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Liu, W., Zhang, Y., Liu, K., Quinn, B., Yang, X., & Peng, Q. (2024). Evolutionary multi-objective optimisation for large-scale portfolio selection with both random and uncertain returns. *IEEE Transactions on Evolutionary Computation*. Advance online publication. https://doi.org/10.1109/TEVC.2023.3349073

Published in:
IEEE Transactions on Evolutionary Computation

Document Version:
Peer reviewed version

This article has been accepted for publication in IEEE Transactions on Evolutionary Computation. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TEVC.2023.3349073

Evolutionary Multi-objective Optimisation for Large-scale Portfolio Selection with Both Random and Uncertain Returns

Weilong Liu, Yong Zhang, Kailong Liu, Barry Quinn, Xingyu Yang, Qiao Peng

***Abstract*—With the advent of Big Data, managing large-scale portfolios of thousands of securities is one of the most challenging tasks in the asset management industry. This study uses an evolutionary multi-objective technique to solve large-scale portfolio optimisation problems with both long-term listed and newly listed securities. The future returns of long-term listed securities are defined as random variables whose probability distributions are estimated based on sufficient historical data, while the returns of newly listed securities are defined as uncertain variables whose uncertainty distributions are estimated based on experts' knowledge. Our approach defines security returns as theoretically uncertain random variables and proposes a three-moment optimisation model with practical trading constraints. In this study, a framework for applying arbitrary multi-objective evolutionary algorithms to portfolio optimisation is established, and a novel evolutionary algorithm based on large-scale optimisation techniques is developed to solve the proposed model. The experimental results show that the proposed algorithm outperforms state-of-the-art evolutionary algorithms in large-scale portfolio optimisation.**

***Index Terms*—Evolutionary computations, Multi-objective optimisation, Portfolio optimisation, Large-scale investment, Uncertain random variable.**

I. INTRODUCTION

In 2022, the global asset management industry hit a new high of 126 trillion of assets under management (AUM). This figure represents 28 percent of global financial assets, up from 23 percent a decade ago (McKinsey, 2022). The computational complexity of optimal portfolio construction, which simultaneously balances risk minimisation with return maximisation, is perhaps the most intrinsic and recurrent financial problem in the asset management industry. Modern Portfolio Theory (MPT), first introduced by Markowitz [1], extols the virtues of the first two moments of the Gaussian distribution (the so-called mean-variance model) as sufficient to solve the problem of optimal portfolio allocation based on practitioners' views on risk and return. In academia, the MPT continues to be challenged from various perspectives. Some scholars have chosen to extend the mean-variance approach in non-trivial directions [2], while others have attacked the statistical validity of the mean-variance model and proposed practical extension to accommodate fat-tailedness of risk factors [3], and or improvements to the distribution of speculative assets [4]. In this paper, we consider the practical challenge of large-scale portfolio optimisation where historical data for some eligible securities is limited.

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Most existing portfolio optimisation models view financial asset returns as random variables whose distributional characteristics can be extracted from historical data. When there is a lack of sufficient historical data for newly listed securities, scholars have proposed the use of fuzzy variables estimated by experts' judgement, leading to the development of fuzzy portfolio optimisation theory [5–7]. Also, Liu [8] propose the uncertainty theory as another alternative tool for modelling indeterministic quantities that are subject to experts' estimates. Based on the uncertainty theory, portfolio optimisation problems with complementary information are able to be solved in uncertain environments [9–12].

This study addresses the real-world challenge of large-scale portfolio optimisation problems when the eligible basket of securities includes both established (long-term listed) and newly listed entities. Since sufficient historical samples are available for the long-term listed securities, it is usually assumed that the security returns are random variables whose distributions are statistically traceable. In contrast, since sufficient historical data is not available for newly listed securities, security returns are usually defined as uncertain variables whose distributions are estimated based on experts' estimates. In this hybrid environment, uncertain random variables are introduced to deal with the complex system with randomness and uncertainty [13, 14]. Qin [15] first proposes a mean-variance portfolio optimisation model to address this problem. Some scholars have considered the asymmetry of uncertain returns on newly listed securities, extending the work of Qin [15] by incorporating skewness as an additional objective [16, 17]. However, the above literature treats returns on long-term listed securities as being randomly normally distributed, ignoring the asymmetry of random returns.

In practise, trading restrictions add fractions to real-world portfolio optimisation and prevent the use of classical MPT approaches. The inclusion of additional objectives and constraints significantly increases the complexity of solving portfolio optimisation models. Multi-Objective Evolutionary Algorithms (MOEAs) have proven to be a promising candidate to

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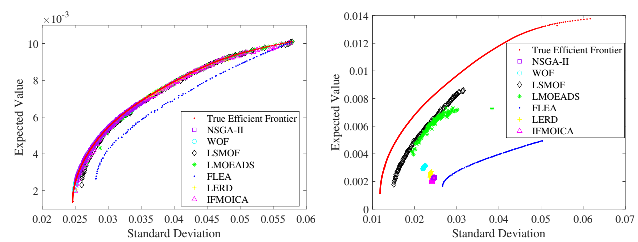
tackle these models and shine in the field of portfolio optimisation [18–21]. The main challenge in dealing with constrained multi-objective optimisation problems is to achieve a balance between convergence, diversity and feasibility. Researchers have made remarkable progress in developing evolutionary algorithmic solutions to constrained optimisation problems by using different constraint handling methods [22–27]. In this study, we focus on a set of real-world constraints included in portfolio optimisation problems, such as cardinality constraints, minimum transaction lot constraints, boundary constraints and short selling prohibition. It is worth mentioning that these constraints are of various types, such as equality constraint constraints, inequality constraints and integer constraints, etc., which brings some challenges when trying to consider them directly with existing evolutionary algorithms. The computational complexity of such constrained portfolio optimisation problems motivates us to develop more efficient techniques for handling constraints when using MOEAs.

As the number of securities to be considered increases significantly, multi-objective portfolio optimisation becomes a typical large-scale challenge. This has led to an increase in academic efforts to solve the large-scale portfolio optimisation problem [28–32]. Various MOEAs have been employed to deal with the complexity of decision scenarios in large-scale investment landscapes, with remarkable results [21, 33–36]. Actually, most existing MOEAs have shown promising performance in solving complex optimisation problems, but their performance may degrade when they handle a large number of decision variables [37]. This challenge has led to the rapid development of the Large-scale Multi-Objective Evolutionary Algorithm (LSMOEA). Three categories of LSMOEAs have been developed based on decision variable grouping [38–40], decision space reduction [41–44] and novel search strategies [45–47]. Our preliminary research has shown that the effectiveness of the prevailing MOEAs in portfolio optimisation studies is inferior by state-of-the-art LSMOEAs when confronted with large-scale portfolio models. This observation encourages us to integrate advanced large-scale optimisation methods into the field of portfolio optimisation. Moreover, even the most advanced LSMOEAs have shown a potential improvement when confronted with portfolio optimisation problems involving thousands of individual assets.

To illustrate the above point, we compare the performance of several state-of-the-art portfolio optimisation algorithms in an example. We chose the Markowitz mean-variance (MV) model, selected for its true efficient frontier can be easily approximated (as indicated by the red data markers in Fig. 1). It is important to note that with the introduction of additional optimisation objectives and real-world constraints, the complexity of the solution problem increases, potentially increasing the performance challenge of MOEAs. Our investigation includes two portfolio optimisation scenarios: one with 30 stocks (small-scale case) and the other with 1000 stocks (large-scale case). We evaluate the performance of seven representative MOEAs: NSGA-II [48], WOF [41], LSMOF [42], LMOEADS [43], FLEA [49], LERD [50] and IFMOICA [21]. These algorithms are employed to find 120 efficient solutions for the MV models. The first algorithm is representative of classical MOEAs, the next five are state-of-the-art LSMOEAs, and the last algorithm is representative of advanced MOEAs for large-scale portfolio optimisation. The parameters of the algorithm follow the original literature, with a termination criterion set at 30000 function evaluations. The efficient frontiers obtained by the different algorithms are presented in Fig. 1. Fig. 1(a) suggests that most MOEAs converge to the true efficient frontier in the small-scale case, confirming the ability of the existing MOEAs to solve small-scale portfolio optimisation problems. However, their performance deteriorates significantly in the larger portfolio case, as shown in Fig. 1(b). This apparent contrast highlights the significant challenges and potential for improvement in applying MOEAs to optimise large-scale portfolios.



(a) Small-scale case (b) Large-scale case

Fig. 1. The efficient fronts of the MV models obtained by the algorithms.

This study addresses a large-scale portfolio optimisation problem involving long-term and newly listed securities in an uncertain random environment. A multi-objective portfolio optimisation model with realistic constraints is proposed. In this model, the mean, variance and skewness of the portfolio return serve as decision criteria and are complemented by constraints such as cardinality, bounding, minimum transaction lot and no short selling to make the model more comprehensive and applicable to real investment scenarios. To solve the proposed model, a novel evolutionary algorithm based on large-scale multi-objective optimisation techniques is proposed. First, an encoder-decoder method is developed to deal with the constraints and convert the proposed model into a model without constraints. Then, an optimisation framework based on variable space reduction is designed to solve the converted model. Finally, a novel search strategy is developed to improve the operational efficiency of the algorithm.

Based on the above discussion, a comparison of the features with some important related works is given in TABLE I. The main highlights and innovations of this study are summarised below: (1) A multi-objective model for large-scale portfolio optimisation problems with long-term listed and newly listed securities has been proposed, taking into account the asymmetry of both random and uncertain returns; (2) An encoder-decoder method is presented to convert the proposed model with realistic constraints into an unconstrained one, allowing the application of arbitrary MOEAs to the proposed constrained model; (3) A novel evolutionary algorithm based on large-scale multi-objective optimisation techniques is developed to solve the proposed model effectively.

