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Universal elements for non-linear operators and their applications

Stanislav Shkarin

Queens's University Belfast, Department of Pure Mathematics, University road, Belfast, BT7 1NN, UK

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ABSTRACT

We prove that under certain topological conditions on the set of universal elements of a continuous map T acting on a topological space X, that the direct sum $T \oplus M_g$ is universal, where M_g is multiplication by a generating element of a compact topological group. We use this result to characterize $\mathbb{R}_+$-supercyclic operators and to show that whenever T is a supercyclic operator and $z_1, \dots, z_n$ are pairwise different non-zero complex numbers, then the operator $z_1 T \oplus \dots \oplus z_n T$ is cyclic. The latter answers affirmatively a question of Bayart and Matheron.

1. Introduction

All topological spaces in this article *are assumed to be Hausdorff* and all vector spaces are supposed to be over the field $\mathbb{K}$ being either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. As usual, $\mathbb{R}_+$ is the set of non-negative real numbers, $\mathbb{Q}$ is the field of rational numbers, $\mathbb{Z}$ is the set of integers, $\mathbb{Z}_+$ is the set of non-negative integers, $\mathbb{N}$ is the set of positive integers and $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$. Symbol $L(X)$ stands for the space of continuous linear operators on a topological vector space X and X^* is the space of continuous linear functionals on X. For each $T \in L(X)$, the dual operator $T^* : X^* \to X^*$ is defined as usual: $(T^* f)(x) = f(Tx)$ for $f \in X^*$ and $x \in X$. It is worth noting that if X is not locally convex, then the elements of X^* may not separate points of X. A family $\mathcal{F} = \{F_a\colon a \in A\}$ of continuous maps from a topological space X to a topological space Y is called *universal* if there is $x \in X$ for which the orbit $O(\mathcal{F}, x) = \{F_a x\colon a \in A\}$ is dense in Y. Such an x is called a *universal element* for $\mathcal{F}$. We use the symbol $\mathcal{U}(\mathcal{F})$ for the set of universal elements for $\mathcal{F}$. If X is a topological space and $T : X \to X$ is a continuous map, then we say that $x \in X$ is *universal* for T if x is universal for the family $\{T^n\colon n \in \mathbb{Z}_+\}$. That is, x is universal for T if the orbit $O(T, x) = \{T^n x\colon n \in \mathbb{Z}_+\}$ is dense in X. We denote the set of universal elements for T by $\mathcal{U}(T)$. That is, $\mathcal{U}(T) = \mathcal{U}(\{T^n\colon n \in \mathbb{Z}_+\})$. In order to formulate the following theorem, we need to recall few topological definitions. A topological space X is called *connected* if it has no subsets different from $\varnothing$ and X, which are closed and open. A topological space X is called *path connected* if for each $x, y \in X$, there is a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. A topological space X is called *simply connected* if for any continuous function $f : \mathbb{T} \to X$, there exist a continuous function $F : \mathbb{T} \times [0, 1] \to X$ and $x_0 \in X$ such that $F(z, 0) = f(z)$ and $F(z, 1) = x_0$ for any $z \in \mathbb{T}$. Next, X is called *locally path connected at* $x \in X$ if for any neighborhood U of x, there exists a neighborhood V of x such that for any $y \in V$, there is a continuous map $f : [0, 1] \to X$ such that $f(0) = x$, $f(1) = y$ and $f([0, 1]) \subseteq U$. A space X is called *locally path connected* if it is locally path connected at every point. Equivalently, X is locally path connected if there is a base of topology of X consisting of path connected sets. Finally, we recall that a topological space X is called *Baire* if for any sequence $\{U_n\}_{n \in \mathbb{Z}_+}$ of dense open subsets of X, the intersection of U_n is dense in X. We say that an element g of a

E-mail address: s.shkarin@qub.ac.uk.

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topological group G is its *generator* if $\{g^n\colon n \in \mathbb{Z}_+\}$ is dense in G. It is worth mentioning that each topological group, which has a generator, is abelian. If $T: X \to X$ and $S: Y \to Y$ are two maps, we use the symbol $T \oplus S$ to denote the map

$$T \oplus S : X \times Y \to X \times Y, \qquad (T \oplus S)(x, y) = (Tx, Sy).$$

The following theorem is a generalization of the Ansari theorem on hypercyclicity of the powers of a hypercyclic operator in several directions.

Theorem 1.1. *Let X be a topological space, $T: X \to X$ be a continuous map and g be a generator of a compact topological group G. Assume also that there is a non-empty subset Y of $\mathcal{U}(T)$ such that $T(Y) \subseteq Y$ and Y is path connected, locally path connected and simply connected. Then the set $\{(T^n x, g^n)\colon n \in \mathbb{Z}_+\}$ is dense in $X \times G$ for any $x \in Y$.*

In the case when X is compact and metrizable, the above theorem follows from Theorem 11.2 in the paper [10] by Furstenburg. Our proof is based on the same general idea as in [10], which is also reproduced in the proofs of main results in [7,17]. We would like to mention the following immediate corollary of Theorem 1.1.

Corollary 1.2. *Let X be a topological space, $T: X \to X$ be a continuous map and g be a generator of a compact topological group G. Assume also that there is a non-empty subset Y of $\mathcal{U}(T)$ such that $T(Y) \subseteq Y$ and Y is path connected, locally path connected and simply connected. Then $Y \times G \subseteq \mathcal{U}(T \oplus M_g)$, where $M_g: G \to G$, $M_g h = gh$.*

If X is a topological vector space and $T \in L(X)$, then a universal element for T is called *a hypercyclic vector* for T and the operator T is called *hypercyclic* if $\mathcal{U}(T) \neq \emptyset$. Moreover, $x \in \mathcal{U}(\{sT^n\colon s \in \mathbb{K},\ n \in \mathbb{Z}_+\})$ is called a *supercyclic vector for* T and T is called *supercyclic* if it has supercyclic vectors. Similarly T is called $\mathbb{R}_+$*-supercyclic* if the family $\mathcal{F} = \{sT^n\colon s \in \mathbb{R}_+,\ n \in \mathbb{Z}_+\}$ is universal and elements of $\mathcal{U}(\mathcal{F})$ are called $\mathbb{R}_+$*-supercyclic vectors* for T. We refer to surveys [13,14,19] for additional information on hypercyclicity and supercyclicity. The question of characterizing of $\mathbb{R}_+$-supercyclic operators on complex Banach spaces was raised in [3,17]. Maria de la Rosa has recently demonstrated that the answer conjectured in [3,17] is indeed true. We obtain the same result in the more general setting of topological vector spaces by means of applying Theorem 1.1.

Theorem 1.3. *A continuous linear operator T on a complex infinite dimensional topological vector space X is $\mathbb{R}_+$-supercyclic if and only if T is supercyclic and either the point spectrum $\sigma_p(T^*)$ of the dual operator T is empty or $\sigma_p(T^*) = \{z\}$, where $z \in \mathbb{C} \setminus \{0\}$ and $z/|z|$ has infinite order in the group $\mathbb{T}$.*

Recall also that a vector x from a topological vector space X is called a *cyclic* vector for $T \in L(X)$ if $\operatorname{span}\{T^n x\colon n \in \mathbb{Z}_+\}$ is dense in X and T is called cyclic if it has cyclic vectors. In [11] it is observed that if T is a hypercyclic operator on a Banach space, then $T \oplus (-T)$ is cyclic. It is shown in [2] that if T is a supercyclic operator on a complex Banach space and $z_1, \dots, z_n$ are pairwise different complex numbers such that $z_1^k = \dots = z_n^k = 1$ for some $k \in \mathbb{N}$, then $z_1 T \oplus \dots \oplus z_n T$ is cyclic. It is also asked in [2] whether the condition $z_1^k = \dots = z_n^k = 1$ can be removed. The next theorem provides an affirmative answer to this question.

Theorem 1.4. *Let $n \in \mathbb{N}$ and T be a supercyclic continuous linear operator on an infinite dimensional topological vector space X. Then for any pairwise different non-zero $z_1, \dots, z_n \in \mathbb{K}$, the operator $z_1 T \oplus \dots \oplus z_n T$ is cyclic.*

Theorem 1.1 is proved in Section 2. In Section 3 we formulate few connectedness related lemmas, needed for application of Theorem 1.1. The proof of these lemmas is postponed until the last section. Section 4 is devoted to some straightforward applications of Theorem 1.1. In particular, it is shown that Ansari's theorems [1] on hypercyclicity of powers of a hypercyclic operator and supercyclicity of powers of supercyclic operators, the León-Saavedra and Müller theorem [17] on hypercyclicity of rotations of hypercyclic operators and the Müller and Peris [7] theorem on hypercyclicity of each operator T_t with $t > 0$ in a strongly continuous universal semigroup $\{T_t\}_{t \geqslant 0}$ of continuous linear operators acting on a complete metrizable topological vector space X, all follow from Theorem 1.1. In Section 5, Theorem 1.3, characterizing $\mathbb{R}_+$-supercyclic operators, is proven. Theorems 1.4 is proved in Section 6. In Section 7 we discuss the structure of supercyclic operators T with non-empty $\sigma_p(T^*)$, prove cyclicity of finite direct sums of operators satisfying the Supercyclicity Criterion with the same sequence $\{n_k\}$ and raise few questions.

2. Proof of Theorem 1.1

Lemma 2.1. *Let X be a topological space with no isolated points and $T: X \to X$ be a continuous map. Then $T(\mathcal{U}(T)) \subseteq \mathcal{U}(T)$. Moreover, each $x \in \mathcal{U}(T)$ belongs to $\overline{O(T, Tx)} \setminus O(T, Tx)$.*

Proof. Let $x \in \mathcal{U}(T)$. Then the orbit $O(T,x) = \{T^k x\colon\ k \in \mathbb{Z}_+\}$ is dense in X. Since X has no isolated points, $O(T,Tx) = O(T,x) \setminus \{x\}$ is also dense in X. Hence $Tx \in \mathcal{U}(T)$. Thus $T(\mathcal{U}(T)) \subseteq \mathcal{U}(T)$. Since $Tx \in \mathcal{U}(T)$, we have $x \in \overline{O(T,Tx)}$. It remains to show that $x \notin O(T,Tx)$. Assume that $x \in O(T,Tx)$. Then $x = T^n x$ for some $n \in \mathbb{N}$ and $O(T,x)$ is finite: $O(T,x) = \{T^k x\colon\ 0 \leqslant k < n\}$. Since any finite set is closed in X, $O(T,x) = \overline{O(T,x)} = X$. Hence X is finite and therefore does have isolated points. This contradiction completes the proof. □

The following two lemmas are well-known and could be found in any textbook treating topological groups, see, for instance [15].

Lemma 2.2. *A closed subsemigroup of a compact topological group is a subgroup.*

Lemma 2.3. *Let G be a compact abelian topological group, $g \in G$ and $g \neq 1_G$. Then there exists a continuous homomorphism $\varphi : G \to \mathbb{T}$ such that $\varphi(g) \neq 1$.*

We would also like to remind the following topological fact, see for instance [8].

Lemma 2.4. *Let X be a topological space, Y be a compact topological space and $\pi : X \times Y \to X$ be the projection: $\pi(x,y) = x$. Then the map π is closed.*

We also need the following lemma, which borrows heavily from the constructions in [7,17].

Lemma 2.5. *Let X be a topological space, Λ be a subsemigroup of the semigroup $C(X)$ of continuous maps from X to X and $\mathcal{U} = \mathcal{U}(\Lambda)$ be the set of universal elements for the family Λ. Let also G be a compact abelian topological group and $\varphi : \Lambda \to G$ be a homomorphism: $\varphi(TS) = \varphi(T)\varphi(S)$ for any $T, S \in \Lambda$. For each $x, y \in X$ we denote*

$$N_x = \overline{\big\{\big(Tx, \varphi(T)\big)\colon\ T \in \Lambda\big\}} \quad \text{and} \quad F_{x,y} = \big\{h \in G\colon\ (y,h) \in N_x\big\}. \tag{2.1}$$

Then

- (a1) *$F_{x,y}$ is closed in G for any $x, y \in X$ and $F_{x,y} \neq \emptyset$ if $x \in \mathcal{U}$,*
- (a2) *$F_{x,y}F_{y,u} \subseteq F_{x,u}$ for any $x, y, u \in X$,*
- (a3) *$F_{x,x} = H$ is a closed subgroup of G for any $x \in \mathcal{U}$, which does not depend on the choice of $x \in \mathcal{U}$ and for any $x, y \in \mathcal{U}$, $F_{x,y}$ is a coset of H,*
- (a4) *the function $f : \mathcal{U} \times \mathcal{U} \to G/H$, $f(x,y) = F_{x,y}$ is separately continuous and satisfies $f(x,y) = f(y,x)^{-1}$, $f(x,y)f(y,u) = f(x,u)$, and $f(x,x) = 1_{G/H}$ for any $x, y, u \in \mathcal{U}$. Moreover, if $T \in \Lambda$ and $x, y, Ty \in \mathcal{U}$, then $f(x,Ty) = \varphi(T)f(x,y)$.*

Proof. (a1) Closeness of $F_{x,y}$ follows from the closeness of N_x and the obvious equality $\{y\} \times F_{x,y} = N_x \cap (\{y\} \times G)$. Assume now that $x \in \mathcal{U}$, $y \in X$ and $F_{x,y} = \emptyset$. Then for any $h \in G$, (y,h) does not belong to the closed set $N_x \subset X \times G$. Hence we can pick open neighborhoods U_h and V_h of y and h in X and G respectively such that $(U_h \times V_h) \cap N_x = \emptyset$. Since G is compact, the open cover $\{V_h\colon\ h \in G\}$ has a finite subcover $\{V_{h_1}, \dots, V_{h_n}\}$. Then

$$(G \times U) \cap N_x \subseteq \bigcup_{j=1}^{n} (U_{h_j} \times V_{h_j}) \cap N_x = \emptyset, \quad \text{where } U = \bigcap_{j=1}^{n} U_{h_j}.$$

From the definition of N_x it follows that the set $\{Tx\colon\ T \in \Lambda\}$ does not intersect U, which is open in X and non-empty since $y \in U$. This contradicts universality of x for Λ.

(a2) Let $a \in F_{x,y}$ and $b \in F_{y,u}$. We have to demonstrate that $ab \in F_{x,u}$. Let U be any neighborhood of u in X and V be any neighborhood of 1_G in G. Pick a neighborhood W of 1_G in G such that $W \cdot W \subseteq V$. Since $b \in F_{y,u}$, there exists $T \in \Lambda$ such that $Ty \in U$ and $b^{-1}\varphi(T) \in W$. Since T is continuous, $T^{-1}(U)$ is a neighborhood of y. Since $a \in F_{x,y}$, there exists $S \in \Lambda$ such that $Sx \in T^{-1}(U)$ and $a^{-1}\varphi(S) \in W$. Then $TSx \in T(T^{-1}(U)) = U$ and $(ab)^{-1}\varphi(TS) = a^{-1}\varphi(S)b^{-1}\varphi(T) \in W \cdot W \subseteq V$ (here we use commutativity of G). That is, $(TSx, \varphi(TS)) \in U \times abV$. Therefore the set $\{(Rx, \varphi(R))\colon\ R \in \Lambda\}$ intersects any neighborhood of (u, ab) in $X \times G$. Thus, $(u, ab) \in N_x$ and $ab \in F_{x,u}$.

