splitting of the possibility distribution associated with a possibilistic knowledge base using Equation 17 and Equation 18 is syntax-independent.

Let \mathcal{B}_1 and \mathcal{B}_2 be two possibilistic bases and $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ be their associated possibility distributions respectively. The idea of the semantic I - S - C method can be described as follows. We

first find a splitting point by the algorithm below and then split $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ into $\langle \pi_{11}, \pi_{12} \rangle$ and $\langle \pi_{21}, \pi_{22} \rangle$ respectively by the splitting point (π_{i1} are obtained by Equation 17 and π_{i2} are obtained by Equation 18). After that, we combine π_{11} and π_{21} using a t-conorm operator (the result is a possibility distribution π_1) and combine π_{12} and π_{22} using a t-norm operator (the result is a possibility distribution π_2). Finally, the resulting possibility distribution of our semantic *I-S-C* operator is defined as $\pi_{I-S-C}(\omega) = \min(\pi_1(\omega), \pi_2(\omega))$. The following algorithm is the semantic counterpart of Algorithm 1.

Algorithm 3

Input: two possibility distributions $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ which are associated with $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, \dots, n\}$ and $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, \dots, m\}$ respectively.

Output: a splitting point γ .

Step 1 Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{(\varphi_i, \gamma_i) : i = 1, \dots, n + m\}$. Rearrange the weights of formulas in \mathcal{B} such that $\gamma_1 > \gamma_2 > \dots > \gamma_{n+m}$. Let $\gamma_{n+m+1} = 0$.

Step 2 Compute $Inc(\mathcal{B})$, assume $Inc(\mathcal{B}) = \gamma_k$. Let $l = k$.

Step 3 Split $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ with regard to γ_l into $\pi_{\mathcal{B}_1} = \langle \pi_{11}, \pi_{12} \rangle$ and $\pi_{\mathcal{B}_2} = \langle \pi_{21}, \pi_{22} \rangle$, where π_{i1} ($i = 1, 2$) are obtained by Equation 17 and π_{i2} ($i = 1, 2$) are obtained by Equation 18.

Step 4 Combine π_{11} and π_{21} using a t-conorm operator ct and combine π_{12} and π_{22} using a t-norm operator tn . The results are $\pi_1(\omega) = tn(\pi_{11}(\omega), \pi_{21}(\omega))$ and $\pi_2(\omega) = ct(\pi_{12}(\omega), \pi_{22}(\omega))$, for all ω .

Step 5 Let $\pi_{S-C}(\omega) = \min(\pi_1(\omega), \pi_2(\omega))$ for all ω .

Step 6 If π_{S-C} is subnormal, let $l = l - 1$ and go to Step 3.

Step 7 Return γ_l .

The semantic *I-S-C* operator is then defined as follows.

Definition 17 Let $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, \dots, n\}$ and $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, \dots, m\}$ be two possibilistic knowledge bases, $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ be their associated possibility distributions. Let γ be the splitting point obtained by Algorithm 3. Suppose $\pi_{\mathcal{B}_i}$ ($i = 1, 2$) are split into $\pi_i = \langle \pi_{i1}, \pi_{i2} \rangle$ w.r.t γ , where π_{i1} and π_{i2} are obtained by Equation 17 and 18 respectively. Suppose π_1 and π_2 are defined by $\pi_1(\omega) = ct(\pi_{11}(\omega), \pi_{21}(\omega))$ and $\pi_2(\omega) = tn(\pi_{12}(\omega), \pi_{22}(\omega))$, for all ω . The resulting possibility distribution of the semantic incremental *S-C* operator, denoted as π_{I-S-C} , is defined as $\pi_{I-S-C}(\omega) = \min(\pi_1(\omega), \pi_2(\omega))$, for all ω .

The following proposition shows that the semantic *I-S-C* operator resulting in a possibility distribution which is equivalent to that associated with the possibilistic knowledge base obtained by the *I-S-C* merging operator.

Proposition 14 Let $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, \dots, n\}$ and $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, \dots, m\}$ be two possibilistic knowledge bases, $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ be their associated possibility distributions. We have $\pi_{\mathcal{B}_{I-S-C}}(\omega) = \pi_{I-S-C}(\omega)$, for all ω .

We have the following relationship between the resulting possibility distributions of the semantic *I-S-C* operator and t-conorm based merging operator.

Proposition 15 Let \mathcal{B}_1 and \mathcal{B}_2 be two possibilistic bases, and their associated possibility distributions are $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ respectively. Let π_{I-S-C} be the resulting possibility distribution of merging by the semantic *I-S-C* operator and π_{ct} be the resulting possibility distribution by the t-conorm, then π_{I-S-C} is more specific than π_{ct} , that is $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$ for all $\omega \in \Omega$.

6 An Alternative Way to Split Possibilistic Knowledge Bases

6.1 An alternative splitting approach

When defining the *I-S-C* operator, given two possibilistic bases \mathcal{B}_1 and \mathcal{B}_2 , we split each of them using a weight obtained by Algorithm 1 such that $\mathcal{B}_1 = \mathcal{C}_1 \cup \mathcal{D}_1$ and $\mathcal{B}_2 = \mathcal{C}_2 \cup \mathcal{D}_2$. We then combine \mathcal{C}_1 and \mathcal{C}_2 by a t-conorm operator weakening conflicting information. Since $\mathcal{C}_1 \cup \mathcal{C}_2$ consists of possibilistic formulas in $\mathcal{B}_1 \cup \mathcal{B}_2$ with necessity degrees less than the splitting point, there may exist some *free formulas* in \mathcal{C}_1 or \mathcal{C}_2 . So, when we combine \mathcal{C}_1 and \mathcal{C}_2 by a t-conorm operator, these *free formulas* are combined with other formulas as disjunctive forms. However, we know *free formulas* will not cause inconsistency, so it is safe to keep them unchanged. Therefore, we propose the following alternative approach to splitting the knowledge bases.

Definition 18 (*free-formulas-based splitting*) *Given a possibilistic base \mathcal{B} , the splitting of \mathcal{B} with regard to $Free(\mathcal{B})$ is a pair $\langle \mathcal{C}_{Con}, \mathcal{D}_{Free} \rangle$ such that $\mathcal{B} = \mathcal{C}_{Con} \cup \mathcal{D}_{Free}$, where*

$$\begin{aligned} \mathcal{D}_{Free} &= \{(\phi, \alpha) \mid (\phi, \alpha) \in Free(\mathcal{B})\}, \\ \mathcal{C}_{Con} &= \mathcal{B} \setminus \mathcal{D}_{Free} = \{(\phi, \alpha) \mid (\phi, \alpha) \notin Free(\mathcal{B})\}. \end{aligned}$$

That is, \mathcal{D}_{Free} contains all the free formulas, and \mathcal{C}_{Con} contains all the conflict formulas in \mathcal{B} .

Lemma 1 *Let \mathcal{B} be a possibilistic knowledge base. Let \mathcal{B} be split by the upper-free-degree approach and free-formulas approach respectively, with the splitting results as $\mathcal{B} = \mathcal{C} \cup \mathcal{D}$ and $\mathcal{B} = \mathcal{C}_{Con} \cup \mathcal{D}_{Free}$. Then $\mathcal{D} \subseteq \mathcal{D}_{Free}$, and $\mathcal{C}_{Con} \subseteq \mathcal{C}$.*

We omitted the proof of Lemma 1, as it is easy to prove.

Now we define the free-formulas-based *S-C* operator.

Definition 19 *Let $\mathcal{B}_1 = \{(\phi_1, \alpha_1), \dots, (\phi_n, \alpha_n)\}$ and $\mathcal{B}_2 = \{(\psi_1, \beta_1), \dots, (\psi_m, \beta_m)\}$ be two possibilistic bases. Let $\langle \mathcal{C}', \mathcal{D}' \rangle$ be a splitting of $\mathcal{B}_1 \cup \mathcal{B}_2$ with regard to $Free(\mathcal{B}_1 \cup \mathcal{B}_2)$. Suppose \mathcal{B}_1 is split into a pair $\langle \mathcal{C}_{Con_1}, \mathcal{D}_{Free_1} \rangle$ such that $\mathcal{C}_{Con_1} = \mathcal{C}' \cap \mathcal{B}_1$ and $\mathcal{D}_{Free_1} = \mathcal{D}' \cap \mathcal{B}_1$, and \mathcal{B}_2 is split into a pair $\langle \mathcal{C}_{Con_2}, \mathcal{D}_{Free_2} \rangle$ such that $\mathcal{C}_{Con_2} = \mathcal{C}' \cap \mathcal{B}_2$ and $\mathcal{D}_{Free_2} = \mathcal{D}' \cap \mathcal{B}_2$. Let \mathcal{C}_{Con} and \mathcal{D}_{Free} be the possibilistic knowledge bases obtained by merging \mathcal{C}_{Con_1} and \mathcal{C}_{Con_2} using a t-conorm operator and merging \mathcal{D}_{Free_1} and \mathcal{D}_{Free_2} using a t-norm operator respectively. The result of the *F-S-C* merging operator, denoted by \mathcal{B}_{F-S-C} , is defined as $\mathcal{B}_{F-S-C} = \mathcal{C}_{Con} \cup \mathcal{D}_{Free}$.*

The resulting possibilistic knowledge base of the *F-S-C* operator is always consistent.

Lemma 2 *The possibilistic base \mathcal{B}_{F-S-C} obtained in *F-S-C* method is consistent.*

The proof of Lemma 2 is similar to that of Proposition 3.

Example 10 *Suppose there are two persons whose beliefs on Tweety the penguin are expressed by two possibilistic knowledge bases \mathcal{B}_1 and \mathcal{B}_2 respectively, where $\mathcal{B}_1 = \{(\text{penguin}(\text{Tweety}), 1), (\text{bird}(\text{Tweety}), 0.8), (\text{eatfish}(\text{Tweety}), 0.8)\}$ and $\mathcal{B}_2 = \{(\text{bird}(\text{Tweety}) \rightarrow \text{fly}(\text{Tweety}), 1), (\neg \text{fly}(\text{Tweety}), 0.8), (\text{eatfish}(\text{Tweety}) \rightarrow \text{swim}(\text{Tweety}), 0.6)\}$. Since $Free(\mathcal{B}_1 \cup \mathcal{B}_2) = \{(\text{penguin}(\text{Tweety}), 1), (\text{eatfish}(\text{Tweety}), 0.8), (\text{eatfish}(\text{Tweety}) \rightarrow \text{swim}(\text{Tweety}), 0.6)\}$, \mathcal{B}_1 and \mathcal{B}_2 are split into $\langle \mathcal{C}_{Con_1}, \mathcal{D}_{Free_1} \rangle$ and $\langle \mathcal{C}_{Con_2}, \mathcal{D}_{Free_2} \rangle$, where $\mathcal{C}_{Con_1} = \{(\text{bird}(\text{Tweety}), 0.8)\}$ and $\mathcal{D}_{Free_1} = \{(\text{penguin}(\text{Tweety}), 1), (\text{eatfish}(\text{Tweety}), 0.8)\}$, and $\mathcal{C}_{Con_2} = \{\text{bird}(\text{Tweety}) \rightarrow \text{fly}(\text{Tweety}), 1), (\neg \text{fly}(\text{Tweety}), 0.8)\}$ and $\mathcal{D}_{Free_2} = \{(\text{eatfish}(\text{Tweety}) \rightarrow \text{swim}(\text{Tweety}), 0.6)\}$. Suppose the t-conorm is the probabilistic*

sum and the t -norm is the product operator. Then combining \mathcal{C}_{Con_1} and \mathcal{C}_{Con_2} using the probabilistic sum and combining \mathcal{D}_{Free_1} and \mathcal{D}_{Free_2} using the product operator we get $\mathcal{C}_{Con} = \{(bird(Tweety) \vee \neg fly(Tweety), 0.64)\}$, and $\mathcal{D}_{Free} = \{(penguin(Tweety), 1), (eat\ fish(Tweety), 0.8), (eat\ fish(Tweety) \rightarrow swim(Tweety), 0.6), (penguin(Tweety) \vee (eat\ fish(Tweety) \rightarrow swim(Tweety))), 1)\}$. So $\mathcal{B}_{F-S-C} = \{(bird(Tweety) \vee \neg fly(Tweety), 0.64), (penguin(Tweety), 1), (eat\ fish(Tweety), 0.8), (eat\ fish(Tweety) \rightarrow swim(Tweety), 0.6), (penguin(Tweety) \vee (eat\ fish(Tweety) \rightarrow swim(Tweety))), 1)\}$. The conflicting formulas $(bird(Tweety), 0.8)$, $(bird(Tweety) \rightarrow fly(Tweety), 1)$, $(\neg fly(Tweety), 0.8)$ are weakened to be $(bird(Tweety) \vee \neg fly(Tweety), 0.64)$. That is, after merging, we are moderately confident that either Tweety is a bird or it cannot fly.