The rest of this paper is structured as follows: Section II reviews the relevant literature. Section III briefly introduces the

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TABLE I
THE FEATURE COMPARISON WITH EXISTING APPROACHES.

Feature	[15]	[20]	[51]	[12]	[52]	[17]	[16]	Our
Asymmetric fuzzy returns	×	✓	✓	×	×	✓	✓	✓
Asymmetric random returns	×	×	×	×	×	×	×	✓
Skewness	×	✓	×	×	×	✓	✓	✓
Environment	Uncertain Random	Uncertain	Uncertain	Uncertain	Uncertain	Uncertain Random	Uncertain Random	Uncertain Random
Cardinality	×	✓	✓	×	×	✓	✓	✓
Transaction Lot	×	×	×	✓	×	×	×	✓
Bounding	×	✓	×	✓	×	✓	×	✓
No Short-selling	×	✓	✓	✓	✓	✓	✓	✓
Risk-free Asset	×	×	×	×	×	×	×	✓
Large-scale	×	×	×	×	×	×	×	✓

uncertainty theory and uncertain random variables. Section IV describes the proposed model. Section V explains the solution algorithm. Section VI conducts case studies. Finally, Section VII provides a conclusion.

II. LITERATURE REVIEW

Markowitz [1] proposes the MV portfolio optimisation model, which formulates the problem in a mathematical framework for the first time. The model uses mean and variance to characterise return and risk, respectively. However, variance treats investment returns above and below the mean as equivalent to increased risk, which is unrealistic when security returns are asymmetrically distributed. To address this challenge, some scholars have replaced variance with a measure of downside risk [53]; others have used skewness, the third central moment, to measure the degree of asymmetry in the distribution of returns [3, 54]. In these approaches, only the randomness of financial markets is taken into account, and security returns are treated as random variables whose distributions can be derived from historical data. However, in some emerging markets, there may be a lack of sufficient trading data. In this case, some researchers consider security returns as fuzzy variables whose distributions are estimated by experts' knowledge. With the development of fuzzy techniques, researchers began to use fuzzy numbers to formulate payoff distributions and to study fuzzy portfolio optimisation problems based on three different approaches: fuzzy set theory [5], possibility theory [55] and credibility theory [6, 56, 57].

However, fuzzy theory has been criticised for the paradoxes associated with describing security returns using fuzzy numbers [58]. To better describe subjectively imprecise quantities, Liu [8] proposes an uncertainty theory that deals with uncertain quantities estimated by experts. Based on uncertainty theory, many works have been done to solve uncertain portfolio optimisation problems [9–12, 51, 52, 59]. In most real-world situations, it is doubtful whether there is sufficient historical data for all securities or for any securities. Consequently, portfolio optimisation problems in the real world are usually simultaneously associated with random and uncertain returns. Liu [13, 14] proposes to use uncertain random variables to model systems with randomness and uncertainty. Qin [15] is the first to study portfolio optimisation problems in hybrid environments using uncertain random variables. Li et al. [16] present a mean-variance-skewness model for the uncertain random portfolio optimisation problem. Mehlawat et al. [17] propose a portfolio optimisation model using higher moments in uncertain random environments.

In addition to uncertainty and randomness, real-world constraints are also important factors in portfolio optimisations. The studies conducted in this area are very active, have various constraints such as cardinality and transaction lots, and are integrated into the existing portfolio optimisation models [16, 17, 60]. However, the introduction of realistic constraints has turned the models into NP-hard problems that can be computationally very challenging. MOEA is a good candidate to solve the models. Numerous works have been carried out with the aim of using MOEAs to solve multi-objective portfolio optimisation models with realistic constraints. Chen et al. [18] design a novel hybrid MOEA to solve the multi-period mean-variance-skewness model. Wang et al. [19] present a fuzzy simulation-based particle swarm optimisation algorithm for the bi-objective portfolio optimisation model. Chen et al. [20] present a novel hybrid ICA-FA algorithm for solving multi-period uncertain portfolio optimisation.

With the development of Big Data technologies, the field of portfolio optimisation in large-scale scenarios has attracted considerable attention. Numerous researchers have focused on the challenge of solving large-scale MV portfolios and have advanced the development of several efficient computational approaches [28, 29, 31]. Nevertheless, there are still challenges in tackling large-scale portfolio optimisation models that involve multiple objectives and complex constraints. In this context, MOEAs have proven to be powerful tools for tackling the computational complexity of large-scale portfolio optimisation paradigms. A notable example is the work of Branke et al. [33], who present an envelope-based MOEA specifically tailored to the field of portfolio optimisation. Their pioneering work shows remarkable superiority over several existing MOEAs for portfolios consisting of hundreds of assets. Similarly, Chen et al. [35] present a Genetic Relation Algorithm with Guided Mutation for tackling large-scale portfolio optimisation problems. Golmakani et al. [34] propose an improved Particle Swarm Optimisation algorithm to overcome the computational challenges of large-scale portfolio optimisation with constraints. Liang et al. [36] introduce a Multi-objective Dynamic Multi-Swarm Particle Swarm Optimiser to meet the requirements of large-scale portfolio optimisation. Li et al. [21] propose an improved Imperialist Competitive Algorithm to the portfolio optimisation problem, and they employ the parallelised optimisation techniques to improve

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computational efficiency in scenarios with large-scale individual assets.

MOEAs have shown promise in solving various optimisation problems [37, 61]. In recent years, significant efforts have been made to solve large multi-objective optimisation problems, with performance improved mainly by three techniques: grouping of decision variables [38–40], decision space reduction [41–44, 51] and novel search strategies [45–47]. In the last three years, a number of constrained multi-objective evolutionary algorithms have been developed for handling constraints, which exhibited outstanding performance on complex constraints, small feasible regions, and large number of decision variables [22–24, 27]. There are various realistic constraints in large-scale portfolio optimisation problems, and solving them using evolutionary algorithms requires the design of effective constraint handling methods.

In summary, portfolio optimisation as a fundamental financial problem has attracted the interest of researchers in various fields. However, using advanced evolutionary algorithms to solve large-scale portfolio optimisation problems with both long-term listed and newly listed securities is still challenging. To meet this challenge, a large-scale uncertain random mean-variance-skewness portfolio optimisation model is proposed. In this model, the future returns of long-term listed and newly listed securities are treated as random and uncertain variables, respectively. Additionlly, the constraints of cardinality, minimum transaction lot, boundary and prohibition of short selling are considered to fit the real investment world. Then, an encoder-decoder method is developed to convert the model into an unconstrained one so that any MOEA can be applied to solve the proposed model. Finally, a novel evolutionary algorithm based on large-scale multi-objective optimisation techniques is proposed to solve the transformed model.

III. PRELIMINARIES

Liu [8] proposes uncertainty theory as a branch of axiomatic mathematics to study uncertainty in relation to the degree of human belief. Let Γ be a non-empty set and ℓ be a σ-algebra over Γ. A function $\mathcal{M} : \ell \rightarrow [0,1]$ is called an uncertain measure if it satisfies: (1) $\mathcal{M}(\Gamma) = 1$; (2) $\mathcal{M}(\Lambda)+\mathcal{M}(\Lambda^c) = 1$ for any event $\Lambda \in \ell$; (3) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have $\mathcal{M}(\bigcup_i^\infty \Lambda_i) \le \sum_i^\infty \mathcal{M}(\Lambda_i)$. The triple $(\Gamma, \ell, \mathcal{M})$ is called an uncertain space. An uncertain variable is a function ξ from an uncertain space $(\Gamma, \ell, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B. The uncertain distribution Φ of an uncertain variable ξ is defined by $\Phi_\xi(x) = \mathcal{M}\{\xi \le x\}$ for any real number x.

***Definition** 1:* [13]. Let $(\Gamma, \ell, \mathcal{M})$ and $(\Omega, \mathcal{A}, P)$ be uncertain space and probability space, respectively. The product $(\Gamma, \ell, \mathcal{M}) \times (\Omega, \mathcal{A}, P)$ is called a chance space. $\Theta \in \ell \times \mathcal{A}$ is called an uncertain random event. The chance measure of Θ is defined by

$$\mathrm{Ch}\{\Theta\} = \int_0^1 P\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \ge x\} \quad (1)$$

***Definition** 2:* [13]. An uncertain random variable is a function ξ from a chance space $(\Gamma, \ell, \mathcal{M}) \times (\Omega, \mathcal{A}, P)$ to the set of real numbers such that $\{\xi \in B\}$ is and event in $\ell \times \mathcal{A}$ for any Borel set B. Its chance distribution is defined by $\Psi(x) = \mathrm{Ch}\{\xi \le x\}$.

***Definition** 3:* [14]. Let ξ be an uncertain random variable. Its expected value is defined by

$$E[\xi] = \int_0^{+\infty} \mathrm{Ch}\{\xi \ge x\} dx - \int_{-\infty}^0 \mathrm{Ch}\{\xi \le x\} dx \quad (2)$$

provided that at least one of the two integrals exists.

***Theorem** 1:* [14]. Let $\eta_1, \ldots, \eta_n$ be independent random variables with probability distributions $\Upsilon_1, \ldots, \Upsilon_n$, respectively, and let $\xi_1, \ldots, \xi_n$ be independent uncertain variables with probability distributions $\Phi_1, \ldots, \Phi_n$, then the uncertain random variable $\tau = f(\eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_n)$ has an expected value

$$E[\tau] = \int_{\mathbb{R}^n} E[f(y_1, \ldots, y_n, \xi_1, \ldots, \xi_n)] d\Upsilon_1 \cdots d\Upsilon_n \quad (3)$$

***Theorem** 2:* [13]. Let ξ_1 be a random variable and ξ_2 be an uncertain variable. ξ_1 or ξ_2 can be regarded as a special uncertain random variable, and $\xi_1 + \xi_2$ and $\xi_1 \cdot \xi_2$ are also an uncertain random variable.