(a3) Let $x \in \mathcal{U}$. By (a1) and (a2) $F_{x,x}$ is closed non-empty and satisfies $F_{x,x}F_{x,x} \subseteq F_{x,x}$. Thus, $F_{x,x}$ is a closed subsemigroup of the compact topological group G. By Lemma 2.2, $F_{x,x}$ is a closed subgroup of G. Let now $x, y \in \mathcal{U}$. According to (a2) $F_{x,y}F_{y,y}F_{y,x} \subseteq F_{x,x}$. By (a1), we can pick $a \in F_{x,y}$ and $b \in F_{y,x}$. It follows that $aF_{y,y}b = abF_{y,y} \subseteq F_{x,x}$. Since $F_{x,x}$ and $F_{y,y}$ are subgroups of G and $abF_{y,y} \subseteq F_{x,x}$, we have $F_{y,y} = abF_{y,y}(abF_{y,y})^{-1} \subseteq F_{x,x}$. Similarly $F_{x,x} \subseteq F_{y,y}$. Hence $F_{x,x} = F_{y,y}$ and therefore the subgroup $H = F_{x,x}$ does not depend on the choice of $x \in \mathcal{U}$. Now by (a2) $F_{x,y}H = F_{x,y}F_{y,y} \subseteq F_{x,y}$. Thus $aH \subseteq F_{x,y}$. On the other hand, by (a2) $F_{x,y}F_{y,x} \subseteq F_{x,x} = H$. Hence $F_{x,y}b = bF_{x,y} \subseteq H$ and therefore $F_{x,y} \subseteq b^{-1}H$. That is, $F_{x,y}$ is contained in a coset of H and contains a coset of H. It follows that $F_{x,y}$ is a coset of H.

(a4) According to (a3) the function $f : \mathcal{U} \times \mathcal{U} \to G/H$, $f(x,y) = F_{x,y}$ is well defined. Let $x, y, u \in \mathcal{U}$. By (a3) $F_{x,y}F_{y,u} \subseteq F_{x,u}$. According to (a4) $F_{x,y}$, $F_{y,u}$ and $F_{x,u}$ are cosets of H. Hence $F_{x,y}F_{y,u} = F_{x,u}$. It follows that $f(x,y)f(y,u) =$

$f(x,u)$. Similarly by (a3) and (a4) $F_{x,y}F_{y,x} = F_{x,x} = H$ and therefore $f(x,y)f(y,x) = 1_{G/H}$. Thus $f(y,x) = f(x,y)^{-1}$. Since $F_{x,x} = H$, we have $f(x,x) = 1_{G/H}$. Assume now that $T \in \Lambda$ and $Ty \in \mathcal{U}$. From the definition of $F_{x,y}$, it immediately follows that $\varphi(T) \in F_{y,Ty}$. Hence $f(x,Ty) = f(x,y)f(y,Ty) = \varphi(T)f(x,y)$.

It remains to demonstrate that f is separately continuous. Since $f(y,x) = f(x,y)^{-1}$, it suffices to verify that for any fixed $x \in \mathcal{U}$, the function $\theta : \mathcal{U} \to G/H$, $\theta(y) = f(x,y)$ is continuous. Let A be a closed subset of G/H and $A_0 = \pi^{-1}(A)$, where $\pi : G \to G/H$ is the canonical projection. Clearly $\theta^{-1}(A) = \{y \in \mathcal{U}\colon\ F_{x,y} \subseteq A_0\}$ coincides with $\pi_1(N_x \cap (\mathcal{U} \times A_0))$, where $\pi_1 : \mathcal{U} \times G \to \mathcal{U}$ is the projection onto $\mathcal{U}$: $\pi_1(v,h) = v$. Since G is compact, Lemma 2.4 implies that the map π_1 is closed and therefore $\theta^{-1}(A) = \pi_1(N_x \cap (\mathcal{U} \times A_0))$ is closed in $\mathcal{U}$. Since A is an arbitrary closed subset of G/H, θ is continuous. □

The following lemma is a particular case of Lemma 2.5.

Lemma 2.6. *Let X be a topological space with no isolated points, $T : X \to X$ be a continuous map, g be an element of a compact abelian topological group G and $\mathcal{U} = \mathcal{U}(T)$. For each $x, y \in X$ we denote*

$$N_x = \overline{\{(T^n x, g^n)\colon\ n \in \mathbb{Z}_+\}} \quad \text{and} \quad F_{x,y} = \{h \in G\colon\ (y,h) \in N_x\}. \tag{2.2}$$

Then conditions (a1)–(a3) *of Lemma* 2.5 *are satisfied and*

(a4′) *the function $f : \mathcal{U} \times \mathcal{U} \to G/H$, $f(x,y) = F_{x,y}$ is separately continuous and satisfies $f(x,y) = f(y,x)^{-1}$, $f(x,Ty) = gf(x,y)$, $f(x,y)f(y,u) = f(x,u)$, $f(x,x) = 1_{G/H}$ for any $x, y, u \in \mathcal{U}$.*

Proof. We apply Lemma 2.5 with $\Lambda = \{T^n\colon\ n \in \mathbb{Z}_+\}$ and $\varphi : \Lambda \to G$, $\varphi(T^n) = g^n$. Conditions (a1)–(a3) follow directly from Lemma 2.5 and so does (a4) since $T(\mathcal{U}) \subseteq \mathcal{U}$ according to Lemma 2.1 and $\varphi(T) = g$. □

Before proving Theorem 1.1, we would like to introduce some notation. Clearly, for any continuous function $f : [a,b] \to \mathbb{T}$, there exists a unique, up to adding an integer, continuous function $\varphi : [a,b] \to \mathbb{R}$ such that $f(t) = e^{2\pi i \varphi(t)}$. Then the number $\varphi(b) - \varphi(a) \in \mathbb{R}$ is uniquely defined and called the *winding number* of f (sometimes also called the index of f or the variation of the argument of f). We denote the winding number of f as $w(f)$. We would also like to remind the following elementary and well-known properties of the winding number.

(w0) If $f : [a,b] \to \mathbb{T}$ is continuous and $f(a) = f(b)$, then $w(f) \in \mathbb{Z}$. Moreover, if also $g : [a,b] \to \mathbb{T}$ is continuous, $g(a) = g(b)$ and there exists continuous $h : [a,b] \times [0,1] \to \mathbb{T}$ satisfying $h(t,0) = f(t)$, $h(t,1) = g(t)$ and $h(a,s) = h(b,s)$ for $t \in [a,b]$ and $s \in [0,1]$, then $w(f) = w(g)$.
(w1) If $a < b < c$ and $f : [a,c] \to \mathbb{T}$ is continuous, then $w(f) = w(f|_{[a,b]}) + w(f|_{[b,c]})$.
(w2) If $f : [a,b] \to \mathbb{T}$, $h : [c,d] \to [a,b]$ are continuous, $h(c) = a$, $h(d) = b$, then $w(f \circ h) = w(f)$.
(w3) If $f : [a,b] \to \mathbb{T}$ is continuous and $u \in \mathbb{T}$, then $w(uf) = w(f)$.
(w4) If $f : [a,b] \to \mathbb{T}$ is continuous and not onto, then $|w(f)| < 1$.

The following lemma is a key ingredient of the proof of Theorem 1.1. It seems also that it has a potential for different applications.

Lemma 2.7. *Let X be a path connected, locally path connected and simply connected topological space, $T : X \to X$ be continuous and g be an element of a compact abelian topological group G. Assume also that there exists a continuous map $f : X \to G$ such that $f(Ty) = gf(y)$ for any $y \in X$. Then either $g = 1_G$ or $O(T,x)$ is closed in X for any $x \in X$.*

Proof. Assume that there exists $x \in X$ such that the orbit $O(T,x)$ is non-closed and $g \neq 1_G$. By Lemma 2.3, there exists a continuous homomorphism $\varphi : G \to \mathbb{T}$ such that $z = \varphi(g) \neq 1$. Since $\mathbb{T}$ has no closed subgroups except $\mathbb{T}$ and $U_n = \{u \in \mathbb{T}\colon\ u^n = 1\}$ for $n \in \mathbb{N}$, we see that $\varphi(G)$, being a non-trivial closed subgroup of $\mathbb{T}$, coincides either with $\mathbb{T}$ or with U_n for some $n \geqslant 2$. Let $r = \varphi \circ f$. Since φ and f are continuous, r is a continuous map from X to $\mathbb{T}$. Moreover, from the properties of φ and f it follows that

$$r(Tv) = zr(v) \quad \text{for any } v \in X. \tag{2.3}$$

Case $\varphi(G) = U_n$ for some $n \geqslant 2$. Since $z \neq 1$, (2.3) implies that $r(Tv) \neq r(v)$ for any $v \in X$. Hence $r(X)$ consists of more than one element. Thus, r is a continuous map from X to $\mathbb{T}$, whose range, being a subset of $\varphi(G)$, is finite and consists of more than one element. The existence of such a map contradicts connectedness of X.

Case $\varphi(G) = \mathbb{T}$. Pick $y \in \overline{O(T,x)} \setminus O(T,x)$ and $z_0 \in \mathbb{T} \setminus \{r(y)\}$. Since r is continuous, $r^{-1}(\mathbb{T} \setminus \{z_0\})$ is a neighborhood of y in X. Since X is locally path connected, we can pick a neighborhood U of y such that U is path connected and $U \subseteq r^{-1}(\mathbb{T} \setminus \{z_0\})$. Since $y \in \overline{O(T,x)}$, we can pick $n \in \mathbb{Z}_+$ such that $T^n x \in U$. Since U is path connected, there is continuous $\gamma_0 : [0,1] \to U$ such that $\gamma_0(0) = y$, $\gamma_0(1) = T^n x$. Since X is path connected, there is continuous $\alpha : [0,1] \to X$ such that

$\alpha(0)=x$ and $\alpha(1)=Tx$. Let $\beta:[0,1]\to\mathbb{T}$, $\beta=r\circ\alpha$. By (2.3), $\beta(1)/\beta(0)=z\neq 1$. Hence the winding number $w(\beta)$ is non-integer and therefore non-zero. Next, since $y\in\overline{O(T,Tx)}\setminus O(T,x)$, we see that the set $\{k\in\mathbb{N}:\ T^kx\in U\}$ is infinite. Thus we can pick $m\in\mathbb{N}$ such that $T^{n+m}x\in U$ and $m|w(\beta_1)|>2$. Since $T^{n+m}x\in U$ and U is path connected, there exists continuous $\gamma_1:[0,1]\to U$ such that $\gamma_1(0)=T^{n+m}x$ and $\gamma_1(1)=y$.

Consider the path $\rho:[0,m+2]\to X$ defined by the formula

$$\rho(t)=\begin{cases}\gamma_0(t) & \text{if } t\in[0,1);\\ T^{n+k-1}\alpha(t-k) & \text{if } t\in[k,k+1),\ 1\leqslant k\leqslant m;\\ \gamma_1(t-m-1) & \text{if } t\in[m+1,m+2].\end{cases}\tag{2.4}$$

Since $T^{n+k-1}\alpha(1)=T^{n+k}\alpha(0)=T^{n+k}x$ for $1\leqslant k\leqslant m$, $\gamma_0(1)=T^n\alpha(0)=T^nx$, $T^{n+m-1}\alpha(1)=\gamma_1(0)=T^{n+m}x$ and $\gamma_0(0)=\gamma_1(1)=y$, we see that ρ is continuous and $\rho(0)=\rho(m+2)$. Since X is simply connected, there exists a continuous map $\tau:[0,m+2]\times[0,1]\to X$ such that $\tau(0,s)=\tau(m+2,s)$, $\tau(t,0)=\rho(t)$ and $\tau(t,1)=x_0\in X$ for any $s\in[0,1]$ and $t\in[0,m+2]$. Thus, $r\circ\tau$ provides a homotopy of the path $r\circ\rho:[0,m+2]\to\mathbb{T}$ and a constant path. According to (w0), $w(r\circ\rho)=0$. Then by (w1),

$$0=w(r\circ\rho)=\sum_{j=0}^{m+1}w(r\circ\rho|_{[j,j+1]}).$$

Since γ_0 and γ_1 take values in $U\subseteq r^{-1}(\mathbb{T}\setminus\{z_0\})$, $r\circ\rho|_{[0,1]}$ and $r\circ\rho|_{[m+1,m+2]}$ take values in $\mathbb{T}\setminus\{z_0\}$. According to (w4), $|w(r\circ\rho|_{[0,1]})|<1$ and $w(r\circ\rho|_{[m+1,m+2]})<1$. Thus, by the last display,

$$\left|\sum_{j=1}^{m}w(r\circ\rho|_{[j,j+1]})\right|<2.$$

On the other hand, from (2.4) and (2.3) we see that for each $j\in\{1,\dots,m\}$,

$$r\circ\rho(t)=r\bigl(T^{n+j-1}\alpha(t-j)\bigr)=z^{n+j-1}r\bigl(\alpha(t-j)\bigr)=z^{n+j-1}\beta(t-j)\quad\text{for any } t\in[j,j+1].$$

Thus, according to (w2) and (w3), $w(r\circ\rho|_{[j,j+1]})=w(\beta)$ for $1\leqslant j\leqslant m$. Hence

$$2>\left|\sum_{j=1}^{m}w(r\circ\rho|_{[j,j+1]})\right|=\bigl|mw(\beta)\bigr|=m\bigl|w(\beta)\bigr|>2.$$

This contradiction completes the proof. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We can disregard the case of one-element X for its triviality. Let $x\in Y$ and N_x be the set defined in (2.2). By Lemma 2.6, the set $H=F_{x,x}=\{h\in G:\ (x,h)\in N_x\}$ is a closed subgroup of G. We shall show that $H=G$. By Lemma 2.6, there exists a separately continuous function $f:Y\times Y\to G/H$ such that $f(v,Ty)=gf(v,y)$ for any $v,y\in Y$. Consider the function $\psi:Y\to G/H$, $\psi(y)=f(x,y)$. Then ψ is continuous and $\psi(Ty)=g\psi(y)=(gH)\psi(y)$ for any $y\in Y$. Since $x\in Y\subseteq\mathcal{U}(T)$, the orbit $O(T,x)$ is dense in X. On the other hand, since $T(Y)\subseteq Y$, $O(T,x)\subseteq Y$ and therefore Y is dense in X. Since X is not one-element, then so is Y. Since Y is connected, Y has no isolated points. By Lemma 2.1, $x\in\overline{O(T,Tx)}\setminus O(T,Tx)$. Hence $O(T,Tx)$ is not closed in Y. Lemma 2.7, applied to the restriction of T to Y, implies that $gH=1_{G/H}$. On the other hand, g is a generator of G and therefore gH is a generator of G/H. It follows that the group G/H is trivial. That is, $H=G$.