The resulting knowledge base of F - S - C operator and that of I - S - C operator are not comparable in general. Let us look at Example 8 again.

Example 11 (Continue Example 8) $Free(\mathcal{B}_1 \cup \mathcal{B}_2) = \{(\phi, 0.7), (\neg\phi, 0.6)\}$. So \mathcal{B}_1 and \mathcal{B}_2 are split as $\langle \mathcal{C}_{Con_1}, \mathcal{D}_{Free_1} \rangle$ and $\langle \mathcal{C}_{Con_2}, \mathcal{D}_{Free_2} \rangle$, where $\mathcal{C}_{Con_1} = \{(\phi, 0.7)\}$, $\mathcal{D}_{Free_1} = \{(\psi, 0.7)\}$, and $\mathcal{C}_{Con_2} = \{(\neg\phi, 0.6)\}$, $\mathcal{D}_{Free_2} = \{(\psi, 0.4)\}$. Combining \mathcal{C}_{Con_1} and \mathcal{C}_{Con_2} using the probabilistic sum and combining \mathcal{D}_{Free_1} and \mathcal{D}_{Free_2} using product operator we get $\mathcal{C}_{Con} = \{(\top, 0.42)\}$ and $\mathcal{D}_{Free} \equiv_s \{(\psi, 0.88)\}$. Therefore $\mathcal{B}_{F-S-C} \equiv_s \{(\psi, 0.88)\}$. Clearly, \mathcal{B}_{F-S-C} is not comparable with \mathcal{B}_{I-S-C} in Example 8.

However, we have the following results.

Proposition 16 Given two possibilistic bases \mathcal{B}_1 and \mathcal{B}_2 , if \mathcal{B}_{F-S-C} is the possibilistic base obtained by the F - S - C method and $\mathcal{B}_{Upper-S-C}$ is the possibilistic base obtained by the U - S - C method, then

$$\mathcal{B}_{F-S-C} \vdash_{\pi} (\phi, \alpha), \quad \text{for all } (\phi, \alpha) \in \mathcal{B}_{Upper-S-C}, \quad (19)$$

but not vice versa.

6.2 Properties

We now consider the logical properties of the F - S - C operator. It is easy to generalize the F - S - C operator to n possibilistic knowledge bases. We denote the generalized F - S - C operator by Δ_{F-S-C} . We have the following proposition for the logical properties of operator Δ_{F-S-C} .

Proposition 17 The F - S - C operator Δ_{F-S-C} satisfies (P1), (P2), (P4). It does not satisfy (P3), (P5) and (P6) in general.

Since the F - S - C operator does not satisfy (P3), it is syntax-dependent. However, it satisfies the following important postulate, which is falsified by all the other possibilistic merging operators.

(P7) Let $(\phi, \alpha) \in \bigcup \mathcal{E}$. If (ϕ, α) is free in $\bigcup \mathcal{E}$, then $\Delta(\mathcal{E}) \vdash_{\pi} (\phi, \beta)$, where $\beta \geq \alpha$.

Proposition 18 Δ_{F-S-C} satisfies (P7). The I - S - C operator, t -conorm based operator, and renormalization based operator do not satisfy (P7).

The computational complexity of the F - S - C operator is worse than other operators (under the usual assumption in complexity theory).

Proposition 19 Generating a knowledge base by the F - S - C method is $F\Sigma_2^P$ -complete.

6.3 Application of the F - S - C operator to merge flat knowledge bases

6.3.1 F - S - C operator: flat case

It has been pointed out in [15] that when the necessity degrees of all the possibilistic formulas are taken as 1, possibilistic logic regresses to classical logic. So classical logic is a special case of possibilistic logic in which all the formulas have the same level of priority. That is, given a set of formulas $K = \{\phi_1, \dots, \phi_n\}$ in classical logic, we can relate it to a set of possibilistic formulas $\mathcal{K} = \{(\phi_1, 1), \dots, (\phi_n, 1)\}$. Therefore, our F - S - C method can be applied to merge flat (or classical) knowledge bases. For notational simplicity, we omit “flat” and use “knowledge base” only in this section.

Definition 20 *Given a knowledge base K , the splitting of K w.r.t $Free(K)$ is a pair $\langle K_{Con}, K_{Free} \rangle$ such that $K_{Free} = \{\phi \mid \phi \in Free(K)\}$, $K_{Con} = K \setminus K_{Free}$.*

It is clear that we have the following proposition.

Proposition 20 *Let $E = \{K_1, \dots, K_n\}$ be a set of n knowledge bases. Let $\langle K', K'' \rangle$ be a splitting of $\cup E$ w.r.t $Free(\cup E)$. Suppose K_i is split into a pair $\langle K_{Con_i}, K_{Free_i} \rangle$ such that $K_{Con_i} = K' \cap K_i$ and $K_{Free_i} = K'' \cap K_i$. Let $K_{Con} = \{(\bigvee_{i=1}^n \phi_i : \phi_i \in K_{Con_i})\}$ and $K_{Free} = \cup_{i=1}^n K_{Free_i}$. Suppose the result of the F - S - C merging operator in the flat case is K_{F-S-C} , we then have $K_{F-S-C} = K_{Con} \cup K_{Free}$.*

In [4], a consequence relation called *free consequence relation* is defined to cope with inconsistency in knowledge bases.

Definition 21 *A formula ϕ is said to be a free consequence of a knowledge base K , denoted $K \models_{Free} \phi$, if and only if ϕ is logically entailed by $Free(K)$, namely,*

$$K \models_{Free} \phi, \quad \text{iff} \quad Free(K) \models \phi$$

Given two knowledge bases K_1 and K_2 , a method was introduced in [4] which concatenated K_1 and K_2 , i.e., the result of merging is $K_1 \cup K_2$. When $K_1 \cup K_2$ was inconsistent, some inconsistency tolerant consequence relations, for example, the free consequence relation, could be used to deal with it.

Proposition 21 *Let K_1 and K_2 be two flat knowledge bases. Let K_{F-S-C} be the knowledge base obtained by merging K_1 and K_2 using the F - S - C merging operator. Then every free consequence of $K_1 \cup K_2$ can be inferred from K_{F-S-C} .*

The proof of Proposition 21 shows that K_{F-S-C} keeps all the free formulas unchanged, and combines all the subbases containing conflict formulas using the maximum. In contrast, if we combine K_1 and K_2 by concatenation and deal with the inconsistency using the free consequence relation, then only free formulas are used and the conflict formulas are ignored. Consequently, the converse of Proposition 21 is false.

Example 12 *Given two flat bases $K_1 = \{\phi, \neg\phi \vee \neg\psi\}$, $K_2 = \{\psi, \neg\phi \vee \delta, \psi \vee \delta\}$, the free base of $K_1 \cup K_2$ is $Free(K_1 \cup K_2) = \{\neg\phi \vee \delta, \psi \vee \delta\}$. Splitting K_1 and K_2 with regard to $Free(K_1 \cup K_2)$, we have $K_1 = K_{Con_1} \cup K_{Free_1}$ such that $K_{Con_1} = \{\phi, \neg\phi \vee \neg\psi\}$ and $K_{Free_1} = \emptyset$, and $K_2 = K_{Con_2} \cup K_{Free_2}$ such that $K_{Con_2} = \{\psi\}$ and $K_{Free_2} = \{\neg\phi \vee \delta, \psi \vee \delta\}$. We then have*

$$K_{Con} = \{\phi \vee \psi\} \text{ and } K_{Free} = \{\neg\phi \vee \delta, \psi \vee \delta\}.$$

Finally, $K_{F-S-C} = \{\phi \vee \psi, \neg\phi \vee \delta, \psi \vee \delta\}$. Clearly, $\phi \vee \psi$ cannot be inferred from $Free(K_1 \cup K_2)$.

Application to the merging of flat knowledge bases is a very important characteristic for the F - S - C merging operator. In the flat case, the I - S - C operator is reduced to the t-conorm based operator. The renormalization based merging operator is not applicable to flat knowledge bases because the resulting knowledge base is equivalent to a knowledge base with no information.

Proposition 22 *Let $E = \{K_1, \dots, K_n\}$ be a set of knowledge bases. Let $K_{I-S-C}(E)$ and $K_{ct}(E)$ be the resulting knowledge base of I - S - C operator and t-conorm based operator respectively. Then $K_{I-S-C}(E) = K_{ct}(E) = \{\bigvee_{i=1}^n \phi_i : \phi_i \in K_i\}$. Let $K_{N,tn}(E)$ be the resulting knowledge base of a renormalization based operator. Then $K_{N,tn}(E) \equiv \top$.*

6.3.2 Comparison with other syntax-based merging methods

Let Δ_{F-S-C} denote our F - S - C merging operator. It is clear that for any knowledge profile E , $\Delta_{F-S-C}(E)$ is consistent, so Δ_{F-S-C} satisfies (A1). We also have the following properties for Δ_{F-S-C} .

Proposition 23 *Let E be a knowledge profile. If E is consistent, then $\Delta_{F-S-C}(E) = \bigwedge E$.*

Proposition 24 *If $K_1 \cup K_2$ is not consistent, then $\Delta_{F-S-C}(\{K_1\} \sqcup \{K_2\}) \not\vdash K_1$.*

Proposition 25 $\Delta_{F-S-C}(E_1) \cup \Delta_{F-S-C}(E_2) \vdash \Delta_{F-S-C}(E_1 \sqcup E_2)$

Propositions 23, 24, 25 show that the operator Δ_{F-S-C} satisfies (A2), (A4), and (A5).

However, our F - S - C merging operator does not satisfy all the other postulates in Definition 14.

Proposition 26 *The F - S - C merging operator Δ_{F-S-C} does not satisfy (A3) and (A6) in general.*

By Proposition 26, in the flat case, our F - S - C operator belongs to syntax-based or formula-based merging operators [1, 24]. Compared with model-based operators, formula-based ones are usually computationally more expensive (inference can be Π_2^p -complete) and satisfy less rationality postulates [27]. However, formula-based operators may outperform model-based ones *w.r.t* other criteria, such as strategy-proofness and discriminating power (see [16, 17, 18]). We now compare F - S - C operator with other syntax-based merging operators. Let us first introduce a merging operator in [1].

Definition 22 *Let K be a knowledge base. A subset $M \subseteq K$ is called a maximal consistent subset of K if it satisfies the following conditions:*

1. $M \not\vdash \perp$,
2. $\forall M' \subseteq K$, if $M \subset M'$, then $M' \vdash \perp$.

The set of all maximal consistent subsets of K is denoted as $MAXCONS(K)$.

Definition 23 *Let E be a knowledge profile. The maximal-consistent-subsets (MCS for short) based merging operator, denoted as Δ_{MC} , is defined as*

$$\Delta_{MC}(E) = \bigvee MAXCONS(\cup E).$$

It was shown in [24] that the MCS-based operator satisfies (A1), (A2), (A4) and (A5). However, it does not satisfy (A3) and (A6).

The resulting knowledge base of our operator is not comparable with that of MSC-based operator in general. Let us look at an example.

Example 13 Let $E = \{K_1, K_2\}$ contains two knowledge bases $K_1 = \{\phi, \neg\phi \vee \neg\psi, \varphi\}$ and $K_2 = \{\psi, \neg\psi, \neg\phi \vee \neg\varphi\}$. Since $\{\phi, \neg\phi \vee \neg\psi, \psi\}$ and $\{\phi, \neg\phi \vee \neg\varphi, \varphi\}$ are two minimal inconsistent subbases of $K_1 \cup K_2$, $Free(K_1 \cup K_2) = \emptyset$. Using the F-S-C merging operator, we get $K_{F-S-C} = \{\phi \vee \psi, \neg\phi \vee \neg\psi \vee \neg\varphi, \psi \vee \varphi\}$. By contrast, $K_1 \cup K_2$ contains five maximal consistent knowledge bases $B_1 = \{\phi, \psi, \varphi\}$, $B_2 = \{\phi, \psi, \neg\phi \vee \neg\varphi\}$, $B_3 = \{\phi, \varphi, \neg\phi \vee \neg\psi\}$, $B_4 = \{\phi, \neg\phi \vee \neg\psi, \neg\phi \vee \neg\varphi\}$, $B_5 = \{\psi, \varphi, \neg\phi \vee \neg\psi, \neg\phi \vee \neg\varphi\}$. So $\Delta_{MC}(E) = \bigvee_{i=1}^5 B_i$. It is easy to check that $\psi \vee \varphi$ can not be inferred from B_4 , therefore, it is not inferred from $\Delta_{MC}(E)$. However, $\psi \vee \varphi \in K_{F-S-C}$, so it can be inferred from K_{F-S-C} . Conversely, $\phi \vee \varphi$ can be inferred from each B_i , so it is inferred from $\Delta_{MC}(E)$. However, $\phi \vee \varphi$ cannot be inferred from K_{F-S-C} .