Let η be a random variable and ξ be an uncertain variable. η or ξ can be regarded as a special uncertain random variable. $\eta + \xi$ and $\eta \cdot \xi$ also can be regarded as uncertain random variables, and their expected value can be obtained by

$$E[\eta + \xi] = E[\eta] + E[\xi] \quad (4)$$
$$E[\eta \cdot \xi] = E[\eta] \cdot E[\xi] \quad (5)$$

Proof Let us first prove Equation (4). Denote the probability distribution of η by Υ. It follows from Theorem 1 that we have

$$\begin{aligned} E[\eta + \xi] &= \int_{\mathbb{R}} E[y + \xi] d\Upsilon = \int_{\mathbb{R}} E[y + E[\xi]] d\Upsilon \\ &= E[\eta] + E[\xi] \end{aligned} \quad (6)$$

The proof of Equation (5) is similar to the above process and will not be repeated. □

***Theorem** 3:* Denote the kth central moment of a variable x by $m_k(x)$. Let η be a random variable and ξ be an uncertain variable. For any $a, b \in \mathbb{R}$, the second and third central moments of $(a\eta + b\xi)$ are

$$m_2(a\eta + b\xi) = a^2 m_2(\eta) + b^2 m_2(\xi) \quad (7)$$
$$m_3(a\eta + b\xi) = a^3 m_3(\eta) + b^3 m_3(\xi) \quad (8)$$

Proof Let us first prove Equation (7). It follows from Theorem 1 that we have

$$E\left[(\eta - E[\eta]) \cdot (\xi - E[\xi])\right] = 0 \quad (9)$$

Then, we have

$$\begin{aligned} m_2(a\eta + b\xi) &= E[(a(\eta - E[\eta]) + b(\xi - E[\xi]))^2] \\ &= a^2 E\left[(\eta - E[\eta])^2\right] + b^2 E\left[(\xi - E[\xi])^2\right] \\ &\quad + 2abE\left[(\eta - E[\eta]) \cdot (\xi - E[\xi])\right] \\ &= a^2 m_2(\eta) + b^2 m_2(\xi) \end{aligned} \quad (10)$$

The proof of Equation (3) is similar to the above process and will not be repeated. □

IV. Model Formulation

We consider a multi-objective portfolio optimisation problem with large-scale securities and a risk-free asset, where long-term listed and newly listed securities are represented simultaneously.

Let's consider a scenario with n long-term listed securities, m newly listed securities, and a risk-free asset. Denote the random return of the i-th long-term listed security as η_i and the uncertain return of the j-th newly listed security as ξ_j, $i = 1, \ldots, n,\ j = 1, \ldots, m$. These returns are encapsulated in the vectors $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)'$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)'$, respectively, with $'$ representing the transpose operator. Denote the decision variables for the proportions allocated to the risk-free asset, the i-th long-term listed security and the j-th newly listed security by x_0, x_i and y_j, respectively. The portfolio vectors are symbolised as $\boldsymbol{x} = (x_1, \ldots, x_n)'$ for the long-term listed securities and $\boldsymbol{y} = (y_1, \ldots, y_m)'$ for the newly listed securities. The portfolio vector for all assets, including the risk-free asset and securities, is represented as $X = (x_0, \boldsymbol{x}', \boldsymbol{y}')'$. Then, the portfolio return is given by

$$R(X) = x_0 r_0 + \boldsymbol{x}'\boldsymbol{\eta} + \boldsymbol{y}'\boldsymbol{\xi} \tag{11}$$

It is worth noting that $\boldsymbol{x}'\boldsymbol{\eta}$ is a random variable and $\boldsymbol{y}'\boldsymbol{\xi}$ is an uncertain variable, so $R(X)$ can be considered an uncertain random variable according to Theorem 2.

Denote the mean vector of long-term listed securities by $\boldsymbol{\mu}$. The expected return of random return $\boldsymbol{x}'\boldsymbol{\eta}$ is $\boldsymbol{x}'\boldsymbol{\mu}$ and the kth central moment of $\boldsymbol{x}'\boldsymbol{\eta}$ is $E[(\boldsymbol{x}'\boldsymbol{\eta} - \boldsymbol{x}'\boldsymbol{\mu})^k]$, which gives us the follows:

(1) the second central moment, i.e., variance, of $\boldsymbol{x}'\boldsymbol{\eta}$ is

$$V(\boldsymbol{x}'\boldsymbol{\eta}) = m_2(\boldsymbol{x}'\boldsymbol{\eta}) = \boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} \tag{12}$$

where $\boldsymbol{\Sigma} = E[(\boldsymbol{\eta} - \boldsymbol{\mu})(\boldsymbol{\eta} - \boldsymbol{\mu})']$ is the covariance matrix.

(2) the third central moment of $\boldsymbol{x}'\boldsymbol{\eta}$ is

$$m_3(\boldsymbol{x}'\boldsymbol{\eta}) = \boldsymbol{x}'\boldsymbol{\Theta}(\boldsymbol{x} \otimes \boldsymbol{x}) \tag{13}$$

where $\boldsymbol{\Theta} = E[(\boldsymbol{\eta}-\boldsymbol{\mu})((\boldsymbol{\eta}-\boldsymbol{\mu})' \otimes (\boldsymbol{\eta}-\boldsymbol{\mu})')]$ is the co-skewness matrix.

To simplify the modelling process, uncertain returns $\xi_1, \ldots, \xi_m$ are usually assumed to follow the same type of uncertain distribution. Here, the uncertain returns ξ_j ($j = 1, \ldots, m$) is assumed to be a zigzag uncertain variable, whose distribution is denoted by$\mathcal{Z}(a_j, b_j, c_j)$.

In this case, the uncertain return $\boldsymbol{y}'\boldsymbol{\xi}$ is also determined as uncertain zigzag variable with distribution $\mathcal{Z}(\boldsymbol{y}'\boldsymbol{a}, \boldsymbol{y}'\boldsymbol{b}, \boldsymbol{y}'\boldsymbol{c})$, where $\boldsymbol{a} = (a_1, \ldots, a_m)'$, $\boldsymbol{b} = (b_1, \ldots, b_m)'$ and $\boldsymbol{c} = (c_1, \ldots, c_m)'$.

From the above discussions, the expected value of the portfolio return $R(X)$ can be determined as follows

$$\begin{aligned} E[R(X)] &= x_0 r_0 + E[\boldsymbol{x}'\boldsymbol{\eta}] + E[\boldsymbol{y}'\boldsymbol{\xi}] \\ &= x_0 r_0 + \boldsymbol{x}'\boldsymbol{\mu} + \boldsymbol{y}' \cdot \frac{\boldsymbol{a} + 2\boldsymbol{b} + \boldsymbol{c}}{4} \end{aligned} \tag{14}$$

It follows from Theorem 3 that the variance of uncertain random variable $R(X)$ is determined by

$$\begin{aligned} V[R(X)] &= m_2(\boldsymbol{x}'\boldsymbol{\eta}) + m_2(\boldsymbol{y}'\boldsymbol{\xi}) \\ &= \boldsymbol{x}'\Sigma\boldsymbol{x} + \frac{5(\boldsymbol{y}'\boldsymbol{b} - \boldsymbol{y}'\boldsymbol{a})^2 + 5(\boldsymbol{y}'\boldsymbol{c} - \boldsymbol{y}'\boldsymbol{b})^2}{48} \\ &\quad + \frac{6(\boldsymbol{y}'\boldsymbol{b} - \boldsymbol{y}'\boldsymbol{a})(\boldsymbol{y}'\boldsymbol{c} - \boldsymbol{y}'\boldsymbol{b})}{48} \end{aligned} \tag{15}$$

and the third central moment of uncertain random variable $R(X)$ is determined by

$$\begin{aligned} &m_3(R(X)) = m_3(\boldsymbol{x}'\boldsymbol{\eta}) + m_3(\boldsymbol{y}'\boldsymbol{\xi}) \\ &= \boldsymbol{x}'\boldsymbol{\Theta}(\boldsymbol{x} \otimes \boldsymbol{x}) + \frac{(\boldsymbol{y}'\boldsymbol{a} - 2\boldsymbol{y}'\boldsymbol{b} + \boldsymbol{y}'\boldsymbol{c})(\boldsymbol{y}'\boldsymbol{c} - \boldsymbol{y}'\boldsymbol{a})^2}{32} \end{aligned} \tag{16}$$

Then, the skewness of the uncertain random variable $R(X)$ is determined by

$$S[R(X)] = \frac{m_3(R(X))}{V[R(X)]^{3/2}} \tag{17}$$

Based on the above discussion, the three-moment portfolio selection model can be constructed by optimising the objectives in Equations (14)-(17) subject to some common constraints, i.e.,

$$P_1 \begin{cases} \min\limits_{X} \ \{-E[R(X)],\ V[R(X)],\ -S[R(X)]\} & (18) \\ \text{s.t.}\ \ \underline{K} \le \sum\limits_{i=1}^{n} \operatorname{sgn}(x_i) + \sum\limits_{j=1}^{m} \operatorname{sign}(y_j) \le \overline{K} & (19) \\ \quad W_0 x_i / P_i^1 \in \mathbb{N} & (20) \\ \quad W_0 y_j / P_j^2 \in \mathbb{N} & (21) \\ \quad \operatorname{sgn}(x_i) l_i^1 \le \operatorname{sgn}(x_i) x_i \le \operatorname{sgn}(x_i) u_i^1 & (22) \\ \quad \operatorname{sgn}(y_j) l_j^2 \le \operatorname{sgn}(y_j) y_j \le \operatorname{sgn}(y_j) u_j^2 & (23) \\ \quad x_0 + \boldsymbol{x}'\boldsymbol{1}_n + \boldsymbol{y}'\boldsymbol{1}_m = 1 & (24) \\ \quad x_0 \ge 0,\ x_i \ge 0,\ y_j \ge 0 & (25) \\ \quad i = 1, \ldots, n,\ j = 1, \ldots, m & (26) \end{cases}$$

In the model, the Equation (19) represents the cardinality constraint, which limits the number of securities held in the portfolio to a certain interval $[K_{\min},\ K_{\max}]$; Equations (20) and (21) represent the minimum transaction lot constraint, which limits the number of transaction lots invested in each security to an integer, where W_0 is the total capital and P_i^1 and P_j^2 are the price of a round lot of i-th long-term listed security and j-th newly listed security, respectively; Equations (22) and (23) represent the boundary constraints that limit the investment share of each security to an interval; Equations (24) represent the budget constraint that ensures that all available capital is used; Equation (25) prohibits short selling.

