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FINITE DOMINATION AND NOVIKOV RINGS. LAURENT POLYNOMIAL RINGS IN TWO VARIABLES

THOMAS HÜTTEMANN AND DAVID QUINN

ABSTRACT. Let C be a bounded cochain complex of finitely generated free modules over the LAURENT polynomial ring $L = R[x, x^{-1}, y, y^{-1}]$. The complex C is called R -finitely dominated if it is homotopy equivalent over R to a bounded complex of finitely generated projective R -modules. Our main result characterises R -finitely dominated complexes in terms of NOVIKOV cohomology: C is R -finitely dominated if and only if eight complexes derived from C are acyclic; these complexes are $C \otimes_L R[[x, y]][(xy)^{-1}]$ and $C \otimes_L R[x, x^{-1}][[y]][y^{-1}]$, and their variants obtained by swapping x and y , and replacing either indeterminate by its inverse.

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Part I. Introduction

I.1. THE MAIN THEOREM

Let $R \subseteq K$ be a pair of unital rings. A cochain complex C of K -modules is called *R -finitely dominated* if C is homotopy equivalent, as an R -module complex, to a bounded complex of finitely generated projective R -modules.

Finite domination is relevant, for example, in group theory and topology. Suppose that G is a group of type (FP) ; this means, by definition, that the trivial G -module \mathbb{Z} admits a finite resolution C by finitely generated projective $\mathbb{Z}[G]$ -modules. Let H be a subgroup of G . Deciding whether H is of type (FP) is equivalent to deciding whether C is $\mathbb{Z}[H]$ -finitely dominated.

In topology, finite domination is considered in the context of homological finiteness properties of covering spaces, or properties of ends of manifolds. See for example RANICKI's article for results and references [Ran95].

Our starting point is the following result of RANICKI:

Theorem I.1.1 ([Ran95, Theorem 2]). *Let C be a bounded complex of finitely generated free modules over $K = R[x, x^{-1}]$. The complex C is R -finitely dominated if and only if the two complexes*

$$C \otimes_K R((x)) \quad \text{and} \quad C \otimes_K R((x^{-1}))$$

are acyclic.

Here $R((x)) = R[[x]][x^{-1}]$ denotes the ring of formal LAURENT series in x , and $R((x^{-1})) = R[[x^{-1}]]x$ denotes the ring of formal LAURENT series in x^{-1} .

For LAURENT polynomial rings in several indeterminates, it is possible to strengthen this result to allow for iterative application, see for example [HQ13]. In particular, writing $L = R[x, x^{-1}, y, y^{-1}]$ for the LAURENT polynomial ring in two variables, one can show that a bounded complex of finitely generated free L -modules is R -finitely dominated if and only if the four complexes

$$\begin{array}{cc} C \otimes_L R[x, x^{-1}]((y)) & C \otimes_L R[x, x^{-1}]((y^{-1})) \\ C \otimes_{R[x, x^{-1}]} R((x)) & C \otimes_{R[x, x^{-1}]} R((x^{-1})) \end{array}$$

are acyclic.

In the present paper, we pursue a different, non-iterative approach, leading to a new characterisation of finite domination. To state the main result, we introduce another bit of notation: we write $R((x, y))$ for the ring

of those formal LAURENT series f in x and y having the property that $x^k y^k \cdot f \in R[[x, y]]$ for k sufficiently large. That is,

$$R((x, y)) = R[[x, y]][(xy)^{-1}] .$$

Main theorem I.1.2. *Let C be a bounded cochain complex of finitely generated free L -modules. Then the following two statements are equivalent:*

- (a) *The complex C is R -finitely dominated, i.e., C is homotopy equivalent, as an R -module cochain complex, to a bounded cochain complex of finitely generated projective R -modules.*
- (b) *The eight cochain complexes listed below are acyclic (all tensor products are taken over L):*

$$\left. \begin{array}{ll} C \otimes R[x, x^{-1}]((y)) & C \otimes R[x, x^{-1}]((y^{-1})) \\ C \otimes R[y, y^{-1}]((x)) & C \otimes R[y, y^{-1}]((x^{-1})) \end{array} \right\} \quad (1a)$$

$$\left. \begin{array}{ll} C \otimes R((x, y)) & C \otimes R((x^{-1}, y^{-1})) \\ C \otimes R((x, y^{-1})) & C \otimes R((x^{-1}, y)) \end{array} \right\} \quad (1b)$$

The proofs of the two implications are quite different in nature. We will establish (a) \Rightarrow (b) in Corollaries III.4.2 and III.8.1 with tools from homological algebra of multi-complexes, generalising techniques used by the first author in [Hüt11], while the reverse implication is treated in §IV.6 by a homotopy theoretic argument, ultimately generalising one half of the proof of [Ran95, Theorem 2].

I.2. RELATION WITH Σ -INVARIANTS

It might be worth explaining how our results are related to the so-called Σ -invariants in the spirit of BIERI, NEUMANN and STREBEL. Let G be a group. For every character $\chi: G \longrightarrow \mathbb{R}$ to the additive group of the reals we have a monoid $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$. Now suppose C is a non-negatively indexed chain complex of $\mathbb{Z}[G]$ -modules. Then C has, by restriction of scalars, the structure of a $\mathbb{Z}[G_\chi]$ -module chain complex. Following FARBER *et.al.* one defines [FGS10, Definition 9] the m th Σ -invariant of C as

$$\Sigma^m(C) = \{\chi \neq 0 \mid C \text{ has finite } m\text{-type over } \mathbb{Z}[G_\chi]\} / \mathbb{R}_+ .$$

So $\Sigma^m(C)$ is a quotient of the set of non-trivial χ for which there is a chain complex C' consisting of finitely generated projective $\mathbb{Z}[G_\chi]$ -modules, and a $\mathbb{Z}[G_\chi]$ -linear chain map $f: C' \longrightarrow C$ with $f_i: H_i(C') \longrightarrow H_i(C)$ an isomorphism for $i < m$ and an epimorphism for $i = m$. Two different characters are identified in the quotient if, and only if, they are positive real multiples of each other.

Theorem I.2.1 ([FGS10, Corollary 4]). *Suppose that C consists of finitely generated free $\mathbb{Z}[G]$ -modules, and is such that $C_i = 0$ for $i > m$. Let N be a normal subgroup of G with ABELIAN quotient G/N . Then the $\mathbb{Z}[N]$ -module complex C is chain homotopy equivalent to a bounded chain complex of finitely generated projective $\mathbb{Z}[N]$ -modules concentrated in degrees $\leq m$ if and only if $\Sigma^m(C)$ contains the equivalence class of every non-trivial character of G that factorises through G/N (i.e., whose kernel contains N). \square*

Note that the set of equivalence classes of non-trivial characters which are trivial on N represents a sphere in the set of equivalence classes of all non-trivial characters.

An explicit link to NOVIKOV homology has been documented by SCHÜTZ. For a character $\chi: G \longrightarrow \mathbb{R}$ we define the NOVIKOV ring

$$\widehat{RG}_\chi = \left\{ f: G \longrightarrow R \mid \forall t \in \mathbb{R}: \#(\text{supp}(f) \cap \chi^{-1}([t, \infty)) < \infty \right\},$$

equipped with the usual involution product; here $\text{supp}(f) = f^{-1}(R \setminus \{0\})$. Note that $R[G]$ is a subring of \widehat{RG}_χ .

Theorem I.2.2 (SCHÜTZ [Sch06, Theorem 4.7]). *Let C be a bounded chain complex of finitely generated free $R[G]$ -modules. Suppose that N is a normal subgroup of G with quotient $G/N \cong \mathbb{Z}^k$ a free ABELIAN group of finite rank. The complex C is $R[N]$ -finitely dominated if and only if for every character $\chi: G \longrightarrow \mathbb{R}$ which is trivial on N the complex $C \otimes_{R[G]} \widehat{RG}_\chi$ is acyclic. \square*

In these terms, our main result Theorem I.1.2 concerns the case of a split group extension

$$1 \longrightarrow N \longrightarrow N \times \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

(with the translation $R = \mathbb{Z}[N]$ and $L = \mathbb{Z}[N \times \mathbb{Z}^2] \cong R[x, x^{-1}, y, y^{-1}]$), or indeed the case $G = \mathbb{Z}^2$ and trivial group N . The sphere which characterises finite domination of C according to Theorem I.2.1 is homeomorphic to a circle S^1 . In our approach, it is replaced by the boundary of a square in \mathbb{R}^2 ; its geometry and combinatorics encode algebraic information that lead to our new set of homological conditions characterising finite domination.

I.3. ALGEBRAIC EXAMPLES

Let us work over the ring $R = \mathbb{Z}$ for simplicity. We omit the verification of the following purely algebraic facts:

- Lemma I.3.1.** (a) *The ring $\mathbb{Z}((x, y))$ is a unique factorisation domain. An element of $\mathbb{Z}((x, y))$ is a unit if and only if it is a product of a monomial $x^k y^\ell$ (for some $k, \ell \in \mathbb{Z}$) with a unit in $\mathbb{Z}[[x, y]]$ (i.e., a power series having the constant term ± 1).*
- (b) *The ring $\mathbb{Z}[x, x^{-1}]((y))$ is a unique factorisation domain. An element of $\mathbb{Z}[x, x^{-1}]((y))$ is a unit if and only if it is a product of a monomial y^ℓ (for some $\ell \in \mathbb{Z}$) with a unit in $\mathbb{Z}[x, x^{-1}][[y]]$ (i.e., a power series with y^0 having coefficient $\pm x^k$ for some $k \in \mathbb{Z}$). \square*

Example I.3.2. Let $\mu \in \mathbb{Z}[x, x^{-1}, y, y^{-1}]$. The two-step cochain complex C given by

$$\dots \longrightarrow 0 \longrightarrow \mathbb{Z}[x, x^{-1}, y, y^{-1}] \xrightarrow{\mu} \mathbb{Z}[x, x^{-1}, y, y^{-1}] \longrightarrow 0 \longrightarrow \dots$$

is \mathbb{Z} -finitely dominated if and only if μ is a LAURENT monomial with coefficient ± 1 (that is, is a unit).

