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A B S T R A C T
This paper discusses the design of gain-scheduled sampled-data controllers for continuous-time polytopic linear parameter-varying systems. The scheduling variables are assumed to be available only at the sampling instants, and a bound on the time-variation of the scheduling parameters is also assumed to be known. The resultant gain-scheduled controllers improve the maximum achievable delay bound over previous constant-gain ones in the literature.

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1. Introduction

Some computer-control and networked control systems have randomly-varying sampling periods. Such problems can be viewed in the context of continuous-time delay systems, allowing stability to be proved using Lyapunov–Razumikhin (Cao, Sun & Cheng, 1998) or Lyapunov–Krasovskii functional (Fridman, Seuret & Richard, 2004; Naghshtabarzi, Hespanha & Teel, 2008). Some of these results in stability analysis and linear controller design naturally generalize to Polytopic Linear Time Varying (PLTV) systems.

Unfortunately, none of the results in the current literature address the problem of gain-scheduled control with non-constant sampling period for a Polytopic Linear Parameter Varying (PLPV) plant. The main drawback of sampled-data gain-scheduling design for continuous-time plants is that the variation of the scheduling variables occurs continuously. However, in practice all measurements are only taken at discrete sampling instants, and the control action must be piecewise constant. This poses some problems in the resulting LMIs that hinder improving the performance over linear non-scheduled regulators.

In this paper, the results of Ariño and Sala (2008) are applied to gain-scheduled control of continuous plants with varying sampling time. The objective is to show that a combination of recent developments in sampled-data analysis via input delay (Fridman et al., 2004; Naghshtabarzi et al., 2008) and Ariño’s results produces gain-scheduled regulators whose maximum delay bounds improve upon the original linear non-scheduled controllers. In order to achieve this improvement, it is necessary to assume a known bound on the rate of variation of the polytopic interpolating functions µi.

2. Preliminaries

Consider a continuous-time PLTV system

\[ \dot{x} = \sum_{i=1}^{r} \mu_i(t)(A_i x(t) + B_i u(t)) \]  \hspace{1cm} (1)

where \( \sum_{i=1}^{r} \mu_i(t) = 1 \) and the values of \( \mu_i(t) \) are assumed to be computable from measurements. The plant state is assumed sampled, possibly irregularly, at time instances \( t_k \). Denote as \( u_k(t_k) \) the discrete sequence of computed control actions at those instants, and define the zero-order hold control action \( u_k(t) = u_k(t_k) \), for \( t_k \leq t < t_{k+1} \). Introducing \( \xi(t) = t - t_k \), we have

\[ u_k(t) = u_k(t_k) = u_k(t - (t - t_k)) = u_k(t - \xi(t)). \]  \hspace{1cm} (2)

If, in a networked setting, the control signal from the sample at \( t_k \) takes \( \delta_k \) time units to reach the actuator, then \( u(t) = u_k(t - \delta_k) = u_k(t - \xi(t - \delta_k)) \). In the following, \( \tau(t) = \xi(t - \delta_k) + \delta_k \), so \( t_k = t - \tau \), for \( t_k + \delta_k \leq t < t_{k+1} + \delta_{k+1} \). Note that the variable sampling period and network delay are integrated in a single input-delay value \( \tau \).
In Fridman et al. (2004), a linear state-feedback controller $u_d(t_k) = Kx(t_k)$ was designed for such a setting. Improved results for the same setting appear in Naghshtabrizi et al. (2008).

3. Gain scheduling

Consider now a discrete gain-scheduled state-feedback controller for the above system, where $K_j$ is a control gain, defined as:

$$u_d(t_k) = \sum_{j=1}^r \mu_j(t_k)K_jx(t_k).$$

Then, the closed loop is described by:

$$\dot{x} = \sum_{j=1}^r \mu_j(t)\eta_j(t)(A_x(t) - BK_jx(t - \tau))$$

where $\eta_j(t) = \mu_j(t) = \mu_j(t - \tau)$. An upper bound on $\tau$, elsewhere referred to as the maximum delay bound, will be represented by $h$.