V. Solution Algorithm

Considering that the proposed model P_1 is a multi-objective programming model with complex constraints, it would be tedious to solve it using conventional optimisation approaches. In this study, an encoder-decoder method is first developed to convert the model P_1 into an unconstrained model. Then, a novel MOEA method based on large-scale optimisation

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techniques is proposed to solve the converted model. The optimisation framework of the proposed solution algorithm is presented in Fig. 2. Next, we introduce the essential components of the solution algorithm.

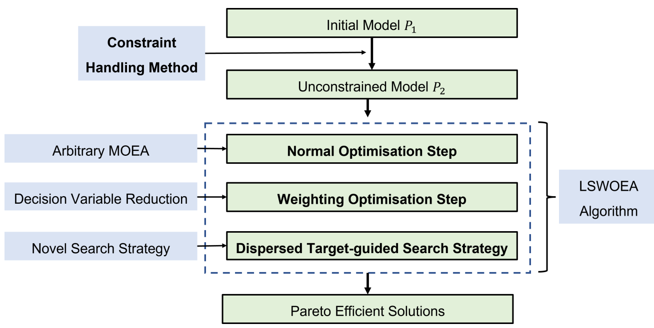

Fig. 2. The optimisation framework of the proposed solution algorithm.

A. Constraint Handling Method

In this section, we introduce a novel approach to deal with some common constraints in portfolio optimisation, which provides a theoretical contribution to the application of MOEA in the field of portfolio optimisation. The innovative constraint handling method is used to transform portfolio optimisation problems with practical constraints into those without constraints. This transformation enables a more efficient application of existing MOEAs to constrained portfolio optimisation problems. Compared to conventional methods for handling constraints in MOEAs, our proposed method avoids the need for repeated evaluations of the feasibility of solutions and mitigates the risk of solution failure due to a large number of infeasible solutions in the population.

A portfolio vector $X = (x_0, \boldsymbol{x}', \boldsymbol{y}')'$ is encoded by a real value vector $\boldsymbol{p}$ in the following search space

$$\begin{aligned}\Delta = \{&\boldsymbol{p} = (p_0, p_1, \ldots, p_{n+m}, p_{n+m+1})' : \\ &0 \le p_k \le 1, k = 0, \ldots, n+m+1\}\end{aligned} \tag{27}$$

Given a representative vector $\boldsymbol{p} \in \Delta$, a unique solution X of the model P_1 satisfying all constraints is obtained by a decoding method, i.e., $X = \text{Decode}(\boldsymbol{p})$. Next, we introduce the decoding method in detail.

First, the element p_{n+m+1} is used to indicate the number of the held securities in the portfolio. Specifically, the cardinality of the portfolio X is formulated as follows

$$K = \text{Round}\left(K_{\min} + p_{n+m+1}\left(K_{\max} - K_{\min}\right)\right) \tag{28}$$

where $\text{Round}(\cdot)$ is the round function.

Second, the elements $p_1, \ldots, p_{n+m}$ are sorted in descending order to $\tilde{p}_1, \ldots, \tilde{p}_{n+m}$. Denote the serial number of the element p_k in the new order sequence by q_k, $k = 1, \ldots, n+m$. The first K elements in the new order sequence, i.e. $\tilde{p}_1, \ldots, \tilde{p}_K$, are selected to form a potential portfolio. The index sets of the selected long-term listed and newly listed securities are thus respectively represented by

$$S_1 = \{k : \; q_k \le K, \; k = 1, \ldots, n\} \tag{29}$$

$$S_2 = \{k - n : \; q_k \le K, \; k = n+1, \ldots, n+m\} \tag{30}$$

Then, a potential portfolio $\tilde{X} = (\tilde{x}_0, \tilde{\boldsymbol{x}}', \tilde{\boldsymbol{y}}')'$ is formulated by the normalisation operation as follows

$$\tilde{x}_0 = \frac{p_0}{p_0 + \sum_{k=1}^{K} \tilde{p}_k} \tag{31}$$

$$\tilde{x}_i = \begin{cases} \dfrac{\tilde{p}_{q_i}}{p_0 + \sum_{k=1}^{K} \tilde{p}_k}, & \text{if } i \in S_1 \\ 0, & \text{if } i \notin S_1 \end{cases} \tag{32}$$

$$\tilde{y}_j = \begin{cases} \dfrac{\tilde{p}_{q_{n+j}}}{p_0 + \sum_{k=1}^{K} \tilde{p}_k}, & \text{if } j \in S_2 \\ 0, & \text{if } j \notin S_2 \end{cases} \tag{33}$$

Note that the potential solution meets the constraints of cardinality, budget and no short sales.

The next step is to check whether the potential solution satisfies the bounding constraints. If it does not, it will be moved into the feasible space. Let $\tilde{l}_i^1 = l_i^1 / (\sum_{i \in S_1} l_i^1 + \sum_{j \in S_2} l_j^2)$ and $\tilde{l}_j^2 = l_j^2 / (\sum_{i \in S_1} l_i^1 + \sum_{j \in S_2} l_j^2)$ for $i \in S_1$ and $j \in S_2$, respectively. The feasibility of potential solution X is tested under the bounding constraints by the following Equation (34)

$$\begin{cases} \theta_1 = \max\limits_{i \in S_1} \left\{ \dfrac{\max\{l_i^1 - \tilde{x}_i, 0\} + \min\{u_i^1 - \tilde{x}_i, 0\}}{\tilde{l}_i^1 - \tilde{x}_i} \right\} \\ \theta_2 = \max\limits_{j \in S_2} \left\{ \dfrac{\max\{l_j^2 - \tilde{y}_j, 0\} + \min\{u_j^2 - \tilde{y}_j, 0\}}{\tilde{l}_j^2 - \tilde{y}_j} \right\} \\ \theta = \max\{\theta_1, \theta_2\} \end{cases} \tag{34}$$

If $\theta = 0$, the potential solution satisfies all the bounding constraints; otherwise, $\theta > 0$, and the potential solution is revised to

$$\begin{cases} \tilde{x}_i \leftarrow \tilde{x}_i + \theta \cdot (\tilde{l}_i^1 - \tilde{x}_i), \; \text{if } i \in S_1 \\ \tilde{y}_j \leftarrow \tilde{y}_j + \theta \cdot (\tilde{l}_j^2 - \tilde{y}_j), \; \text{if } j \in S_2 \end{cases} \tag{35}$$

Finally, to satisfy the minimum transaction lot constraint, the potential solution $\tilde{X}$ is transformed into a feasible solution of Model P_1 as follows:

$$\begin{cases} x_i = \dfrac{P_i^1}{W_0} \cdot \left\lfloor \dfrac{W_0 \tilde{x}_i}{P_i^1} \right\rfloor, \; i = 1, \ldots, n \\ y_j = \dfrac{P_j^2}{W_0} \cdot \left\lfloor \dfrac{W_0 \tilde{y}_j}{P_j^2} \right\rfloor, \; j = 1, \ldots, m \\ x_0 = \tilde{x}_0 + \sum\limits_{i=1}^{n} (\tilde{x}_i - x_i) + \sum\limits_{j=1}^{m} (\tilde{y}_j - y_j) \end{cases} \tag{36}$$

where $\lfloor \cdot \rfloor$ is the floor function. Hereby, we assume $W_0 l_i^1$ and $W_0 l_i^2$ are an integer multiple of P_i^1 and P_i^2, respectively, to ensure the cardinality constraint always holds[1].

In summary, the model P_1 can be transformed into the following model:

$$P_2 \begin{cases} \min\limits_{\boldsymbol{p}} \; \{-E[R(X)], V[R(X)], -S[R(X)]\} \\ \text{s.t.} \quad X = \text{Decoder}(\boldsymbol{p}) \\ \qquad\;\; 0 \le p_k \le 1, \; k = 0, \ldots, n+m+1 \end{cases}$$

[1]This assumption ensures that when a security is held, its investment amount after adjusted by the floor function (see Equation (36)) is still greater than the given lower bound.

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where $\mathrm{Decoder}()$ is the decoding function that maps a representation vector $\boldsymbol{p} \in \Delta$ to a feasible solution for the constrained model P_1. It is clear that the transformed model is an unconstrained optimisation model in the search space Δ. In the next subsection, a novel MOEA based on large-scale optimisation techniques called LMWOEA is proposed to solve it in the case of large-scale investment.

B. Large-scale Evolutionary Algorithm

Although the constraint handling method allows for the easy application of arbitrary MOEAs to the proposed models, there are still significant challenges in terms of computational accuracy and time, especially given the large-scale securities. To efficiently solve the model P_2, a novel evolutionary algorithm named LSWOEA based on large-scale multi-objective optimisation techniques is designed. The algorithm first applies a weighting optimisation-based decision variable reduction technique and then develops a novel dispersed target-guided search strategy to improve the search performance. LSWOEA is briefly introduced in Algorithm 1.