The MCS merging operator does not take into account the source of information in the combination process. That is, it simply conjoins the original knowledge bases first and then takes the disjunction of all the maximal consistent subsets as the resulting knowledge base. In [24], several selection functions were defined to choose among maximal consistent subsets those subsets that best fit a “merging criteria”. Generally, the merging operators based on selection functions do not satisfy (A3) and (A6). One exception is a merging operator called intersection operator, which is defined as follows.

Definition 24 [24] Let E be a knowledge set, M and K be two knowledge bases. Let us denote $dist_{\cap}(M, K) = |K \cap M|$ and $dist_{\cap}(M, E) = \sum_{K \in E} dist_{\cap}(M, K)$. Let $dist_{max} = \max_{M_i \in MAXCONS(\cup E)} (dist_{\cap}(M_i, E))$. Then the intersection operator is defined as

$$\Delta^{\cap, \Sigma}(E) = \{M \in MAXCONS(\cup E) : dist_{\cap}(M, E) = dist_{max}\}.$$

The intersection operator selects those maximal consistent subsets that fit the knowledge bases on a maximum of formulas. It was shown that operator $\Delta^{\cap, \Sigma}$ satisfies (A1),(A2),(A5),(A6). However, it does not satisfy (A3) and (A4) in general.

Let us look at Example 13 again. It is easy to check that $dist_{max} = 4$ and $\Delta^{\cap, \Sigma}(E) = B_5$. So $\Delta^{\cap, \Sigma}(E) \models \phi$ for all $\phi \in K_{F-S-C}$. However, this conclusion is not always true.

Example 14 Let $E = \{K_1, K_2\}$, where $K_1 = \{\phi, \varphi\}$ and $K_2 = \{\psi, \neg\phi \vee \neg\psi, \neg\varphi \vee \neg\delta, \delta\}$. $K_1 \cup K_2 = \{\phi, \varphi, \psi, \neg\phi \vee \neg\psi, \neg\varphi \vee \neg\delta, \delta\}$. There are nine maximal consistent subsets of $K_1 \cup K_2$, for example, $B_1 = \{\psi, \neg\phi \vee \neg\psi, \neg\varphi \vee \delta, \delta\}$ and $B_2 = \{\phi, \neg\phi \vee \neg\psi, \neg\varphi \vee \neg\delta, \delta\}$. It is easy to check that $dist_{\cap}(B_i, E) = 4$ for all i . So $dist_{max} = 4$ and $\Delta^{\cap, \Sigma}(E) = \bigvee_{i=1}^9 B_i$. A formula can be inferred from $\Delta^{\cap, \Sigma}(E)$ iff it can be inferred from all B_i 's. Next, we consider F-S-C operator. The set of free formulas of $K_1 \cup K_2$ is \emptyset . So K_1 and K_2 are split into $K_{Con1} = \{\phi, \varphi\}$ and $K_{Free1} = \emptyset$, and $K_{Con2} = \{\psi, \neg\phi \vee \neg\psi, \neg\varphi \vee \neg\delta, \delta\}$ and $K_{Free2} = K_1 \vee K_2$. We then have $K_{Con} = \{\phi \vee \psi\}$ and $K_{Free} = \emptyset$. Thus, $K_{F-S-C} = \{\phi \vee \psi, \phi \vee \neg\varphi \vee \neg\delta, \phi \vee \delta, \varphi \vee \psi, \neg\phi \vee \neg\psi \vee \varphi, \varphi \vee \delta\}$. It is clear that $\phi \vee \neg\varphi \vee \neg\delta$ is not inferred from B_1 and $\psi \vee \varphi$ is not inferred from B_2 , so both $\phi \vee \neg\varphi \vee \neg\delta$ and $\psi \vee \varphi$ cannot be inferred from $\Delta^{\cap, \Sigma}(E)$.

From the analysis above we conclude that Δ_{F-S-C} satisfies the most important postulates for merging operators in the flat cases. It is a good alternative to syntax-based operators in classical logic. More importantly, Δ_{F-S-C} can be applied to merge uncertain knowledge bases in the

framework of possibilistic logic, where the formulas in conflict are weakened to resolve inconsistency and the weights of free formulas are reinforced. So it provides a very good choice to deal with the problem of combining sources of information, especially when uncertain and incomplete information exists.

7 Related Work

Many possibilistic merging operators have been proposed [3, 6, 5, 7, 9, 10], which are very powerful for dealing with the merging problem when information is pervaded with uncertainty. A common deficiency for existing merging operators is that they are defined by a single operator, so it is not possible to differentiate different classes of information, such as *free* and *conflict* information, during the process of merging. In contrast, our merging operators split each knowledge base into two subbases and then merge different classes of subbases using different merging operators.

Possibilistic logic is closely related to Spohn's ordinal conditional functions [7, 14]. It was pointed out in [7] that semantic representation in ordinal conditional functions is basically the same as that in possibilistic logic. The only difference is that we associate each interpretation ω with an integer $\kappa(\omega)$ in ordinal conditional functions, and the lower $\kappa(\omega)$ is, the more preferred it is. So the possibilistic merging problem is related to merging problem in ordinal conditional functions. Much work has been done on merging sources of information in the framework of ordinal conditional functions [7, 30, 31, 32]. Similar to merging operators in possibilistic logic, merging in the ordinal conditional functions framework can be done semantically by aggregating kappa distributions (see [7]) or epistemic states (see [30]), which are functions from the set of interpretations to the set of integers or naturals. The syntactical representations of semantic merging operators are then obtained as in possibilistic logic (see Equation 6). The problem for merging operators in possibilistic logic is also applied to merging operator in ordinal conditional functions. That is, they also use a single operator to aggregate kappa distributions or epistemic states. In a future work, we will consider applying our split-combination operators to merging sources of information in ordinal conditional functions to solve this problem.

In [33], some aggregation operators have been proposed to combine the *belief states* of a set of information sources, which are defined as modular, transitive relations over possible worlds. The authors also differentiate conflict and conflict-free belief states in the context of merging. Our work differs from theirs in that our merging operators are defined on possibilistic knowledge bases, whilst their aggregation operators are defined on a set of belief states.

8 Conclusions and Further Work

In this paper we first proposed an incremental split-combination (*I-S-C*) merging operator which resolves inconsistency between individually consistent possibilistic knowledge bases. The *I-S-C* operator uses a weight obtained by an incremental algorithm to split each possibilistic knowledge base \mathcal{B}_i into two subbases \mathcal{C}_i and \mathcal{D}_i and then combines \mathcal{C}_i 's and \mathcal{D}_i 's by a t-conorm based merging operator and a t-norm based merging operator respectively. We proved that the resulting knowledge base of the *I-S-C* operator contains more important information than that of the t-conorm based operator. We then proposed another splitting approach, called a *free-formulas-based* splitting. The split-combination merging operator based on this splitting approach, called the *F-S-C* operator, can be applied to merge knowledge bases which are flat. In the flat case, the *F-S-C* operator satisfies most of the postulates in [23].

When discussing the logical properties of I - S - C operator, we adapted the set of postulates for merging propositional knowledge bases in [23] to possibilistic logic. However, there is no possibilistic merging operator satisfying all the postulates (our I - S - C merging operator satisfies most of the postulates). A future work is to propose such a possibilistic merging operator. Another future work is to propose more postulates for merging possibilistic knowledge bases. Postulate (P2) is a postulate which takes the weights of formulas into account. Because possibilistic logic is a kind of *weighted logic*, we may expect to find some additional postulates for possibilistic merging operators which utilize weights.

9 Appendix

Proposition 3 The resulting possibilistic base $\mathcal{B}_{Upper-S-C}$ of U - S - C operator is consistent.

Proof. Suppose $\mathcal{B}_{Upper-S-C}$ is inconsistent, then we have $(\mathcal{C} \cup \mathcal{D})^* \models false$. By Equation 10, we have $\mathcal{C}_1^* \models \mathcal{C}^*$. Therefore $(\mathcal{C}_1 \cup \mathcal{D})^* \models false$. However we have assumed that \mathcal{B}_1 is consistent, so \mathcal{C}_1^* must be consistent. Since $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \{(\phi_i \vee \psi_j, ct(\alpha_i, \beta_j)) \mid (\phi_i, \alpha_i) \in \mathcal{D}_1 \text{ and } (\psi_j, \beta_j) \in \mathcal{D}_2\}$, $\mathcal{D}^* \equiv \mathcal{D}_1^* \cup \mathcal{D}_2^*$. Therefore there must exist some formulas in $\mathcal{D}_1^* \cup \mathcal{D}_2^*$ which are in conflict with formulas in \mathcal{C}^* . This is a contradiction, because all formulas in $\mathcal{D}_1^* \cup \mathcal{D}_2^*$ are free in $\mathcal{B}_1^* \cup \mathcal{B}_2^*$. This completes our proof.

Proposition 4 Given two possibilistic knowledge bases \mathcal{B}_1 and \mathcal{B}_2 , suppose $\mathcal{B}_{Upper-S-C}$ and \mathcal{B}_{I-S-C} are the merging results of \mathcal{B}_1 and \mathcal{B}_2 using U - S - C operator and I - S - C operator respectively, then we have

$$\mathcal{B}_{I-S-C} \vdash (\phi, \alpha), \quad \text{for all } (\phi, \alpha) \in \mathcal{B}_{Upper-S-C},$$

but not vice versa.

Proof: Suppose α_i is the splitting point obtained by Algorithm 1. It is clear that $\alpha_i \leq \alpha_m$, where $\alpha_m = Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$. Suppose \mathcal{B}_1 and \mathcal{B}_2 are split by α_m into $\mathcal{B}_1 = \langle \mathcal{C}_1, \mathcal{D}_1 \rangle$ and $\mathcal{B}_2 = \langle \mathcal{C}_2, \mathcal{D}_2 \rangle$ respectively. Suppose \mathcal{B}_1 and \mathcal{B}_2 are split by α_i into $\mathcal{B}_1 = \langle \mathcal{C}'_1, \mathcal{D}'_1 \rangle$ and $\mathcal{B}_2 = \langle \mathcal{C}'_2, \mathcal{D}'_2 \rangle$ respectively. We have $\mathcal{C}'_j \subseteq \mathcal{C}_j$ and $\mathcal{D}'_j \subseteq \mathcal{D}_j$, where $j = 1, 2$. Suppose $(\phi, \alpha) \in \mathcal{B}_{Upper-S-C}$, we need to consider two cases:

Case 1: $(\phi, \alpha) \in \mathcal{D}_1 \cup \mathcal{D}_2$, then we have $(\phi, \alpha) \in \mathcal{D}'_1 \cup \mathcal{D}'_2$. So $\mathcal{B}_{I-S-C} \vdash (\phi, \alpha)$.

Case 2: $(\phi, \alpha) = (\phi_1 \vee \phi_2, \min(\alpha_1, \alpha_2))$, where $(\phi_1, \alpha_1) \in \mathcal{C}_1$ and $(\phi_2, \alpha_2) \in \mathcal{C}_2$. If (ϕ_1, α_1) or (ϕ_2, α_2) belong to $\mathcal{D}'_1 \cup \mathcal{D}'_2$, then we have $\mathcal{B}_{I-S-C} \vdash (\phi, \alpha)$. Otherwise, $(\phi_1, \alpha_1) \in \mathcal{C}'_1$ and $(\phi_2, \alpha_2) \in \mathcal{C}'_2$. Then $(\phi, \alpha) \in \mathcal{B}_{I-S-C}$, so $\mathcal{B}_{I-S-C} \vdash (\phi, \alpha)$.