Proof. The case $\mu = 0$ is trivial. Otherwise, C is \mathbb{Z} -finitely dominated if and only if the chain complexes listed in (1a) and (1b) are acyclic, which happens if and only if multiplication by μ is surjective after tensoring with

the eight rings $R[x, x^{-1}](y^{\pm 1})$, $R[y, y^{-1}](x^{\pm 1})$ and $R((x^{\pm 1}, y^{\pm 1}))$; note that injectivity is automatic. This happens if and only if μ becomes a unit in all the rings in questions, which happens if and only if μ is a LAURENT monomial with coefficient ± 1 , by Lemma I.3.1. \square

Example I.3.3. Let $\mu = 1 + x(y^2 + x(1 + y))$ and $\nu = x + y(1 + y(1 + x^2))$. The non-acyclic cochain complex C with non-trivial part

$$\mathbb{Z}[x, x^{-1}, y, y^{-1}] \xrightarrow{\alpha} \begin{array}{c} \mathbb{Z}[x, x^{-1}, y, y^{-1}] \\ \oplus \\ \mathbb{Z}[x, x^{-1}, y, y^{-1}] \end{array} \xrightarrow{\beta} \mathbb{Z}[x, x^{-1}, y, y^{-1}]$$

given by

$$\alpha = \begin{pmatrix} \mu \\ \nu \end{pmatrix} \quad \text{and} \quad \beta = (-\nu, \mu)$$

is \mathbb{Z} -finitely dominated.

Proof. Up to isomorphisms and re-indexing, the cochain complex can be obtained by computing the iterated mapping cone of the square diagram

$$\begin{array}{ccc} \mathbb{Z}[x, x^{-1}, y, y^{-1}] & \xrightarrow{\mu} & \mathbb{Z}[x, x^{-1}, y, y^{-1}] \\ \nu \downarrow & & \nu \downarrow \\ \mathbb{Z}[x, x^{-1}, y, y^{-1}] & \xrightarrow{\mu} & \mathbb{Z}[x, x^{-1}, y, y^{-1}] \end{array},$$

i.e., by taking algebraic mapping cones horizontally (*resp.*, vertically) first, and then taking the mapping cone of the resulting vertical (*resp.*, horizontal) map. Using Lemma I.3.1 we see that the map μ becomes an isomorphism after tensoring (over $\mathbb{Z}[x, x^{-1}, y, y^{-1}]$) with the rings $\mathbb{Z}((x, y))$, $\mathbb{Z}((x^{-1}, y))$, $\mathbb{Z}[x, x^{-1}](y^{-1})$, and $\mathbb{Z}[y, y^{-1}](x)$, while ν becomes an isomorphism after tensoring with any of the rings $\mathbb{Z}((x, y^{-1}))$, $\mathbb{Z}((x^{-1}, y^{-1}))$, $\mathbb{Z}[x, x^{-1}](y)$, and $\mathbb{Z}[y, y^{-1}](x^{-1})$. Consequently in all cases the iterated mapping cone, which is isomorphic to C tensored with the ring under consideration, will be an acyclic complex. The claim now follows from the Main Theorem. \square

I.4. FINITELY DOMINATED COVERING SPACES

Proposition I.4.1. *Let X be a connected finite CW complex with universal covering space \tilde{X} and fundamental group $\pi_1 X = G \times \mathbb{Z}^2$ for some group G . Let $Y \rightarrow X$ be the covering determined by the projection $\pi_1 X \rightarrow \mathbb{Z}^2$, and let C denote the cochain complex which is, up to the re-indexing $C^k = C_{-k}$, the cellular $\mathbb{Z}[\pi_1 X]$ -free chain complex of \tilde{X} . Then Y is a finitely dominated space (*i.e.*, is a retract up to homotopy of a finite CW-complex) if and only if all the complexes listed in (1a) and (1b) are acyclic.*

Proof. A connected CW complex Z is finitely dominated if and only if $\pi_1 Z$ is finitely presented, and the cellular chain complex of the universal covering space of Z is homotopy equivalent to a bounded complex of finitely generated projective $\mathbb{Z}[\pi_1 Z]$ -modules, cf. [Ran95, §3].

We apply this criterion to the space $Z = Y$, noting that $\pi_1 Y = G$, and that \tilde{X} is the universal covering space of Y . Since X is a finite CW complex, its fundamental group $G \times \mathbb{Z}^2$ is finitely presented; consequently, its retract G is finitely presented as well [Wal65, Lemma 1.3].

We identify the rings $\mathbb{Z}[G \times \mathbb{Z}^2]$ and $\mathbb{Z}[G][x, x^{-1}, y, y^{-1}]$ by saying that first and second unit vector in \mathbb{Z}^2 correspond to the indeterminates x and y , respectively. Since G is finitely presented, it is enough to show that the cochain complex C is $\mathbb{Z}[G]$ -finitely dominated, which can be detected cohomologically by the Main Theorem applied to the complex C and the ring $R = \mathbb{Z}[G]$. \square

Example I.4.2. Let Y_0 be the real plane \mathbb{R}^2 , considered as a CW complex with 0-cells the integral points, 1-cells joining vertically or horizontally adjacent 0-cells, and 2-cells the integral unit squares. The group \mathbb{Z}^2 acts evenly¹ by translation. By suitably attaching \mathbb{Z}^2 -indexed collections of cells of dimensions 4, 5 and 6 we obtain a simply-connected space Y with an even action of \mathbb{Z}^2 such that its cellular cochain complex is

- the cellular complex of \mathbb{R}^2 in chain levels 0, 1 and 2;
- the chain complex of Example I.3.3 in chain levels 4, 5 and 6;
- trivial otherwise.

Let $X = Y/\mathbb{Z}^2$; the projection map $Y \longrightarrow X$ is the universal cover, and $\pi_1 X = \mathbb{Z}^2$. It follows from Example I.3.3 and Proposition I.4.1 that Y is a finitely dominated CW complex.

Part II. Conventions. Multi-complexes

Throughout the paper we let R denote a unital associative ring (possibly non-commutative). Modules will be tacitly understood to be unital right modules, unless a different convention is specified. We will use cochain complexes of R -modules, or of modules over a ring related to R ; that is, our differentials *increase* the degree.

II.1. LOWER TRIANGULAR COCHAIN COMPLEXES

Definition II.1.1. Let $n \geq 1$. A *lower n -triangular cochain complex* consists of a cochain complex C and for each $q \in \mathbb{Z}$ a module isomorphism $\phi_q: C^q \cong \bigoplus_{p=1}^n C^{p,q}$ such that for each q the composition $d_{\ell,k}$

$$C^{k,q} \longrightarrow \bigoplus_{p=1}^n C^{p,q} \xrightarrow{\phi_q^{-1}} C^q \longrightarrow C^{q+1} \xrightarrow{\phi_{q+1}} \bigoplus_{p=1}^n C^{p,q+1} \longrightarrow C^{\ell,q+1}$$

is the zero map whenever $n \geq k > \ell \geq 1$.

In other words, under the isomorphisms ϕ_q the differential of C becomes a lower triangular matrix of the form

$$D = \begin{pmatrix} d_{1,1} & 0 & 0 & 0 & \cdots & 0 \\ d_{2,1} & d_{2,2} & 0 & 0 & \cdots & 0 \\ d_{3,1} & d_{3,2} & d_{3,3} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ d_{n-1,1} & d_{n-1,2} & \cdots & \cdots & d_{n-1,n-1} & 0 \\ d_{n,1} & d_{n,2} & \cdots & \cdots & d_{n,n-1} & d_{n,n} \end{pmatrix}$$

¹An *even* action is, by definition, a free and properly discontinuous action.

with $d_{\ell,k}: C^{k,q} \longrightarrow C^{\ell,q+1}$ satisfying $D \circ D = 0$. We have $d_{k,k} \circ d_{k,k} = 0$ for $1 \leq k \leq n$ so that $C^{k,*}$ is a cochain complex with differential $d_{k,k}$.

Definition II.1.2. Let C and D be lower n -triangular cochain complexes, with structure isomorphisms ϕ_q and ψ_q , respectively. A map $f: C \longrightarrow D$ of lower triangular cochain complexes is a map of cochain complexes f such that the composition $f_{\ell,k}$

$$C^{k,q} \longrightarrow \bigoplus_{p=1}^n C^{p,q} \xrightarrow{\phi_q^{-1}} C^q \xrightarrow{f} D^q \xrightarrow{\psi_q} \bigoplus_{p=1}^n D^{p,q} \longrightarrow D^{\ell,q}$$

is the zero map whenever $n \geq k > \ell \geq 1$.

In other words, under the isomorphisms ϕ_q and ψ_q the map f becomes a lower triangular matrix of the form

$$F = \begin{pmatrix} f_{1,1} & 0 & 0 & 0 & \cdots & 0 \\ f_{2,1} & f_{2,2} & 0 & 0 & \cdots & 0 \\ f_{3,1} & f_{3,2} & f_{3,3} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1,1} & f_{n-1,2} & \cdots & \cdots & f_{n-1,n-1} & 0 \\ f_{n,1} & f_{n,2} & \cdots & \cdots & f_{n,n-1} & f_{n,n} \end{pmatrix}$$

with $f_{\ell,k}: C^{k,q} \longrightarrow D^{\ell,q}$. Then $f_{k,k}$ is a cochain map from $C^{k,*}$ to $D^{k,*}$.

Lemma II.1.3. Let $f: C \longrightarrow D$ be a map of lower n -triangular cochain complexes. Suppose that, in the notation used above, the maps

$$f_{k,k}: C^{k,*} \longrightarrow D^{k,*} \quad (1 \leq k \leq n)$$

are quasi-isomorphisms of cochain complexes. Then f is a quasi-isomorphism.

Proof. The Lemma is a tautology for $n = 1$. — For $1 \leq k \leq n$ define a cochain complex $C(k)$ by setting

$$C(k)^q = \bigoplus_{p=k}^n C^{p,q},$$

equipped with differential

$$\begin{pmatrix} d_{k,k} & 0 & 0 & 0 & \cdots & 0 \\ d_{k+1,k} & d_{k+1,k+1} & 0 & 0 & \cdots & 0 \\ d_{k+2,k} & d_{k+2,k+1} & d_{k+2,k+2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ d_{n-1,k} & d_{n-1,k+1} & \cdots & \cdots & d_{n-1,n-1} & 0 \\ d_{n,k} & d_{n,k+1} & \cdots & \cdots & d_{n,n-1} & d_{n,n} \end{pmatrix}$$

(this is the lower right-hand part of the matrix D above). We note that $C(k+1)$ is a subcomplex of $C(k)$, and that the quotient $C(k)/C(k+1)$ is nothing but $C^{k,*}$. Clearly $C(n) = C^{n,*}$, and $C(1)$ is isomorphic to C via the structure isomorphisms ϕ_q .