Algorithm 1 LSWOEA(P_2, B, A_1)

Require: Problem P_2, Multi-objective optimiser A, Single-objective optimiser B.
Ensure: Solution population S.
FE$_{\max}$,g_1, ps ← Parameter Setting. //FE$_{\max}$: Maximum number of function evaluations; g_1: Number of maximum iterations in the normal optimisation step; ps: Population size.
S ← Randomly initialise population of size ps for Problem P_2.
repeat
  S ← Weighting_Optimisation(S, P_2, B). //See Algorithm 2.
  $g \leftarrow 1$.
  while $g \leq g_1$ **do**
    S_1 ← Generate an offspring population of S by the multi-objective optimiser A.
    S ← Conduct the environment selection on $S \cup S_1$.
    R ← Create a set of reference vectors. //See Subsection V-B2.
    S ← Dispersed_Target_Guided_Strategy(S, P_2, R). //See Algorithm 3.
    $g \leftarrow g + 1$.
  end while
until All function evaluations are used.
return S

It should be noted that a population-based multi-objective optimiser and a single-objective optimiser must be included in the optimisation process. In this study, at each generation, the multi-objective optimiser A is randomly selected from the traditional MOEAs, including NSGA-II [48], MOEA-D [62], SMPSO [63] and NSGA-III [64], and the single-objective optimiser B is implemented by the widely used DE [65].

1) Decision Space Reduction Method: In recent years, large-scale multi-objective evolutionary algorithms (LSMOEAs) have become a vibrant area of research in which a large number of advanced techniques have emerged. We hypothesise that the use of these techniques could effectively solve complex large-scale portfolio optimisation problems. Therefore, we evaluate the effectiveness of various advanced LSMOEAs for large-scale portfolio optimisation. Our findings show that LSMOF [42] excels in large-scale portfolio optimisation. This finding is an important motivation to integrate the decision space reduction methods of LSMOF [42] into our proposed algorithm. From a practical point of view, the application of the decision space reduction technique greatly improves the solution performance of MOEAs in large-scale portfolio optimisation and provides valuable insights to overcome the challenges posed by such problems.

Let $N = n + m + 1$ be the dimension of decision variables. Taking a fixed solution $\tilde{\boldsymbol{p}}$ as a reference point, two direction vectors are defined as follows

$$\begin{cases} \boldsymbol{v}_l = \tilde{\boldsymbol{p}} - \mathbf{0}_N \\ \boldsymbol{v}_u = \mathbf{1}_N - \tilde{\boldsymbol{p}} \end{cases} \tag{37}$$

where $\mathbf{0}_N$ and $\mathbf{1}_N$ are the lower and upper boundary points of the search space. Given two weight variables w_1 and w_2 between 0 and 0.5, two corresponding points in the search space are determined by

$$\begin{cases} \boldsymbol{p}_1^{new} = \mathbf{0}_N + w_1 \dfrac{\boldsymbol{v}_l}{\|\boldsymbol{v}_l\|} l_{\max} \\ \boldsymbol{p}_2^{new} = \mathbf{1}_N - w_2 \dfrac{\boldsymbol{v}_u}{\|\boldsymbol{v}_u\|} l_{\max} \end{cases} \tag{38}$$

where $l_{\max} = \|\mathbf{1}_N - \mathbf{0}_N\| = \sqrt{N}$ is the maximum diagram length in the search space. Then the objective values associated with the weight variables w_1 and w_2 can be calculated as follows

$$\begin{cases} \boldsymbol{g}_l(\tilde{\boldsymbol{x}}, w_1) = \boldsymbol{f}(\boldsymbol{x}_1^{new}) \\ \boldsymbol{g}_2(\tilde{\boldsymbol{x}}, w_2) = \boldsymbol{f}(\boldsymbol{x}_2^{new}) \end{cases} \tag{39}$$

Given a set of reference solutions of size h, once each of them is associated with two weighting variables, a total number of $2h$ new solutions can be constructed. Specifically, denote the set of reference solutions by $\tilde{P} = \{\tilde{\boldsymbol{p}}_1', \ldots, \tilde{\boldsymbol{p}}_h'\}$ and the weight vector by $\boldsymbol{w} = (w_{11}, w_{12}, \ldots, w_{h1}, w_{h2})'$. The corresponding $2h$ solutions, denoted by $\boldsymbol{p}_{11}^{new}, \boldsymbol{p}_{12}^{new}, \ldots, \boldsymbol{p}_{h1}^{new}, \boldsymbol{p}_{h2}^{new}$, can be generated according to Equations (37) and (38) and their objective values can be calculated according to Equation (39). Assume that the set of reference points $\tilde{P}$ is given. In this case, the optimisation of the decision vector $\boldsymbol{p}$ in the original problem can be converted into an optimisation of the weight vector $\boldsymbol{w}$ to find a set of superior solutions for the original problem. Here the metric of hypervolume (HV) [66] is used to evaluate the quality of a set of solutions. Denote the HV of $2h$ solutions associated with the weight vector $\boldsymbol{w}$ and the reference point set $\tilde{X}$ by $H(\boldsymbol{w}, \tilde{X})$. Then, for an arbitrary but fixed set of reference solutions $\tilde{P}$, the original problem P can be reconstructed as the following one-objective model

$$P_3(\tilde{X}) \begin{cases} \max\limits_{\boldsymbol{w}} \ f(\boldsymbol{w}) = H(\boldsymbol{w}, \tilde{X}) \\ \text{s.t. } \boldsymbol{w} = (w_{11}, w_{12}, \ldots, w_{h1}, w_{h2})' \in \mathbb{R}^{2h} \\ \qquad 0 \leq w_{j1}, w_{j2} \leq 0.5, \ j = 1, \ldots, h \end{cases}$$

It can be seen that the weighting optimisation problem $P_3(\tilde{X})$ has only $2h < N$ decision variables, which serves the purpose of decision space reduction by bounding the search space. Model $P_3(\tilde{X})$ is optimised with a single objective optimiser

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with population size ps_w. In each iteration, up to $2h \cdot ps_w$ new solutions (de-duplicated) to the original problem P can be obtained, which are collected as candidate solutions.

Obviously, the original problem P and the weight optimisation problem $P_3(\tilde{X})$ are complementary. On the one hand, Model P can reach all possible solutions, but it can converge very slowly in a large-dimensional space. On the other hand, Model $P_3(\tilde{X})$ has the disadvantage of restricting the search space and the advantage of searching a smaller space more thoroughly. To exploit the synergy of these two formulations, two different optimisation phases are alternated: a normal optimisation step and a weight optimisation step. The original problem P is optimised in the normal optimisation step for fixed function evaluations of g_1. Then, the weighting optimisation step is performed for h different reference solutions, as shown in Algorithm 2.

Algorithm 2 Weighting_Optimisation(S, P, B)

Require: Population S, Original Problem P, Single-objective optimiser B.
Ensure: New Population S.

```
h, ps_w, g_2 ← Parameter setting. //h: Number of reference solutions; ps_w: Population size; g_2: Maximum number of iterations.
X̃ = {x̃_1, ..., x̃_h} ← Conduct the environment selection on S to select h reference solutions.
P_3(X̃) ← Construct a weighting optimisation problem.
H ← Randomly initialise population of size ps_w for problem P_3(X̃).
S_1 ← Collect the generated candidate solutions based on the initial population H. //See Equations (37) and (38).
for all g = 1, ..., g_2 do
    H ← Optimize Problem P_3(X̃) with optimiser B.
    S_2 ← Collect the generated candidate solutions based on the current population H. //See Equations (37) and (38).
    S_1 ← S_1 ∪ S_2.
end for
S ← Conduct the environment selection on S ∪ S_1 to select ps solutions.
return S
```

2) A novel search strategy: We introduce a novel search strategy based on a distributed goal-directed approach to further improve the solution performance of the proposed algorithm. This search strategy consists of assigning a unique reference point to each parent solution, followed by deriving a unique single-objective search goal for each parent solution based on their respective reference points. This technique improves the convergence and dispersion of the resulting solution set, which is facilitated by the decentralised search directions for each parent solution.

First, a set of reference vectors $R = \{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{N_f}\}$ on a normalised $(M-1)$-dimensional hyperplane is established for the M-objective problem. In this study, a method from [67] is used to generate the N_f reference vectors widely distributed on the entire normalised hyperplane.

Second, the objective vectors of the parent population are normalised. Suppose that N_f solutions are selected from the current population to form the parent population denoted by $S_p = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{N_f}\}$. The ideal point and the anti-ideal point of the population are determined by the minimum and maximum values for each objective function, respectively, i.e., $\boldsymbol{z}^{\min} = (z_1^{\min}, \ldots, z_M^{\min})'$ and $\boldsymbol{z}^{\max} = (z_1^{\max}, \ldots, z_M^{\max})'$. Then the normalised objective functions for each solution are defined by

$$f_i^n(\boldsymbol{p}|S_p) = \frac{f_i(\boldsymbol{p}) - z_i^{\min}}{z_i^{\max} - z_i^{\min}},\ i = 1, \ldots, M \tag{40}$$

Then each member of the parent population is associated with a unique reference vector. The matching problem is considered as a classical assignment problem where the objective is to minimise the total distance between individuals and relevant reference points. Denote the decision variable as $D = \{d_{ij}\} \in \mathbb{R}^{N_f \times N_f}$. Then the assignment model is formulated as follows

$$P_4 \begin{cases} \min\limits_{D} \ \sum_{i=1}^{N_f}\sum_{j=1}^{N_f} d_{ij} \cdot \|\boldsymbol{f}^n(\boldsymbol{x}_i|S_p) - \boldsymbol{r}_j\| \\ \text{s.t.} \sum_{i=1}^{N_f} d_{ij} = 1,\ j = 1, \ldots, N_f \\ \sum_{j=1}^{N_f} d_{ij} = 1,\ i = 1, \ldots, N_f \\ d_{ij} \in \{0,\ 1\} \end{cases} \tag{41}$$

where $\|\cdot\|$ is the L_2-norm function and $d_{ij} = 1$ indicates that the solution $\boldsymbol{x}_i$ is ordered to the reference point $\boldsymbol{r}_j$. It can be seen that the model P_3 is an integer linear programming problem that can be solved with the function 'intlinprog' in MATLAB.