\mathcal{B}_{I-S-C} is not always equivalent to $\mathcal{B}_{Upper-S-C}$. By Example 4 and Example 6, we have $\mathcal{B}_{I-S-C} = \{(\phi \vee \delta, 0.9), (\neg \phi \vee \varphi, 0.8), (\neg \psi \vee \delta, 0.7), (\neg \phi \vee \psi, 0.6), (\delta \vee \varphi, 0.5), (\phi, 0.5)\}$ and $\mathcal{B}_{Upper-S-C} = \{(\phi \vee \delta, 0.9), (\neg \phi \vee \varphi, 0.8), (\neg \psi \vee \delta, 0.7), (\neg \phi \vee \psi \vee \delta \vee \varphi, 0.5), (\phi \vee \delta \vee \varphi, 0.5), (\phi \vee \psi, 0.3), (\neg \phi \vee \psi, 0.3)\}$. It is clear that $(\neg \phi \vee \psi, 0.6)$ in \mathcal{B}_{I-S-C} cannot be inferred from $\mathcal{B}_{Upper-S-C}$.

Proposition 5 Given two possibilistic bases \mathcal{B}_1 and \mathcal{B}_2 , let $\mathcal{B}_{Upper-S-C}$ be the possibilistic base obtained by the U - S - C method (which is based on a t-norm tn and a t-conorm ct) and \mathcal{B}_{ct} be the resulting possibilistic base of merging using ct , then

$$\mathcal{B}_{Upper-S-C} \vdash_{\pi} (\phi, \alpha), \quad \text{for all } (\phi, \alpha) \in \mathcal{B}_{ct} \quad (20)$$

Proof. By Equation (5), every formula in \mathcal{B}_{dm} has the form $(\phi_i \vee \psi_j, \min(\alpha_i, \beta_j))$, where $(\phi_i, \alpha) \in \mathcal{B}_1$ and $(\psi_j, \beta_j) \in \mathcal{B}_2$, so we consider four cases:

Case 1: $\alpha_i, \beta_j \leq Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, then we have $(\phi_i \vee \psi_j, \min(\alpha_i, \beta_j)) \in \mathcal{B}_{Upper-S-C}$. So $\mathcal{B}_{Upper-S-C} \vdash_{\pi} (\phi_i \vee \psi_j, \min(\alpha_i, \beta_j))$.

Case 2: $\alpha_i > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, and $\beta_j \leq Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, then $min(\alpha_i, \beta_j) = \beta_j$ and $\mathcal{B}_{Upper-S-C} \vdash_{\pi} (\phi_i, \alpha_i)$. Since $\phi_i \models (\phi_i \vee \psi_j)$ and $\alpha_i \geq \beta_j$, we have $\mathcal{B}_{Upper-S-C} \vdash_{\pi} (\phi_i \vee \psi_j, min(\alpha_i, \beta_j))$.

Case 3: $\alpha_i \leq Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, and $\beta_j > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, this case is a dual to case 2.

Case 4: $\alpha_i > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$ and $\beta_j > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, we can suppose $\alpha_i > \beta_j$, then $min(\alpha_i, \beta_j) = \beta_j$. Since $\phi_i \models (\phi_i \vee \psi_j)$, we have $\mathcal{B}_{Upper-S-C} \vdash_{\pi} (\phi_i \vee \psi_j, min(\alpha_i, \beta_j))$.

Proposition 8 The $G-I-S-C$ operator $\Delta_{G-I-S-C}$ satisfies (P1)-(P3). It does not satisfy (P4)-(P6) in general.

Proof: It is clear that (P1) holds.

(P2) If $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$ is consistent, then $Inc(\mathcal{B}) = 0$. So $Inc(\mathcal{B}) = \gamma_{n_1}$ and $l = n_1$. Then $\mathcal{C}_i = \emptyset$ and $\mathcal{D}_i = \mathcal{B}_i$ for $i = 1, 2$. So $\Delta_{G-I-S-C}(\mathcal{E}) = \Delta_{ct}(\mathcal{E})$, where $\Delta_{tn}(\mathcal{E})$ is the resulting possibilistic knowledge base of merging \mathcal{B}_i using the t-norm tn . By Equation 9, it is clear that (P2) holds.

(P3) Before the proof of (P3), we need two lemmas.

Lemma 3 Suppose $\mathcal{E}_1 \equiv_s \mathcal{E}_2$, then $\cup \mathcal{E}_1 \equiv_s \cup \mathcal{E}_2$.

Lemma 4 Let $\mathcal{E}_1 = \{\mathcal{B}_1, \dots, \mathcal{B}_n\}$ and $\mathcal{E}_2 = \{\mathcal{B}'_1, \dots, \mathcal{B}'_n\}$, where \mathcal{B}_i and \mathcal{B}'_i are individual consistent. Suppose $\mathcal{E}_1 \equiv_s \mathcal{E}_2$, and \mathcal{B}_i and \mathcal{B}'_i ($i = 1, \dots, n$) are split w.r.t a weight γ into $\mathcal{B}_i = \langle \mathcal{C}_i, \mathcal{D}_i \rangle$ and $\mathcal{B}'_i = \langle \mathcal{C}'_i, \mathcal{D}'_i \rangle$ respectively, then $\mathcal{C}_i \equiv_s \mathcal{C}'_i$ and $\mathcal{D}_i \equiv_s \mathcal{D}'_i$.

The proofs of Lemma 3 and Lemma 4 are clear from the definitions of equivalence of possibilistic knowledge profiles and possibilistic knowledge bases, we do not provide them here.

We now continue the proof of (P3).

Suppose $\gamma^{\mathcal{E}_1}$ and $\gamma^{\mathcal{E}_2}$ are splitting points of \mathcal{E}_1 and \mathcal{E}_2 obtained by Algorithm 2 respectively, we now prove that $\Delta_{G-I-S-C}(\mathcal{E}_1) \equiv_s \Delta_{G-I-S-C}(\mathcal{E}_2)$ by induction over number t of times that the algorithm goes to Step 3. It is clear that if $\cup \mathcal{E}_1$ or $\cup \mathcal{E}_2$ is consistent, the conclusion holds. We assume $\cup \mathcal{E}_i$ ($i = 1, 2$) are inconsistent.

1) When $t = 1$, by Lemma 3, $\cup \mathcal{E}_1 \equiv_s \cup \mathcal{E}_2$, we have $Inc(\cup \mathcal{E}_1) = Inc(\cup \mathcal{E}_2) = \gamma_k$ and then $l_{\mathcal{E}_1} = l_{\mathcal{E}_2} = k - 1$. Suppose $\mathcal{B}_{S-C}^{\mathcal{E}_i}$ ($i = 1, 2$) are resulting possibilistic knowledge bases of merging \mathcal{E}_i by the $S-C$ operator which uses γ_{k-1} as the splitting point. By Lemma 4, it is clear that $\mathcal{B}_{S-C}^{\mathcal{E}_1} \equiv_s \mathcal{B}_{S-C}^{\mathcal{E}_2}$. If $\mathcal{B}_{S-C}^{\mathcal{E}_1}$ (or $\mathcal{B}_{S-C}^{\mathcal{E}_2}$) is consistent, then the algorithm stops and we can conclude that $\Delta_{G-I-S-C}(\mathcal{E}_1) \equiv_s \Delta_{G-I-S-C}(\mathcal{E}_2)$. Otherwise, the Algorithm goes to Step 3.

2) Suppose $t = j$. Let γ_{l_1} and γ_{l_2} be the splitting point for \mathcal{E}_1 and \mathcal{E}_2 respectively, $\mathcal{B}_{S-C}^{\mathcal{E}_1, j}$ and $\mathcal{B}_{S-C}^{\mathcal{E}_2, j}$ be resulting possibilistic knowledge bases of merging \mathcal{E}_1 and \mathcal{E}_2 by the $S-C$ operator which uses γ_{l_1} and γ_{l_2} as the splitting point respectively. Let us assume that $\gamma_{l_1} = \gamma_{l_2}$, $\mathcal{B}_{S-C}^{\mathcal{E}_1, j} \equiv_s \mathcal{B}_{S-C}^{\mathcal{E}_2, j}$, and $\mathcal{B}_{S-C}^{\mathcal{E}_i, j}$ ($i = 1, 2$) are inconsistent. Since $\gamma_{l_1} = \gamma_{l_2}$, we have $l_1 = l_2$. Suppose $t = j + 1$. It is clear that $\gamma_{l_1-1} = \gamma_{l_2-1}$. Let $\mathcal{B}_{S-C}^{\mathcal{E}_1, j+1}$ and $\mathcal{B}_{S-C}^{\mathcal{E}_2, j+1}$ be resulting possibilistic knowledge bases of merging \mathcal{E}_1 and \mathcal{E}_2 by the $S-C$ operator which uses γ_{l_1-1} and γ_{l_2-1} as the splitting point respectively. We only need to prove that $\mathcal{B}_{S-C}^{\mathcal{E}_1, j+1} \equiv_s \mathcal{B}_{S-C}^{\mathcal{E}_2, j+1}$. This is easy to follow from Lemma 4. If $\mathcal{B}_{S-C}^{\mathcal{E}_1, j+1}$ is consistent, then the algorithm stops and we can conclude that $\Delta_{G-I-S-C}(\mathcal{E}_1) \equiv_s \Delta_{G-I-S-C}(\mathcal{E}_2)$. Otherwise, the Algorithm goes to Step 3.

By 1) and 2) we have proven that $\Delta_{G-I-S-C}(\mathcal{E}_1) \equiv_s \Delta_{G-I-S-C}(\mathcal{E}_2)$.

(P4)-(P6): Let us consider the following counterexample.

Let $\mathcal{E}_1 = \{\mathcal{B}_1, \mathcal{B}_2\}$ and $\mathcal{E}_2 = \{\mathcal{B}_3, \mathcal{B}_4\}$, where $\mathcal{B}_1 = \{(\phi, 0.9), (\neg\psi, 0.7)\}$, $\mathcal{B}_2 = \{(\psi, 0.8), (\neg\phi, 0.6)\}$, $\mathcal{B}_3 = \{(\varphi, 0.4)\}$ and $\mathcal{B}_4 = \{(\chi, 0.4)\}$. Let $ct = max$ and $tn = min$. It is easy to check that $\Delta_{G-I-S-C}(\mathcal{E}_1) = \{(\phi, 0.9), (\neg\phi \vee \neg\psi, 0.6)\}$, $\Delta_{G-I-S-C}(\mathcal{E}_2) = \{(\varphi, 0.4), (\chi, 0.4)\}$. Since $\Delta_{G-I-S-C}(\mathcal{E}_1) \cup \Delta_{G-I-S-C}(\mathcal{E}_2)$ is consistent, $\Delta_{G-I-S-C}(\{\Delta_{G-I-S-C}(\mathcal{E}_1)\}, \{\Delta_{G-I-S-C}(\mathcal{E}_2)\}) = \{(\phi, 0.9), (\neg\phi \vee \neg\psi, 0.6), (\varphi, 0.4), (\chi, 0.4)\}$. In contrast, we have $\Delta_{G-I-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2) = \{(\phi, 0.9), (\psi, 0.8), (\neg\phi \vee \neg\psi \vee$

$\varphi \vee \chi, 0.4\}$. $(\chi, 0.4) \in \Delta_{G-I-S-C}(\{\Delta_{G-I-S-C}(\mathcal{E}_1)\}, \{\Delta_{G-I-S-C}(\mathcal{E}_2)\})$, but it cannot be inferred from $\Delta_{G-I-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2)$. Conversely, $(\psi, 0.8) \in \Delta_{G-I-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2)$, but it cannot be inferred from $\Delta_{G-I-S-C}(\{\Delta_{G-I-S-C}(\mathcal{E}_1)\}, \{\Delta_{G-I-S-C}(\mathcal{E}_2)\})$. So (P4) and (P5) are not satisfied. We consider (P6). Since $\Delta_{G-I-S-C}(\mathcal{E}_1)^* \wedge \Delta_{G-I-S-C}(\mathcal{E}_2)^* \equiv \phi \wedge \neg\psi \wedge \varphi \wedge \chi$ and $(\Delta_{G-I-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2))^* \equiv \phi \wedge \psi \wedge (\varphi \vee \chi)$, so $\Delta_{G-I-S-C}(\mathcal{E}_1)^* \wedge \Delta_{G-I-S-C}(\mathcal{E}_2)^* \not\equiv (\Delta_{G-I-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2))^*$. However, $\Delta_{G-I-S-C}(\mathcal{E}_1) \cup \Delta_{G-I-S-C}(\mathcal{E}_2)$ is consistent, so (P6) is not satisfied.

Proposition 9 The t-conorm based merging operator Δ_{ct} satisfies (P1),(P3)-(P5). It doesn't not satisfy (P2) in general. The operator $\Delta_{N,tn}$ satisfies (P1)-(P3). It does not satisfy (P4) and (P5) in general.

Proof: We first consider operator Δ_{ct} .

It is clear that (P1) is satisfied by Δ_{ct} . We give an counterexample for (P2).