When applying the developments in Fridman et al. (2004, Lemma 2.3) and Naghshtabrizi et al. (2008, Theorem 6), the result is a set of LMs which can be easily generalized to the proposed gain-scheduled controller by replacing $K$ with $K_j$. Conditions for a decreasing derivative of the Lyapunov–Krasovskii functional, in both cases, can be expressed as two expressions in the form:

$$\sum_{j=1}^r \sum_{l=1}^r \mu_j(t)\eta_j(t)\phi_{ij} > 0.$$  

(5)

Details are omitted for brevity, and only the final $\phi_{ij}$ appears in the lemmas below.

**Lemma 1.** Following Fridman et al. (2004) with system (4) then stabilizing gains $K_j = \tilde{Y}_jQ^{-1}$ can be found if there exists a matrix $Q_1 > 0$, matrices $Q_2^{(i), t}$, $\tilde{Z}_1^{(i), t}$, $\tilde{Z}_2^{(i), t}$, $\tilde{Z}_3^{(i), t}$ $\in \mathbb{R}^{n \times n}$, a positive scalar $\epsilon$ and matrices $\phi_y$ and $\phi_y^T$ below which fulfill (5).

$$\phi_y = -\begin{bmatrix} \epsilon Q_1 & 0 & \epsilon \tilde{Y}_1^T B_1 \\ \epsilon \tilde{Z}_1^{(i), t} & \epsilon \tilde{Z}_2^{(i), t} & \epsilon \tilde{Z}_3^{(i), t} \\ * & * & * \end{bmatrix}$$

(6)

$$\phi_y^T = -\begin{bmatrix} Q_1^{(i), t} - \epsilon \tilde{Z}_1^{(i), t} \tilde{Z}_2^{(i), t} \tilde{Z}_3^{(i), t} \\ -Q_1^{(i), t} + \epsilon \tilde{Z}_1^{(i), t} \tilde{Z}_2^{(i), t} \tilde{Z}_3^{(i), t} \\ * & * & * \end{bmatrix}$$

(7)

The above results have been improved in later literature. The result below will be also used in the examples.

**Lemma 2.** From Naghshtabrizi et al. (2008), if there exists a symmetrical positive definite matrix $Q$, positive scalars $\epsilon_1, \epsilon_2$ and matrices of the appropriate sizes $N_d$ and $Y$, such that $\phi_y$ and $\phi_y^T$ below fulfill (5)

$$\phi_y = -\begin{bmatrix} M_{1d,i} + hM_{2d,ij} & hF_{1y}^T \\ * & -he_{1}^T Q \end{bmatrix}$$

(9)

$$\phi_y^T = -\begin{bmatrix} M_{1d,i} & hF_{1y}^T \\ * & -he_{1}^T Q \end{bmatrix}$$

(10)

where

$$F_{1y} := [A_0Q, B_1Y]$$

$$M_{1d,ij} := \begin{bmatrix} I & F_{1y} \end{bmatrix}F_{1y} + F_{1y}^T [I - \epsilon_2]Q[I - I] - N_d[I - I] - [I - I]N_d^T$$

$$M_{2d,ij} := \epsilon_2 \begin{bmatrix} I & F_{1y} \end{bmatrix}F_{1y} + \epsilon_2 F_{1y}^T[I - I]$$

then, the system (1) can be stabilized by feedback gains $K_j = Y_jQ^{-1}$.

Gain scheduling results do not appear in current literature because, if $\mu_j$ is simply considered to be “independent” from $\eta_j$, i.e., if (5) is required to hold for any $\mu_i$ and for any $\eta_j$, it does if and only if $\Phi_{ij} > 0$ for all $i, j$. Hence, the resulting $K_j$ must fulfill the original conditions for the non-scheduled regulator so no extra performance can be extracted from gain scheduling.