Next, the search direction for each parent solution is constructed. Define a target function for the parent solution $\boldsymbol{x}_i$

$$f_i^t(\boldsymbol{x}|D) = \sum_{j=1}^{N_f} d_{i,j}\boldsymbol{r}_j'\boldsymbol{f}^n(\boldsymbol{x}|S_p),\ i = 1, \ldots, N_f \tag{42}$$

where $\boldsymbol{f}^n(\boldsymbol{x}|S_p) = (f_1^n(\boldsymbol{x}|S_p), \ldots, f_M^n(\boldsymbol{x}|S_p))'$. Then we randomly generate a direction vector $\boldsymbol{v}_i = (v_{i,1}, \ldots, v_{i,N})'$ for the parent solution $\boldsymbol{x}_i$, where $v_{i,1} \in \{-1, 1\}$. Let

$$g_i = f_i^t(\boldsymbol{p}_i + \delta\boldsymbol{v}_i|D) - f_i^t(\boldsymbol{p}_i - \delta\boldsymbol{v}_i|D), i = 1, \ldots, N_f \tag{43}$$

where δ is a fully small positive number. $g_i > 0$ indicates that a small step in $\boldsymbol{x}_i$ towards direction $\boldsymbol{v}_i$ will cause the target value $f_i^t(\boldsymbol{x}_i|D)$ to increase, and vice versa.

Finally, the offspring solution $\boldsymbol{p}_i^{new}$ of the parent solution $\boldsymbol{p}_i$ is generated along the search direction that reduces the target function:

$$\boldsymbol{p}_i^{new} = \begin{cases} \boldsymbol{p}_i + rand \cdot (0.5 + 0.5\boldsymbol{v}_i - \boldsymbol{p}_i), \text{ if } g_i \le 0 \\ \boldsymbol{p}_i + rand \cdot (0.5 - 0.5\boldsymbol{v}_i - \boldsymbol{p}_i), \text{ if } g_i > 0 \end{cases} \tag{44}$$

where $rand \in [0,\ 1]$ is a random coefficients. Equation (44) drives solution $\boldsymbol{x}_i^{new}$ in the direction of decreasing the objective value and ensures that its elements remain between $[0,\ 1]$.

In summary, Algorithm 3 represents the pseudocode of the dispersed target-guided search strategy.

VI. Numerical Experiments

In this section, a numerical experiment is presented to illustrate the applicability and effectiveness of the proposed model and algorithm.

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Algorithm 3 Dispersed_Target_Guided_Strategy(S, P, R)

Require: Population S, Problem P, A Set of Reference Solutions $R = (\boldsymbol{r}'_1, \ldots, \boldsymbol{r}'_{N_f})$.
Ensure: New Population S.

$S_p \leftarrow$ Randomly select N_f solutions from Population S as parents.
$f_1^n(\boldsymbol{x}|S_p), \ldots, f_M^n(\boldsymbol{x}|S_p) \leftarrow$ Define the normalize functions for the parent population. //See Equation (40).
$D \leftarrow$ Assign each parent to a unique reference vector by solving Problem P_4.
for all $i = 1, \ldots, N_f$ **do**
  $f_i^t(\boldsymbol{x}|D) \leftarrow$ Define the target objective function for the parent solution $\boldsymbol{x}_i$. //See Equation (42).
  $\boldsymbol{v}_i \leftarrow$ Randly generate a direction vector for the parent solution $\boldsymbol{x}_i$ and construct the judgment indicator g_i. //See Equation (43).
  $\boldsymbol{x}_i^{new} \leftarrow$ Generate a offspring population for the parent solution $\boldsymbol{x}_i$. //See Equation (44).
end for
$S \leftarrow$ Conduct the environment selection on $S \cup \{\boldsymbol{x}_i^{new}\}_{i=1}^{N_f}$.
return S

A. Implementation in A Small Case

Now consider a portfolio optimisation problem with 20 long-term listed securities (A1-A20) and 10 newly listed securities (A21-A30). The securities are randomly selected from the Shanghai Stock Exchange (SSE). We use historical data of 614 weekly returns to estimate the random returns and historical data of only 51 weekly returns to the uncertain returns.

The initial capital W_0 is set at CNY 1 million. A round lot of securities on the SSE is 100 shares. The minimum and maximum number of securities held is set at $K_{\min} = 9$ and $K_{\max} = 21$, respectively. The minimum investment amount of securities is 0.01, and the upper limits of investment shares are set at 0.6 for all securities. Additionally, some general parameters for the proposed LSWOEA are given as follows: $\text{FE}_{\max} = 30000$, $ps = 120$, $g_1 = 50$, $h = 10$, $ps_w = 10$ and $g_2 = 50$.

The LSWOEA algorithm is used to address the portfolio optimisation problem and obtain a set of effective solutions. Investors can review the options generated and select the investment strategy that best fits their objectives. For example, if an investor's primary objective is to achieve a high return, the first step would be to identify a set of proposed solutions where the target for risk and skewness exceeds the 30th percentile. The investment strategy with the highest expected return would then be selected from this group of solutions. We show three investment strategies with different target preferences in Fig. 3. Note that securities with an investment share of 0 have not been included for reasons of space. The number shown above each bar represents the number of transaction lots (100 shares/lot). As can be seen in Fig. 3, the solutions satisfy all realistic constraints, such as the cardinality constraint, the minimum transaction lot constraint and the bounding constraint. The return preference and skewness preference investment strategies allocate more funds to A30 and A18, respectively, while the risk preference investment strategy tends to hold the risk-free asset. The results indicate that the proposed approach is practical to solve the multi-objective portfolio optimisation problem with realistic constraints.

B. Algorithm Comparison

Next, we gradually increase the number of securities in the experiments and examine six different data sets composed of long-term listed securities with quantities of n = 20, 150, 300, 450, 600 and 750 and newly listed securities with quantities of m = 10, 50, 100, 150, 200 and 250.

To empirically investigate the performance of LSWOEA, nine existing MOEAs are selected as the baselines for the experiments, namely, SparseEA [22], SECSO [68], MSKEA [69], MPMMEA [70], LSWOF-NSGA-II [42], LMOEADS [43], FLEA [49], LERD [50]) and IFMOICA [21]. For fair comparisons, the population size and the maximum number of objective evaluations for all algorithms are set to 120 and 30000, respectively. Other recommended parameter settings for the compared algorithms are taken from the original literature. All compared algorithms are reproduced based on the PlatEMO [71] and the original literature. To evaluate the algorithms directly, we use the indicator hypervolume (HV) [66] to quantify the performance of each solution set. This indicator is well known for measuring convergence and diversity in multi-objective optimisation and is therefore popular when the true Pareto front of the problem is unknown. For a more accurate calculation, the objective space is normalised to [0, 1] by the max-min method based on all solutions in a test. The reference point for calculating HV is set to the maximum values of the normalised objectives, i.e. $(1, 1, 1)'$.

In the comparisons, we repeated the comparisons for each algorithm in 200 runs to obtain statistical results. The box-plots of the resulting HV scores are shown in Fig. 4. This analysis shows that our LSWOEA algorithm outperforms the benchmark algorithms in all six datasets. It is noteworthy that among the benchmark algorithms, three that specialise in large-scale sparse MOEAs (SparseEA, SECSO and MSKEA) show commendable performance, which is consistent with the sparse nature of our proposed portfolio model. However, our LSWOEA algorithm significantly outperforms these alternatives in terms of solution efficiency and stability, confirming its superior performance capabilities.

For a more intuitive presentation, we listed the best, 75th quantile, median, 25th quantile, worst and mean HV values achieved by each algorithm in the 200 tests, as summarised in TABLE II. We also performed statistical tests to illustrate the statistical significance of our experimental results. To this end, we used the Mann-Whitney U test, a widely recognised non-parametric statistical method, to assess significant differences in the HV values generated by each pair of algorithms [72, 73]. Given the 45 pairwise comparisons between the ten algorithms, the Bonferroni correction method is applied to adjust the significance level and the new alpha level is set at $\alpha = 0.05/45 = 0.0011$. We list the p values resulting from these statistical tests for the proposed LSWOEA algorithm compared to each comparison algorithm in the last row of each database in TABLE II.

TABLE II shows that all p values are much smaller than α, indicating a significant difference between the HV values