Example 15 Let $\mathcal{E} = \{\mathcal{B}_1, \mathcal{B}_2\}$, where $\mathcal{B}_1 = \{(p, 0.7)\}$ and $\mathcal{B}_2 = \{(q, 0.9)\}$, where p, q are two propositional symbols. Let ct be a t-conorm, By Equation 10, $\Delta_{ct}(\mathcal{E}) = \{(\phi \vee \psi, \alpha)\}$, where $\alpha = tn(0.7, 0.9)$. It is easy to check that $\Delta_{ct}(\mathcal{E}) \not\vdash_{\pi} (p, 0.7)$, so $\exists \beta \geq 0.7$ such that $\Delta_{ct}(\mathcal{E}) \not\vdash_{\pi} (p, \beta)$.

(P3) holds because Δ_{ct} is semantically defined by aggregation of possibility distributions associated \mathcal{B}_i .

(P4) and (P5): Let π_{ct, \mathcal{E}_1} , π_{ct, \mathcal{E}_2} and $\pi_{ct, \mathcal{E}_1 \sqcup \mathcal{E}_2}$ be possibility distributions obtained by aggregating possibility distribution associated with possibilistic bases of \mathcal{E}_1 , \mathcal{E}_2 and $\mathcal{E}_1 \sqcup \mathcal{E}_2$ using ct respectively. Let π_{ct} be the possibility distribution obtained by aggregating π_{ct, \mathcal{E}_1} and π_{ct, \mathcal{E}_2} using ct . To prove $\Delta(\{\Delta_{ct}(\mathcal{E}_1)\}, \{\Delta_{ct}(\mathcal{E}_2)\}) \equiv_s \Delta(\mathcal{E}_1 \sqcup \mathcal{E}_2)$, we only need to prove $\pi_{ct}(\omega) = \pi_{ct, \mathcal{E}_1 \sqcup \mathcal{E}_2}(\omega)$, for all ω . This equation holds because ct is a associative and commutative operator.

It is clear that Δ_{ct} does not satisfy (P6).

Next we consider operator $\Delta_{N,tn}$.

(P1): By Equation 11, (P1) holds clearly.

(P2): If $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$ is consistent, $\Delta_{N,tn}(\mathcal{E}) = \Delta_{tn}(\mathcal{E})$, where $\Delta_{tn}(\mathcal{E})$ is the resulting possibilistic knowledge base of merging \mathcal{B}_i using the t-norm tn . By Equation 9, it is clear that (P2) holds.

(P3): By Equation 11, it is clear that (P3) holds due to $\Delta_{tn}(\mathcal{E}_1) \equiv_s \Delta_{tn}(\mathcal{E}_2)$.

(P4) and (P5): Let us consider the following counterexample.

Let $\mathcal{E}_1 = \{\mathcal{B}_1, \mathcal{B}_2\}$ and $\mathcal{E}_2 = \{\mathcal{B}_3, \mathcal{B}_4\}$, where $\mathcal{B}_1 = \{(\phi, 0.8), (\psi, 0.4), (\varphi, 0.4)\}$, $\mathcal{B}_2 = \{(\phi \rightarrow \neg\psi, 0.6)\}$, $\mathcal{B}_3 = \{(\varphi, 0.4)\}$ and $\mathcal{B}_4 = \{(\chi, 0.4)\}$. Let tn be the product operator. By Equation 9, we have $\Delta_{tn}(\mathcal{E}_1) = \{(\phi, 0.8), (\neg\phi \vee \neg\psi \vee \neg\varphi, 0.76), (\phi \rightarrow \neg\psi, 0.6), (\psi, 0.4), (\varphi, 0.4)\}$ and $\Delta_{tn}(\mathcal{E}_2) = \{(\varphi \vee \chi, 0.64), (\varphi, 0.4), (\chi, 0.4)\}$. So $\Delta_{N,tn}(\mathcal{E}_1) = \{(\phi, 0.8), (\neg\phi \vee \neg\psi \vee \neg\varphi, 0.76), (\phi \rightarrow \neg\psi, 0.6)\}$ and $\Delta_{N,tn}(\mathcal{E}_2) = \{(\varphi \vee \chi, 0.64), (\varphi, 0.4), (\chi, 0.4)\}$. Combining $\Delta_{N,tn}(\mathcal{E}_1)$ and $\Delta_{N,tn}(\mathcal{E}_2)$ by tn we get $\Delta_{tn}(\{\Delta_{N,tn}(\mathcal{E}_1)\}, \{\Delta_{N,tn}(\mathcal{E}_2)\}) = \{(\phi \vee \varphi \vee \chi, 0.93), (\phi \vee \varphi, 0.88), (\phi \vee \chi, 0.88), (\neg\phi \vee \neg\psi \vee \chi, 0.86), (\neg\phi \vee \neg\psi \vee \varphi \vee \chi, 0.85), (\phi, 0.8), (\neg\phi \vee \neg\psi \vee \varphi, 0.76), (\neg\phi \vee \neg\psi \vee \chi, 0.76), (\neg\phi \vee \neg\psi \vee \neg\varphi, 0.76), (\varphi \vee \chi, 0.64), (\phi \rightarrow \neg\psi, 0.6), (\varphi, 0.4), (\chi, 0.4)\}$. Since $\Delta_{tn}(\{\Delta_{N,tn}(\mathcal{E}_1)\}, \{\Delta_{N,tn}(\mathcal{E}_2)\})$ is consistent, we have $\Delta_{N,tn}(\{\Delta_{N,tn}(\mathcal{E}_1)\}, \{\Delta_{N,tn}(\mathcal{E}_2)\}) = \Delta_{tn}(\{\Delta_{N,tn}(\mathcal{E}_1)\}, \{\Delta_{N,tn}(\mathcal{E}_2)\})$. In contrast, since $\mathcal{E}_1 \sqcup \mathcal{E}_2 = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\}$, we have $\Delta_{tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2) = \{(\phi \vee \varphi \vee \chi, 0.93), (\phi \vee \varphi, 0.88), (\phi \vee \chi, 0.88), (\neg\phi \vee \neg\psi \vee \chi, 0.86), (\neg\phi \vee \neg\psi \vee \varphi \vee \chi, 0.85), (\phi, 0.8), (\varphi \vee \chi, 0.78), (\psi \vee \varphi \vee \chi, 0.78), (\neg\phi \vee \neg\psi \vee \varphi, 0.76), (\neg\phi \vee \neg\psi \vee \chi, 0.76), (\neg\phi \vee \neg\psi \vee \neg\varphi, 0.76), (\varphi \vee \chi, 0.64), (\psi \vee \varphi, 0.64), (\psi \vee \chi, 0.64), (\varphi, 0.64), (\varphi \vee \psi, 0.64), (\phi \rightarrow \neg\psi, 0.6), (\varphi, 0.4), (\chi, 0.4), (\psi, 0.4)\}$. The inconsistency degree of $\Delta_{tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2)$ is 0.4, so $\Delta_{N,tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2) = \{(\phi \vee \varphi \vee \chi, 0.93), (\phi \vee \varphi, 0.88), (\phi \vee \chi, 0.88), (\neg\phi \vee \neg\psi \vee \chi, 0.86), (\neg\phi \vee \neg\psi \vee \varphi \vee \chi, 0.85), (\phi, 0.8), (\varphi \vee \chi, 0.78), (\psi \vee \varphi \vee \chi, 0.78), (\neg\phi \vee \neg\psi \vee \varphi, 0.76), (\neg\phi \vee \neg\psi \vee \chi, 0.76), (\neg\phi \vee \neg\psi \vee \neg\varphi, 0.76), (\varphi \vee \chi, 0.64), (\psi \vee \varphi, 0.64), (\psi \vee \chi, 0.64), (\varphi, 0.64), (\varphi \vee \psi, 0.64), (\phi \rightarrow \neg\psi, 0.6)\}$. It is clear that $(\varphi, 0.4)$ in $\Delta_{N,tn}(\{\Delta_{N,tn}(\mathcal{E}_1)\}, \{\Delta_{N,tn}(\mathcal{E}_2)\})$ cannot be inferred from $\Delta_{N,tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2)$ and $(\varphi, 0.64)$ in $\Delta_{N,tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2)$ cannot be inferred from $\Delta_{N,tn}(\{\Delta_{N,tn}(\mathcal{E}_1)\}, \{\Delta_{N,tn}(\mathcal{E}_2)\})$.

(P6) Let us look at the counterexample in the proof of Proposition 8. Let $tn = \min$. $\Delta_{N,tn}(\mathcal{E}_1) = \{(\phi, 0.9), (\psi, 0.8)\}$ and $\Delta_{N,tn}(\mathcal{E}_2) = \{(\varphi, 0.4), (\chi, 0.4)\}$. Clearly, $\Delta_{N,tn}(\mathcal{E}_1) \cup \Delta_{N,tn}(\mathcal{E}_2)$ is consistent. We also have $\Delta_{N,tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2) = \{(\phi, 0.9), (\psi, 0.8)\}$. It is clear that $\Delta_{N,tn}(\mathcal{E}_1)^* \wedge \Delta_{N,tn}(\mathcal{E}_2)^* \neq (\Delta_{N,tn}(\mathcal{E}_1 \sqcup \mathcal{E}_2))^*$.

Proposition 10 Generating a consistent possibilistic knowledge base by I - S - C operator is in $F\Delta_2^p(\mathcal{O}(n))$, where $\Delta_2^p(\mathcal{O}(n))$ denotes the set of decision problems decidable in polynomial time with no more than $\mathcal{O}(n)$ calls to an NP oracle, where n is the number of different valuations involved in \mathcal{B} . [22], and “F” in $F\Delta_2^p(\mathcal{O}(n))$ stands for *function* and is intended to turn a complexity class for decision problem into one for *search problem*, i.e., problems that have answers.

Proof: In Step 1, rearranging the weights of formulas in \mathcal{B} is a *sort problem*, which can be solved in polynomial time. By Proposition 13 in [20], computing $Inc(\mathcal{B})$ is NP -hard and requires $\lceil \log_2 n \rceil$ satisfiability checks. Steps 3, 4 and 5 can be carried out in polynomial time. Step 6 needs a satisfiability check, so it is NP -hard. Since $Inc(\mathcal{B})$ and $Free_{upp}(\mathcal{B})$ are lower and upper bound respectively for the splitting point, we know that Steps 3-6 will be repeated at most $k-m+1$ times, where k and m are such that $a_k = Inc(\mathcal{B})$ and $a_m = Free_{upp}(\mathcal{B})$. Therefore, computation of I - S - C operator needs at most $k-m+\lceil \log_2 n \rceil+1$ satisfiability checks. This proves the proposition.

Proposition 11 Generating a consistent possibilistic knowledge base by a t-cornorm based merging operator is in FP and generating a consistent possibilistic knowledge base by a renormalization based merging operator is in $F\Delta_2^p$.

Proof: To generate a possibilistic knowledge base by a t-cornorm based merging operator, we only need to take the disjunctions of formulas in \mathcal{B}_i 's and compute the weight associated with the disjunctions. Both computations can be done in polynomial time.

The renormalization based merging operator is computed in two steps. The first step is to combine the original possibilistic knowledge bases using a t-norm. This step can be done in polynomial time. In the second step, we need to compute the inconsistency degree of the possibilistic knowledge base obtained by the first step. Then only those possibilistic formulas whose weights are greater than the inconsistency degree are kept in the resulting possibilistic knowledge base. By Proposition 13 in [20], computing the inconsistency degree of a possibilistic knowledge base is NP -hard and requires $\lceil \log_2 n \rceil$ satisfiability checks, where n is the number of different valuations involved in \mathcal{B} . Therefore, computation of renormalization based operator needs at most $\lceil \log_2 n \rceil$ satisfiability checks. This proves the proposition.

Proposition 14 Let $\mathcal{B}_1 = \{(\phi_i, \alpha_i) : i = 1, \dots, n\}$ and $\mathcal{B}_2 = \{(\psi_j, \beta_j) : j = 1, \dots, m\}$ be two possibilistic knowledge bases, $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ be their associated possibility distributions. We have $\pi_{\mathcal{B}_{I-S-C}}(\omega) = \pi_{I-S-C}(\omega)$, for all ω .

Proof: Let us compare Algorithm 1 and Algorithm 3 step by step.