By Lemma 2.6, for each $y\in Y$, the set $F_{x,y}=\{h\in G:\ (y,h)\in N_x\}$ is a coset of H in G. Since $H=G$, we have $F_{x,y}=G$ for any $y\in Y$. Hence $Y\times G\subseteq N_x$. Since Y is dense in X, the last inclusion implies that $N_x=X\times G$. Thus, $\{(T^nx,g^n):\ n\in\mathbb{Z}_+\}$ is dense in $X\times G$. □

3. Connectedness

Let X be a topological vector space. We can consider the corresponding projective space $\mathbb{P}X$. As a set $\mathbb{P}X$ consists of one-dimensional linear subspaces of X. We define the natural topology on $\mathbb{P}X$ by declaring the map $\pi:X\setminus\{0\}\to\mathbb{P}X$, $\pi(x)=\langle x\rangle$ continuous and open, where $\langle x\rangle$ is the linear span of the one-element set $\{x\}$. If $X=\mathbb{K}^{n+1}$, then $\mathbb{P}X$ is the usual n-dimensional (real or complex) projective manifold. Note that if $T\in L(X)$ is injective, it induces the continuous map $T_P:\mathbb{P}X\to\mathbb{P}X$, $T_P\langle x\rangle=\langle Tx\rangle$. Finally, we can consider the space $\mathbb{P}_+X$ of the rays in X. Namely, the elements of $\mathbb{P}_+X$ are the rays $[x]=\{tx:\ t\geqslant 0\}$ for $x\in X\setminus\{0\}$ and the topology on $\mathbb{P}_+X$ is defined by declaring the map $\pi_+:X\setminus\{0\}\to\mathbb{P}_+X$, $\pi(x)=[x]$ continuous and open. Clearly, $\mathbb{P}_+X$ is homeomorphic to the unit sphere in X if X is a normed space.

In order to apply Theorem 1.1, we need the following lemmas.

Lemma 3.1. *Any topological vector space is path connected, locally path connected and simply connected.*

Lemma 3.2. *If X is a topological vector space of real dimension $\geqslant 3$, then $X \setminus \{0\}$ is path connected, locally path connected and simply connected.*

Lemma 3.3. *If X is a complex topological vector space, then $\mathbb{P}X$ is path connected, locally path connected and simply connected.*

Lemma 3.4. *If X is a topological vector space of real dimension $\geqslant 3$, then $\mathbb{P}_+X$ is path connected, locally path connected and simply connected.*

The proof of the above lemmas comes as a combination of well-known facts and application of standard techniques of infinite dimensional topology. We include the complete proofs for convenience of the reader, but ban them to the last section.

Remark. It is worth noting that if X is a real topological vector space of dimension $\geqslant 2$, then $\mathbb{P}X$ fails to be simply connected.

4. First applications of Theorem 1.1

In this section, we derive a handful of known results and their slight generalizations from Theorem 1.1. We start with proving of a few corollaries of Theorem 1.1.

We need the following theorem, proved in [25], which is a generalization to arbitrary topological vector spaces of a result known previously for locally convex topological vector spaces.

Theorem W. *Let T be a continuous linear operator on an infinite dimensional topological vector space X. If T is hypercyclic, then $p(T)$ has dense range for any non-zero polynomial p. If T is supercyclic and the space X is complex, then the point spectrum $\sigma_p(T^*)$ is either empty or a one-element set $\{z\}$ with $z \in \mathbb{C} \setminus \{0\}$. If $\sigma_p(T^*) = \varnothing$, then $p(T)(X)$ is dense in X for each non-zero polynomial p. If $\sigma_p(T^*) = \{z\}$, then $p(T)(X)$ is dense in X for any polynomial p such that $p(z) \neq 0$. Finally, if T is supercyclic and the space X is real, then anyway, there exists $z_0 \in \mathbb{C} \setminus \{0\}$ such that $p(T)(X)$ is dense in X for any real polynomial p such that $p(z_0) \neq 0$.*

Corollary 4.1. *Let T be a hypercyclic continuous linear operator on a topological vector space X and g be a generator of a compact topological group G. Then $\{(T^n x, g^n)\colon n \in \mathbb{Z}_+\}$ is dense in $X \times G$ for any $x \in \mathcal{U}(T)$. In other words, $\mathcal{U}(T \oplus M_g) = \mathcal{U}(T) \times G$.*

Proof. Let $\mathcal{P} = \mathbb{K}[z]$ be the space of polynomials on one variable. By Theorem W, $p(T)$ has dense range for any $p \in \mathcal{P} \setminus \{0\}$. Thus $O(T, p(T)x) = p(T)(O(T,x))$ is dense for any $x \in \mathcal{U}(T)$. It follows that the set $\mathcal{U}(T)$ is invariant under $p(T)$ for any $p \in \mathcal{P} \setminus \{0\}$. Let $x \in \mathcal{U}(T)$ and $X_0 = \{p(T)x\colon p \in \mathcal{P}\}$. Since each $p(T)x$ with non-zero p belongs to $\mathcal{U}(T)$, we see that the linear map $p \mapsto p(T)x$ from $\mathcal{P}$ to X_0 is one-to-one and onto and the restriction of T to the invariant subspace X_0 is injective. According to Lemma 3.2, $Y = X_0 \setminus \{0\}$ is path connected, locally path connected and simply connected. Since $Y \subseteq \mathcal{U}(T)$ and $T(Y) \subseteq Y$, from Theorem 1.1 it follows that $\{(T^n x, g^n)\colon n \in \mathbb{Z}_+\}$ is dense in $X \times G$. □

Similar property can be proven for supercyclic operators. First, we observe that supercyclicity of an injective operator $T \in L(X)$ can be interpreted as universality of the induced map T_P, $T_P(\langle x \rangle) = \langle Tx \rangle$ on the projective space $\mathbb{P}X$. Namely, $x \in X$ is supercyclic for T if and only if $\langle x \rangle$ is universal for T_P.

Corollary 4.2. *Let X be an infinite dimensional complex topological vector space and $T \in L(X)$ be supercyclic. Then for any supercyclic vector x for T and any generator g of a compact topological group G, $\{(zT^n x, g^n)\colon n \in \mathbb{Z}_+,\ z \in \mathbb{C}\}$ is dense in $X \times G$.*

Proof. Let x be a supercyclic vector for T.

Case $\sigma_p(T^*) = \varnothing$. Consider the linear space $X_0 = \{p(T)x\colon p \in \mathcal{P}\}$. Exactly as in the proof of Corollary 4.1, we have that X_0 is infinite dimensional and dense in X, the restriction of T to the invariant subspace X_0 is injective and each non-zero element of X_0 is a supercyclic vector for T. It follows that each element of the projective space $\mathbb{P}X_0$ is universal for the induced map T_P. By Lemma 3.3, $\mathbb{P}X_0$ is path connected, locally path connected and simply connected. Theorem 1.1 implies that $\{(T_P^n \langle x \rangle, g^n)\colon n \in \mathbb{Z}_+\}$ is dense in $\mathbb{P}X_0 \times G$ and therefore in $\mathbb{P}X \times G$. The latter density means that $\{(zT^n x, g^n)\colon n \in \mathbb{Z}_+,\ z \in \mathbb{C}\}$ is dense in $X \times G$.

Case $\sigma_p(T^*) \neq \varnothing$. By Theorem W, there exists $w \in \mathbb{C} \setminus \{0\}$ such that $\sigma_p(T^*) = \{w\}$. Multiplying T by w^{-1}, we can, without loss of generality, assume that $w = 1$. Pick non-zero $f \in X^*$ such that $T^* f = f$. Clearly $f(x) \neq 0$. Indeed, otherwise the orbit of x lies in $\ker f$ and x is not even a cyclic vector for T. Multiplying f by a non-zero constant, if necessary, we may assume that $f(x) = 1$. Let $Z = \{x \in X\colon f(x) = 1\}$. It is straightforward to verify that $T(Z) \subseteq Z$ and $u \in Z$ is a supercyclic vector for T if and only if u is universal for the restriction $T|_Z$ of T to Z. Now, the set $Y = \{p(T)x\colon p \in \mathcal{P},\ p(1) = 1\}$ is a subset of Z, $x \in Y$ and $T(Y) \subseteq Y$. From Theorem W it follows that each $p(T)x$ with $p(1) \neq 0$ is a supercyclic vector for T. Hence each element of Y is universal for $T|_Z$. Moreover, according to Lemma 3.1, Y being an affine subspace of X, is path connected, locally path connected and simply connected. By Theorem 1.1, $\{(T^n x, g^n)\colon n \in \mathbb{Z}_+\}$ is dense in $Z \times G$. Hence $\{(zT^n x, g^n)\colon n \in \mathbb{Z}_+,\ z \in \mathbb{C}\}$ is dense in $X \times G$. □

It is worth noting that an analog of Corollary 4.2 fails for supercyclic operators on real topological vector spaces. Indeed, let $t \in \mathbb{R} \setminus \mathbb{Q}$, A be a linear operator on $\mathbb{R}^2$ with the matrix

$$A = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix}$$

and B be any hypercyclic operator on a real topological vector space X. It is easy to verify that $T = A \oplus B$ is a supercyclic operator on $\mathbb{R}^2 \times X$, $g = e^{2\pi i t}$ is a generator of $\mathbb{T}$, while $\{(zT^n x, g^n)\colon n \in \mathbb{Z}_+,\ z \in \mathbb{R}\}$ is not dense in $(\mathbb{R}^2 \times X) \times \mathbb{T}$ for any $x \in \mathbb{R}^2 \times X$. On the other hand if we impose the additional condition that $p(T)$ has dense range for any non-zero polynomial p, the result extends to the real case. One has to use the fact that a vector is $\mathbb{R}$-supercyclic if and only if it is $\mathbb{R}_+$-supercyclic [3] and apply Theorem 1.1 exactly as in the first case of the above proof with $\mathbb{P}X_0$ replaced by $\mathbb{P}_+X_0$ and T_P by $T_{P_+}[x] = [Tx]$. Of course, one has to use Lemma 3.4 instead of Lemma 3.3 to ensure the required connectedness. This leads to the following corollary.

Corollary 4.3. *Let X be a real topological vector space and $T \in L(X)$ be a supercyclic operator such that $p(T)$ has dense range for any non-zero $p \in \mathcal{P}$. Then for any supercyclic vector x for T and any generator g of a compact topological group G, $\{(sT^n x, g^n)\colon n \in \mathbb{Z}_+,\ s > 0\}$ is dense in $X \times G$.*

4.1. The Ansari theorem

Applying Corollary 4.1 in the case when g is a generator of an n-element cyclic group G, carrying the discrete topology, we immediately obtain the following statement, which is the Ansari theorem [1] on hypercyclicity of powers of hypercyclic operators.

Corollary 4.4. *Let X be a topological vector space, $T \in L(X)$ and $n \in \mathbb{N}$. Then $\mathcal{U}(T^n) = \mathcal{U}(T)$. In particular T^n is hypercyclic if and only if T is hypercyclic.*

The supercyclicity version of the Ansari theorem for operators on a complex topological vector space follows similarly from Corollary 4.2. In order to incorporate the real case, we need the following non-linear analog of the Ansari theorem. The advantage compared to the direct application of Theorem 1.1 is that in the case of finite group G, one can significantly relax the topological assumptions on the set of universal elements.

Proposition 4.5. *Let X be a topological space with no isolated points, $n \in \mathbb{N}$ and $T : X \to X$ be a continuous map such that $\mathcal{U}(T)$ is connected. Then $\mathcal{U}(T^n) = \mathcal{U}(T)$.*

Proof. Let g be a generator of the cyclic group G of order n (carrying the discrete topology) and $x \in \mathcal{U}(T)$. By Lemma 2.6 the set $H = F_{x,x}$ defined in 2.2 is a subgroup of G, each $F_{x,y}$ for $y \in \mathcal{U}(T)$ is a coset of H, the map $\psi : \mathcal{U}(T) \to G/H$, $\psi(y) = F_{x,y}$ is continuous and $\psi(T^k x) = g^k H$. Since g is a generator of G, it follows that ψ is onto. Since G/H is finite, ψ is continuous and $\mathcal{U}(T)$ is connected, it follows that G/H is trivial. That is, $H = G$. Hence $F_{x,y} = H$ for any $y \in \mathcal{U}(T)$ and therefore $\mathcal{U}(T) \times G$ is contained in the closure of the orbit $O(T \oplus M_g, (x, 1))$. From Lemma 2.1 we see that $\mathcal{U}(T)$ contains $O(T, x)$ and therefore is dense in X. Hence $(x, 1)$ is universal for $T \oplus M_g$. Since G carries the discrete topology, $\{T^k x\colon g^k = 1\}$ is dense in X. The latter set is exactly $O(T^n, x)$. Thus $x \in \mathcal{U}(T^n)$. That is, $\mathcal{U}(T) \subseteq \mathcal{U}(T^n)$. The opposite inclusion is obvious. □

It is worth noting that the original proof of Ansari [1] can also be adapted to prove the last proposition. Our point was to show that the result follows also from Lemma 2.6.

Corollary 4.6. *Let X be a topological vector space, $T \in L(X)$ and $n \in \mathbb{N}$. Then the sets of supercyclic vectors for T^n and T coincide. In particular T^n is supercyclic if and only if T is supercyclic.*

Proof. It is well known that if X is finite dimensional, then supercyclic operators do exist on X if and only if the real dimension of X does not exceed 2. In this case we can easily see that the required property is satisfied. Thus we can assume that X is infinite dimensional. Consider the space $X_0 = \{p(T)x\colon p \in \mathcal{P}\}$. Since X_0 is dense in X, it is infinite dimensional. Moreover, the restriction of T to X_0 is injective. By Theorem W, there exists $z_0 \in \mathbb{C}$ such that the set $\{p(T)x\colon p(z_0) \neq 0\}$ consists of supercyclic vectors for T. Thus the set $Y = \{\langle p(T)x\rangle\colon p(z_0) \neq 0\}$ consists of universal elements of the induced map

$T_P : \mathbb{P}X_0 \to \mathbb{P}X_0$. One can easily see that Y is connected and dense in $\mathbb{P}X_0$. Hence $\mathcal{U}(T_P)$ is connected. By Proposition 4.5, $\langle x\rangle \in \mathcal{U}(T_P^n)$. It follows that x is a supercyclic vector for T^n. □

4.2. The León–Müller theorem

Let X be a topological vector space. Recall that the pointwise convergence topology on $L(X)$ is called the *strong operator topology*. We say that an operator $S \in L(X)$ is *a generalized rotation* if the strong operator topology closure G of $\{S^n\colon\ n \in \mathbb{Z}_+\}$ in $L(X)$ is a compact *subgroup* of the semigroup $L(X)$ and the map $(x, A) \mapsto Ax$ from $X \times G$ to X is continuous.