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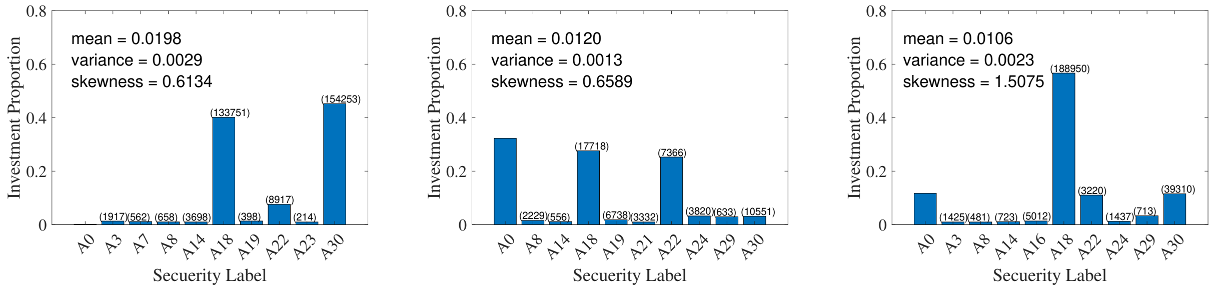
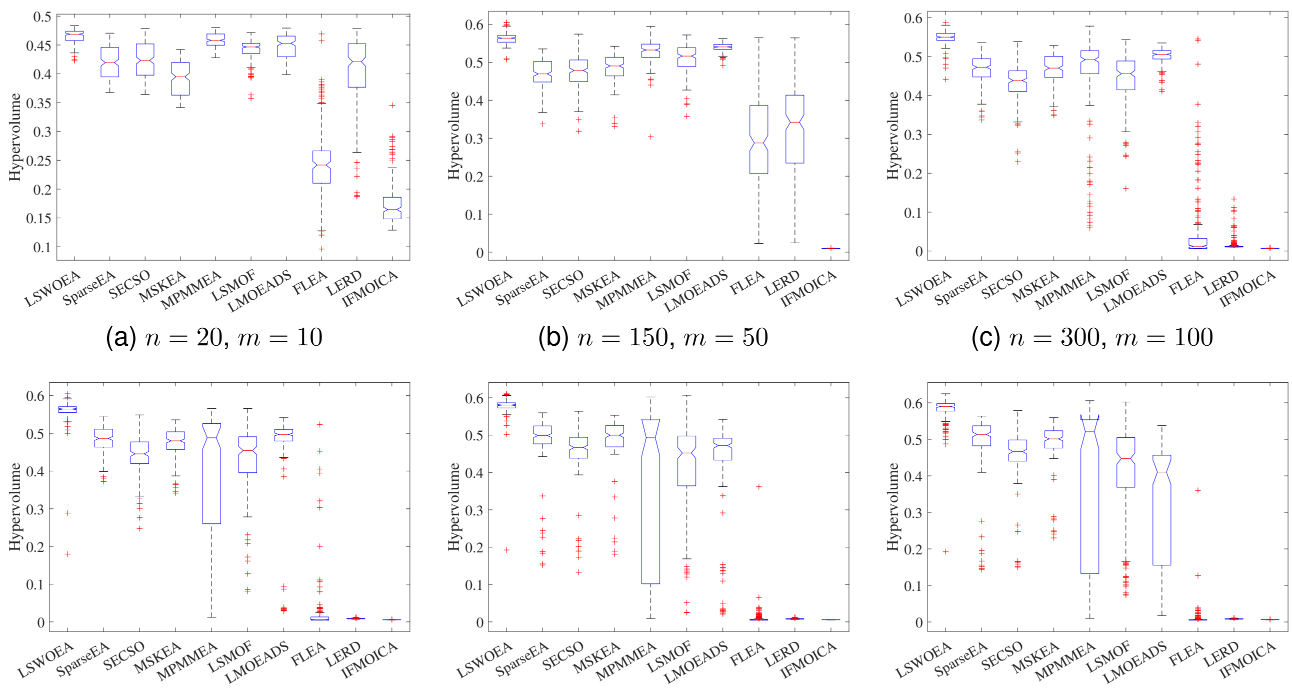


(a) Return preference (b) Risk preference (c) Skewness preference

Fig. 3. The three investment strategies with different objective preferences.



(a) $n = 20$, $m = 10$ (b) $n = 150$, $m = 50$ (c) $n = 300$, $m = 100$

(d) $n = 450$, $m = 150$ (e) $n = 600$, $m = 200$ (f) $n = 750$, $m = 250$

Fig. 4. The boxplots of the HVs of the algorithms.

generated by the LSWOEA algorithm and the values generated by the other algorithms. As can be seen from the table, LSWOEA achieves higher HV values than other algorithms in the most cases. Although the worst HV of LSWOEA is slightly lower than the other algorithms at some datasets, it consistently outperforms the other algorithms in terms of other quantiles and mean performance. For example, LSWOEA consistently achieves the optimal HV value of the 25th percentile for all datasets, demonstrating its robust performance even under less favourable conditions. In summary, LSWOEA exhibits a remarkable ability to tackle large-scale portfolio optimisation problems, indicating its potential to improve the performance of evolutionary algorithms in optimisation.

To verify the effectiveness of the proposed constraint handling method, we have conducted extensive experiments to evaluate this approach in the context of portfolio optimisation. For comparison, we use some SoTA constrained multi-objective evolutionary algorithms (CMOEAs) to directly solve the proposed model with 1000 securities. In particular, integer constraints in the model are handled by the encoding method, while other inequality or equality constraints are handled by the constraint violation method. We select six SoTA CMOEAs as follows: CCMO [22], cDPEA [23], ICMA [24], MOEAD-DAE [25], SparseEA [74], and MSKEA [69]. Moreover, we apply the above algorithms to the model after eliminating the constraints to verify the effectiveness of the proposed constraint handling method. We ran each algorithm 50 times independently and plot the mean HV values in Fig. 5.

Fig. 5 shows that, with the exception of the ICMA algorithm, applying the evaluated CMOEAs to the model after eliminating constraints achieves higher HV values than applying them directly to the model before eliminating constraints. This observation shows that our proposed constraint handling method effectively improves the solution performance of most algorithms for solving portfolio optimisation problems with constraints. Moreover, Fig. 5 suggests that our proposed LSWOEA algorithm achieves higher HV values than all six CMOEAs after constraint elimination, indicating that the proposed solution algorithm has significant advantages in solving large-scale portfolio optimisation problems with constraints.

This article has been accepted for publication in IEEE Transactions on Evolutionary Computation. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TEVC.2023.3349073

TABLE II
THE BEST, 75TH QUANTILE, MEDIAN, 25TH QUANTILE, WORST AND MEAN HVS OF THE ALGORITHMS AND THE RESULTS OF THE STATISTICAL TESTS.

$(n,\ m)$	-	LSWOEA	SparseEA	SECSO	MSKEA	MPMMEA	LSMOF	LMOEADS	FLEA	LERD	IFMOICA
(20, 10)	Best	**0.4841**	0.4705	0.4789	0.4422	0.4805	0.4715	0.4794	0.4692	0.4784	0.3454
	75th quantile	**0.4739**	0.4458	0.4518	0.4198	0.4691	0.4528	0.4655	0.2663	0.4520	0.1858
	Median	**0.4687**	0.4194	0.4231	0.3949	0.4579	0.4466	0.4527	0.2417	0.4210	0.1646
	25th quantile	**0.4576**	0.3940	0.3977	0.3630	0.4495	0.4354	0.4293	0.2101	0.3715	0.1482
	Worst	0.4226	0.3676	0.3645	0.3415	**0.4278**	0.3573	0.3986	0.0960	0.1873	0.1288
	Mean	**0.4654**	0.4196	0.4238	0.3930	0.4583	0.4424	0.4496	0.2415	0.3965	0.1722
	p-value	-	4.83E-67	4.83E-67	4.90E-67	2.47E-09	2.52E-42	1.71E-19	4.69E-66	1.62E-36	4.83E-67
(150, 50)	Best	**0.6048**	0.5352	0.5743	0.5423	0.5945	0.5720	0.5631	0.5647	0.5643	0.0110
	75th quantile	**0.5706**	0.5023	0.5062	0.5132	0.5474	0.5385	0.5474	0.3861	0.4085	0.0099
	Median	**0.5633**	0.4690	0.4783	0.4902	0.5323	0.5161	0.5405	0.2872	0.3421	0.0096
	25th quantile	**0.5530**	0.4479	0.4490	0.4641	0.5127	0.4888	0.5342	0.2052	0.2310	0.0093
	Worst	**0.5076**	0.3379	0.3185	0.3315	0.3041	0.3578	0.4913	0.0230	0.0243	0.0085
	Mean	**0.5627**	0.4734	0.4757	0.4861	0.5294	0.5108	0.5396	0.2872	0.3183	0.0096
	p-value	-	1.12E-66	5.61E-67	3.51E-47	1.22E-36	1.33E-52	3.42E-47	1.62E-64	3.09E-65	4.83E-67
(300, 100)	Best	**0.5875**	0.5355	0.5391	0.5285	0.5782	0.5435	0.5350	0.5452	0.1337	0.0080
	75th quantile	**0.5594**	0.4941	0.4631	0.4998	0.5143	0.4883	0.5153	0.0314	0.0121	0.0069
	Median	**0.5497**	0.4724	0.4384	0.4702	0.4913	0.4557	0.5054	0.0121	0.0110	0.0067
	25th quantile	**0.5417**	0.4469	0.4103	0.4457	0.4559	0.4143	0.4935	0.0066	0.0102	0.0065
	Worst	**0.4416**	0.3370	0.2296	0.3491	0.0597	0.1606	0.4107	0.0054	0.0081	0.0061
	Mean	**0.5490**	0.4702	0.4340	0.4721	0.4596	0.4440	0.5017	0.0477	0.0152	0.0067
	p-value	-	1.71E-64	1.51E-66	1.94E-43	3.44E-52	1.92E-63	1.49E-60	2.91E-66	4.83E-67	4.83E-67
(450, 150)	Best	**0.6050**	0.5458	0.5489	0.5358	0.5657	0.5659	0.5415	0.5236	0.0125	0.0065
	75th quantile	**0.5707**	0.5109	0.4767	0.5041	0.5260	0.4910	0.5103	0.0130	0.0090	0.0058
	Median	**0.5644**	0.4865	0.4455	0.4801	0.4883	0.4532	0.4967	0.0056	0.0086	0.0057
	25th quantile	**0.5554**	0.4636	0.4194	0.4573	0.2563	0.3956	0.4797	0.0051	0.0080	0.0055
	Worst	0.1797	0.3721	0.2476	**0.3414**	0.0122	0.0814	0.0292	0.0045	0.0069	0.0051
	Mean	**0.5596**	0.4856	0.4446	0.4780	0.4029	0.4357	0.4713	0.0240	0.0086	0.0057
	p-value	-	1.14E-63	2.40E-64	6.65E-52	6.01E-58	8.10E-62	2.54E-63	6.14E-67	4.83E-67	4.83E-67
(600, 200)	Best	**0.6107**	0.5598	0.5640	0.5532	0.6022	0.6068	0.5423	0.3618	0.0123	0.0063
	75th quantile	**0.5865**	0.5245	0.4941	0.5259	0.5405	0.4969	0.4917	0.0066	0.0087	0.0059
	Median	**0.5805**	0.4987	0.4660	0.4996	0.4928	0.4499	0.4720	0.0054	0.0082	0.0058
	25th quantile	**0.5727**	0.4763	0.4381	0.4686	0.0924	0.3609	0.4322	0.0050	0.0077	0.0056
	Worst	**0.1925**	0.1521	0.1326	0.1810	0.0090	0.0248	0.0210	0.0046	0.0064	0.0052
	Mean	**0.5775**	0.4897	0.4591	0.4932	0.3589	0.4148	0.4362	0.0100	0.0082	0.0058
	p-value	-	4.61E-65	1.35E-65	5.41E-51	1.09E-56	1.28E-63	1.59E-65	4.90E-67	4.83E-67	4.83E-67
(750, 250)	Best	**0.6247**	0.5636	0.5790	0.5592	0.6056	0.6021	0.5375	0.3600	0.0111	0.0071
	75th quantile	**0.5974**	0.5365	0.4980	0.5228	0.5526	0.5043	0.4558	0.0063	0.0090	0.0064
	Median	**0.5895**	0.5133	0.4663	0.5010	0.5206	0.4461	0.4102	0.0056	0.0085	0.0063
	25th quantile	**0.5767**	0.4820	0.4402	0.4761	0.1322	0.3684	0.1537	0.0053	0.0081	0.0062
	Worst	0.1922	0.1434	0.1502	**0.2301**	0.0098	0.0735	0.0175	0.0049	0.0067	0.0058
	Mean	**0.5827**	0.4977	0.4626	0.4933	0.3825	0.4130	0.3333	0.0096	0.0086	0.0063
	p-value	-	2.25E-49	1.51E-53	5.49E-39	1.23E-45	9.24E-51	3.55E-57	5.61E-67	4.83E-67	4.83E-67