Step 1 and Step 2: two algorithms have the same Step 1 and Step 2;

Step 3: by Proposition 12, we have $\pi_{11} = \pi_{\mathcal{C}_1}$ and $\pi_{12} = \pi_{\mathcal{D}_1}$, and $\pi_{21} = \pi_{\mathcal{C}_2}$ and $\pi_{22} = \pi_{\mathcal{D}_2}$;

Step 4: by Equation 6 and discussions in Section 2, we have $\pi_1 = \pi_{\mathcal{C}}$ and $\pi_2 = \pi_{\mathcal{D}}$;

Step 5: by Equation 6 and discussions in Section 2, we have $\pi_{I-S-C} = \pi_{\mathcal{C} \cup \mathcal{D}}$, that is, $\pi_{I-S-C} = \pi_{\mathcal{B}_{I-S-C}}$;

Step 6: Since $\pi_{I-S-C} = \pi_{\mathcal{B}_{I-S-C}}$, π_{I-S-C} is subnormal if and only if \mathcal{B}_{I-S-C} is inconsistent.

By the comparison above, we can infer that Algorithm 1 and Algorithm 3 output the same splitting point γ . Therefore, by Proposition 12, it is clear that $\pi_{\mathcal{B}_{I-S-C}} = \pi_{I-S-C}$.

Proposition 15 Let \mathcal{B}_1 and \mathcal{B}_2 be two possibilistic bases, and their associated possibility distributions are π_1 and π_2 respectively. Let π_{I-S-C} be the resulting possibility distribution of merging

by the semantic I - S - C operator and π_{ct} be the resulting possibility distribution by the t-conorm, then π_{I-S-C} is more specific than π_{ct} , that is $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$ for all $\omega \in \Omega$.

Proof: Let ω be an arbitrary possible world. By Definition 17, we have $\pi_{I-S-C}(\omega) = \min(ct(\pi_{11}(\omega), \pi_{21}(\omega)), tn(\pi_{12}(\omega), \pi_{22}(\omega)))$. In contrast, $\pi_{ct}(\omega) = ct(\pi_{B_1}(\omega), \pi_{B_2}(\omega))$. Since $\pi_{B_1}(\omega) = \min(\pi_{11}(\omega), \pi_{12}(\omega))$ and $\pi_{B_2}(\omega) = \min(\pi_{21}(\omega), \pi_{22}(\omega))$, we have $\pi_{ct}(\omega) = ct(\min(\pi_{11}(\omega), \pi_{12}(\omega)), \min(\pi_{21}(\omega), \pi_{22}(\omega)))$. We now prove that $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$. Since $tn(\pi_{12}(\omega), \pi_{22}(\omega)) \leq \min(\pi_{12}(\omega), \pi_{22}(\omega))$, $\pi_{I-S-C}(\omega) \leq \min(ct(\pi_{11}(\omega), \pi_{21}(\omega)), \min(\pi_{12}(\omega), \pi_{22}(\omega)))$. We consider the following two cases:

Case 1: Suppose $\min(\pi_{12}(\omega), \pi_{22}(\omega)) \geq ct(\pi_{11}(\omega), \pi_{21}(\omega))$. Then $\pi_{I-S-C} \leq ct(\pi_{11}(\omega), \pi_{21}(\omega))$. Since $ct(\pi_{11}(\omega), \pi_{21}(\omega)) \geq \max(\pi_{11}(\omega), \pi_{21}(\omega))$, we have $\min(\pi_{12}(\omega), \pi_{22}(\omega)) \geq \max(\pi_{11}(\omega), \pi_{21}(\omega))$. So $\pi_{11}(\omega) \leq \min(\pi_{12}(\omega), \pi_{22}(\omega))$ and $\pi_{21}(\omega) \leq \min(\pi_{12}(\omega), \pi_{22}(\omega))$. We then have $\pi_{ct}(\omega) = ct(\pi_{11}(\omega), \pi_{21}(\omega))$. Therefore, $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$.

Case 2: Suppose $\min(\pi_{12}(\omega), \pi_{22}(\omega)) < ct(\pi_{11}(\omega), \pi_{21}(\omega))$. Then $\pi_{I-S-C} \leq \min(\pi_{12}(\omega), \pi_{22}(\omega))$. We can prove that $\min(\pi_{12}(\omega), \pi_{22}(\omega)) \leq \pi_{ct}(\omega)$. Suppose either $\pi_{11}(\omega) \geq \pi_{12}(\omega)$ or $\pi_{21}(\omega) \geq \pi_{22}(\omega)$, then $\pi_{ct}(\omega) \geq \min(\pi_{12}(\omega), \pi_{22}(\omega))$. So $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$. Otherwise, suppose $\pi_{11}(\omega) < \pi_{12}(\omega)$ and $\pi_{21}(\omega) < \pi_{22}(\omega)$, then $\pi_{ct}(\omega) = ct(\pi_{11}(\omega), \pi_{21}(\omega))$. By assumption, we have $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$.

According to Case 1 and Case 2, we can conclude that $\pi_{I-S-C}(\omega) \leq \pi_{ct}(\omega)$. This completes the proof.

Proposition 16 Given two possibilistic bases \mathcal{B}_1 and \mathcal{B}_2 , if \mathcal{B}_{F-S-C} is the possibilistic base obtained by F - S - C method and $\mathcal{B}_{Upper-S-C}$ is the possibilistic base obtained by U - S - C method, then

$$\mathcal{B}_{F-S-C} \vdash_{\pi} (\phi, \alpha), \quad \text{for all } (\phi, \alpha) \in \mathcal{B}_{Upper-S-C}, \quad (21)$$

but not vice versa.

Proof. Let $(\varphi, \delta) \in \mathcal{B}_{Upper-S-C}$, since $\mathcal{B}_{Upper-S-C} = \mathcal{C} \cup \mathcal{D}$, where \mathcal{C} is obtained by Equation (7) and \mathcal{D} is obtained by Equation (8), we have $(\varphi, \delta) \in \mathcal{C} \cup \mathcal{D}$. On the one hand, suppose $(\varphi, \delta) \in \mathcal{D}$, then $(\varphi, \delta) \in \mathcal{D}_1 \cup \mathcal{D}_2$. By Lemma 2, $\mathcal{D}_1 \subseteq \mathcal{D}_{Free_1}$ and $\mathcal{D}_2 \subseteq \mathcal{D}_{Free_2}$. So $(\varphi, \delta) \in \mathcal{D}_{Free_1} \cup \mathcal{D}_{Free_2} = \mathcal{D}_{Free}$. Since $\mathcal{B}_{F-S-C} = \mathcal{C}_{Con} \cup \mathcal{D}_{Free}$, we have $(\varphi, \delta) \in \mathcal{B}_{F-S-C}$, so $\mathcal{B}_{F-S-C} \vdash_{\pi} (\varphi, \delta)$. On the other hand, suppose $(\varphi, \delta) \in \mathcal{C}$, then (φ, δ) has the form $(\phi_i \vee \psi_j, \min(\alpha_i, \beta_j))$, where $(\phi_i, \alpha_i) \in \mathcal{C}_1$ and $(\psi_j, \beta_j) \in \mathcal{C}_2$. By Lemma 2, $\mathcal{C}_{Con_1} \subseteq \mathcal{C}_1$ and $\mathcal{C}_{Con_2} \subseteq \mathcal{C}_2$. We consider following two cases:

Case 1: $(\phi_i, \alpha_i) \in \mathcal{C}_{Con_1}$ and $(\psi_j, \beta_j) \in \mathcal{C}_{Con_2}$. In this case, we have $(\varphi, \delta) = (\phi_i \vee \psi_j, \min(\alpha_i, \beta_j)) \in \mathcal{C}_{Con}$. So $(\varphi, \delta) \in \mathcal{B}_{F-S-C}$ and $\mathcal{B}_{F-S-C} \vdash_{\pi} (\varphi, \delta)$.

Case 2: $(\phi_i, \alpha_i) \notin \mathcal{C}_{Con_1}$ or $(\psi_j, \beta_j) \notin \mathcal{C}_{Con_2}$. Assume $(\psi_j, \beta_j) \notin \mathcal{C}_{Con_2}$ (for $(\phi_i, \alpha_i) \notin \mathcal{C}_{Con_1}$, the proof is similar). In this case, $(\psi_j, \beta_j) \in \mathcal{D}_{Free}$, so $(\psi_j, \beta_j) \in \mathcal{B}_{F-S-C}$. Since $\psi_j \models \varphi$, and $\beta_j \geq \min(\alpha_i, \beta_j) = \delta$, we have $\mathcal{B}_{F-S-C} \vdash_{\pi} (\varphi, \delta)$.

Conversely, let us look at Example 8 again. $Free_{upper}(\mathcal{B}_1 \cup \mathcal{B}_2) = 0.7$. So \mathcal{B}_1 is split into $\mathcal{C}_1 = \{(\phi, 0.7), (\psi, 0.7)\}$ and $\mathcal{D}_1 = \emptyset$, and \mathcal{B}_2 is split into $\mathcal{C}_2 = \{(\neg\phi, 0.6), (\psi, 0.4)\}$ and $\mathcal{D}_2 = \emptyset$. So $\mathcal{B}_{Upper-S-C} = \{(\phi \vee \psi, 0.4), (\neg\phi \vee \psi, 0.6), (\psi, 0.4)\}$. By Example 11, $\mathcal{B}_{F-S-C} = \{(\psi, 0.88)\}$. So every possibilistic formula in $\mathcal{B}_{Upper-S-C}$ can be inferred from \mathcal{B}_{F-S-C} . In contrast, $(\psi, 0.88)$ in \mathcal{B}_{F-S-C} cannot be inferred from $\mathcal{B}_{Upper-S-C}$.

Proposition 17 The F - S - C operator Δ_{F-S-C} satisfies (P1), (P2), (P4). It does not satisfy (P3), (P5) and (P6) in general.

Proof: By Lemma 2, (P1) is satisfied. (P2) is satisfied because F - S - C operator is equivalent to the t-norm based operator when $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$ is consistent. (P3) is falsified. Let us look at an counterexample.

Example 16 Let $\mathcal{E}_1 = \{\mathcal{B}_1, \mathcal{B}_2\}$, where $\mathcal{B}_1 = \{(\phi, 0.6), (\psi, 0.6)\}$ and $\mathcal{B}_2 = \{(\neg\psi, 0.9)\}$. Let $\mathcal{E}_2 = \{\mathcal{B}_3, \mathcal{B}_4\}$, where $\mathcal{B}_3 = \{(\phi \wedge \psi, 0.6)\}$ and $\mathcal{B}_4 = \{(\neg\psi, 0.9)\}$. It is clear that $\mathcal{E}_1 \equiv \mathcal{E}_2$. Suppose

$ct = \max$ and $tn = \min$. We have $\Delta_{F-S-C}(\mathcal{E}_1) = \{(\phi, 0.6)\}$ and $\Delta_{F-S-C}(\mathcal{E}_2) = \{(\phi \vee \neg\psi, 0.6)\}$. So $\Delta_{F-S-C}(\mathcal{E}_1) \not\equiv_s \Delta_{F-S-C}(\mathcal{E}_2)$.