We define the analogous objects $D(k)$ associated to the cochain complex D ; the remarks on the $C(k)$ apply *mutatis mutandis*.

The map f (or rather, its matrix representation F) restricts to cochain complex maps $f(k): C(k) \longrightarrow D(k)$, with $f(n) = f_{n,n}$ and $f(1)$ being isomorphic to f . The maps $f(k)$ fit into commutative ladder diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(k+1) & \longrightarrow & C(k) & \longrightarrow & C^{k,*} & \longrightarrow & 0 \\ & & \downarrow f(k+1) & & \downarrow f(k) & & \downarrow f_{k,k} & & \\ 0 & \longrightarrow & D(k+1) & \longrightarrow & D(k) & \longrightarrow & D^{k,*} & \longrightarrow & 0 \end{array}$$

for $1 \leq k < n$, inducing a ladder diagram of long exact cohomology sequences as usual. Using the Five Lemma, and the fact that the $f_{k,k}$ are known to be quasi-isomorphisms by hypothesis, we conclude in turn that the maps $f(n-1)$, $f(n-2)$, \dots , $f(1) \cong f$ are quasi-isomorphisms, thereby proving the Lemma. \square

II.2. DOUBLE COMPLEXES

A *double complex* $D^{*,*}$ is a $\mathbb{Z} \times \mathbb{Z}$ -indexed collection $(D^{p,q})_{p,q \in \mathbb{Z}}$ of right R -modules together with “horizontal” and “vertical” differentials

$$d_h: D^{p,q} \longrightarrow D^{p+1,q} \quad \text{and} \quad d_v: D^{p,q} \longrightarrow D^{p,q+1}$$

which satisfy the conditions

$$d_h \circ d_h = 0, \quad d_v \circ d_v = 0, \quad d_h \circ d_v + d_v \circ d_h = 0.$$

Note that the differentials anti-commute. We will in general consider unbounded double complexes so that $D^{p,q} \neq 0$ may occur for $|p|$ and $|q|$ arbitrarily large.

Definition II.2.1. Let $D^{*,*}$ be a double complex. We define its *direct sum totalisation* to be the cochain complex $\text{T\oet } D^{*,*}$ which in cochain level n is given by the direct sum

$$(\text{T\oet } D^{*,*})^n = \bigoplus_p D^{p,n-p};$$

the differential is given by $d_h + d_v$, where d_h and d_v are the “horizontal” and “vertical” differentials of $D^{*,*}$ respectively.

We will make use of the following standard result when comparing double complexes and the direct sum totalisation of each.

Lemma II.2.2. *Let $h: D^{*,*} \longrightarrow E^{*,*}$ be a map of double complexes which are concentrated in finitely many columns. If h is a quasi-isomorphism on each column or on each row, then the induced map*

$$\text{T\oet } (h): \text{T\oet } D^{*,*} \longrightarrow \text{T\oet } E^{*,*}$$

is a quasi-isomorphism.

Proof. Let us first deal with the case that h is a quasi-isomorphism on each column. Note that $\text{T\oet } D^{*,*}$ and $\text{T\oet } E^{*,*}$ can be given the structure of lower n -triangular complexes in the sense of Definition II.1.1, for the same n , such that $\text{T\oet } (h)$ is a map of lower triangular complexes in the sense of Definition II.1.2. In more detail, let us assume that $D^{*,*}$ and $E^{*,*}$ are

concentrated in columns 1 to n , for ease of indexing. Then $(\text{T\oet } D^{*,*})^q = \bigoplus_{p=1}^n D^{p,q-p}$ so that $C = \text{T\oet } D^{*,*}$ has the lower triangular decomposition

$$C^q = \bigoplus_{p=1}^n C^{p,q} \quad \text{where} \quad C^{p,q} = D^{p,p-q} .$$

A similar decomposition can be defined for $\text{T\oet } E^{*,*}$. The hypothesis that h is a quasi-isomorphism on each column translates into the hypothesis of Lemma II.1.3 which thus implies that $\text{T\oet } (h)$ is a quasi-isomorphism as claimed.

If h is a quasi-isomorphism on each row we can apply the same reasoning with the roles of rows and columns reversed, provided our original complex is bounded in the vertical direction as well. In the general case, one can appeal to a spectral sequence argument. More precisely, ‘‘filtration by rows’’ gives rise to a convergent spectral sequence

$$E_1^{*,*} = H^v(D^{*,*}) \implies H^*(\text{T\oet } D^{*,*}) ,$$

and similarly for the bicomplex $E^{*,*}$; the map f induces a map of spectral sequences which is an isomorphism on E^1 -terms, and thus induces a quasi-isomorphism on abutments as claimed. \square

Lemma II.2.3. *Suppose that the double complex $E^{*,*}$ is concentrated in the first quadrant (that is, suppose that $E^{p,q} = 0$ if $p < 0$ or $q < 0$). Suppose further that we are given a cochain complex C with $C^q = 0$ if $q < 0$, and maps $h_q: C^q \longrightarrow E^{0,q}$ such that for each $q \geq 0$ the sequence*

$$0 \longrightarrow C^q \xrightarrow{h_q} E^{0,q} \longrightarrow E^{1,q} \longrightarrow E^{2,q} \longrightarrow \dots$$

is exact (i.e., is an acyclic cochain complex), and such that the composites

$$C^q \longrightarrow E^{0,q} \longrightarrow E^{0,q+1} \quad \text{and} \quad C^q \longrightarrow C^{q+1} \longrightarrow E^{0,q+1}$$

agree up to sign. Then there is a quasi-isomorphism

$$C \longrightarrow \text{T\oet } E^{*,*}$$

induced by the h_q .

Proof. The proof given by BOTT and TU [BT82, p. 97] applies to the current situation; see the remark following Proposition 8.8 of *loc.cit.*. \square

II.3. TRIPLE COMPLEXES

Definition II.3.1. A triple complex $T^{*,*,*}$ is a \mathbb{Z}^3 -indexed family of modules $T^{x,y,z}$ together with three anti-commuting differentials

$$d_x: T^{x,y,z} \longrightarrow T^{x+1,y,z} , \quad d_y: T^{x,y,z} \longrightarrow T^{x,y+1,z} , \quad d_z: T^{x,y,z} \longrightarrow T^{x,y,z+1} .$$

‘‘Anti-commuting’’ means that the differentials satisfy $d_i d_j = (\delta_{ij} - 1) d_j d_i$.

The *direct sum totalisation* $\text{T\oet } T^{*,*,*}$ is the cochain complex with

$$(\text{T\oet } T^{*,*,*})^n = \bigoplus_{x+y+z=n} T^{x,y,z}$$

and differential $d = d_x + d_y + d_z$. We use the same notation as for the direct sum totalisation of double complexes.

Definition II.3.2. For a triple complex $T^{*,*,*}$ we denote by $\text{Tot}_{x,y} T^{*,*,*}$ the *partial totalisation* of $T^{*,*,*}$ with respect to x and y . We define this to be the double complex given by

$$(\text{Tot}_{x,y} T^{*,*,*})^{p,q} = \bigoplus_{x+y=p} T^{x,y,q}$$

with “horizontal” differential $d_h = d_x + d_y$ and “vertical” differential $d_v = d_z$.

It is easy to verify that

$$\text{T\oet}(\text{Tot}_{x,y} T^{*,*,*}) = \text{T\oet} T^{*,*,*},$$

where T\oet denotes the direct sum totalisation of either a double or a triple complex as determined by context.

Lemma II.3.3. *Let $f: T^{*,*,*} \longrightarrow U^{*,*,*}$ be a map of triple complexes concentrated in a finite cubical region of \mathbb{Z}^3 . Suppose that f is a quasi-isomorphism of all cochain complexes in z -direction (resp., in y -direction, resp., in x -direction), that is, suppose that*

$$f: T^{x,y,*} \longrightarrow U^{x,y,*} \quad (\text{resp., } T^{x,*,z} \longrightarrow U^{x,*,z}, \text{ resp., } T^{*,y,z} \longrightarrow U^{*,y,z})$$

is a quasi-isomorphism for all $x, y \in \mathbb{Z}$ (resp., all $x, z \in \mathbb{Z}$, resp., all $y, z \in \mathbb{Z}$). Then

$$\text{T\oet}(f): \text{T\oet} T^{*,*,*} \longrightarrow \text{T\oet} U^{*,*,*}$$

is a quasi-isomorphism.