Proposition 4.7. *Let X be a topological vector space, $T \in L(X)$ and $S \in L(X)$ be a generalized rotation. Then T is hypercyclic if and only if $\{S^nT^n\colon\ n \in \mathbb{Z}_+\}$ is universal and $\mathcal{U}(T) \subseteq \mathcal{U}(\{S^nT^n\colon\ n \in \mathbb{Z}_+\})$. If additionally $TS = ST$, then T is hypercyclic if and only if ST is hypercyclic and $\mathcal{U}(T) = \mathcal{U}(ST)$.*

Proof. Let G be the strong operator topology closure of $\{S^n\colon\ n \in \mathbb{Z}_+\}$ in $L(X)$. Since S is a generalized rotation, G is compact and the map $\Phi : X \times G \to X$, $\Phi(x, A) = Ax$ from $X \times G$ to X is continuous. Let $x \in \mathcal{U}(T)$. By Corollary 4.1, the set $\Omega = \{(T^nx, S^n)\colon\ n \in \mathbb{Z}_+\}$ is dense in $X \times G$. Since Φ is onto and continuous $\Phi(\Omega) = \{S^nT^nx\colon\ n \in \mathbb{Z}_+\}$ is dense in X. Hence $x \in \mathcal{U}(\{S^nT^n\colon\ n \in \mathbb{Z}_+\})$. Thus $\mathcal{U}(T) \subseteq \mathcal{U}(\{S^nT^n\colon\ n \in \mathbb{Z}_+\})$.

Assume now that $TS = ST$. Then $S^nT^n = (ST)^n$ for each $n \in \mathbb{Z}_+$. Hence $\mathcal{U}(T) \subseteq \mathcal{U}(ST)$. Clearly S is invertible and S^{-1} is also a generalized rotation. Hence $\mathcal{U}(ST) \subseteq \mathcal{U}(S^{-1}(ST)) = \mathcal{U}(T)$. Thus $\mathcal{U}(T) = \mathcal{U}(ST)$. □

If X is a complex topological vector space and $z \in \mathbb{T}$, then obviously, the rotation operator zI is a generalized rotation. Applying Proposition 4.7 with $S = zI$, we immediately obtain the León-Saavedra and Müller theorem [17] on hypercyclicity of rotations of hypercyclic operators.

Corollary 4.8. *If T is a hypercyclic continuous linear operator on a complex topological vector space X and $z \in \mathbb{T}$, then zT is hypercyclic and $\mathcal{U}(zT) = \mathcal{U}(T)$.*

If a $X_1, \dots, X_n$ are complex topological vector spaces, $X = X_1 \times \cdots \times X_n$ and $z_1, \dots, z_n \in \mathbb{T}$, then the operator $S \in L(X)$, $S = z_1I \oplus \cdots \oplus z_nI$ is a generalized rotation. Applying Proposition 4.7 with this S, we get the following slight generalization of the León–Müller theorem.

Proposition 4.9. *Let $T_j \in L(X_j)$ for $1 \leqslant j \leqslant n$ be such that $T = T_1 \oplus \cdots \oplus T_n$ is hypercyclic. Then for any $z = (z_1, \dots, z_n) \in \mathbb{T}^n$, the operator $T_z = z_1T_1 \oplus \cdots \oplus z_nT_n$ is also hypercyclic and $\mathcal{U}(T) = \mathcal{U}(T_z)$.*

Similar proof with application of Corollary 4.2 instead of Corollary 4.1 gives the following supercyclic analog of Proposition 4.7.

Proposition 4.10. *Let X be a complex topological vector space, $T \in L(X)$ and $S \in L(X)$ be a generalized rotation. Then T is supercyclic if and only if $\mathcal{F} = \{sS^nT^n\colon\ n \in \mathbb{Z}_+,\ s \in \mathbb{C}\}$ is universal and any supercyclic vector for T is universal for $\mathcal{F}$. If additionally $TS = ST$, then T is supercyclic if and only if ST is supercyclic and the operators T and ST have the same sets of supercyclic vectors.*

Applying Proposition 4.10 with $S = z_1I \oplus \cdots \oplus z_nI$, we also get the following corollary.

Proposition 4.11. *Let X_j be complex topological vector spaces and $T_j \in L(X_j)$ for $1 \leqslant j \leqslant n$ be such that $T = T_1 \oplus \cdots \oplus T_n$ is supercyclic. Then for any $z = (z_1, \dots, z_n) \in \mathbb{T}^n$, the operator $T_z = z_1T_1 \oplus \cdots \oplus z_nT_n$ is also supercyclic and the sets of supercyclic vectors for T and T_z coincide.*

Remark. The same argument allows to replace finite direct sums of operators in Propositions 4.9 and 4.11 by, for instance, countable c_0 or ℓ_p $(1 \leqslant p < \infty)$ sums of Banach spaces.

4.3. Hypercyclicity of members of a continuous universal semigroup

Recently, Conejero, Müller and Peris [7] have proven that if $\{T_t\}_{t \geqslant 0}$ is a strongly continuous semigroup of continuous linear operators acting on a complete metrizable topological vector space X and the family $\{T_t\colon\ t \in \mathbb{R}_+\}$ is universal, then each T_t with $t > 0$ is hypercyclic. We show now that this theorem is also a corollary of Theorem 1.1.

Proposition 4.12. *Let $\{T_t\}_{t \geqslant 0}$ be semigroup of continuous linear operators acting on a topological vector space X. Assume also that the map $(t, x) \mapsto T_tx$ from $\mathbb{R}_+ \times X$ to X is continuous. Then $\mathcal{U}(T_t) = \mathcal{U}(T_s)$ for any $t, s > 0$. That is, either T_t for $t > 0$ are all non-hypercyclic or T_t for $t > 0$ are all hypercyclic and have the same set of hypercyclic vectors.*

Proof. Let $t,s>0$. We have to show that $\mathcal{U}(T_t)=\mathcal{U}(T_s)$. Clearly, it is enough to demonstrate that $\mathcal{U}(T_s)\subseteq\mathcal{U}(T_t)$ for any $s,t>0$. If $t/s\in\mathbb{Q}$, then $t/s=m/n$ for some $m,n\in\mathbb{N}$. Then $T_t^n=T_s^m$ and according to the Ansari theorem (Corollary 4.4), we have $\mathcal{U}(T_t)=\mathcal{U}(T_t^n)=\mathcal{U}(T_s^m)=\mathcal{U}(T_s)$. It remains to consider the case $t/s\notin\mathbb{Q}$. In this case $z=e^{2\pi is/t}$ is a generator of $\mathbb{T}$. Let $x\in\mathcal{U}(T_s)$ and $y\in X$. By Corollary 4.1, the set $\{(T_{ns}x,z^n)\colon n\in\mathbb{Z}_+\}$ is dense in $X\times\mathbb{T}$. Hence, we can pick a net $\{n_\alpha\}_{\alpha\in D}$ of non-negative integers such that $T_{n_\alpha s}x\to y$ and $z^{n_\alpha}\to e^{-2\pi is}$. Since $z^{n_\alpha}=e^{2\pi in_\alpha s/t}$ and $z^{n_\alpha}\to e^{-2\pi is}$, we can pick a net $\{m_\alpha\}_{\alpha\in D}$ of non-negative integers such that $s(n_\alpha+1)-tm_\alpha\to 0$. That is, $tm_\alpha=sn_\alpha+b_\alpha$, where $\{b_\alpha\}$ is a net in $\mathbb{R}_+$, converging to s. Using continuity of the map $(a,x)\mapsto T_ax$, we see that $T_{tm_\alpha}x=T_{b_\alpha}T_{sn_\alpha}x\to T_sy$ since $b_\alpha\to s$ and $T_{sn_\alpha}x\to y$. Hence $T_sy\in\overline{O(T_t,x)}$ for any $y\in X$. Since T_s is hypercyclic, it has dense range and therefore $O(T_t,x)$ is dense in X and $x\in\mathcal{U}(T_t)$. Thus, $\mathcal{U}(T_s)\subseteq\mathcal{U}(T_t)$. □

The following observation belongs to Oxtoby and Ulam [21]. We present it in a slightly more general form, although the proof does not need any changes.

Proposition 4.13. *Let M be a Baire separable metrizable space and $\{T_t\}_{t\geqslant 0}$ be a semigroup of continuous maps from M to itself such that the map $(t,x)\mapsto T_tx$ from $\mathbb{R}_+\times M$ to M is continuous and the family $\{T_t\colon t\in\mathbb{R}_+\}$ is universal. Then for almost all $t\in\mathbb{R}_+$ in the Baire category sense, T_t is universal. In particular, there exists $t>0$ such that T_t is universal.*

Taking into account that the map $(t,x)\mapsto T_tx$ is continuous for any strongly continuous semigroup $\{T_t\}_{t\geqslant 0}$ of continuous linear operators on a Baire topological vector space and combining Proposition 4.13 with Proposition 4.12, we immediately obtain the following corollary, which is exactly the Conejero–Müller–Peris theorem.

Corollary 4.14. *Let $\{T_t\}_{t\geqslant 0}$ be a strongly continuous semigroup of continuous linear operators on a Baire separable metrizable topological vector space such that the family $\{T_t\colon t\in\mathbb{R}_+\}$ is universal. Then each T_t for $t>0$ is hypercyclic and T_t for $t>0$ share the set of hypercyclic vectors.*

5. A characterization of $\mathbb{R}_+$-supercyclic operators

The proof of Theorem 1.3 naturally splits in two cases: when $\sigma_p(T^*)$ is empty and when $\sigma_p(T^*)$ is a singleton. The case $\sigma_p(T)=\emptyset$ is covered by the following proposition by León-Saavedra and Müller [17]. They prove it for operators on Banach spaces, however, virtually the same proof works for operators acting on arbitrary topological vector spaces (just replace sequence convergence by net convergence).

Proposition LM. *Let X be a complex topological vector space and $T\in L(X)$ be a supercyclic operator such that $\sigma_p(T^*)=\emptyset$. Then T is $\mathbb{R}_+$ supercyclic. Moreover the sets of supercyclic and $\mathbb{R}_+$-supercyclic vectors for T coincide.*

It remains to consider the case when $\sigma_p(T^*)$ is a singleton.

Proposition 5.1. *Let X be a complex topological vector space and $T\in L(X)$ be a supercyclic operator such that $\sigma_p(T^*)=\{z\}$ for some $z\in\mathbb{C}\setminus\{0\}$ such that $z/|z|$ has infinite order in $\mathbb{T}$. Then T is $\mathbb{R}_+$ supercyclic. Moreover the sets of supercyclic and $\mathbb{R}_+$-supercyclic vectors for T coincide.*

Proof. Since any $\mathbb{R}_+$-supercyclic vector for T is supercyclic, it suffices to verify that any supercyclic vector for T is $\mathbb{R}_+$-supercyclic. Replacing T be $|z|^{-1}T$, if necessary, we can, without loss of generality, assume that $|z|=1$. Let $x\in X$ be a supercyclic vector for T. It suffices to show that x is $\mathbb{R}_+$-supercyclic for T. Let $f\in X^*$ be a non-zero functional such that $T^*f=zf$ and $f(x)=1$ (the last condition is just a normalization). Consider the affine hyperplane $Z=\{u\in X\colon f(u)=1\}$ and a map $S:Z\to Z$, $S(u)=z^{-1}Tu$. The map S is indeed taking values in Z since $f(Su)=z^{-1}f(Tu)=z^{-1}(T^*f)(u)=z^{-1}zf(u)=f(u)=1$ for any $u\in Z$. It is also clear that S is continuous. Let $\mathcal{P}_1=\{p\in\mathcal{P}\colon p(z)=1\}$ and $Y=\{p(T)x\colon p\in\mathcal{P}_1\}$. First, we shall demonstrate that Y is a dense subset of Z, $S(Y)\subseteq Y$ and $Y\subseteq\mathcal{U}(S)$. Indeed, let $y\in Y$. Then $y=p(T)x$ for some $p\in\mathcal{P}_1$. Since $f(y)=f(p(T)x)=(p(T^*)f)(x)=p(z)f(x)=1$, we have $y\in Y$ and therefore $Y\subseteq Z$. Next, $Sy=z^{-1}Ty=q(T)x$, where $q(t)=z^{-1}tp(t)$. Since $q(z)=z^{-1}zp(z)=p(z)=1$, we have $Sy\in Y$ and therefore $S(Y)\subseteq Y$. Next, since x is supercyclic for T, the set $A=\{sT^nx\colon n\in\mathbb{Z}_+,\ s\in\mathbb{C}\setminus\{0\}\}$ is dense in X. By Theorem W, $p(T)(A)=\{sT^ny\colon n\in\mathbb{Z}_+,\ s\in\mathbb{C}\setminus\{0\}\}$ is also dense in X. Since $A\subset X\setminus\ker f$ and the map $\Phi:X\setminus\ker f\to Z$, $\Phi(y)=y/f(y)$ is continuous and onto, we have that $\Phi(p(T)(A))$ is dense in Z. On the other hand, it is easy to see that $\Phi(p(T)(A))=O(S,y)$. Thus $y\in\mathcal{U}(S)$ and therefore $Y\subseteq\mathcal{U}(S)$. Now, Y is an affine subspace of X and therefore, Lemma 3.1 implies that Y is path connected, locally path connected and simply connected. Since z has infinite order in $\mathbb{T}$, z is a generator of $\mathbb{T}$. Applying Theorem 1.1, we see that $\{(S^ny,z^n)\colon n\in\mathbb{Z}_+\}$ is dense in $Z\times\mathbb{T}$ for any $y\in Y$. In particular, $\{(S^nx,z^n)\colon n\in\mathbb{Z}_+\}$ is dense in $Z\times\mathbb{T}$. By definition of S, $\{(z^{-n}T^nx,z^n)\}$ is dense in $Z\times\mathbb{T}$. Consider the map $\Psi:Z\times\mathbb{T}\to X$, $\Psi(u,s)=su$. Clearly Ψ is continuous and $\Psi(Z\times\mathbb{T})=Z_0=\{u\in X\colon |f(u)|=1\}$. Therefore the set $\Psi(\{(z^{-n}T^nx,z^n)\colon n\in\mathbb{Z}_+\})=\{T^nx\colon n\in\mathbb{Z}_+\}$ is dense in Z_0. Since $\{su\colon s\in\mathbb{R}_+,\ u\in Z_0\}$ is dense in X, we see that $\{sT^nx\colon n\in\mathbb{Z}_+,\ s\in\mathbb{R}_+\}$ is also dense in X. That is, x is an $\mathbb{R}_+$-supercyclic vector for T. □

Proof of Theorem 1.3. If $\sigma_p(T^*)=\emptyset$ or $\sigma_p(T^*)=\{z\}$ with $z/|z|$ being an infinite order element of $\mathbb{T}$, then according to Propositions LM and 5.1, T is $\mathbb{R}_+$-supercyclic. It remains to show that if $\sigma_p(T^*)=\{z\}$ with $z\in\mathbb{C}\setminus\{0\}$ and $z/|z|$ having finite order in $\mathbb{T}$, then T is not $\mathbb{R}_+$-supercyclic. Replacing T be $|z|^{-1}T$, if necessary, we can, without loss of generality, assume that $|z|=1$. Then $z\in\mathbb{T}$ has finite order. Let G be the finite subgroup of $\mathbb{T}$ generated by z. Pick a non-zero $f\in X^*$ such that $T^*f=zf$. Let $x\in X$. Then for any $s\in\mathbb{R}_+$ and $n\in\mathbb{Z}_+$, $f(sT^nx)=sz^nf(x)\in A=f(x)G\mathbb{R}_+$. The set A, being a finite union of rays, is nowhere dense in $\mathbb{C}$. Since $f:X\to\mathbb{C}$ is open, the set $f^{-1}(A)$ is nowhere dense in X. Hence $\{sT^nx\colon n\in\mathbb{Z}_+,\ s\in\mathbb{R}_+\}$, being a subset of $f^{-1}(A)$, is nowhere dense in X. Thus x is not an $\mathbb{R}_+$-supercyclic vector for T. Since $x\in X$ is arbitrary, T is not $\mathbb{R}_+$-supercyclic. □

6. Cyclic direct sums of scalar multiples of a supercyclic operator

We shall repeatedly use the following elementary observation.