(P4): Let $\mathcal{E}_1 = \{\mathcal{B}_1, \dots, \mathcal{B}_m\}$ and $\mathcal{E}_2 = \{\mathcal{B}_{m+1}, \dots, \mathcal{B}_n\}$. Let $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$. Suppose \mathcal{B}_i ($i = 1, \dots, n$) are split *w.r.t* $Free(\mathcal{B})$ into \mathcal{C}_i and \mathcal{D}_i . Let \mathcal{C} and \mathcal{D} be the possibilistic knowledge bases obtained by merging \mathcal{C}_i by a t-conorm ct (its dual t-norm is tn) and \mathcal{D}_i by t-norm tn' (its dual t-conorm is ct'). By Equation 10 and Equation 9, we have $\mathcal{C} = \{(\phi_1 \vee \dots \vee \phi_n, tn(\alpha_1, \dots, \alpha_n)) : (\phi_i, \alpha_i) \in \mathcal{C}_i\}$ and $\mathcal{D} = \mathcal{D}_1 \cup \dots \cup \mathcal{D}_n \cup \{(D_j, 1 - tn'(x_1, \dots, x_n)) : j = 2, \dots, n\}$ (see Proposition 2 for the definition of D_j). Then $\Delta_{F-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2) = \mathcal{C} \cup \mathcal{D}$. Next, suppose \mathcal{B}_i ($i = 1, \dots, m$) are split *w.r.t* $Free(\mathcal{B}_1 \cup \dots \cup \mathcal{B}_m)$ into \mathcal{C}'_i and \mathcal{D}'_i , and \mathcal{B}_i ($i = m+1, \dots, n$) are split *w.r.t* $Free(\mathcal{B}_{m+1} \cup \dots \cup \mathcal{B}_n)$ into \mathcal{C}''_i and \mathcal{D}''_i . It is clear that $\mathcal{C}'_i \subseteq \mathcal{C}_i$ and $\mathcal{D}_i \subseteq \mathcal{D}'_i$. Let \mathcal{C}' and \mathcal{D}' be the possibilistic knowledge bases obtained by merging \mathcal{C}'_i ($i = 1, \dots, m$) by ct and \mathcal{D}'_i ($i = 1, \dots, m$) by tn' , and \mathcal{C}'' and \mathcal{D}'' be the possibilistic knowledge bases obtained by merging \mathcal{C}''_i ($i = m+1, \dots, n$) by ct and \mathcal{D}''_i ($i = m+1, \dots, n$) by tn' . By Equations 10 and 9, we have $\mathcal{C}' = \{(\phi_1 \vee \dots \vee \phi_m, tn(\alpha_1, \dots, \alpha_m)) : (\phi_i, \alpha_i) \in \mathcal{C}'_i, i = 1, \dots, m\}$ and $\mathcal{D}' = \mathcal{D}'_1 \cup \dots \cup \mathcal{D}'_m \cup \{(D'_j, 1 - tn'(x_1, \dots, x_m)) : j = 2, \dots, m\}$, and $\mathcal{C}'' = \{(\phi_{m+1} \vee \dots \vee \phi_n, tn(\alpha_{m+1}, \dots, \alpha_n)) : (\phi_i, \alpha_i) \in \mathcal{C}''_i, i = m+1, \dots, n\}$ and $\mathcal{D}'' = \mathcal{D}''_{m+1} \cup \dots \cup \mathcal{D}''_n \cup \{(D''_j, 1 - tn'(x_1, \dots, x_n)) : j = 2, \dots, n\}$. By Definition 7, $\Delta_{F-S-C}(\mathcal{E}_1) = \mathcal{C}' \cup \mathcal{D}'$ and $\Delta_{F-S-C}(\mathcal{E}_2) = \mathcal{C}'' \cup \mathcal{D}''$. Suppose $\Delta_{F-S-C}(\mathcal{E}_1)$ is split *w.r.t* $Free(\Delta_{F-S-C}(\mathcal{E}_1) \cup \Delta_{F-S-C}(\mathcal{E}_2))$ into $\mathcal{C}_{\mathcal{E}_1}$ and $\mathcal{D}_{\mathcal{E}_1}$, and $\Delta_{F-S-C}(\mathcal{E}_2)$ is split *w.r.t* $Free(\Delta_{F-S-C}(\mathcal{E}_1) \cup \Delta_{F-S-C}(\mathcal{E}_2))$ into $\mathcal{C}_{\mathcal{E}_2}$ and $\mathcal{D}_{\mathcal{E}_2}$. Let $\mathcal{C}_{\mathcal{E}_1, \mathcal{E}_2}$ and $\mathcal{D}_{\mathcal{E}_1, \mathcal{E}_2}$ be obtained by merging $\mathcal{C}_{\mathcal{E}_1}$ and $\mathcal{C}_{\mathcal{E}_2}$ by ct and merging $\mathcal{D}_{\mathcal{E}_1}$ and $\mathcal{D}_{\mathcal{E}_2}$ by tn respectively. By Equations 10 and 9, we have $\mathcal{C}_{\mathcal{E}_1, \mathcal{E}_2} = \{(\phi_i \vee \psi_j, tn(\alpha_i, \beta_j)) : (\phi_i, \alpha_i) \in \mathcal{C}_{\mathcal{E}_1} \text{ and } (\psi_j, \beta_j) \in \mathcal{C}_{\mathcal{E}_2}\}$ and $\mathcal{D}_{\mathcal{E}_1, \mathcal{E}_2} = \mathcal{D}_{\mathcal{E}_1} \cup \mathcal{D}_{\mathcal{E}_2} \cup \{(\phi_i \vee \psi_j, ct'(\alpha_i, \beta_j)) : (\phi_i, \alpha_i) \in \mathcal{D}_{\mathcal{E}_1} \text{ and } (\psi_j, \beta_j) \in \mathcal{D}_{\mathcal{E}_2}\}$. So $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\}, \{\Delta_{F-S-C}(\mathcal{E}_2)\}) = \mathcal{C}_{\mathcal{E}_1, \mathcal{E}_2} \cup \mathcal{D}_{\mathcal{E}_1, \mathcal{E}_2}$. We now prove that for every $(\phi, \alpha) \in \Delta_{F-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2)$, $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\}, \{\Delta_{F-S-C}(\mathcal{E}_2)\}) \vdash_{\pi} (\phi, \alpha)$.

Suppose $(\phi, \alpha) \in \mathcal{D}$, then we have the following two cases:

Case 1: Suppose $(\phi, \alpha) \in \mathcal{D}_1 \cup \dots \cup \mathcal{D}_n$, without loss of generality, we assume that $(\phi, \alpha) \in \mathcal{D}_i$ for some $i \leq m$. Since $\mathcal{D}_i \subseteq \mathcal{D}'$ and $\mathcal{D}' \subseteq \Delta_{F-S-C}(\mathcal{E}_1)$, $(\phi, \alpha) \in \Delta_{F-S-C}(\mathcal{E}_1)$. We must have $(\phi, \alpha) \in Free(\Delta_{F-S-C}(\mathcal{E}_1) \cup \Delta_{F-S-C}(\mathcal{E}_2))$. Otherwise, there exist $\mathcal{C}'_s \subseteq \mathcal{C}'$, $\mathcal{D}'_s \subseteq \mathcal{D}'$, $\mathcal{C}''_s \subseteq \mathcal{C}''$ and $\mathcal{D}''_s \subseteq \mathcal{D}''$, such that $(\phi, \alpha) \in \mathcal{D}'_s$ and $(\mathcal{C}'_s \cup \mathcal{D}'_s \cup \mathcal{D}''_s \cup \mathcal{D}''_s)^*$ is a minimally inconsistent subbase of $(\Delta_{F-S-C}(\mathcal{E}_1) \cup \Delta_{F-S-C}(\mathcal{E}_2))^*$. It follows that (ϕ, α) is in conflict in \mathcal{B} . So $(\phi, \alpha) \in \mathcal{C}_i$ for some i , which is a contradiction. Therefore, $(\phi, \alpha) \in \mathcal{D}_{\mathcal{E}_1}$ and so $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\}, \{\Delta_{F-S-C}(\mathcal{E}_2)\}) \vdash_{\pi} (\phi, \alpha)$.

Case 2: Suppose $(\phi, \alpha) = (D_j, 1 - tn'(x_1, \dots, x_n))$ for some $k \geq 2$. That is, $(\phi, \alpha) = (\phi_{i_1} \vee \dots \vee \phi_{i_k}, ct'(\alpha_{i_1}, \dots, \alpha_{i_k}))$, where $(\phi_{i_j}, \alpha_{i_j}) \in \mathcal{D}_{i_j}$ and $i_j \in \{1, \dots, n\}$. Without loss of generality, we assume that $i_1 < \dots < i_l < m < \dots < i_k$. According to discussions in Case 1, we have $(\phi_{i_1} \vee \dots \vee \phi_{i_l}, ct'(\alpha_{i_1}, \dots, \alpha_{i_l})) \in \mathcal{D}'$ and $(\phi_{i_{l+1}} \vee \dots \vee \phi_{i_k}, ct'(\alpha_{i_{l+1}}, \dots, \alpha_{i_k})) \in \mathcal{D}''$. So $(\phi_{i_1} \vee \dots \vee \phi_{i_k}, ct'(\alpha_{i_1}, \dots, \alpha_{i_l}), ct'(\alpha_{i_{l+1}}, \dots, \alpha_{i_k})) \in \Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\}, \{\Delta_{F-S-C}(\mathcal{E}_2)\})$ and then $(\phi_{i_1} \vee \dots \vee \phi_{i_k}, ct'(\alpha_{i_1}, \dots, \alpha_{i_k})) \in \Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\}, \{\Delta_{F-S-C}(\mathcal{E}_2)\})$. Therefore, we have $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\}, \{\Delta_{F-S-C}(\mathcal{E}_2)\}) \vdash_{\pi} (\phi, \alpha)$.

Suppose $(\phi, \alpha) \in \mathcal{C}$, then $(\phi, \alpha) = (\phi_1 \vee \dots \vee \phi_n, tn(\alpha_1, \dots, \alpha_n))$, where $(\phi_i, \alpha_i) \in \mathcal{C}_i$ for all i . Suppose $(\phi_i, \alpha_i) \in \mathcal{C}'_i$ for all i , then $(\phi_1 \vee \dots \vee \phi_m, tn(\alpha_1, \dots, \alpha_m)) \in \mathcal{C}'$ and $(\phi_{m+1} \vee \dots \vee \phi_n, tn(\alpha_{m+1}, \dots, \alpha_n)) \in \mathcal{C}''$. If $(\phi_1 \vee \dots \vee \phi_m, tn(\alpha_1, \dots, \alpha_m)) \in \mathcal{D}_{\mathcal{E}_1}$ or $(\phi_{m+1} \vee \dots \vee \phi_n, tn(\alpha_{m+1}, \dots, \alpha_n)) \in \mathcal{D}_{\mathcal{E}_2}$, then it is clear that $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\}, \{\Delta_{F-S-C}(\mathcal{E}_2)\}) \vdash_{\pi} (\phi, \alpha)$. Otherwise, we have $(\phi_1 \vee \dots \vee \phi_n, tn(tn(\alpha_{m+1}, \dots, \alpha_n), tn(\alpha_{m+1}, \dots, \alpha_n))) \in \Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\}, \{\Delta_{F-S-C}(\mathcal{E}_2)\})$. So $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\}, \{\Delta_{F-S-C}(\mathcal{E}_2)\}) \vdash_{\pi} (\phi, \alpha)$. Suppose, without loss of generality, $(\phi_i, \alpha_i) \in \mathcal{D}'_i$ for some $i < m$ and $(\phi_j, \alpha_j) \in \mathcal{C}'_i$ for all $j \neq i$. Then $(\phi_i, \alpha_i) \in \mathcal{D}'$, $(\phi_1 \vee \dots \vee \phi_{i-1} \vee \phi_{i+1} \vee \dots \vee \phi_m, tn(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_m)) \in \mathcal{C}'$ and $(\phi_{m+1} \vee \dots \vee \phi_n, tn(\alpha_1, \dots, \alpha_n)) \in \mathcal{C}''$. So $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\}, \{\Delta_{F-S-C}(\mathcal{E}_2)\}) \vdash_{\pi} (\phi, \alpha)$.

Therefore, we have proven that $\Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\}, \{\Delta_{F-S-C}(\mathcal{E}_2)\}) \vdash_{\pi} (\phi, \alpha)$.

(P5): Let us look at a counterexample.