Proof. If $f: T^{x,y,*} \longrightarrow U^{x,y,*}$ is a quasi-isomorphism for all $x, y \in \mathbb{Z}$, then $\text{Tot}_{x,y}(f): \text{Tot}_{x,y} T^{*,*,*} \longrightarrow \text{Tot}_{x,y} U^{*,*,*}$ is a map of double complexes satisfying the hypotheses of Lemma II.2.2. Consequently,

$$\text{T\oet}(f) = \text{T\oet} \text{Tot}_{x,y}(f): \text{T\oet} T^{*,*,*} \longrightarrow \text{T\oet} U^{*,*,*}$$

is a quasi-isomorphism. — The other cases can be proved in a similar manner. \square

Part III. Finite domination implies acyclicity

III.1. TRUNCATED PRODUCTS. ONE VARIABLE

Let R be a ring with unit, and let z be an indeterminate. We have obvious R -module isomorphisms

$$R[z] \cong \bigoplus_{\mathbb{N}} R, \quad R[z, z^{-1}] \cong \bigoplus_{\mathbb{Z}} R, \quad R[[z]] \cong \prod_{\mathbb{N}} R$$

mapping rz^k to the element r of the k th summand or factor on the right. Using these isomorphisms, we may write elements of these infinite sums or products as polynomials, LAURENT polynomials and formal power series in z , respectively. Similarly, a formal LAURENT power series in z (involving finitely many negative powers of z) corresponds to an element of a “truncated product” via the obvious R -module isomorphism

$$R[[z]][z^{-1}] \cong \bigoplus_{i < 0} R \oplus \prod_{i \geq 0} R.$$

This ring is known as a NOVIKOV ring, as is its counterpart with z^{-1} in place of z ; we reserve the notation

$$R((z)) = R[[z]][z^{-1}] \quad \text{and} \quad R((z^{-1})) = R[[z^{-1}]] [z].$$

There is a module theoretic version corresponding to the construction of the NOVIKOV ring $R((z))$. Given a \mathbb{Z} -indexed family of modules M_i we define the *left truncated product* to be the module

$$\text{lt} \prod_i M_i = \bigoplus_{i < 0} M_i \oplus \prod_{i \geq 0} M_i;$$

the elements of this truncated product will be written as formal LAURENT series $\sum_{i \geq k} m_i z^i$ with $m_i \in M_i$. We let $M((z))$ denote the module of formal LAURENT series with coefficients in M ,

$$M((z)) = \text{lt} \prod M = \left\{ \sum_{i \geq k} m_i z^i \mid k \in \mathbb{Z}, m_i \in M \right\}.$$

Note that $M((z))$ carries a canonical $R((z))$ -module structure described by $z \cdot \sum_{i \geq k} m_i z^i = \sum_{i \geq k} m_i z^{i+1}$. — Dually we define the *right truncated product* to be the module

$$\prod_i^{\text{rt}} M_i = \prod_{i \leq 0} M_i \oplus \bigoplus_{i > 0} M_i$$

of formal LAURENT series which are finite to the right, and define $M((z^{-1}))$ by setting

$$M((z^{-1})) = \prod^{\text{rt}} M = \left\{ \sum_{i \leq k} m_i z^i \mid k \in \mathbb{Z}, m_i \in M \right\}.$$

The module $M((z^{-1}))$ carries an obvious $R((z^{-1}))$ -module structure described by $z^{-1} \cdot \sum_{i \leq k} m_i z^i = \sum_{i \leq k} m_i z^{i-1}$.

Lemma III.1.1. *Suppose that M is a finitely presented right R -module. There is a natural isomorphism of $R((z))$ -modules*

$$\Phi_M: M \otimes_R R((z)) \xrightarrow{\cong} M((z)), \quad m \otimes \sum_{i \geq k} r_i z^i \mapsto \sum_{i \geq k} m r_i z^i,$$

and a similar isomorphism $\Psi_M: M \otimes_R R((z^{-1})) \xrightarrow{\cong} M((z^{-1}))$.

Proof. The proof is standard, details can be found, for example, in [Hüt11, Lemma 2.1]. One establishes the result for finitely generated free R -modules first, and then passes to the general case by considering a two-step resolution of M by finitely generated free modules. \square

III.2. TRUNCATED PRODUCT TOTALISATION OF DOUBLE COMPLEXES

We will consider non-standard totalisation functors for double complexes formed by using truncated products; this technology has been used in [Hüt11] to analyse NOVIKOV cohomology of algebraic mapping tori. We begin by recalling some definitions and results useful for our present purposes; later we will extend these ideas to LAURENT rings with two variables.

Definition III.2.1. Let $D^{*,*}$ be a double complex. We define its *left truncated totalisation* to be the cochain complex ${}^{\text{lt}}\text{Tot } D^{*,*}$ which in cochain level n is given by the left truncated product

$$({}^{\text{lt}}\text{Tot } D^{*,*})^n = \prod_p^{\text{lt}} D^{p,n-p};$$

the differential d is induced in an obvious way by the horizontal differential $d_h: D^{p,q} \longrightarrow D^{p+1,q}$ and the vertical differential $d_v: D^{p,q} \longrightarrow D^{p,q+1}$: the component mapping into the p th factor of $({}^{\text{lt}}\text{Tot } D^{*,*})^{n+1}$ is the sum of the horizontal differential coming from the $(p-1)$ st factor of $({}^{\text{lt}}\text{Tot } D^{*,*})^n$, and the vertical differential coming from the p th factor of $({}^{\text{lt}}\text{Tot } D^{*,*})^n$. Explicitly, for an element $x = \sum_{p \geq k} a_p z^p$ of $({}^{\text{lt}}\text{Tot } D^{*,*})^n$ we have

$$d(x) = \sum_{p \geq k} (d_h(a_{p-1}) + d_v(a_p)) z^p$$

(where we set $a_{k-1} = 0$ for convenience).

Dually, we define the *right truncated totalisation* to be the cochain complex $\text{Tot}^{\text{rt}} D^{*,*}$ which in cochain level n is given by the right truncated product

$$(\text{Tot}^{\text{rt}} D^{*,*})^n = \prod_p^{\text{rt}} D^{p,n-p}$$

with differential described by

$$d: \sum_{p \leq k} a_p z^p \mapsto \sum_{p \leq k+1} (d_h(a_{p-1}) + d_v(a_p)) z^p.$$

Proposition III.2.2 ([Ber12, Corollary 6.7], [Hüt11, Proposition 1.2]). *Suppose the double complex $D^{*,*}$ has exact columns. Then ${}^{\text{lt}}\text{Tot } D^{*,*}$ is acyclic. Dually, if $D^{*,*}$ has exact rows then $\text{Tot}^{\text{rt}} D^{*,*}$ is acyclic.*

Proof. This can be proved by an elementary diagram chase, associating to each cocycle m in $({}^{\text{lt}}\text{Tot } D^{*,*})^n$ (resp., in $(\text{Tot}^{\text{rt}} D^{*,*})^n$) an element in the module $({}^{\text{lt}}\text{Tot } D^{*,*})^{n-1}$ (resp., in $(\text{Tot}^{\text{rt}} D^{*,*})^{n-1}$) with coboundary m . Details can be found in the given references. \square

III.3. ALGEBRAIC MAPPING 1-TORI

Definition III.3.1. Let C be a cochain complex of right R -modules, and let $h: C \longrightarrow C$ be a cochain map. The *mapping 1-torus* $\mathcal{T}(h)$ of h is the $R[z, z^{-1}]$ -module cochain complex

$$\mathcal{T}(h) = \text{Cone} \left(C \otimes_R R[z, z^{-1}] \xrightarrow{h \otimes \text{id} - \text{id} \otimes z} C \otimes_R R[z, z^{-1}] \right)$$

where the map “ z ” denotes the self map of $R[z, z^{-1}]$ given by multiplication by the indeterminate z .

In this definition “Cone” stands for the algebraic mapping cone; if a map of cochain complexes $f: X \longrightarrow Y$ is considered as a double complex $D^{*,*}$ concentrated in columns $p = -1, 0$ with horizontal differential f and the

differential of X modified by the factor -1 , then $\text{Cone}(f) = \text{Tot } D^{*,*}$. Explicitly, we have $\text{Cone}(f)^n = X^{n+1} \oplus Y^n$, and the differential is given by the following formula:

$$\begin{aligned} \text{Cone}(f)^n = X^{n+1} \oplus Y^n &\longrightarrow X^{n+2} \oplus Y^{n+1} = \text{Cone}(f)^{n+1} \\ (x, y) &\mapsto (-d(x), f(x) + d(y)) \end{aligned}$$

Proposition III.3.2. (1) For homotopic maps $f, g: C \longrightarrow C$, the mapping 1-tori $\mathcal{T}(f)$ and $\mathcal{T}(g)$ are isomorphic.

(2) MATHER trick: For maps $f: C \longrightarrow D$ and $g: D \longrightarrow C$ of cochain complexes, the mapping 1-tori $\mathcal{T}(fg)$ and $\mathcal{T}(gf)$ are homotopy equivalent.

(3) If C is a bounded above complex of $R[z, z^{-1}]$ -modules which are projective as R -modules, then C and $\mathcal{T}(z)$ are homotopy equivalent as $R[z, z^{-1}]$ -module complexes. Here “ z ” denotes the R -linear self map given by multiplication with the indeterminate z , and the mapping 1-torus is formed by considering C as an R -module complex.

Proof. This can be found (using the language of chain complexes rather than cochain complexes), for example, in [HQ13, §2]. \square

III.4. MAPPING 1-TORI AND TOTALISATION. APPLICATIONS

Let $h: C \longrightarrow C$ be a self map of a cochain complex C . Observe that by additivity of tensor products we have an equality of cochain complexes

$$\mathcal{T}(h) \otimes_{R[z, z^{-1}]} R((z)) = \text{Cone} \left(C \otimes_R R((z)) \xrightarrow{h \otimes \text{id} - \text{id} \otimes z} C \otimes_R R((z)) \right). \quad (2)$$

In the mapping cone on the right we notice that while the cochain modules are of the form $M \otimes_R R((z))$ the differential is *not* of the form $f \otimes_R \text{id}_{R((z))}$ for an R -linear map f ; the reason is the presence of the self map $\text{id} \otimes z$ which “raises the z -degree”, and this cannot happen for maps of the form $f \otimes \text{id}$.

However, if C consists of finitely presented R -modules we can identify the cochain complex $C \otimes_R R((z))$ with $C((z))$ by Lemma III.1.1, and we can further identify the right hand side of (2) with ${}^{\text{lt}}\text{Tot}(D^{*,*})$ for the following bicomplex:

$$\left. \begin{aligned} D^{p,q} &= C^{p+q+1} \oplus C^{p+q} \\ d_h: D^{p,q} &\longrightarrow D^{p+1,q} \\ (x, y) &\mapsto (0, -x) \\ d_v: D^{p,q} &\longrightarrow D^{p,q+1} \\ (x, y) &\mapsto (-d_C(x), h(x) + d_C(y)) \end{aligned} \right\} \quad (3)$$

Here d_C denotes the differential of the complex C . We have $d_h \circ d_h = 0$, and the p th column $D^{p,*}$ of $D^{*,*}$ is the p th shift of $\text{Cone}(h)$, with unchanged differential, so that $d_v \circ d_v = 0$. Finally, horizontal and vertical differentials anti-commute: for a typical element $(x, y) \in D^{p,q} = C^{p+q+1} \oplus C^{p+q}$ we have

$$\begin{aligned} d_v \circ d_h(x, y) &= d_v(0, -x) = (0, d_C(-x)) = -(0, d_C(x)) \\ &= -d_h((-d_C(x), h(x) + d_C(y))) = -d_h \circ d_v(x, y). \end{aligned}$$

To complete the identification we postulate that the p th column of $D^{*,*}$ corresponds to the terms with coefficient z^p in the formal LAURENT series notation for the truncated product ${}^{\text{lt}}\text{Tot}(D^{*,*})$.