Lemma 6.1. *Let T be a continuous linear operator on a topological vector space X, $x\in X$ and L be a closed linear subspace of X such that $T(L)\subseteq L$, L is contained in the cyclic subspace $C(T,x)=\overline{\mathrm{span}}\{T^kx\colon k\in\mathbb{Z}_+\}$ and $x+L$ is a cyclic vector for the quotient operator $\widetilde{T}\in L(X/L)$, $\widetilde{T}(u+L)=Tu+L$. Then x is a cyclic vector for T.*

Proof. Let U be a non-empty open subset of X and $\pi:X\to X/L$ be the canonical map $\pi(x)=x+L$. Since $x+L$ is a cyclic vector for $\widetilde{T}$, there exists $p\in\mathcal{P}$ such that $p(\widetilde{T})(x+L)\in\pi(U)$. Hence we can pick $w\in L$ such that $w+p(T)x\in U$. That is, w belongs to the open set $-p(T)x+U$. Since $w\in L\subseteq C(T,x)$, we can find $q\in\mathcal{P}$ such that $q(T)x\in -p(T)x+U$. That is, $(p+q)(T)x\in U$. Thus $\{r(T)x\colon r\in\mathcal{P}\}$ is dense in X and therefore x is a cyclic vector for T. □

The proof of Theorem 1.4 is different in the cases $\mathbb{K}=\mathbb{R}$ and $\mathbb{K}=\mathbb{C}$. We start with the more difficult complex case.

6.1. Proof of Theorem 1.4. Case $\mathbb{K}=\mathbb{C}$

Lemma 6.2. *Let $n\in\mathbb{N}$, $z=(z_1,\dots,z_n)\in\mathbb{T}^n$, G be the closure in $\mathbb{T}^n$ of $\{z^m\colon m\in\mathbb{Z}_+\}$, T be s supercyclic operator on a complex topological vector space X, u be a supercyclic vector for T, $x=(u,\dots,u)\in X^n$ and $S=z_1T\oplus\cdots\oplus z_nT$. Then $\overline{A}=B$, where*

$$A=\{sS^kx\colon k\in\mathbb{Z}_+,\ s\in\mathbb{C}\}\quad\text{and}\quad B=\{(w_1y,\dots,w_ny)\colon y\in X,\ w\in G\}.$$

Proof. By Lemma 2.2, G is a closed subgroup of $\mathbb{T}^n$. By Corollary 4.2, the set $C=\{(sT^ku,z^k)\colon k\in\mathbb{Z}_+,\ s\in\mathbb{C}\}$ is dense in $X\times G$. Consider the map

$$\varphi:X\times G\to B,\qquad \varphi(v,w)=(w_1v,\dots,w_nv).$$

Clearly φ is continuous and onto. Hence $\varphi(C)=A$ is dense in B. Thus, $\overline{A}=B$. □

Corollary 6.3. *Let $n,k\in\mathbb{N}$, $k\leqslant n$ and $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ be such that $|z_j|=1$ for $1\leqslant j\leqslant k$ and $|z_j|<1$ if $j>k$, $z'=(z_1,\dots,z_k)\in\mathbb{T}^k$ and G be the closure in $\mathbb{T}^k$ of $\{(z')^m\colon m\in\mathbb{Z}_+\}$, T be s supercyclic operator on a complex topological vector space X, u be a supercyclic vector for T, $x=(u,\dots,u)\in X^n$ and $S=z_1T\oplus\cdots\oplus z_nT$. Then $\overline{A}\supseteq B$, where*

$$A=\{sS^kx\colon k\in\mathbb{Z}_+,\ s\in\mathbb{C}\}\quad\text{and}\quad B=\{(w_1y,\dots,w_ky,0,\dots,0)\colon y\in X,\ w\in G\}.$$

If additionally, the numbers $z_1,\dots,z_k$ are pairwise different, then the cyclic subspace $C(S,x)$ contains the space

$$L_k=\{v\in X^n\colon v_j=0\text{ for }j>k\}.$$

Proof. The inclusion $\overline{A}\supseteq B$ follows immediately from Lemma 6.2 and the inequalities $|z_j|<1$ for $j>k$. Assume now that $z_1,\dots,z_k$ are pairwise different. Clearly $\overline{A}\subseteq C(S,x)$. Hence $B\subseteq C(S,x)$. Since $C(S,x)$ is a linear subspace of X^n, $\mathrm{span}(B)\subseteq C(S,x)$. It remains to demonstrate that $\mathrm{span}(B)\supseteq L_k$. Let $j\in\{1,\dots,n\}$, $v\in X$ and $v^j\in X^n$ be defined as $v^j_j=v$ and $v^j_l=0$ for $l\neq j$. Since vectors $\{v^j\colon v\in X,\ 1\leqslant j\leqslant k\}$ span L_k, it suffices to show that $v^j\in\mathrm{span}(B)$. Let $e^j\in\mathbb{C}^k$ be the jth basic vector: $e^j_j=1$ and $e^j_l=0$ for $l\neq j$. Since $z_1,\dots,z_k$ are pairwise different, the matrix $\{z^l_j\}^k_{j,l=1}$ is invertible: its determinant is the Van der Monde one. Hence, there exist $\alpha_1,\dots,\alpha_k\in\mathbb{C}$ such that

$$e^j=\sum_{j=1}^k\alpha_j(z')^j\quad\text{and therefore}\quad v^j=\sum_{j=1}^k\alpha_j\bigl(z_1^jv,\dots,z_k^jv\bigr).$$

Since each $(z_1^jv,\dots,z_k^jv)$ belongs to B, we see that $v^j\in\mathrm{span}(B)$. Thus, $L_k\subseteq\mathrm{span}(B)$. □

Proof of Theorem 1.4. Case $\mathbb{K}=\mathbb{C}$. Let $n\in\mathbb{N}$, T be a supercyclic continuous linear operator on a complex topological vector space X, u be a supercyclic vector for T and $z_1,\dots,z_n$ be pairwise different non-zero complex numbers. Let also

$S = z_1T \oplus \cdots \oplus z_nT$. It suffices to show that $x = (u, \dots, u)$ is a cyclic vector for S. We shall use induction with respect to n. The case $n = 1$ is trivial. Without loss of generality, we may assume that $1 = |z_1| \geqslant |z_2| \geqslant \cdots \geqslant |z_n|$. Let $m \in \{1, n\}$ be the maximal number for which $|z_1| = \cdots = |z_m|$. Then $z_1, \dots, z_m$ are pairwise different elements of $\mathbb{T}$. By Corollary 6.3, the space $L_m = \{v \in X^n\colon\ v_j = 0 \text{ for } j > m\}$ is contained in the cyclic subspace $C(S, x)$. It is also clear that $S(L_m) \subseteq L_m$ and X^n/L_m is naturally isomorphic to X^{n-m} and the quotient operator $\widetilde{S}(v + L_m) = Sv + L_m$, acting on X^n/L_m is naturally similar to $z_{m+1}T_{m+1} \oplus \cdots \oplus z_nT_n$. From the induction hypothesis it follows that $x + L_m$ is a cyclic vector for $\widetilde{S}$. According to Lemma 6.1, x is a cyclic vector for S. □

6.2. Proof of Theorem 1.4. Case $\mathbb{K} = \mathbb{R}$

Lemma 6.4. *Let $0 < t_1 < \cdots < t_n$ and T be a supercyclic operator on a real topological vector space X. Then $S = t_1T \oplus \cdots \oplus t_nT$ is cyclic.*

Proof. Without loss of generality, we can assume that $t_n = 1$. Let x be a supercyclic vector for T. We shall demonstrate that $u = (x, \dots, x)$ is a cyclic vector for S. We use the induction with respect to n. The case $n = 1$ is trivial. Assume that $n \geqslant 2$ and the statement is true for a sum $n - 1$ positive scalar multiples of a supercyclic operator. Since T is supercyclic and $t_j < 1$ for $j < n$, we see that

$$L = \{0\} \times \cdots \times \{0\} \times X \subset \overline{\{sS^nu\colon\ s \in \mathbb{R},\ n \in \mathbb{Z}_+\}}.$$

In particular, L is contained in the cyclic subspace $C(S, u)$. Obviously $S(L) \subseteq L$ and X^n/L is naturally isomorphic to X^{n-1} and the quotient operator $\widetilde{S}(v + L) = Sv + L$, acting on X^n/L is naturally similar to $t_1T \oplus \cdots \oplus t_{n-1}T$. From the induction hypothesis it follows that $u + L_m$ is a cyclic vector for $\widetilde{S}$. According to Lemma 6.1, u is a cyclic vector for S. □

In order to incorporate negative multiples, we need the following elementary observation.

Lemma 6.5. *Let X be a topological vector space and $T \in L(X)$ be such that $T(X)$ is dense in X and T^2 is cyclic. Then $S = T \oplus (-T)$ is cyclic.*

Proof. Let x be a cyclic vector for T^2. We shall demonstrate that (x, x) is a cyclic vector for S. Indeed, for any $p \in \mathcal{P}$, $p(S^2)(x, x) = (p(T^2)x, p(T^2)x)$ and $Sp(S^2)(x, x) = (Tp(T^2)x, -Tp(T^2)x)$. Since T^2 is cyclic and T has dense range, we see that the cyclic space $C(S, (x, x))$ contains the spaces $L_0 = \{(u, u)\colon\ u \in X\}$ and $L_1 = \{(u, -u)\colon\ u \in X\}$. Since $X \times X = L_0 \oplus L_1$, we see that $C(S, (x, x)) = X \times X$ and therefore (x, x) is a cyclic vector for S. □

Proof of Theorem 1.4. Case $\mathbb{K} = \mathbb{R}$. Let $A = \{|z_j|\colon\ 1 \leqslant j \leqslant n\}$ and $t_1, \dots, t_k$ be such that $t_1 < \cdots < t_k$ and $A = \{t_1, \dots, t_k\}$. Since z_j are real, we see that $\{z_1, \dots, z_n\} \subseteq \{t_1, \dots, t_k, -t_1, \dots, -t_k\}$. Thus, it is enough to show that $R_0 = R \oplus (-R)$ is cyclic, where $R = t_1T \oplus \cdots \oplus t_kT$. The Ansari theorem (Corollary 4.6) implies that T^2 is supercyclic. By Lemma 6.4, the operator $R^2 = t_1^2T^2 \oplus \cdots \oplus t_n^2T^2$ is cyclic. Since T is supercyclic, the range of T is dense and therefore the range of R is dense. By Lemma 6.5, $R_0 = R \oplus (-R)$ is cyclic. □

7. Concluding remarks

It would be interesting to investigate possibilities of extension of Lemma 2.5 to the case of non-commutative group G. The latter could be useful in studying of universality of non-commuting families of operators.

7.1. Universal semigroups

The Conejero–Müller–Peris theorem on hypercyclicity of members of a strongly continuous universal semigroup of linear operators on a complete metrizable topological vector space fails for semigroups labeled by $\mathbb{R}_+^n$ for $n \geqslant 2$.

Example 7.1. Take a compact weighted backward shift S on ℓ_2. The operator e^{tS} is hypercyclic for each $t \in \mathbb{R}$, $t \neq 0$ (see, for instance, [12]). Take $c > \|e^S\|$ and consider the operators $T_{t,s} = c^{s-t}e^{tS} \oplus c^{s-t}I$ acting on $\ell_2 \oplus \mathbb{K}$. From hypercyclicity of S it follows that the family $\{T_{t,s}\colon\ t \in \mathbb{Z}_+,\ s \in \mathbb{R}_+\}$ is universal and therefore the strongly continuous semigroup $\{T_{t,s}\}_{t,s \in \mathbb{R}_+^2}$ is universal. On the other hand, $T_{t,s} - c^{s-t}I$ has non-dense range for any $(t, s) \in \mathbb{R}_+$. By Theorem W each $T_{t,s}$ is non-hypercyclic. It is also easy to see that $\{T_{t,s}\colon\ (t, s) \in \mathbb{Z}_+^2\}$ is non-universal.

Thus, members of a universal strongly continuous semigroup $\{T_t\}_{t \in \mathbb{R}_+^2}$ may be all non-hypercyclic and $\{T_t\}_{t \in \mathbb{Z}_+^2}$ may be non-universal. This means that in order to extend the theorem on hypercyclicity of members of a strongly continuous universal semigroup, we need some extra assumptions about the semigroup. The next question seems to be natural and interesting.

Question 7.2. *Let X be a separable infinite dimensional complex Banach space and $z \mapsto T_z$ be a holomorphic operator valued function from $\mathbb{C}$ to $L(X)$ such that $T_0 = I$ and $T_{z+w} = T_z T_w$ for any $z, w \in \mathbb{C}$. Assume also that the family $\{T_z\colon z \in \mathbb{C}\}$ is universal. Is it true that each T_z with $z \neq 0$ is hypercyclic? What about holomorphic semigroups, labeled by a sector in $\mathbb{C}$?*

Holomorphic universal semigroups of bounded linear operators on Banach spaces have been treated in [4], where it is shown that every complex separable infinite dimensional Banach space X supports a holomorphic uniformly continuous mixing semigroup $\{T_z\}_{z\in\Delta(\frac{\pi}{2})}$, where $\Delta(\frac{\pi}{2})$ is the sector $\{re^{i\varphi}\colon r \geqslant 0,\ |\varphi| < \frac{\pi}{2}\}$, and $\{T_z\}_{z\in\Delta(\frac{\pi}{2})}$ is said to be (topologically) mixing if, for any pair (U, V) of non-empty open subsets of X, there is $r_0 > 0$ such that $T(z)(U) \cap V \neq \emptyset$ as soon as $|z| > r_0$. It is worth noting that $\{T_z\}$ is mixing if and only if $\{T_{z_n}\colon n \in \mathbb{N}\}$ is universal for any sequence $\{z_n\}$ such that $|z_n| \to \infty$. It is possible to show that the above result admits a stronger form. Namely, any separable infinite dimensional complex Banach space supports a mixing group $\{T_z\}_{z\in\mathbb{C}}$.