However, a related result by some of the authors (Arifio & Sala, 2008) can improve performance over non-scheduled settings if knowledge in the form:

$$|\mu_i - \eta_i| \leq \delta \quad i = 1, \ldots, r$$

(11)

is available. In the case under consideration, if a bound $\gamma$ on the rate of change of the scheduling parameters is known, i.e., $\left| \frac{d\mu_i}{dt} \right| \leq \gamma$, it then follows from the mean-value theorem that:

$$|\mu_i - \eta_i| = |\mu_i(t) - \mu_i(t - \tau)| \leq \gamma \tau \leq \gamma h \quad i = 1, \ldots, r$$

(12)

The results from Arifio and Sala (2008) regarding additive bounds on $|\mu_k - \eta_k|$ can now be applied to the stabilization conditions expressed as (5). The additive bound result is:

**Corollary 3.** If the scheduling functions $\mu_i, \eta_i$ satisfy

$$|\mu_i - \eta_i| \leq \gamma h \quad i = 1, \ldots, r$$

(13)

then (5) holds if there exists $R_j > 0, N_j > 0, X_y = X_{y,0}$ and $X_{y,i+r}$ for $i = 1, \ldots, r$ such that

$$M_j = R_j - N_j, \quad \tilde{M}_y = \tilde{R}_j + N_j$$

(14)

$$M_j + \tilde{M}_y \geq X_y + X_{y,i}$$

(15)

$$\Phi_{y} - 2\tilde{M}_y - \sum_{i=1}^{r} \gamma h \left( M_{ik}^{+} + M_{ik}^{-} \right) \geq X_{y,i+r} + X_{y,i+r,0}$$

(16)

$$W_{11} = \begin{bmatrix} X_{11} & \cdots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \cdots & X_{rr} \end{bmatrix}$$

(17)

$$W_{12} = \begin{bmatrix} X_{1r+1} & \cdots & X_{12r} \\ \vdots & \ddots & \vdots \\ X_{r(r+1)} & \cdots & X_{r(2r)} \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{11} \end{bmatrix} > 0.$$
A bisection algorithm was used to find the maximum value of $h$ (the sum of the sampling period bound plus maximum round-trip delay bound) a feasible gain-scheduled regulator for the polytopic system stated above. The rate of variation of $\mu_i$ is assumed to be bounded with $\gamma = 0.1$. The solver was SeDuMi (Sturm, 1999) with default options.

**Fridman’s approach.** A line-search was performed to find an optimum value for $\epsilon$ between 0.05 and 1. For Fridman’s original method $h = 0.3777$ (for $\epsilon = 0.5$) was the optimum result, whereas our method gave an improved value of $h = 0.5867$ (for $\epsilon = 0.3$).

**Naghshtabrizi’s approach.** Their non-scheduling controller gives a $h$ of 0.4553 s with non-scheduled-gains of $[-2.5829 - 0.6413]$ with $\epsilon_1 = 1.1$ and $\epsilon_2 = 2.5$, where the $\epsilon$ values were found by a grid search. In the gain-scheduled case $h$ was increased to 0.8230 s with scheduled gains of $[-2.5979 - 0.6579]$ and $[-1.1801 - 0.2891]$ where $\epsilon_1 = 2$ and $\epsilon_2 = 2.8$.

As can be clearly seen in Table 1 the gain-scheduling results have significantly increased the maximum allowable delay bound/sample interval with respect to linear non-scheduled controllers, both with the same tuning parameter and with the optimum ones.

The last table row shows that redoing the computations for a large value of the derivative bound $\gamma$, the resulting maximum delay bound converges to that in the non-scheduled case, as expected intuitively no improvement over Naghshtabrizi would be obtained with $\gamma \geq 1/0.4553 = 2.19$ as $\delta \geq 1$ in (11) is a trivial bound so Corollary 3 becomes useless.

## 5. Conclusions

This paper has explained how the results in Ariño and Sala (2008) may be integrated into a gain-scheduling sampled-data setup or a networked control system with varying sampling time and network delay. This allows maximum delay bounds for stability to be improved, when compared to robust constant-gain state feedback regulators in literature. A numerical example has shown the efficacy of the proposed method.

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