Lemma III.4.1. *Suppose that C is a bounded above cochain complex of projective right $R[x, x^{-1}, y, y^{-1}]$ -modules. Suppose further that C is homotopy equivalent, as an R -module complex, to a bounded complex B of finitely generated projective right R -modules. Then the induced cochain complex $C \otimes_{R[x, x^{-1}, y, y^{-1}]} R[y, y^{-1}]((x))$ is acyclic.*

Proof. Let $f: C \longrightarrow B$ and $g: B \longrightarrow C$ be mutually inverse homotopy equivalences of R -module complexes. There are $R[y, y^{-1}]$ -module homotopy equivalences

$$C \simeq \mathcal{T}(y) \simeq \mathcal{T}(ygf) \simeq \mathcal{T}(fyg) =: A$$

where the symbol “ y ” denotes the R -linear self map given by multiplication by y , cf. Proposition III.3.2; note that all mapping 1-tori here are mapping 1-tori of R -linear maps. Now since B is bounded and consists of finitely generated projective right R -modules, the complex $A = \mathcal{T}(fyg)$ is bounded and consists of finitely generated projective right $R[y, y^{-1}]$ -modules.

Let $\alpha: C \longrightarrow A$ and $\beta: A \longrightarrow C$ be mutually inverse chain homotopy equivalences of $R[y, y^{-1}]$ -module complexes. There are chain homotopy equivalences of $R[x, x^{-1}, y, y^{-1}]$ -module complexes

$$C \simeq \mathcal{T}(x) \simeq \mathcal{T}(x\beta\alpha) \simeq \mathcal{T}(\alpha x\beta)$$

where “ x ” denotes the $R[y, y^{-1}]$ -linear self map given by multiplication by x , cf. Proposition III.3.2 applied to the ring $R[y, y^{-1}]$ instead of R ; note that all mapping 1-tori here are mapping 1-tori of $R[y, y^{-1}]$ -linear maps.

It follows that the two complexes

$$C \otimes_{R[x, x^{-1}, y, y^{-1}]} R[y, y^{-1}]((x)) \quad \text{and} \quad \mathcal{T}(\alpha x\beta) \otimes_{R[x, x^{-1}, y, y^{-1}]} R[y, y^{-1}]((x)) \quad (4)$$

are homotopy equivalent. Moreover, $\mathcal{T}(\alpha x\beta)$ is bounded and consists of finitely generated projective $R[x, x^{-1}, y, y^{-1}]$ -modules by our results on A . So we can identify the second complex in (4) with ${}^{\text{lt}}\text{Tot}(D^{*,*})$ of a certain double complex of $R[y, y^{-1}]$ -modules. In fact, we are dealing with the bi-complex construction given in (3) above, working over the ring $R[y, y^{-1}]$ instead of R and applied to the mapping 1-torus

$$\mathcal{T}(\alpha x\beta) = \text{Cone} \left(A \otimes_{R[y, y^{-1}]} R[x, x^{-1}, y, y^{-1}] \xrightarrow{(\alpha x\beta) \otimes \text{id} - \text{id} \otimes x} C \otimes_{R[y, y^{-1}]} R[x, x^{-1}, y, y^{-1}] \right).$$

Now the map $\alpha x\beta: A \longrightarrow A$ is a homotopy equivalence so that its mapping cone is acyclic. It follows that the columns of $D^{*,*}$, which are shifted copies of this mapping cone, are exact whence ${}^{\text{lt}}\text{Tot}(D^{*,*})$ is acyclic by Proposition III.2.2. In view of the homotopy equivalence of the complexes in (4) this proves the Lemma. \square

Corollary III.4.2. *Under the conditions of the Main Theorem part (a), all the cochain complexes listed under (1a) are acyclic.*

Proof. For the complex $C \otimes_L R[y, y^{-1}]((x))$ this is the content of the previous Lemma. By a change of variables, leaving y fixed and replacing x by x^{-1} we get the result for $C \otimes_L R[y, y^{-1}]((x^{-1}))$. Finally, by a further change of variables swapping x and y we cover the remaining two cases. \square

Remark III.4.3. The change of variables $x \mapsto x^{-1}$, $y \mapsto y$ executed in the proof can be avoided by using right truncated totalisations, and by changing the definition of the mapping torus to involve $\text{id} \otimes z^{-1}$ rather than $\text{id} \otimes z$.

III.5. TRUNCATED PRODUCTS. TWO VARIABLES

For a ring R we define the NOVIKOV ring $R((x, y)) = R[[x, y]][(xy)^{-1}]$ to be the ring of those formal LAURENT power series f in two variables which have the property that for some $\ell \geq 0$ we have $x^\ell y^\ell f \in R[[x, y]]$. That is,

$$R((x, y)) = \left\{ \sum_{p, q \geq k} a_{p, q} x^p y^q \mid k \in \mathbb{Z}, a_{p, q} \in R \right\},$$

equipped with the obvious multiplication of power series.

There is a related module theoretic construction, to be described next.

Definition III.5.1. Let $M_{j, k}$ be a $\mathbb{Z} \times \mathbb{Z}$ -indexed family of R -modules. We define their *truncated product* (more precisely, their *below-and-left truncated product*) by

$$\text{blt} \prod_{p, q} M_{p, q} = \left\{ \sum_{p, q \geq k} m_{p, q} x^p y^q \mid k \in \mathbb{Z}, m_{p, q} \in M_{p, q} \right\} \subseteq \prod_{p, q \in \mathbb{Z}} M_{p, q},$$

and if $M_{p, q} = M$ for all $p, q \in \mathbb{Z}$ we use the notation

$$M((x, y)) = \text{blt} \prod_{p, q} M.$$

Note that $M((x, y))$ has an obvious $R((x, y))$ -module structure given by ‘‘multiplication of LAURENT series’’.

Lemma III.5.2. *Suppose that M is a finitely presented right R -module. There is a natural isomorphism*

$$\begin{aligned} M \otimes_R R((x, y)) &\xrightarrow{\cong} M((x, y)), \\ m \otimes \sum_{j, k \geq \ell} r_{j, k} x^j y^k &\mapsto \sum_{j, k \geq \ell} m r_{j, k} x^j y^k. \end{aligned}$$

Proof. This is similar to Lemma III.1.1. We omit the details. \square

III.6. BELOW AND LEFT TRUNCATED TOTALISATION OF TRIPLE COMPLEXES

Let $D^{*,*,*}$ be a triple complex, cf. §II.3. To it we associate its *below-and-left truncated totalisation*, denoted $\text{blt Tot}(D^{*,*,*})$; this is the cochain complex given by

$$\left(\text{blt Tot } D^{*,*,*} \right)^n = \text{blt} \prod_{p, q} D^{p, q, n-p-q}$$

with differential $d = d_x + d_y + d_z$. Explicitly,

$$d: \sum_{p,q \geq k} m_{p,q} x^p y^q \mapsto \sum_{p,q \geq k} (d_x(m_{p-1,q}) + d_y(m_{p,q-1}) + d_z(m_{p,q})) x^p y^q$$

where $m_{p,q} = 0$ if $p < k$ or $q < k$.

Proposition III.6.1. *Suppose the triple complex $D^{*,*,*}$ is exact in z -direction; that is, suppose that the chain complexes $D^{x,y,*}$ are acyclic for all $x, y \in \mathbb{Z}$. Then $\text{Tot}(D^{*,*,*})$ is acyclic.*

Proof. This is elementary: we prove directly that any cocycle in the totalisation is a coboundary. Let $m = \sum_{p,q \geq k} m_{p,q} x^p y^q \in (\text{Tot } D^{*,*,*})^n$ be such that $d(m) = 0$. By definition of d this means that $d_z(m_{k,k}) = 0$; by exactness in z -direction we find an element $b_{k,k} \in D^{k,k,n-1-2k}$ with $d_z(b_{k,k}) = m_{k,k}$. Now iteratively for $\ell = 1, 2, 3, \dots$ we note that

$$\begin{aligned} d_z(m_{k+\ell,k} - d_x(b_{k+\ell-1,k})) &= d_z(m_{k+\ell,k}) + d_x d_z(b_{k+\ell-1,k}) \\ &= d_z(m_{k+\ell,k}) + d_x(m_{k+\ell-1,k}) = 0, \end{aligned}$$

using that $d_x d_z = -d_z d_x$ and $d(m) = 0$. By exactness in z -direction we find $b_{k+\ell,k}$ with $d_z(b_{k+\ell,k}) = m_{k+\ell,k} - d_x(b_{k+\ell-1,k})$ so that

$$d_x(b_{k+\ell-1,k}) + d_z(b_{k+\ell,k}) = m_{k+\ell,k}.$$

Similarly, we note that

$$\begin{aligned} d_z(m_{k,k+\ell} - d_y(b_{k,k+\ell-1})) &= d_z(m_{k,k+\ell}) + d_y d_z(b_{k,k+\ell-1}) \\ &= d_z(m_{k,k+\ell}) + d_y(m_{k,k+\ell-1}) = 0, \end{aligned}$$

using that $d_y d_z = -d_z d_y$ and $d(m) = 0$. By exactness in z -direction we find $b_{k,k+\ell}$ with $d_z(b_{k,k+\ell}) = m_{k,k+\ell} - d_y(b_{k,k+\ell-1})$ so that

$$d_y(b_{k,k+\ell-1}) + d_z(b_{k,k+\ell}) = m_{k,k+\ell}.$$

So far we have constructed elements of the form $b_{*,k}$ and $b_{k,*}$, and proceed now to iterate the construction for $j = k+1, k+2, \dots$ as follows. First, $d(m) = 0$ implies that $d_x(m_{j-1,j}) + d_y(m_{j,j-1}) + d_z(m_{j,j}) = 0$. By construction of the previous $b_{p,q}$ we have $m_{j-1,j} = d_x(b_{j-2,j}) + d_y(b_{j-1,j-1}) + d_z(b_{j-1,j})$ so that