7.2. Remarks on Theorem 1.4

For completeness of the picture we include a generalization of Lemma 6.5 in the case $\mathbb{K} = \mathbb{C}$.

Lemma 7.3. *Let $n \in \mathbb{N}$ and T be a continuous linear operator with dense range on a complex topological vector space X such that T^n is cyclic. Let also $z = e^{2\pi i/n}$. Then the operator $S = T \oplus zT \oplus z^2T \oplus \cdots \oplus z^{n-1}T$ is cyclic.*

Proof. Let x be a cyclic vector for T^n. Then $L = \{r(T^n)x\colon r \in \mathcal{P}\}$ is dense in X. Since T has dense range, the spaces $T(L), \ldots, T^{n-1}(L)$ are also dense in X. It suffices to verify that $u = (x, \ldots, x) \in X^n$ is a cyclic vector for S. Let $M = C(S, u)$, $0 \leqslant k \leqslant n-1$ and $r \in \mathcal{P}$. Then

$$S^k r(S^n)u = \bigl(T^k r(T^n)x, z^k T^k r(T^n)x, \ldots, z^{k(n-1)} T^k r(T^n)x\bigr) \in M.$$

Thus, M contains the vectors of the shape $(a, z^k a, \ldots, z^{k(n-1)}a)$ for $a \in T^k(L)$ and $0 \leqslant k \leqslant n-1$. Since M is closed and $T^k(L)$ is dense in X, we see that

$$M \supseteq N_k = \bigl\{\bigl(a, z^k a, \ldots, z^{k(j-1)}a\bigr)\colon a \in X\bigr\} \quad \text{for } 0 \leqslant k \leqslant n-1.$$

Finally, the matrix $\{z^{kl}\}_{k,l=0}^{n-1}$ is invertible since its determinant is a Van der Monde one. Invertibility of the latter matrix implies that the union of N_k for $0 \leqslant k \leqslant n-1$ spans X^n. Hence $M = X^n$ and therefore u is a cyclic vector for S. □

Lemma 7.3 shows that under the condition $z_1^k = \cdots = z_n^k = 1$, the requirements on the operator T in Theorem 1.4 can be weakened. Namely, instead of supercyclicity of T it is enough to require cyclicity of T^k and the density of the range of T. For general z_j this is however not true. For instance, the Volterra operator $V \in L(L^2[0,1])$, $Vf(t) = \int_0^t f(s)\,ds$ has dense range, all powers of V are cyclic, while $V \oplus 2V$ is non-cyclic [20]. The latter example also provides means for an elementary proof of the fact that the Volterra operator V is not weakly supercyclic (= not supercyclic on $L_2[0,1]$ with weak topology), which also follows from a rather involved general theorem in [20], providing a sufficient condition for a bounded linear operator on a Banach space to be not weakly supercyclic.

Corollary 7.4. *The Volterra operator V acting on $L^2[0,1]$ is not weakly supercyclic.*

Proof. Consider the operator $J : L_2[0,1] \to L_2[0,1]$, $Jf(x) = f(\frac{1-x}{2})$. It is easy to see that $2JV = V^*J$. We show that $V \oplus 2V$ is not cyclic. Indeed, assume that (f, g) is a cyclic vector for $V \oplus 2V$. Then $f \neq 0$ and $J^*f \neq 0$ since J^* is injective. Using the equation $2JV = V^*J$, it is easy to verify that the orbit $\{(V \oplus 2V)^n(f, g)\colon n \in \mathbb{Z}_+\}$ is lying in the kernel of the continuous functional $\Phi(u, v) = \langle u, Jg\rangle - \langle v, J^*f\rangle$ on $H = L^2[0,1] \times L^2[0,1]$, where $\langle x, y\rangle = \int_0^1 x(t)y(t)\,dt$. Since $J^*f \neq 0$, we have $\Phi \neq 0$, which contradicts cyclicity of (f, g) for $V \oplus 2V$. Since cyclicity of an operator on a Banach space is equivalent to cyclicity with respect to the weak topology, $V \oplus 2V$ is non-cyclic on H with weak topology. By Theorem 1.4, V is not weakly supercyclic. □

It seems to be an appropriate place to reproduce the following question by Sophie Grivaux.

Question 7.5. *Let T be a continuous linear operator on a Banach space X such that $T \oplus T$ is cyclic. Does it follow that T^2 is cyclic?*

7.3. Cyclicity of direct sums of operators satisfying the Supercyclicity Criterion

The following important sufficient condition of supercyclicity is provided in [19] for Banach space operators. In the general setting it is a corollary of the following result of Bés and Peris [5, Theorem 2.3 and Remark 2.6]. Recall that an infinite family $\mathcal{F}$ of continuous maps from a topological space X to a topological space Y is called *hereditarily universal* if each infinite subfamily of $\mathcal{F}$ is universal.

Theorem BP. *Let $\{T_n\}_{n\in\mathbb{Z}_+}$ be a sequence of pairwise commuting continuous linear operators with dense range on a separable Baire metrizable topological vector space X. Then the following conditions are equivalent*:

(p1) *The family $\{T_n \oplus T_n\colon n \in \mathbb{Z}_+\}$ is universal*;
(p2) *There is an infinite set $A \subseteq \mathbb{Z}_+$ such that $\{T_n\colon n \in A\}$ is hereditarily universal*;
(p3) *There exist a strictly increasing sequence $\{n_k\}$ of non-negative integers, dense subsets E and F of X and maps $S_k : F \to X$ for $k \in \mathbb{Z}_+$ such that $T_{n_k}x \to 0$, $S_k y \to 0$ and $T_{n_k}S_k y \to y$ as $k \to \infty$ for any $x \in E$ and $y \in F$.*

We say that a continuous linear operator T on a separable Baire metrizable topological vector space X satisfies the *Supercyclicity Criterion* [19] if there exist a strictly increasing sequence $\{n_k\}_{k\in\mathbb{Z}_+}$ of positive integers, a sequence $\{s_k\}_{k\in\mathbb{Z}_+}$ of positive numbers, dense subsets E and F of X and maps $S_k : F \to X$ for $k \in \mathbb{Z}_+$ such that $T^{n_k}S_k y \to y$, $s_k T^{n_k}x \to 0$ and $s_k^{-1}S_k y \to 0$ as $k \to \infty$ for any $x \in E$ and $y \in F$. From Theorem BP it immediately follows that any operator T satisfying the Supercyclicity Criterion is supercyclic. Moreover it follows that T satisfies the Supercyclicity Criterion if and only if $T \oplus T$ is supercyclic.

Lemma 7.6. *Let T_j for $1 \leqslant j \leqslant k$ be continuous linear operators on separable Baire metrizable topological vector spaces X_j all satisfying the Supercyclicity Criterion with the same sequence $\{n_l\}_{l\in\mathbb{Z}_+}$. Then there exists an infinite subset A of $\mathbb{N}$ and sequences $\{s_{j,l}\}_{l\in A,\ 1\leqslant j\leqslant n}$ of positive numbers such that the family $\mathcal{F} = \{s_{1,l}T_1^l \oplus \cdots \oplus s_{k,l}T_k^l\colon l \in A\}$ is hereditarily universal.*

Proof. Let $\{n_l\}_{l\in\mathbb{Z}_+}$ be a strictly increasing sequence of positive integers such that each T_j satisfies the Supercyclicity Criterion with this sequence. Then, for each $j \in \{1,\dots,k\}$, we can pick a sequence $\{s_{j,n_l}\}_{l\in\mathbb{Z}_+}$ of positive numbers, dense subsets E_j and F_j of X_j and maps $S_{j,l} : F_j \to X_j$ for $l \in \mathbb{Z}_+$ such that $T_j^{n_l}S_{j,l}y \to y$, $s_{j,n_l}T_j^{n_l}x \to 0$ and $s_{j,n_l}^{-1}S_{j,l}y \to 0$ as $l \to \infty$ for any $x \in E_j$ and $y \in F_j$. Then $F = F_1 \times \cdots \times F_k$ and $E = E_1 \times \cdots \times E_k$ are dense in $X = X_1 \times \cdots \times X_k$. Let

$$S_l = s_{1,n_l}^{-1}S_{1,n_l} \oplus \cdots \oplus s_{k,n_l}^{-1}S_{k,n_l} : F \to X \quad \text{and} \quad T_l = s_{1,n_l}T_1^{n_l} \oplus \cdots \oplus s_{k,n_l}T_k^{n_l} \in L(X).$$

From the above properties of s_{j,n_l} and $S_{j,l}$ it follows that $T_lS_ly \to y$, $T_lx \to 0$ and $S_ly \to 0$ as $l \to \infty$ for any $x \in E$ and $y \in F$. It is also easy to see that T_l have dense range and commute with each other. By Theorem BP, there is an infinite subset B of $\mathbb{Z}_+$ such that the family $\{T_l\colon l \in B\}$ is hereditarily universal. Clearly this family has the required shape. □

The following result shows that if we restrict ourselves to operators, satisfying the Supercyclicity Criterion, then cyclicity of finite direct sums is satisfied not only for scalar multiples of the same operator.

Theorem 7.7. *Let T_j for $1 \leqslant j \leqslant k$ be continuous linear operators on separable Baire metrizable topological vector spaces X_j all satisfying the Supercyclicity Criterion with the same sequence $\{n_l\}_{l\in\mathbb{Z}_+}$. Then the direct sum $T_1 \oplus \cdots \oplus T_k$ is cyclic.*

Proof. By Lemma 7.6, there exists an infinite subset A of $\mathbb{Z}_+$ and positive numbers $\{s_{j,l}\}_{l\in A,\ 1\leqslant j\leqslant k}$ such that the family $\mathcal{F} = \{S_l = s_{1,l}T_1^l \oplus \cdots \oplus s_{k,l}T_k^l\colon l \in A\}$ is hereditarily universal. Denote $s_l = (s_{1,l},\dots,s_{k,l}) \in \mathbb{R}_+^k$. Replacing A by a smaller infinite subset of $\mathbb{Z}_+$, if necessary, may assume that $s_l/\|s_l\|_\infty \to s \in \mathbb{R}_+^k$ as $l \to \infty$, $l \in A$. Clearly $\|s\|_\infty = 1$. Without loss of generality, we can also assume that $1 = s_1 \geqslant s_2 \geqslant \cdots \geqslant s_k$. Let $m \in \mathbb{N}$ be such that $1 \leqslant m \leqslant k$, $s_m \neq 0$ and $s_j = 0$ if $m < j \leqslant k$. Let $x = (x_1,\dots,x_k)$ be a universal vector for the family $\mathcal{F}$. It suffices to show that x is a cyclic vector for $S = T_1 \oplus \cdots \oplus T_k$. We shall use induction with respect to k. For $k = 1$, the statement is trivial. Assume that $k \geqslant 2$ and it is true for the direct sums of less than k operators. Denote $X = X_1 \times \cdots \times X_k$. First, we shall show that $N = \{u \in X\colon u_j = 0 \text{ for } j > m\}$ is contained in the cyclic subspace $C(S,x)$. Let $w \in N$. Since x is universal for $\mathcal{F}$, there exists a strictly increasing sequence $\{n_l\}_{l\in\mathbb{Z}_+}$ of elements of A such that $s_{j,n_l}T_j^{n_l}x_j \to w_j/s_j$ if $1 \leqslant j \leqslant m$ and $s_{j,n_l}T_j^{n_l}x_j \to 0$ if $j > m$. Consider the polynomials $p_l(z) = \|s_{n_l}\|_\infty z^{n_l}$. Since $s_k/\|s_k\|_\infty \to s \in \mathbb{R}_+^n$ as $k \to \infty$, $k \in A$, we obtain $p_l(S)x \to (w_1,\dots,w_m,0,\dots,0) = w$ as $l \to \infty$, $l \in A$. Hence $w \in C(S,x)$ and therefore $N \subseteq C(S,x)$. Clearly $S(N) \subseteq N$. On the other hand X/N is naturally isomorphic to $X_{m+1} \times \cdots \times X_n$ and the quotient operator $\widetilde{S}(u+N) = Su + N$, acting on X/N is naturally similar to $T_{m+1} \oplus \cdots \oplus T_n$. From the induction hypothesis it follows that $x+N$ is a cyclic vector for $\widetilde{S}$. According to Lemma 6.1, x is a cyclic vector for S. □

7.4. Strongly n-supercyclic operators

Recently Feldman [9] has introduced the notion of an n-supercyclic operator for $n \in \mathbb{N} \cup \{\infty\}$. A continuous linear operator T on a topological vector space X is called *n-supercyclic* for $n \in \mathbb{N}$ if there exists an n-dimensional linear subspace L of X such that its orbit $\{T^nx\colon n \in \mathbb{Z}_+,\ x \in L\}$ is dense in X. Such a space L is called an *n-supercyclic subspace* for T. Clearly, 1-supercyclicity coincides with the usual supercyclicity. In [9], for any $n \in \mathbb{N}$, $n \geqslant 2$, a bounded linear operator T on ℓ_2 is constructed, which is n-supercyclic and not $(n-1)$-supercyclic. In [6], the question is raised whether powers of n-supercyclic operators are n-supercyclic. The question remains open. It is worth mentioning that the answer becomes affirmative if we replace n-supercyclicity by a slightly stronger property.

Namely, if X is a topological vector space of dimension $\geqslant n \in \mathbb{N}$, then the set $\mathbb{P}_n X$ of all n-dimensional linear subspaces of X can be endowed with the natural topology. That is we consider the (open) subset X_n of all linearly independent n-tuples $x = (x_1, \dots, x_n) \in X^n$ with the topology induced from X^n and declare the map $\pi_n : X_n \to \mathbb{P}_n X$, $\pi_n(x) = \mathtt{span}\{x_1, \dots, x_n\}$ continuous and open. Observe that $\mathbb{P}_n \mathbb{K}^m$ for $m \geqslant n$ is the classical Grassmanian manifold. We say that $L \in \mathbb{P}_n X$ is a *strongly n-supercyclic* subspace for $T \in L(X)$ if each $T^k(L)$ is n-dimensional and the sequence $\{T^k(L)\colon k \in \mathbb{Z}_+\}$ is dense in $\mathbb{P}_n X$. An operator is *strongly n-supercyclic* if it has a strongly n-supercyclic subspace. Clearly, strong 1-supercyclicity is equivalent to supercyclicity and strong n-supercyclicity implies n-supercyclicity. The advantage of strong n-supercyclicity lies in the fact that it reduces to universality of the self-map of $\mathbb{P}_n X$ induced by T. Thus, using connectedness of $\mathbb{P}_n X$ we can prove that the powers of a strongly n-supercyclic operators are strongly n-supercyclic by means of applying Proposition 4.5 in pretty much the same way as we did it for supercyclic operators. It is worth noting that n-supercyclic operators, constructed by Feldman in [9], are in fact strongly n-supercyclic. This leads to the following question.

Question 7.8. *Are n-supercyclicity and strong n-supercyclicity equivalent?*

We would also like to reproduce the following interesting question raised in [6,9].