Let $\mathcal{E}_1 = \{\mathcal{B}_1, \mathcal{B}_2\}$ and $\mathcal{E}_2 = \{\mathcal{B}_3, \mathcal{B}_4\}$, where $\mathcal{B}_1 = \{(\phi, 0.8), (\varphi, 0.4)\}$, $\mathcal{B}_2 = \{(\psi, 0.4)\}$, $\mathcal{B}_3 = \{(\neg\phi \vee \neg\psi, 0.6)\}$ and $\mathcal{B}_4 = \{(\varphi, 0.6)\}$. Suppose the t-norm is the product operator and the t-norm is the probabilistic sum. We then have $\Delta_{F-S-C}(\mathcal{E}_1) = \{(\phi \vee \psi, 0.88), (\phi, 0.8), (\psi \vee \varphi, 0.64), (\varphi, 0.4), (\psi, 0.4)\}$ and $\Delta_{F-S-C}(\mathcal{E}_2) = \{(\neg\phi \vee \neg\psi \vee \varphi, 0.84), (\neg\phi \vee \neg\psi, 0.6), (\varphi, 0.6)\}$. $\Delta_{F-S-C}(\mathcal{E}_1)$ is split *w.r.t* $Free(\Delta_{F-S-C}(\mathcal{E}_1) \cup \Delta_{F-S-C}(\mathcal{E}_2))$ into $\mathcal{C}_{\mathcal{E}_1} = \{(\phi, 0.8), (\psi, 0.4)\}$ and $\mathcal{D}_{\mathcal{E}_1} = \{(\phi \vee \psi, 0.88), (\psi \vee \varphi, 0.64), (\varphi, 0.4)\}$, and $\Delta_{F-S-C}(\mathcal{E}_2)$ is split *w.r.t* $Free(\Delta_{F-S-C}(\mathcal{E}_1) \cup \Delta_{F-S-C}(\mathcal{E}_2))$ into $\mathcal{C}_{\mathcal{E}_2} = \{(\neg\phi \vee \neg\psi, 0.6)\}$ and $\mathcal{D}_{\mathcal{E}_2} = \{(\neg\phi \vee \neg\psi \vee \varphi, 0.84), (\varphi, 0.6)\}$. So $(\psi \vee \varphi, 0.856) \in \Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\}, \{\Delta_{F-S-C}(\mathcal{E}_2)\})$. In contrast, $\mathcal{E}_1 \sqcup \mathcal{E}_2 = \{\mathcal{B}_1, \dots, \mathcal{B}_4\}$. Let $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_4$. \mathcal{B}_i ($i = 1, 2, 3, 4$) are split *w.r.t* $Free(\mathcal{B})$ into $\mathcal{C}_1 = \{(\phi, 0.8)\}$ and $\mathcal{D}_1 = \{(\varphi, 0.4)\}$, $\mathcal{C}_2 = \{(\psi, 0.4)\}$ and $\mathcal{D}_2 = \emptyset$, $\mathcal{C}_3 = \{(\neg\phi \vee \neg\psi, 0.6)\}$ and $\mathcal{D}_3 = \emptyset$, and $\mathcal{C}_4 = \emptyset$ and $\mathcal{D}_4 = \{(\varphi, 0.6)\}$. Combining \mathcal{C}_i using the probabilistic sum we get $\mathcal{C} = \emptyset$, and combining \mathcal{D}_i using the product operator we get $\mathcal{D} = \{(\varphi, 0.76), (\varphi, 0.6), (\varphi, 0.4)\}$, which is equivalent to $\{(\varphi, 0.76)\}$. So $\Delta_{F-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2) \equiv_s \{(\varphi, 0.76)\}$. It is clear that $\Delta_{F-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2) \not\vdash_{\pi} (\psi \vee \varphi, 0.856)$. Therefore, $\Delta_{F-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2) \not\vdash_{\pi} \Delta_{F-S-C}(\{\Delta_{F-S-C}(\mathcal{E}_1)\}, \{\Delta_{F-S-C}(\mathcal{E}_2)\})$.

(P6): We have the following counterexample.

Let $\mathcal{E}_1 = \{\mathcal{B}_1, \mathcal{B}_2\}$ and $\mathcal{E}_2 = \{\mathcal{B}_3, \mathcal{B}_4\}$, where $\mathcal{B}_1 = \{(\phi, 0.8), (\varphi, 0.4)\}$, $\mathcal{B}_2 = \{(\psi, 0.4)\}$, $\mathcal{B}_3 = \{(\neg\phi \wedge \neg\psi, 0.6)\}$ and $\mathcal{B}_4 = \{(\psi, 0.6)\}$. Suppose the t-norm is the product operator and the t-norm is the probabilistic sum. We then have $\Delta_{F-S-C}(\mathcal{E}_1) = \{(\phi \vee \psi, 0.88), (\phi, 0.8), (\psi \vee \varphi, 0.64), (\varphi, 0.4), (\psi, 0.4)\}$ and $\Delta_{F-S-C}(\mathcal{E}_2) = \{(\neg\phi \vee \psi, 0.64)\}$. It is clear that $\Delta_{F-S-C}(\mathcal{E}_1) \cup \Delta_{F-S-C}(\mathcal{E}_2)$ is consistent. However, we have $\Delta_{F-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2) = \{(\varphi, 0.4)\}$. Clearly, $\Delta_{F-S-C}(\mathcal{E}_1)^* \wedge \Delta_{F-S-C}(\mathcal{E}_2)^* \neq (\Delta_{F-S-C}(\mathcal{E}_1 \sqcup \mathcal{E}_2))^*$. So Δ_{F-S-C} does not satisfy (P6).

Proposition 18 Δ_{F-S-C} satisfies (P7). The $G-I-S-C$ operator, t-conorm based operator, and renormalization based operator do not satisfy (P7).

Proof: It is clear that Δ_{F-S-C} satisfies (P7). Clearly, t-conorm based operator, and renormalization based operator do not satisfy (P7). To show that $G-I-S-C$ operator does not satisfy (P6), we consider the following counterexample. Let $\mathcal{B}_1 = \{(\phi, 0.8), (\psi, 0.7)\}$ and $\mathcal{B}_2 = \{(\neg\psi, 0.6), (\varphi, 0.5)\}$. It is easy to check that $\mathcal{B}_{I-S-C} = \mathcal{B}_1$. $(\varphi, 0.5)$ is a free formula in $\mathcal{B}_1 \cup \mathcal{B}_2$, however, it is deleted after merging.

Proposition 19 Generating a knowledge base by the $F-S-C$ method is $F\Sigma_2^p$ -complete.

Proof: It has been proven in [11] that the computation of all the free formulas and conflict formulas is Σ_2^p -complete. So the splitting of $\mathcal{B}_1 \cup \mathcal{B}_2$ is Σ_2^p -complete. The combination step can be done in polynomial time. The proposition follows clearly.

Proposition 21 Let K_1 and K_2 be two flat knowledge bases. Let K_{F-S-C} be the knowledge base obtained by merging K_1 and K_2 using the $F-S-C$ merging operator. Then every free consequence of $K_1 \cup K_2$ can be inferred from K_{F-S-C} .

Proof. By Proposition 20, $K_{F-S-C} = D_1 \cup D_2 \cup \{\phi \vee \psi \mid \phi \in K_1, \psi \in K_2, \phi, \psi \notin Free(K_1 \cup K_2)\}$. Since $D_1 \cup D_2 = Free(K_1 \cup K_2)$, we have

$$K_{F-S-C} = Free(K_1 \cup K_2) \cup \{\phi \vee \psi \mid \phi \in K_1, \psi \in K_2, \phi, \psi \notin Free(K_1 \cup K_2)\}.$$

So $Free(K_1 \cup K_2) \subseteq K_{F-S-C}$. If φ is a free consequence of $K_1 \cup K_2$, then $Free(K_1 \cup K_2) \vdash \varphi$. Therefore, $K_{F-S-C} \vdash \varphi$.

Proposition 24 If $K_1 \cup K_2$ is not consistent, then $\Delta_{F-S-C}(K_1 \sqcup K_2) \not\vdash K_1$.

Proof: Suppose $\Delta_{F-S-C}(K_1 \sqcup K_2) \vdash K_1$, then we have $\Delta_{F-S-C}(K_1 \sqcup K_2) \vdash K_{Con_1}$, which is equivalent to $K_{Free_1} \cup K_{Free_2} \cup K_{Con} \vdash K_{Con_1}$, where $K_{Con} = \{\phi \vee \psi : \phi \in K_{Con_1}, \psi \in K_{Con_2}\}$. Since

$K_{Con_2} \vdash K_{Con}$, $K_{Free_1} \cup K_{Free_2} \cup K_{Con_2} \vdash K_{Con_1}$. However, $K_{Free_1} \cup K_{Free_2} \cup K_{Con_2}$ is consistent, so $K_{Free_1} \cup K_{Free_2} \cup K_{Con_2} \cup K_{Con_1}$ is consistent, which is contradictory to the assumption.

Proposition 25 $\Delta_{F-S-C}(E_1) \cup \Delta_{F-S-C}(E_2) \vdash \Delta_{F-S-C}(E_1 \sqcup E_2)$

Proof: If $\Delta_{F-S-C}(E_1) \cup \Delta_{F-S-C}(E_2)$ is inconsistent, the conclusion clearly holds, so we assume that it is consistent. Let $K' = \cup(E_1)$, $K'' = \cup(E_2)$ and $K = \cup(E_1 \sqcup E_2)$. Suppose K' , K'' and K are split as K'_{Con} and K'_{Free} , K''_{Con} and K''_{Free} , and K_{Con} and K_{Free} respectively. It is clear $K'_{Con} \cup K''_{Con} \subseteq K_{Con}$ and $K'_{Free} \cup K''_{Free} \subseteq K_{Free}$. Suppose $E_1 = \{K_{11}, \dots, K_{1k}\}$, $E_2 = \{K_{21}, \dots, K_{2m}\}$, and $E = \{K_1, \dots, K_n\}$, where $n = k + m$. So K_{1i} in E_1 are split into $K_{1i,Con} = K'_{Con} \cap K_{1i}$ and $K_{1i,Free} = K'_{Free} \cap K_{1i}$, K_{2j} in E_2 are split into $K_{2j,Con} = K''_{Con} \cap K_{2j}$ and $K_{2j,Free} = K''_{Free} \cap K_{2j}$, and K_k in E are split into $K_{k,Con} = K_{Con} \cap K_k$ and $K_{k,Free} = K_{Free} \cap K_k$. So $\Delta_{F-S-C}(E_1) = \vee_i(K_{1i,Con}) \cup K'_{Free}$ ³, $\Delta_{F-S-C}(E_2) = \vee_j(K_{2j,Con}) \cup K''_{Free}$, and $\Delta_{F-S-C}(E_1 \sqcup E_2) = \vee_j(K_{k,Con}) \cup K_{Free}$. We now need to prove $\vee_i(K_{1i,Con}) \cup \vee_j(K_{2j,Con}) \cup K'_{Free} \cup K''_{Free} \vdash \vee_k(K_{k,Con}) \cup K_{Free}$. Since $K_{Free} \subseteq K'_{Free} \cup K''_{Free}$, it is clear every formula in K_{Free} can be inferred from $\vee_i(K_{1i,Con}) \cup \vee_j(K_{2j,Con}) \cup K'_{Free} \cup K''_{Free}$. For any formula $\phi \in \vee_k(K_{k,Con})$ such that $\phi = \phi_1 \vee \dots \vee \phi_n$, if there exists a ϕ_i such that $\phi_i \in K'_{Free}$ or $\phi_i \in K''_{Free}$, then it is clear ϕ can be inferred from $\vee_i(K_{1i,Con}) \cup \vee_j(K_{2j,Con}) \cup K'_{Free} \cup K''_{Free}$. Otherwise, there must exist a formula $\psi \in \vee_i(K_{1i,Con})$ and a formula $\psi' \in \vee_j(K_{2j,Con})$ such that $\phi = \psi \vee \psi'$. So ϕ is inferred from $\vee_i(K_{1i,Con}) \cup \vee_j(K_{2j,Con}) \cup K'_{Free} \cup K''_{Free}$. Therefore, every formula in $\vee_k(K_{k,Con}) \cup K_{Free}$ can be inferred from $\vee_i(K_{1i,Con}) \cup \vee_j(K_{2j,Con}) \cup K'_{Free} \cup K''_{Free}$.

Proposition 26 The $F-S-C$ merging operator Δ_{F-S-C} does not satisfy (A3) and (A6) in general.

Proof: For (A3), let us consider the following counter-example:

$$E_1 = \{\{\phi, \psi\}, \{\neg\phi\}\}, \quad E_2 = \{\{\phi \wedge \psi\}, \{\neg\phi\}\}.$$

It is clear $E_1 \leftrightarrow E_2$. However, $\Delta_{F-S-C}(E_1) = \{\psi\}$, whilst $\Delta_{F-S-C}(E_2) = \{\neg\phi \vee \psi\}$. So (A3) does not hold.

For (A6), let us consider the following counter-example:

$$E_1 = \{\{\phi \wedge \neg\psi\}, \{\psi\}\}, \quad E_2 = \{\{\phi \rightarrow \neg\psi\}, \{\gamma\}\}.$$

We have $\Delta_{F-S-C}(E_1) = \{\phi \vee \psi\}$ and $\Delta_{F-S-C}(E_2) = \{\phi \rightarrow \neg\psi, \gamma\}$. It is clear that $\Delta_{F-S-C}(E_1)$ and $\Delta_{F-S-C}(E_2)$ are consistent. So $\Delta_{F-S-C}(E_1) \cup \Delta_{F-S-C}(E_2) = \{\phi \vee \psi, \phi \rightarrow \neg\psi, \gamma\}$. However, $E_1 \sqcup E_2 = \{\{\phi \wedge \neg\psi\}, \{\psi\}, \{\phi \rightarrow \neg\psi\}, \{\gamma\}\}$, so $\Delta_{F-S-C}(E_1 \sqcup E_2) = \{\gamma\}$. $\phi \vee \psi$ cannot be inferred from $\Delta_{F-S-C}(E_1 \sqcup E_2)$, so (A6) is not satisfied.

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³For simplicity, given a set of knowledge bases $\{K_i : i = 1, \dots, n\}$, we use $\vee_i(K_i)$ to denote all the disjunctions among K_i , i.e., $\vee_i(K_i) = \{\phi_1 \vee \dots \vee \phi_n : \phi_i \in K_i\}$.

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