$$d_x(m_{j-1,j}) = d_x d_y(b_{j-1,j-1}) - d_z d_x(b_{j-1,j})$$

and, in the same way,

$$d_y(m_{j,j-1}) = -d_x d_y(b_{j-1,j-1}) - d_z d_y(b_{j,j-1})$$

which together results in

$$d_z(m_{j,j} - d_x(b_{j-1,j}) - d_y(b_{j,j-1})) = 0.$$

(We have used the usual convention that $b_{p,q} = 0$ if $p < k$ or $q < k$.) By exactness in z -direction there is $b_{j,j}$ with $d_z(b_{j,j}) = m_{j,j} - d_x(b_{j-1,j}) - d_y(b_{j,j-1})$ or, re-arranged,

$$d_x(b_{j-1,j}) + d_y(b_{j,j-1}) + d_z(b_{j,j}) = m_{j,j}.$$

Still keeping j fixed, we now iterate over $\ell = 1, 2, \dots$ by observing that

$$d_z(m_{j+\ell,j} - d_x(b_{j+\ell-1,j}) - d_y(b_{j+\ell,j-1})) = 0$$

so we find $b_{j+\ell,j}$ with $d_z(b_{j+\ell,j}) = m_{j+\ell,j} - d_x(b_{j+\ell-1,j}) - d_y(b_{j+\ell,j-1})$ or, re-arranged,

$$d_x(b_{j+\ell-1,j}) + d_y(b_{j+\ell,j-1}) + d_z(b_{j+\ell,j}) = m_{j+\ell,j} .$$

Again, for the same j we find in a similar manner an element $b_{j,j+\ell}$ with

$$d_x(b_{j-1,j+\ell}) + d_y(b_{j,j+\ell-1}) + d_z(b_{j,j+\ell}) = m_{j,j+\ell} .$$

This finishes both iterations. It remains to observe that, by construction, we have shown that

$$d\left(\sum_{p,q \geq k} b_{p,q} x^p y^q\right) = \sum_{p,q \geq k} m_{p,q} x^p y^q = m$$

so that m is a coboundary as claimed. \square

III.7. THE MAPPING 2-TORUS

Suppose we have a cochain complex C of R -modules with differential denoted d_C , and two self maps $f, g: C \rightarrow C$ which commute up to homotopy. Suppose moreover we are given a specific choice of homotopy $H: fg \simeq gf$ such that $d_C H + H d_C = fg - gf$. To such data we associate a cochain complex of $R[x, x^{-1}, y, y^{-1}]$ -modules, the *mapping 2-torus* $\mathcal{T}(f, g; H)$. The module in degree n is the direct sum

$$\begin{aligned} \mathcal{T}(f, g; H)^n = & C^{n+2} \otimes_R R[x, x^{-1}, y, y^{-1}] \oplus (C^{n+1} \otimes_R R[x, x^{-1}, y, y^{-1}] \\ & \oplus C^{n+1} \otimes_R R[x, x^{-1}, y, y^{-1}]) \oplus C^n \otimes_R R[x, x^{-1}, y, y^{-1}] , \end{aligned} \quad (5)$$

and the coboundary map $d_{\mathcal{T}}: \mathcal{T}(f, g; H)^n \rightarrow \mathcal{T}(f, g; H)^{n+1}$ is given by the following matrix:

$$d_{\mathcal{T}} = \begin{pmatrix} d_C \otimes \text{id} & 0 & 0 & 0 \\ -(g \otimes \text{id} - \text{id} \otimes x) & -d_C \otimes \text{id} & 0 & 0 \\ f \otimes \text{id} - \text{id} \otimes y & 0 & -d_C \otimes \text{id} & 0 \\ H \otimes \text{id} & f \otimes \text{id} - \text{id} \otimes y & g \otimes \text{id} - \text{id} \otimes x & d_C \otimes \text{id} \end{pmatrix} \quad (6)$$

By construction, $\mathcal{T}(f, g; H)$ is a lower 4-triangular complex in the sense of Definition II.1.1.

Remark III.7.1. If $fg = gf$ we may choose $H = 0$, and the mapping 2-torus $\mathcal{T}(f, g; 0)$ is nothing but the total complex of the following twofold cochain complex concentrated in columns -2 , -1 and 0 :

$$C \otimes_R R[x, x^{-1}, y, y^{-1}] \xrightarrow{\alpha} \begin{array}{c} C \otimes_R R[x, x^{-1}, y, y^{-1}] \\ \oplus \\ C \otimes_R R[x, x^{-1}, y, y^{-1}] \end{array} \xrightarrow{\beta} C \otimes_R R[x, x^{-1}, y, y^{-1}] \quad (7)$$

Here the maps α and β are given by the matrices

$$\alpha = \begin{pmatrix} -(g \otimes \text{id} - \text{id} \otimes x) \\ f \otimes \text{id} - \text{id} \otimes y \end{pmatrix} \quad \text{and} \\ \beta = (f \otimes \text{id} - \text{id} \otimes y, \quad g \otimes \text{id} - \text{id} \otimes x) .$$

(This twofold cochain complex features *commuting* differentials and thus should be converted to a double complex by changing the “vertical” differential coming from C in the middle summand by a sign.)

Properties of the mapping 2-torus. For later applications we need to record that the mapping 2-torus has properties analogous to those of the mapping 1-torus of §III.3. We keep the notation from above.

Theorem III.7.2. (1) *Suppose that the maps f and g commute, and suppose that we are given a self map $h: C \longrightarrow C$ together with a homotopy $A: h \simeq \text{id}$ so that $d_C A + Ad_C = h - \text{id}$. Then $h(fAg - gAf)$ is a homotopy from $(hf)(hg)$ to $(hg)(hf)$, and the matrix*

$$\Phi = \begin{pmatrix} h \otimes \text{id} & 0 & 0 & 0 \\ -hgA \otimes \text{id} & h \otimes \text{id} & 0 & 0 \\ hfA \otimes \text{id} & 0 & h \otimes \text{id} & 0 \\ h(fAg - gAf)A \otimes \text{id} & -hfA \otimes \text{id} & -hgA \otimes \text{id} & h \otimes \text{id} \end{pmatrix}$$

defines a quasi-isomorphism

$$\Phi: \mathcal{T}(f, g; 0) \longrightarrow \mathcal{T}(hf, hg; h(fAg - gAf)) .$$

If C is a bounded above complex of projective modules over R then $\mathcal{T}(f, g; 0)$ and $\mathcal{T}(hf, hg; h(fAg - gAf))$ are homotopy equivalent.

(2) *Suppose that the maps f and g commute as before. Given cochain maps $\alpha: B \longrightarrow C$ and $\beta: C \longrightarrow B$ and a homotopy $A: \alpha\beta \simeq \text{id}$ so that $d_C A + Ad_C = \alpha\beta - \text{id}$, the composite map $\beta(fAg - gAf)\alpha$ is a homotopy from $(\beta f\alpha)(\beta g\alpha)$ to $(\beta g\alpha)(\beta f\alpha)$, the composite map $\alpha\beta(fAg - gAf)$ is a homotopy from $(\alpha\beta f)(\alpha\beta g)$ to $(\alpha\beta g)(\alpha\beta f)$, and the diagonal matrix with entries $\alpha \otimes \text{id}$ on the main diagonal defines a cochain map*

$$\begin{aligned} \alpha_*: \mathcal{T}(\beta f\alpha, \beta g\alpha; \beta(fAg - gAf)\alpha) \\ \longrightarrow \mathcal{T}(\alpha\beta f, \alpha\beta g; \alpha\beta(fAg - gAf)) . \end{aligned}$$

If α is a quasi-isomorphism so is α_* . If in addition B and C are bounded above complexes of projective R -modules then α_* is a homotopy equivalence.

(3) *If C is actually a complex of $R[x, x^{-1}, y, y^{-1}]$ -modules, then there is an $R[x, x^{-1}, y, y^{-1}]$ -linear quasi-isomorphism $\mathcal{T}(y, x; 0) \longrightarrow C$. If C is a bounded above complex of projective modules over the LAURENT ring $R[x, x^{-1}, y, y^{-1}]$ then C and $\mathcal{T}(y, x; 0)$ are homotopy equivalent.*

Proof. (1) First we calculate

$$\begin{aligned} d_C h(fAg - gAf) + h(fAg - gAf)d_C &= hf(d_C A + Ad_C)g - hg(d_C A + Ad_C)f \\ &= hf(h - \text{id})g - hg(h - \text{id})f = hfhg - hghf \end{aligned}$$

(recall that $fg = gf$) so that $h(fAg - gAf)$ is a homotopy from $(hf)(hg)$ to $(hg)(hf)$ as claimed. Next, we need to check that Φ defines a cochain map. Let $\hat{d}_{\mathcal{T}}$ denote the differential of $\mathcal{T}(hf, hg; h(fAg - gAf))$; it is given by a matrix similar to (6) with f and g replaced by hf and hg , respectively, and the homotopy H replaced by $h(fAg - gAf)$. We need to verify that $\hat{d}_{\mathcal{T}}\Phi = \Phi d_{\mathcal{T}}$. The calculation is tedious but straightforward; we concentrate on the first entry in the fourth row of the product, leaving the others as

exercises to the reader. In the case at hand we have to show that the two sums

$$\begin{aligned} (\Phi d_{\mathcal{T}})_{4,1} &= h(fAg - gAf)Ad_C \otimes \text{id} + hfAg \otimes \text{id} - hfA \otimes x \\ &\quad - hgAf \otimes \text{id} + hgA \otimes y \end{aligned}$$

on the one hand, and

$$\begin{aligned} (\hat{d}_{\mathcal{T}}\Phi)_{4,1} &= h(fAg - gAf)h \otimes \text{id} - hfhgA \otimes \text{id} + hgA \otimes y \\ &\quad + hghfA \otimes \text{id} - hfA \otimes x + d_C h(fAg - gAf)A \otimes \text{id} \end{aligned}$$

on the other hand, are equal. By cancelling equal terms and re-arranging, remembering that f , g and h commute with d_C , this amounts to verifying the equality

$$\begin{aligned} hfAgAd_C - hfd_CAgA + hgd_CAfA - hgAfAd_C \\ = hgAf - hfAg - hfhgA + hghfA + hfAgh - hgAfh . \end{aligned}$$

Since f and g are cochain maps we can now add the zero-term

$$hfA(gd_C - d_Cg)A + hgA(d_Cf - fd_C)A$$

on the left hand side, and collect terms on the right, to get the equivalent equation

$$\begin{aligned} hfAg(Ad_C + d_CA) - hf(Ad_C + d_CA)gA \\ + hg(d_CA + Ad_C)fA - hgAf(Ad_C + d_CA) \\ = hfAg(h - \text{id}) - hgAf(h - \text{id}) - (hfhg - hghf)A \end{aligned}$$

which is satisfied since $Ad_C + d_CA = h - \text{id}$ and $fg = gf$.