Question 7.9. *Let $n \in \mathbb{N}$ and T be an n-supercyclic operator on a complex topological vector space X such that $\sigma_p(T^*) = \varnothing$. Is it true that T is cyclic?*

7.5. Supercyclic operators with non-empty point spectrum of the dual

Our final remark concerns the following claim made in [16]:

(L) any supercyclic operator T on a complex Banach space X satisfying $\sigma_p(T^*) = \{z\}$ with $z \in \mathbb{C} \setminus \{0\}$ is similar to the operator of the shape $z(S \oplus I_{\mathbb{C}})$, where S is a hypercyclic operator and $I_{\mathbb{C}}$ is the identity operator on the one-dimensional space $\mathbb{C}$.

If (L) was true, we could have significantly simplified the proof of the characterization of $\mathbb{R}_+$-supercyclicity in Section 5. Unfortunately, this statement is false. It will be shown along with a characterization of supercyclic operators with non-empty $\sigma_p(T^*)$.

If T is a supercyclic operator on a complex topological vector space, then by Theorem W, either $\sigma_p(T^*) = \varnothing$ or $\sigma_p(T^*) = \{z\}$ with $z \in \mathbb{C} \setminus \{0\}$. In the latter case, there is a non-zero $f \in X^*$ such that $T^* f = zf$. It is clear that $Y = \ker f$ is a closed invariant subspace for T of codimension 1. Pick any $e \in X$ such that $f(e) = 1$. Then $X = Y \oplus \langle e \rangle \simeq Y \times \mathbb{C}$. Since $T^* f = zf$, we have $Te = z(e + u)$, where $u \in Y$. Thus, T is naturally similar to the operator $zS_u \in L(Y \times \mathbb{C})$, where $S_u(y, t) = (Sy + tu, t)$ and $S = z^{-1} T|_Y \in L(Y)$. Thus, any supercyclic operator, whose dual has non-empty point spectrum, is similar to a scalar multiple of an operator of the shape S_u. It remains to figure out when an operator S_u is supercyclic.

Lemma 7.10. *Let Y be a topological vector space $S \in L(Y)$, $u \in Y$ and*

$$S_u \in L(Y \times \mathbb{K}), \quad S_u(y, t) = (Sy + tu, t). \tag{7.1}$$

Then (x, s) is a supercyclic vector for S_u if and only if $s \neq 0$ and $u - s^{-1}(I - S)x$ is a universal vector for the family $\{p_n(S)\colon n \in \mathbb{N}\}$, where

$$p_n(z) = \sum_{j=0}^{n-1} z^j. \tag{7.2}$$

Proof. If $s = 0$, then $O(S_u, (x, s))$ is contained in $Y \times \{0\}$ and (x, s) can not even be a cyclic vector for S_u. Assume now that $s \neq 0$. Then (x, s) is a supercyclic vector for S_u if and only if $(y, 1)$ is a supercyclic vector for S_u, where $y = s^{-1} x$. Direct calculation shows that for any $n \in \mathbb{N}$,

$$S_u^n(y, 1) = \bigl(S^n y + p_n(S)u, 1\bigr) = \bigl(y + p_n(S)\bigl(u - (I - S)y\bigr), 1\bigr).$$

It immediately follows that $\{zS_u^n(y, 1)\colon n \in \mathbb{Z}_+,\ z \in \mathbb{K}\}$ is dense in $Y \times \mathbb{K}$ if and only if $\{y + p_n(S)(u - (I - S)y)\colon n \in \mathbb{N}\}$ is dense in Y. Since the translation by y is a homeomorphism from Y to itself, the latter happens if and only if $\{p_n(S)(u - (I - S)y)\colon n \in \mathbb{N}\}$ is dense in Y. Thus, $(y, 1)$ is a supercyclic vector for S_u if and only if $u - (I - S)y$ is a universal vector for $\{p_n(S)\colon n \in \mathbb{N}\}$. It remains to recall that $y = s^{-1} x$. □

Corollary 7.11. *Let Y be a topological vector space $S \in L(Y)$, $u \in Y$ and $S_u \in L(Y \times \mathbb{K})$ be defined by (7.1). Then S_u is supercyclic if and only if the coset $u + (I - S)(Y)$ contains a universal vector for the family $\{p_n(S)\colon n \in \mathbb{N}\}$, where p_n are polynomials defined in (7.2).*

It becomes a natural question to study universality of the family $\{p_n(S)\colon n \in \mathbb{N}\}$.

Lemma 7.12. *Let Y be a topological vector space, $S \in L(Y)$ and $\mathcal{F} = \{p_n(S)\colon n \in \mathbb{N}\}$, where p_n are polynomials defined by* (7.2)*. Then $\mathcal{F}$ is universal if and only if S is hypercyclic. Moreover, $(I - S)(\mathcal{U}(S)) \subseteq \mathcal{U}(\mathcal{F}) \subseteq \mathcal{U}(S)$.*

Proof. If S is hypercyclic, then by Theorem W, $I - S$ has dense range. The same holds true if $\mathcal{F}$ is universal. Indeed, assume that there is $x \in \mathcal{U}(\mathcal{F})$ and $I - S$ has non-dense range. Then the closure Z in Y of $I - S(Y)$ is invariant for S and $Z \neq Y$. Consider the operator $\widetilde{S} : Y/Z \to Y/Z$, $\widetilde{S}(y + Z) = Sy + Z$. Clearly $x + Z$ is universal for $\mathcal{A} = \{p_n(\widetilde{S})\colon n \in \mathbb{N}\}$. On the other hand, from the definition of Z it follows that $\widetilde{S}$ is the identity operator on Y/Z and therefore $\mathcal{A} = \{nI\colon n \in \mathbb{N}\}$. The latter system is obviously non-universal. This contradiction shows that $I - S$ has dense range if $\mathcal{F}$ is universal. Thus, we can assume from the beginning that $(I - S)(Y)$ is dense in Y. It follows that if $x \in \mathcal{U}(\mathcal{F})$, then $(I - S)x \in \mathcal{U}(\mathcal{F})$. Since $p_n(S)(I - S)x = x - S^n x$, universality of $(I - S)x$ for $\mathcal{F}$ implies that x is hypercyclic for S. Hence universality of $\mathcal{F}$ implies hypercyclicity of S and $\mathcal{U}(\mathcal{F}) \subseteq \mathcal{U}(S)$. Now if $x \in \mathcal{U}(S)$, then the sequence $p_n(S)(I - S)x = x - S^n x$ is dense in X and therefore $(I - S)x \in \mathcal{U}(\mathcal{F})$. Thus hypercyclicity of S implies universality of $\mathcal{F}$ and $(I - S)(\mathcal{U}(S)) \subseteq \mathcal{U}(\mathcal{F})$. □

Lemma 7.13. *Let Y be a topological vector space, $S \in L(Y)$, $u \in Y$ and $S_u \in L(Y \times \mathbb{K})$ be defined by* (7.1)*. Assume also that $1 \notin \sigma_p(S^*)$. Then S_u is similar to $S_0 = S \oplus I_{\mathbb{K}}$ if and only if $u \in (I - S)(Y)$.*

Proof. First, assume that $u \in (I - S)(Y)$. Then $u = v - Sv$ for some $v \in Y$. Consider the operator $\Lambda \in L(Y \times \mathbb{K})$, $\Lambda(x, s) = (x + sv, s)$. Clearly Λ is invertible and $\Lambda^{-1} \in L(Y \times \mathbb{K})$, $\Lambda^{-1}(x, s) = (x - sv, s)$. It is easy to see that $\Lambda^{-1} S_u \Lambda = S_0$. Thus, S_u is similar to S_0.

Assume now that S_u is similar to S_0. That is, there exists $\Lambda \in L(Y \times \mathbb{K})$ such that Λ is invertible, Λ^{-1} is continuous and $\Lambda^{-1} S_u \Lambda = S_0$. Since $\sigma_p(S^*) = \varnothing$, $\ker(S_0^* - I)$ is the one-dimensional space spanned by the functional $f_0 \in (Y \times \mathbb{K})^*$, $f_0(y, t) = t$. Since $(S_u^* - I) f_0 = 0$, we see that $\ker f_0 = Y \times \{0\}$ must be Λ-invariant. Since $S_0(0, 1) = (0, 1)$ and $(0, 1) \notin Y \times \{0\}$, Λ_u must have an eigenvector $(x, t) \notin Y \times \{0\}$ corresponding to the eigenvalue 1. Since $t \neq 0$, we can, without loss of generality, assume that $t = 1$. Thus, $S_u(x, 1) = (x, 1)$. It follows that $Sx + u = x$. That is, $u = (I - S)x \in (I - S)(X)$. □

Corollary 7.14. *Let Y be a topological vector space $S \in L(Y)$, $u \in Y$ and $S_u \in L(Y \times \mathbb{K})$ be defined by* (7.1)*. If u is universal for $\{p_n(S)\colon n \in \mathbb{N}\}$, where p_n are polynomials defined by* (7.2) *and $u \notin (I - S)(Y)$, then S_u is supercyclic and not similar to $S_0 = S \oplus I_{\mathbb{K}}$.*

Proof. Supercyclicity of S_u follows from Corollary 7.11. By Lemma 7.12, S is hypercyclic and therefore by Theorem W, $\sigma_p(S^*) = \varnothing$. By Lemma 7.13, S_u is not similar to $S_0 = S \oplus I_{\mathbb{K}}$. □

Corollary 7.15. *Let Y be a Baire separable metrizable topological vector space and $S \in L(Y)$ be a hypercyclic operator such that $(I - S)(Y) \neq Y$. Then there is $u \in Y$ such that the operator S_u defined by* (7.1) *is supercyclic and not similar to $S_0 = S \oplus I_{\mathbb{K}}$.*

Proof. Since S is hypercyclic, Lemma 7.12 implies that the family $\mathcal{F} = \{p_n(S)\colon n \in \mathbb{N}\}$ with p_n being the polynomials defined in (7.2) has dense set of universal elements. Since the set of universal elements for any family of maps taking values in a second countable topological space is a G_δ-set [13], $\mathcal{U}(\mathcal{F})$ is a dense G_δ-subset of the Baire space Y. Hence $\mathcal{U}(\mathcal{F})$ is not contained in $(I - S)(Y)$. Indeed, assume that $\mathcal{U}(\mathcal{F}) \subseteq (I - S)(Y)$. Since $(I - S)(Y) \neq Y$, there is $w \in Y$ such that $w \notin (I - S)(Y)$. Since $(I - S)(Y)$ is a linear subspace of Y, $(I - S)(Y) \cap (w + (I - S)(Y)) = \varnothing$. Hence $\mathcal{U}(\mathcal{F}) \cap (w + \mathcal{U}(\mathcal{F})) = \varnothing$. Thus $\mathcal{U}(\mathcal{F})$ and $w + \mathcal{U}(\mathcal{F})$ are disjoint dense G_δ-subsets in Y. The existence of such subsets is impossible since Y is Baire. Thus, we can choose $u \in \mathcal{U}(\mathcal{F}) \setminus (I - S)(Y)$. According to Corollary 7.14, S_u is supercyclic and not similar to $S_0 = S \oplus I_{\mathbb{K}}$. □

To illustrate the above corollary, we consider the weighted backward shifts on ℓ_2. If $\{w_n\}_{n \in \mathbb{N}}$ is a bounded sequence of positive numbers, then the operator $T_w : \ell_2 \to \ell_2$ acting on the canonical basis as $T_w e_0 = 0$, $T_w e_n = w_n e_{n-1}$ for $n > 0$ is called an *backward weighted shift*. Clearly T_w is compact if and only if $w_n \to 0$ as $n \to \infty$. According to Salas [22], any operator $S = I + T_w$ is hypercyclic. On the other hand, $I - S = -T_w$ is not onto whenever the sequence w is not bounded from below by a positive constant. In particular, for any compact backward weighted shift T, $S = I + T$ is hypercyclic and $I - S$ is not onto. Thus, by Corollary 7.15, we can choose $u \in \ell_2$ such that S_u is supercyclic and not similar to $S \oplus I_{\mathbb{K}}$. Thus, the statement (L) is indeed false.

From Lemmas 7.10 and 7.12 it follows that supercyclicity of S_u implies hypercyclicity of S. Moreover, supercyclicity of S_0 is equivalent to hypercyclicity of S. It is natural therefore to consider the question whether hypercyclicity of S implies supercyclicity of S_u for any vector u. The following example provides a negative answer to this question even in the friendly situation when Y is a Hilbert space.

Proposition 7.16. *There exists a hypercyclic operator $S \in L(\ell_2)$ and $u \in \ell_2$ such that the operator $S_u \in L(\ell_2 \times \mathbb{K})$ defined by* (7.1) *is not supercyclic.*

Proof. Consider the backward weighted shift $T \in L(\ell_2)$ with the weight sequence $w_n = e^{-2n}$, $n \in \mathbb{N}$ and let $S = I + T$. Let also $u \in \ell_2$, $u_n = (n + 1)^{-1}$ for $n \in \mathbb{Z}_+$. According the above cited theorem of Salas, S is hypercyclic. It remains to demonstrate that S_u is not supercyclic.

Assume that S_u is supercyclic. By Corollary 7.11, the set $u+T(\ell_2)$ contains a universal vector for the family $\mathcal{F}=\{p_n(S):\ n\in\mathbb{N}\}$ with p_n being the polynomials defined in (7.2). By Lemma 7.12, $\mathcal{U}(\mathcal{F})\subseteq\mathcal{U}(S)$ and therefore $(u+T(\ell_2))\cap\mathcal{U}(S)\neq\varnothing$. That is, there exists $x\in\ell_2$ such that $u+Tx$ is a hypercyclic vector for S. We are going to obtain a contradiction by showing that $\|S^n(u+Tx)\|\to\infty$ as $n\to\infty$.