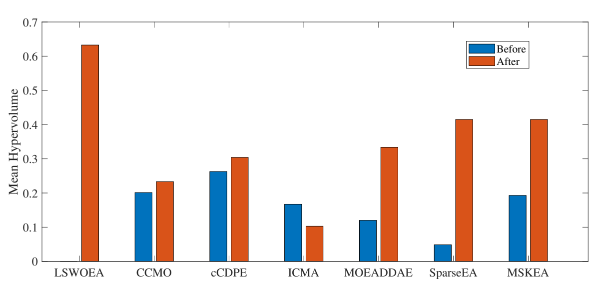


Fig. 5. The mean HVs obtained by the algorithms before and after the elimination of constraints.

C. Further Analysis

The running time is an important factor to explain the functionality of the algorithm in practical applications. The relevant codes of our experiments are implemented in MATLAB R2021a, with Windows 11 systems configured as Intel (R) Core (TM) i7-12700H CPU with 2.30 GHz. We present the average CPU time of the comparative algorithms with different numbers of securities in Fig. 6. Fig. 6 shows the LSWOEA has a slightly higher running time than the comparative algorithms. As the number of securities increases, the running time of the LSWOEA also increases slightly. In the case of 1000 securities, the LSWOEA takes about 15 seconds to execute, which is within an acceptable range and indicates that it is implementable and practical in large-scale portfolio optimisation.

Moreover, we conducted an experiment to evaluate and compare the performance of the algorithms at different numbers of maximum function evaluations FE$\max$. Specifically, we applied the algorithms to solve the portfolio optimisation problem with $(n = 750, m = 250)$ dimensions 50 times and recorded the efficient solutions obtained by the algorithms

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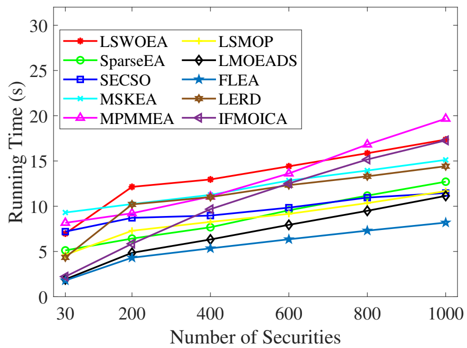


Fig. 6. The average CPU time of the comparative algorithms with different numbers of securities.

at FEmax $= 20000, 25000, \ldots, 60000$. We then plotted the mean HV values of the algorithms over these 50 tests in Fig. 7. Fig. 7 shows that The average HV values of FLEA, LERD and IFMOICA almost overlap near 0, indicating that they perform poorly in large-scale portfolio optimisation. The MPMMEA, LSMOF and LMOEADS algorithms exhibit a significant increase in mean HV at higher FE_{max} values. This observation indicates that these algorithms benefit significantly from a higher number of iterations. In contrast, the LSWOEA, SparseEA and MSKEA algorithms show a slight increase in mean HV with increasing FE_{max}. This observation indicates that our proposed LSWOEA algorithm already achieves good convergence results at 20000 function evaluations, which means that further increasing the number of iterations does not significantly increase the performance of the algorithm. The experimental results show the accelerated convergence of our proposed LSWOEA algorithm to a superior set of approximate Pareto-optimal solutions. This is a convincing proof of the superior solution performance and excellent convergence efficiency of the proposed algorithm, which enhances the robustness and credibility of our study.

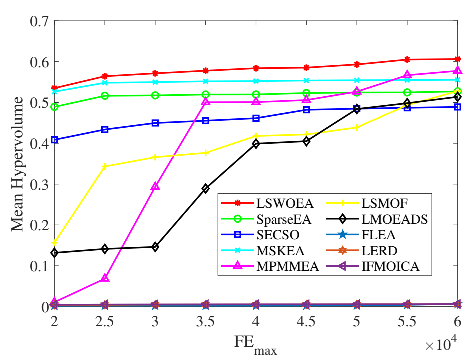
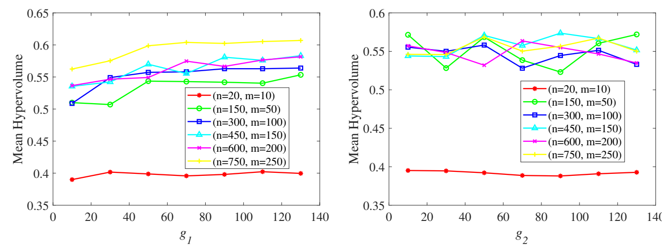


Fig. 7. The mean HVs obtained by the algorithms under different numbers of FE_{max}.

In our proposed LSWOEA algorithm, the parameters g_1 and g_2 are used to control the number of iterations during the normal optimisation and weight optimisation steps in a main loop. We performed an empirical analysis of the influence of these two parameters in the revised manuscript. To do so, we used LSWOEA with different values for g_1 and g_2 to handle portfolio optimisation problems with different security dimensions. Specifically, We set g_1 to $10, 30, \ldots, 130$ and kept g_2 constant at 50. Conversely, g_1 is set to 50 and g_2 is set to $10, 30, \ldots, 130$. For each parameter combination, the algorithm is run 50 times, followed by the calculation of the mean hypervolume (HV) values for the resulting solution sets. The summarised results are shown in Fig. 8.



(a) $g_2 = 50$ (b) $g_1 = 50$

Fig. 8. The mean HVs obtained by LSWOEA with different settings of g_1 and g_2.

Fig. 8(a) shows that the influence of g_1 on the performance of the algorithm is negligible when dealing with scenarios characterised by a low security dimension. However, the performance of the algorithm shows an increasing trend when g_1 increases in the context of higher security dimensions. We therefore recommend setting g_1 above 50 when dealing with large portfolio optimisation problems. Fig. 8(b) shows that the performance of the proposed algorithm does not seem to be affected by different settings of g_2. This phenomenon is mainly due to the minimal number of function evaluations required during the weighting optimisation step to achieve convergence, which is consistent with the observations of the LSMOF algorithm [42].

VII. Conclusion

This study discusses the application of evolutionary multi-objective optimisation to a large-scale portfolio selection problem with long-term listed and newly listed securities. A model of multi-objective portfolio optimisation with real constraints is proposed. In terms of solution algorithms, this study addresses two dilemmas of MOEA in large-scale portfolio optimisation problems. On the one hand, an encoder-decoder method is developed to deal with the complex constraints and to provide a solution framework for applying arbitrary MOEAs to portfolio selection problems. On the other hand, a novel MOEA for large-scale portfolio optimisation is proposed enrich the practise of evolutionary algorithms in the portfolio optimisation community.

To evaluate the effectiveness of the proposed model and algorithm, a numerical experiment analysis is performed for a large-scale portfolio optimisation problem with hundreds of securities. To illustrate the practicality of the proposed approach, the experiment first systematically examines a small-scale problem. The number of securities is then gradually increased to allow a comparison of the proposed LSWOEA with some state-of-the-art MOEAs. The experimental results show that LSWOEA maintains excellent performance and stability when the number of securities significantly increases, while all other compared algorithms drop sharply. Despite a slightly longer run time compared to other benchmark MOEAs, the LSWOEA still demonstrates superior solution performance, enabling it to solve portfolio selection problems involving thousands of securities effectively. In summary, the proposed approach provides an effective solution framework for large-scale portfolio optimisation and enriches the practical applicability of evolutionary algorithms and portfolio optimisation in the context of Big Data.

Considering some limitations of this study, the main ways in which it can be extended in the future are as follows. Firstly, the algorithm presented in the novel search strategy uses a modified pseudo-gradient method for single-objective optimisation. The potential for further improvements can be realised by incorporating more advanced and effective optimisation techniques. Secondly, it should be noted that our proposed algorithm does not have a significant advantage in terms of running time. Our future research may focus on improving the solution efficiency. Finally, in future work we may look at the economic significant of the proposed method by running a number of financial backtests on historical financial data.

Acknowledgments

The authors gratefully acknowledge the financial support provided by the Humanities and Social Science Fund of the Ministry of Education [grant number 21YJA630117], the National Natural Science Foundation of China [grant number 72371080; 62373224], the Guangdong Basic and Applied Basic Research Foundation [grant number 2023A1515012840], and the Special Project of Guangzhou Basic and Applied Basic Research [grant number SL2024A04J02640].

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