Now Φ is a lower triangular matrix which has quasi-isomorphisms on its diagonal, and both the differentials $d_{\mathcal{T}}$ and $\hat{d}_{\mathcal{T}}$ are lower triangular as well. It follows from Lemma II.1.3 that Φ is a quasi-isomorphism. — If C is a bounded above complex of projective R -modules, then both source and target of Φ are bounded above complexes of projective $R[x, x^{-1}, y, y^{-1}]$ -modules, and it is well known that a quasi-isomorphism of such complexes must be a homotopy equivalence.

(2) This is similar to the proof of part (1). We omit the details.

(3) Abbreviate $R[x, x^{-1}, y, y^{-1}]$ by L . The map

$$\gamma: C \otimes_R L \longrightarrow C, \quad z \otimes p \mapsto zp$$

induces a cochain complex map $\mathcal{T}(y, x; 0) \longrightarrow C$. To show that this is a quasi-isomorphism it is enough to verify that its mapping cone is acyclic; but this mapping cone is, up to isomorphism, the total complex of the double complex

$$\begin{array}{ccccc} & & C \otimes_R L & & \\ & & \oplus & & \\ C \otimes_R L & \xrightarrow{\alpha} & & \xrightarrow{\beta} & C \otimes_R L \xrightarrow{\gamma} C \\ & & C \otimes_R L & & \end{array}$$

with C sitting in column $p = 1$, cf. (7) with $f = y$ and $g = x$, so that it is sufficient (using Lemma II.2.2) to demonstrate that this double complex has exact rows.

Since L is a free R -module, an element $z \in C \otimes_R L$ can be expressed uniquely as

$$z = \sum_{i,j \in \mathbb{Z}} m_{i,j} \otimes x^i y^j,$$

with $m_{i,j} = 0$ for almost all pairs (i, j) . We say that z has *x-amplitude in the interval $[a, b]$* if $m_{i,j} = 0$ for $i \notin [a, b]$, and that z has *y-amplitude in the interval $[a, b]$* if $m_{i,j} = 0$ for $j \notin [a, b]$. The *support* of z is the set of all pairs (i, j) with $m_{i,j} \neq 0$.

To show that α is injective we note that

$$\alpha(z) = \left(\begin{array}{c} \sum_{i,j \in \mathbb{Z}} (m_{i,j} x \otimes x^i y^j - m_{i,j} \otimes x^{i+1} y^j) \\ \sum_{i,j \in \mathbb{Z}} (-m_{i,j} y \otimes x^i y^j + m_{i,j} \otimes x^i y^{j+1}) \end{array} \right),$$

so $\alpha(z) = 0$ implies in particular

$$m_{i,j} x - m_{i-1,j} = 0 \tag{8}$$

for all pairs (i, j) . The support of z , if non-empty, is well ordered by the order $(i, j) < (k, l)$ whenever $j < l$, or $j = l$ and $i < k$. When (i, j) is the element of the support which is minimal with respect to this order we have $m_{i,j} \neq 0$, by definition of support, and $m_{i-1,j} = 0$. Using (8) we have $m_{i,j} x = 0$, and since x is a unit in L we arrive at the contradiction $m_{i,j} = 0$. We conclude that $\alpha(z) = 0$ forces $z = 0$.

To show that $\text{im } \alpha = \ker \beta$ we will take an element (z_1, z_2) of $\ker \beta$ and show that it can be reduced to 0 by subtracting a sequence of elements of $\text{im } \alpha \subseteq \ker \beta$. This clearly implies that $(z_1, z_2) \in \text{im } \alpha$ as required.

So let $(z_1, z_2) \in \ker \beta$, where

$$z_1 = \sum_{i,j \in \mathbb{Z}} m_{i,j} \otimes x^i y^j \quad \text{and} \quad z_2 = \sum_{i,j \in \mathbb{Z}} n_{i,j} \otimes x^i y^j.$$

Choose integers $a \leq b$ such that both z_1 and z_2 have *x-amplitude in $[a, b]$* . Let $k \in [a, b]$ be maximal such that $m_{k,j} \neq 0$, for some j , so that z_1 actually has *x-amplitude in $[a, k]$* . If $k > a$ we define

$$u = \sum_{j \in \mathbb{Z}} m_{k,j} \otimes x^{k-1} y^j \in C \otimes_R L$$

and set $(z'_1, z'_2) = (z_1, z_2) - \alpha(u)$. Then z'_1 has *x-amplitude in $[a, k-1]$* while the *x-amplitude of z'_2* remains in $[a, b]$. The element (z'_1, z'_2) will be in $\text{im } \alpha$ if and only if $(z_1, z_2) \in \text{im } \alpha$.

By iteration, we may thus assume that our initial pair (z_1, z_2) is such that the *x-amplitude of z_1* is in $\{a\} = [a, a]$, and z_2 has *x-amplitude in $[a, b]$* for some $b \geq a$. Note that then $\beta((z_1, 0))$ will also have *x-amplitude in $\{a\}$* , and that $\beta((z_1, z_2))$ will have *x-amplitude in $[a, b+1]$* .

Write $\beta((z_1, z_2)) = \sum_{i=a}^{b+1} \sum_{j \in \mathbb{Z}} r_{i,j} \otimes x^i y^j$. Then $r_{i+1,j} = n_{i,j}$ for every $i \geq a$ as $m_{i+1,\ell} = 0$ for any ℓ . Since $\beta((z_1, z_2)) = 0$ we must have $n_{i,j} = r_{i+1,j} = 0$ for every $i \geq a$, so that in fact $z_2 = 0$. This in turn implies that $\beta((z_1, 0)) = 0$ so that $m_{a,\ell+1} \cdot y - m_{a,\ell} = r_{a,\ell+1} = 0$. If z_1 is non-zero, we let ℓ be maximal with $m_{a,\ell} \neq 0$. Then $m_{a,\ell+1} = 0$ and consequently

$0 = r_{a,\ell+1} = m_{a,\ell}$, a contradiction. We conclude that $(z_1, z_2) = 0 \in \text{im } \alpha$ as required.

To show $\text{im } \beta = \ker \gamma$ we proceed in a similar manner. For a given $z \in \ker \gamma$ may be converted, by subtracting elements of $\text{im } \beta$, to an element of the form $m_{a,a} \otimes x^a y^a$ which lies in $\text{im } \beta$ if and only if z does. The conversion involves a systematic reduction of both the x -amplitude and the y -amplitude; we omit the details. Now $\gamma(z) = 0$ implies $m_{a,a} x^a y^a = \gamma(m_{a,a} \otimes x^a y^a) = 0$, and since neither x nor y is a zero divisor this gives us $m_{a,a} \otimes x^a y^a = 0 \in \text{im } \beta$.

Finally, let us remark that γ is surjective since $\gamma(p \otimes 1) = p$. \square

The mapping 2-torus analogue. For later use we record a construction somewhat similar to the mapping 2-torus. Suppose we have a cochain complex C of R -modules with differential denoted d_C , and two self maps $f, g: C \longrightarrow C$ which commute up to homotopy. Suppose moreover we are given a specific choice of homotopy $H: fg \simeq gf$ such that $d_C H + H d_C = fg - gf$. To such data we associate a cochain complex of R -modules, the *mapping 2-torus analogue* $\mathcal{A}(f, g; H)$. The module in degree n is the direct sum

$$\mathcal{A}(f, g; H)^n = C^{n+2} \oplus (C^{n+1} \oplus C^{n+1}) \oplus C^n, \quad (9)$$

and the boundary map $d_{\mathcal{A}}: \mathcal{A}(f, g; H)^n \longrightarrow \mathcal{A}(f, g; H)^{n+1}$ is given by the matrix

$$d_{\mathcal{A}} = \begin{pmatrix} d_C & 0 & 0 & 0 \\ -g & -d_C & 0 & 0 \\ f & 0 & -d_C & 0 \\ H & f & g & d_C \end{pmatrix}. \quad (10)$$

Remark III.7.3. (a) If $fg = gf$ we may choose $H = 0$, and the mapping 2-torus analogue $\mathcal{A}(f, g; 0)$ is nothing but the total complex of the twofold chain complex

$$C \xrightarrow{\alpha} C \oplus C \xrightarrow{\beta} C$$

concentrated in columns -2 , -1 and 0 . Here the maps α and β are given by the matrices

$$\alpha = \begin{pmatrix} -g \\ f \end{pmatrix} \quad \text{and} \quad \beta = (f, g).$$

(b) Suppose in addition that one of f or g is a quasi-isomorphism. Then $\mathcal{A}(f, g; 0)$ is acyclic. For $\mathcal{A}(f, g; 0)$ can be obtained by taking iterated mapping cones in a commuting square of cochain complexes:

$$\begin{array}{ccc} C & \xrightarrow{f} & C \\ \uparrow g & & \uparrow g \\ C & \xrightarrow{f} & C \end{array}$$

That is, $\mathcal{A}(f, g; 0)$ is isomorphic to the cochain complex obtained by applying the algebraic mapping cone functor to the horizontal maps in the square first, and then again to the resulting “vertical” map (or, by symmetry, with roles of horizontal and vertical exchanged). But the mapping cone of a quasi-isomorphism is acyclic, as is the mapping cone of a map of acyclic complexes, whence the assertion.