Taking into account that $S^n=(I+T)^n=\sum_{k=0}^n\binom{n}{k}T^k$, we see that for each $y\in\ell_2$,

$$\left(S^ny\right)_j=\sum_{k=0}^n\binom{n}{k}y_{j+k}e^{-k(k+1)}\quad\text{for any } j,n\in\mathbb{Z}_+. \tag{7.3}$$

Applying this formula to $y=u$, we obtain

$$\left\|S^nu\right\|\geqslant\left|\left(S^nu\right)_0\right|=A_n,\quad\text{where } A_n=\sum_{k=0}^n\binom{n}{k}(k+1)^{-1}e^{-k(k+1)}\text{ for any } n\in\mathbb{Z}_+. \tag{7.4}$$

Now since $x\in\ell_2$, $c=\sup_{j\in\mathbb{Z}_+}|x_j|<\infty$. Hence $|(Tx)_j|\leqslant ce^{-2j-2}$. Substituting these inequalities into (7.3), we see that

$$\left|\left(S^nTx\right)_j\right|\leqslant c\sum_{k=0}^n\binom{n}{k}e^{-2j-2k-2}e^{-k(k+1)}=ce^{-2j}B_n\quad\text{for any } n,j\in\mathbb{Z}_+,\ \text{where } B_n=\sum_{k=0}^n\binom{n}{k}e^{-(k+1)(k+2)}. \tag{7.5}$$

Summing up the inequalities in (7.5), we obtain

$$\left\|S^nTx\right\|\leqslant\sum_{j=0}^\infty\left|\left(S^nTx\right)_j\right|\leqslant cB_n\sum_{j=0}^\infty e^{-2j}<2cB_n\quad\text{for any } n\in\mathbb{Z}_+. \tag{7.6}$$

In order to demonstrate that $\|S^n(u+Tx)\|\to\infty$ as $n\to\infty$, it is enough to show that $A_n\to\infty$ and $B_n=o(A_n)$ as $n\to\infty$. Indeed, then from (7.4) and (7.6) it immediately follows that $\|S^n(u+Tx)\|\to\infty$. Applying the Stirling formula to estimate $\binom{n}{k}(k+1)^{-1}e^{-k(k+1)}$, we see that there exist positive constants α and β such that whenever $1\leqslant k\leqslant n^{1/2}$,

$$\alpha k^{-3/2}\left(nk^{-1}e^{-k}\right)^k\leqslant\binom{n}{k}(k+1)^{-1}e^{-k(k+1)}\leqslant\beta k^{-3/2}\left(nk^{-1}e^{-k}\right)^k. \tag{7.7}$$

From (7.7) it follows that if $\{k_n\}$ is a sequence of positive integers such that $2k_n-\ln n=O(1)$, then for any $a<1/4$,

$$A_n\geqslant\binom{n}{k_n}(k_n+1)^{-1}e^{-k_n(k_n+1)}\geqslant e^{a(\ln n)^2}\quad\text{for all sufficiently large } n.$$

Hence $A_n\to\infty$. Using (7.7), we immediately see that

$$A_n'=\sum_{0\leqslant k<(\ln n)/4}\binom{n}{k}(k+1)^{-1}e^{-k(k+1)}=o\left(e^{b(\ln n)^2}\right)\quad\text{for any } b>3/16.$$

From the last two displays we have $A_n'=o(A_n)$. Then

$$B_n'=\sum_{0\leqslant k<(\ln n)/4}\binom{n}{k}e^{-(k+1)(k+2)}\leqslant A_n'=o(A_n).$$

On the other hand, for $(\ln n)/4\leqslant k\leqslant n$, we have $(k+1)e^{-2k-2}<4n^{-2}\ln n$ and therefore

$$\binom{n}{k}e^{-(k+1)(k+2)}\leqslant\binom{n}{k}(k+1)^{-1}e^{-k(k+1)}4n^{-2}\ln n.$$

Hence

$$(B_n-B_n')\leqslant 4n^{-1}\ln n(A_n-A_n')\leqslant 4n^{-2}\ln nA_n=o(A_n).$$

Thus, $B_n=B_n'+(B_n-B_n')=o(A_n)$, which completes the proof. □

Appendix A. Proof of Lemmas 3.1–3.4

Recall that a subset U of a topological vector space X is called *balanced* if $tx\in U$ for any $x\in U$ and $t\in\mathbb{K}$ with $|t|\leqslant 1$. It is well known [23] that any topological vector space has a base of neighborhoods of zero consisting of balanced sets.

Proof of Lemma 3.1. Let $x,y\in X$. Then $f_{x,y}:[0,1]\to X$, $f_{x,y}(t)=(1-t)x+ty$ is continuous, $f_{x,y}(0)=x$ and $f_{x,y}(1)=y$. If $\mathcal{W}$ is a base of neighborhoods of zero in X consisting of balanced sets, then $\{x+W:\ x\in X,\ W\in\mathcal{W}\}$ is a base of topology of X consisting of path connected sets. Indeed, for any $x\in X$, $W\in\mathcal{W}$ and $w\in W$ the continuous path $f_{x,x+w}$ connects x

and $x+w$ and never leaves $x+W$. Hence X has a base of topology consisting of path connected sets and therefore X is locally path connected. Finally, let $f:\mathbb{T}\to X$ be continuous. Then $h:[0,1]\times\mathbb{T}\to X$, $h(t,s)=tf(s)$ is a contraction of f. Thus, X is simply connected. □

We shall use the following well-known properties of connectedness.

(c1) A path connected topological space X is simply connected if and only if the space $C(\mathbb{T},X)$ of continuous maps from $\mathbb{T}$ to X with the compact-open topology is path connected;
(c2) If X is a locally connected (has a base of topology consisting of connected sets), then any connected component of X is closed and open.

Lemma A.1. *Let X be a topological vector space and A be the subset of $C(\mathbb{T},X\setminus\{0\})$ consisting of continuous functions $f:\mathbb{T}\to X\setminus\{0\}$ with finite dimensional $\mathtt{span}(f(\mathbb{T}))$. Then A is dense in $C(\mathbb{T},X\setminus\{0\})$.*

Proof. Let $f\in C(\mathbb{T},X\setminus\{0\})$. For any $n\in\mathbb{N}$ consider the function $f_n:\mathbb{T}\to X$ such that $f_n(e^{2\pi it})=(k+1-nt)f(e^{2\pi ik/n})+(nt-k)f(e^{2\pi i(k+1)/n})$ for $t\in[k/n,k+1/n]$, $0\leqslant k\leqslant n$. It is straightforward to see that f_n are well defined, continuous and the sequence f_n is convergent to f in $C(\mathbb{T},X)$. Taking into account that f_n does not take value zero for sufficiently large n and $\mathtt{span}(f_n(\mathbb{T}))$ is at most n-dimensional, we obtain density of A in $C(\mathbb{T},X\setminus\{0\})$. □

Proof of Lemma 3.2. Let $\mathcal{W}$ be a base of neighborhoods of zero in X consisting of balanced sets. Consider the family $\mathcal{B}=\{x+W\colon\ x\in X,\ W\in\mathcal{W},\ 0\notin x+W\}$. Clearly $\mathcal{B}$ is a base of topology of $X\setminus\{0\}$. As in the proof of Lemma 3.1, we see that $\mathcal{B}$ consists of path connected sets and therefore $X\setminus\{0\}$ is locally path connected. Next, for any $x,u\in X\setminus\{0\}$, we can pick $y\in X$ such that $0\notin[x,y]\cup[y,u]$. Then $g_{x,y,u}:[0,1]\to X\setminus\{0\}$,

$$g_{x,y,u}(t)=\begin{cases}(1-2t)x+2ty & \text{if } 0\leqslant t\leqslant 1/2,\\ (2-2t)y+(2t-1)u & \text{if } 1/2<t\leqslant 1\end{cases}$$

is continuous, $g(0)=x$ and $g(1)=u$. Thus, $X\setminus\{0\}$ is path connected.

Next, let $f\in C(\mathbb{T},X\setminus\{0\})$. Pick $W\in\mathcal{W}$ such that $0\notin f(\mathbb{T})+W$. Then the set $\Omega_{f,W}=\{g\in C(\mathbb{T},X\setminus\{0\})\colon\ (g-f)(\mathbb{T})\subseteq W\}$ is a neighborhood of f in $C(\mathbb{T},X\setminus\{0\})$. Moreover, $\Omega_{f,W}$ is path connected. Indeed, for any $g\in\Omega_{f,W}$, the continuous path $F:[0,1]\to C(\mathbb{T},x+W)$, $F(t)(s)=(1-t)g(s)+tf(s)$ connects g and f and never leaves $\Omega_{f,W}$. Since the family $\{\Omega_{f,W}\colon\ 0\notin f(\mathbb{T})+W\}$ is a base of topology of $C(\mathbb{T},X\setminus\{0\})$, we see that $C(\mathbb{T},X\setminus\{0\})$ is locally path connected. Assume that $X\setminus\{0\}$ is not simply connected. According to (c1), $C(\mathbb{T},X\setminus\{0\})$ is not path connected. Thus, there is $f\in C(\mathbb{T},X\setminus\{0\})$ that can not be connected with a constant map by a continuous path. By (c2) the connected component Ω of f in $C(\mathbb{T},X\setminus\{0\})$ is open. According to Lemma A.1, the family A of $g\in C(\mathbb{T},X\setminus\{0\})$ with finite dimensional $\mathtt{span}(g(\mathbb{T}))$ is dense in $C(\mathbb{T},X\setminus\{0\})$. Hence we can pick $g\in\Omega$ and an $\mathbb{R}$-linear subspace L of X such that $3\leqslant\dim L<\infty$ and $g(\mathbb{T})\subseteq L$. On the other hand, it is well known [24] that $\mathbb{R}^n\setminus\{0\}$ is simply connected for $n\geqslant 3$. Thus, we can connect g with a constant map by a continuous path x_0 lying within $C(\mathbb{T},L\setminus\{0\})\subseteq C(\mathbb{T},X\setminus\{0\})$. We have obtained a contradiction. □

Lemma A.2. *Let X be a topological vector space and $f:[0,1]\to\mathbb{P}X$ be continuous. Then there exists continuous $g:[0,1]\to X\setminus\{0\}$ such that $f=\pi\circ g$. Similarly, for any continuous $f_0:[0,1]\to\mathbb{P}_+X$, there exists continuous $g_0:[0,1]\to X\setminus\{0\}$ such that $f_0=\pi_+\circ g_0$.*

Proof. Taking into account that $f([0,1])$ and $f_0([0,1])$ are metrizable and compact, we see that $\pi^{-1}(f([0,1]))$ and $\pi_+^{-1}(f_0([0,1]))$ are complete metrizable subsets of X. Consider multivalued functions $\widetilde{f}:[0,1]\to 2^X$ and $\widetilde{f}_0:[0,1]\to 2^X$, $\widetilde{f}(t)$ being $f(t)$ considered as a subset (one-dimensional subspace) of X and $\widetilde{f}_0(t)$ being $f_0(t)$ considered as a subset (ray) of X. It is easy to see that these multivalued maps satisfy all the conditions of Theorem 1.2 [18] by Michael and therefore for any $t_0\in[0,1]$ and any $x_0\in\widetilde{f}(t_0)$ (respectively, $x_0\in\widetilde{f}_0(t_0)$) there exists a continuous map $h_{t_0,x_0}:[0,1]\to X$ such that $h_{t_0,x_0}(t_0)=x_0$ and $h_{t_0,x_0}(t)\in\widetilde{f}(t)$ (respectively, $h_{t_0,x_0}(t)\in\widetilde{f}_0(t)$) for any $t\in[0,1]$. Now it is a routine exercise to show that required functions g and g_0 can be obtained as finite sums of the functions of the shape $\varphi\cdot h_{t_0,x_0}$ with continuous $\varphi:[0,1]\to(0,\infty)$. □

Lemma A.3. *Let X be a complex topological vector space. Then for any continuous $f:\mathbb{T}\to\mathbb{P}X$, there exists continuous $g:\mathbb{T}\to X\setminus\{0\}$ such that $f=\pi\circ g$.*

Proof. By Lemma A.2, we can find continuous $h:[0,1]\to X\setminus\{0\}$ such that $f(e^{2\pi it})=\pi(h(t))$ for any $t\in[0,1]$. Since $h(0)$ and $h(1)$ are both non-zero elements of the one-dimensional space $f(1)$, there is $z\in\mathbb{C}\setminus\{0\}$ such that $h(1)=z^{-1}h(0)$. Let $r>0$ and $s\in\mathbb{R}$ be such that $z=re^{is}$. Now it is easy to see that the function $g:\mathbb{T}\to X\setminus\{0\}$ defined by the formula $g(e^{2\pi it})=(1+(r-1)t)e^{ist}h(t)$ for $0\leqslant t<1$ satisfies the required conditions. □

Lemma A.4. *Let X be a topological vector space. Then for any continuous $f : \mathbb{T} \to \mathbb{P}_+X$, there exists continuous $g : \mathbb{T} \to X \setminus \{0\}$ such that $f = \pi_+ \circ g$.*

Proof. By Lemma A.2, there is a continuous $h : [0,1] \to X \setminus \{0\}$ such that $f(e^{2\pi it}) = \pi(h(t))$ for any $t \in [0,1]$. Since $h(0)$ and $h(1)$ are both non-zero elements of the ray $f(1)$, there is $r > 0$ such that $h(1) = r^{-1}h(0)$. Clearly, the function $g : \mathbb{T} \to X \setminus \{0\}$ defined as $g(e^{2\pi it}) = (1 + (r-1)t)h(t)$ for $0 \leqslant t < 1$ satisfies the required conditions. □

Proof of Lemma 3.3. For every linearly independent $x, y \in X$, we consider the path $h_{x,y} : [0,1] \to \mathbb{P}X$, $h_{x,y}(t) = \langle (1-t)x + ty \rangle$. Clearly each $h_{x,y}$ is continuous, $h_{x,y}(0) = \langle x \rangle$ and $h_{x,y}(1) = \langle y \rangle$. Thus, any two distinct points of $\mathbb{P}X$ can be connected by a continuous path and therefore $\mathbb{P}X$ is path connected. If W is a balanced neighborhood of zero and $x \in X$ are such that $0 \notin x + W$, then for any $y \in x + W$ with $\langle y \rangle \neq \langle x \rangle$, the path $h_{x,y}$ connects $\langle x \rangle$ and $\langle y \rangle$ and never leaves $\pi(x+W)$. Thus, $\pi(x+W)$ is path connected. Since the family of $\pi(x+W)$ forms a base of topology of $\mathbb{P}X$, we see that $\mathbb{P}X$ is locally path connected. Let now $f : \mathbb{T} \to \mathbb{P}X$ be a continuous map. If X is finite dimensional, then $\mathbb{P}X$ is homeomorphic to $\mathbb{PC}^n$ for some $n \in \mathbb{N}$. The latter spaces are known to be simply connected [24]. If X is infinite dimensional, we use Lemma A.3 to find a continuous map $g : \mathbb{T} \to X \setminus \{0\}$ such that $f = \pi \circ g$. According to Lemma 3.2, there is continuous $\varphi : [0,1] \times \mathbb{T} \to X \setminus \{0\}$ and $x_0 \in X \setminus \{0\}$ such that $\varphi(0,s) = g(s)$ and $\varphi(1,s) = x_0$ for any $s \in \mathbb{T}$. The map $\pi \circ \varphi$ provides the required contraction of the closed path f in $\mathbb{P}X$. □

Proof of Lemma 3.4. The proof of path connectedness, local path connectedness of $\mathbb{P}_+X$ is exactly the same as the above proof for $\mathbb{P}X$. Simple connectedness of $\mathbb{P}_+X$ in the case of finite dimensional X follows from the fact that then $\mathbb{P}_+X$ is homeomorphic to the $(n-1)$-dimensional sphere S^{n-1} if the real dimension of X is n. Since S^k is simply connected for $k \geqslant 2$ [24], $\mathbb{P}_+X$ is simply connected if X has finite real dimension $\geqslant 3$. If X is infinite dimensional, the proof of simple connectedness of $\mathbb{P}_+X$ is the same as for $\mathbb{P}X$ for complex X with the only difference that we use Lemma A.4 instead of Lemma A.3. □

Remark. The relative difficulty of some of the above lemmas is due to the fact that we consider general, not necessarily locally convex, topological vector spaces. The locally convex case is indeed elementary because of the guaranteed rich supply of continuous linear functionals.

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