Lemma III.7.4. (1) *Suppose that the maps f and g commute, and suppose that we are given a self map $h: C \longrightarrow C$ together with a homotopy $A: h \simeq \text{id}$ so that $d_C A + A d_C = h - \text{id}$. Then $h(fAg - gAf)$ is a homotopy from $(hf)(hg)$ to $(hg)(hf)$, and the matrix*

$$\begin{pmatrix} h & 0 & 0 & 0 \\ -hgA & h & 0 & 0 \\ hfA & 0 & h & 0 \\ h(fAg - gAf)A & -hfA & -hgA & h \end{pmatrix}$$

defines a quasi-isomorphism

$$\mathcal{A}(f, g; 0) \longrightarrow \mathcal{A}(hf, hg; h(fAg - gAf)) .$$

If C is a bounded above complex of projective modules over R then $\mathcal{A}(f, g; 0)$ and $\mathcal{A}(hf, hg; h(fAg - gAf))$ are homotopy equivalent.

(2) *Suppose that the maps f and g commute as before. Given cochain maps $\alpha: B \longrightarrow C$ and $\beta: C \longrightarrow B$ and a homotopy $A: \alpha\beta \simeq \text{id}$ so that $d_C A + A d_C = \alpha\beta - \text{id}$, the map $\beta(fAg - gAf)\alpha$ is a homotopy from $(\beta f\alpha)(\beta g\alpha)$ to $(\beta g\alpha)(\beta f\alpha)$, the map $\alpha\beta(fAg - gAf)$ is a homotopy from $(\alpha\beta f)(\alpha\beta g)$ to $(\alpha\beta g)(\alpha\beta f)$, and the diagonal matrix with entries α on the diagonal defines a cochain complex map*

$$\begin{aligned} \alpha_*: \mathcal{A}(\beta f\alpha, \beta g\alpha; \beta(fAg - gAf)\alpha) \\ \longrightarrow \mathcal{A}(\alpha\beta f, \alpha\beta g; \alpha\beta(fAg - gAf)) . \end{aligned}$$

If α is a quasi-isomorphism so is α_* . If in addition B and C are bounded above complexes of projective R -modules then α_* is a homotopy equivalence.

Proof. This is almost identical to the proof of Theorem III.7.2, but slightly easier due to the absence of tensor products. We omit the details. \square

III.8. MAPPING 2-TORI AND TOTALISATION. APPLICATIONS

Now suppose that C is an R -finitely dominated bounded above cochain complex of projective $R[x, x^{-1}, y, y^{-1}]$ -modules. We will prove now that $C \otimes_{R[x, x^{-1}, y, y^{-1}]} R((x, y))$ is acyclic.

By our hypothesis there exists a bounded cochain complex B of finitely generated projective R -modules together with mutually inverse homotopy equivalences

$$\alpha: B \longrightarrow C \quad \text{and} \quad \beta: C \longrightarrow B$$

of R -module cochain complexes. Choose a homotopy $A: \alpha\beta \simeq \text{id}_C$. By Theorem III.7.2 there are $R[x, x^{-1}, y, y^{-1}]$ -linear homotopy equivalences

$$\begin{aligned} C \longleftarrow \mathcal{T}(y, x; 0) \longrightarrow \mathcal{T}(\alpha\beta y, \alpha\beta x; \alpha\beta(yAx - xAy)) \\ \longleftarrow^{\alpha_*} \mathcal{T}(\beta y\alpha, \beta x\alpha; \beta(yAx - xAy)\alpha) =: Z \end{aligned}$$

so that $C \otimes_{R[x, x^{-1}, y, y^{-1}]} R((x, y)) \simeq Z \otimes_{R[x, x^{-1}, y, y^{-1}]} R((x, y))$.

Now note that we have isomorphisms

$$\begin{aligned} B \otimes_R R[x, x^{-1}, y, y^{-1}] &\otimes_{R[x, x^{-1}, y, y^{-1}]} R((x, y)) \\ &\cong B \otimes_R R((x, y)) \cong B((x, y)), \end{aligned}$$

the second one being due to the fact that B consists of finitely generated projective R -modules, so that Lemma III.5.2 applies. We can thus identify the n th cochain module of $Z \otimes_{R[x, x^{-1}, y, y^{-1}]} R((x, y))$ with

$$B^{n+2}((x, y)) \oplus (B^{n+1}((x, y)) \oplus B^{n+1}((x, y))) \oplus B^n((x, y)), \quad (11)$$

and the differential with the matrix

$$\begin{pmatrix} d_B((x, y)) & 0 & 0 & 0 \\ -(\beta x \alpha((x, y)) - x) & -d_B((x, y)) & 0 & 0 \\ \beta y \alpha((x, y)) - y & 0 & -d_B((x, y)) & 0 \\ \beta(yAx - xAy)\alpha((x, y)) & \beta y \alpha((x, y)) - y & \beta x \alpha((x, y)) - x & d_B((x, y)) \end{pmatrix}.$$

(The symbol d_B denotes the differential of B . Note that the letter “ x ” in the term $\beta x \alpha$ denotes a self map of C , while the symbol x by itself denotes a self map of $B((x, y))$; similarly with y in place of x .)

This cochain complex arises as the realisation of a triple complex. Indeed, for $x, y, z \in \mathbb{Z}$ let

$$T^{x, y, z} = B^{x+y+z+2} \oplus B^{x+y+z+1} \oplus B^{x+y+z+1} \oplus B^{x+y+z}$$

and define differentials

$$\begin{aligned} d_x: T^{x, y, z} &\longrightarrow T^{x+1, y, z}, & (r, s, t, u) &\mapsto (0, r, 0, -t) \\ d_y: T^{x, y, z} &\longrightarrow T^{x, y+1, z}, & (r, s, t, u) &\mapsto (0, 0, -r, -s) \end{aligned}$$

as well as

$$d_z = \begin{pmatrix} d_B & 0 & 0 & 0 \\ -\beta x \alpha & -d_B & 0 & 0 \\ \beta y \alpha & 0 & -d_B & 0 \\ H & \beta y \alpha & \beta x \alpha & d_B \end{pmatrix} : T^{x, y, z} \longrightarrow T^{x, y, z+1}$$

where $H = \beta(yAx - xAy)\alpha$. That is, $T^{x, y, *}$ is $\mathcal{A}(\beta y \alpha, \beta x \alpha, H)$ shifted down $x + y$ times. — With these definitions $\text{Tot}(T^{*, *, *})$ is precisely the chain complex

$$Z \otimes_{R[x, x^{-1}, y, y^{-1}]} R((x, y))$$

under the identification made above (11). Now the complex $\mathcal{A}(\beta y \alpha, \beta x \alpha; H)$ is acyclic by Lemma III.7.4 and Remark III.7.3 (b); in more detail, we have a chain of homotopy equivalences

$$\mathcal{A}(\beta y \alpha, \beta x \alpha; H) \xrightarrow{\simeq} \mathcal{A}(\alpha \beta y, \alpha \beta x; \alpha \beta (yAx - xAy)) \xleftarrow{\simeq} \mathcal{A}(y, x; 0)$$

with the cochain complex on the right being acyclic. We conclude from Proposition III.6.1 that $Z \otimes_{R[x, x^{-1}, y, y^{-1}]} R((x, y))$ is acyclic.

Corollary III.8.1. *If C is R -finitely dominated, all four of the chain complexes listed in (1b) of Theorem I.1.2 are acyclic.*

Proof. For $C \otimes_R R((x, y))$ this has just been shown. The other cases follow by a “change of coordinates” replacing x by x^{-1} and/or y by y^{-1} . \square

Part IV. Acyclicity implies finite domination

We begin by giving a quick proof of the second implication. Suppose that the chain complexes listed in (1a) and (1b) are acyclic. We will use Theorem I.2.2, applied to the case $G = \mathbb{Z}^2$ and $N = 0$, to conclude that C is R -finitely dominated. A non-trivial character $\chi: G \rightarrow \mathbb{R}$ can be identified with a non-zero vector $\chi = (a, b) \in \mathbb{R}^2$, using the standard inner product of \mathbb{R}^2 . If $\chi = (a, 0)$, for $a > 0$, then $\widehat{RG}_\chi = R[y, y^{-1}]((x))$ so that acyclicity in (1a) implies acyclicity of $C \otimes_{R[G]} \widehat{RG}_\chi$. The situation is similar for $a < 0$ with $\widehat{RG}_\chi = R[y, y^{-1}]((x^{-1}))$, or for $\chi = (0, b)$ with $b \neq 0$ in which case $\widehat{RG}_\chi = R[x, x^{-1}]((y^{\text{sgn } b}))$.

For $\chi = (a, b)$ with $a, b \neq 0$, acyclicity in (1b) implies acyclicity of $C \otimes_{R[G]} \widehat{RG}_\chi$. For example, if $a, b > 0$ then \widehat{RG}_χ contains $R((x, y))$ so that

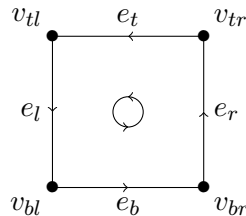
$$C \otimes_{R[G]} \widehat{RG}_\chi \cong C \otimes_{R[G]} R((x, y)) \otimes_{R((x, y))} \widehat{RG}_\chi \simeq 0$$

as the chain complexes in (1b) are actually contractible. In general, for $a, b \neq 0$ we have $\widehat{RG}_\chi \supset R((x^{\text{sgn } a}, y^{\text{sgn } b}))$, and the argument applies *mutatis mutandis*.

We will nevertheless give a different and new proof of the implication in the remainder of the paper. We eschew the use of controlled algebra and inductive arguments in favour of standard homological algebra combined with homotopy-theoretic arguments. While not short our proof is conceptually simple, and lends itself to generalisations for LAURENT rings with many indeterminates. These generalisations can be done at the expense of introducing more elaborate combinatorics; this topic will be taken up in a separate paper.

IV.1. DIAGRAMS INDEXED BY THE FACE LATTICE OF A SQUARE

Let S denote the square $[-1, 1]^2$, a convex polytope in \mathbb{R}^2 . In what follows a face of S will always assumed to be non-empty, unless specified otherwise. We will orient the edges of S , and S itself, counter-clockwise as indicated in the following picture which also shows our labelling for the faces of S :



The orientation gives us a choice of incidence numbers $[F : G] \in \{-1, 0, 1\}$ for all faces F and G of S . For example, $[e : S] = 1$ for every edge e , while

$$[v_{bl} : e_b] = -1 \quad \text{and} \quad [v_{br} : e_b] = 1$$

for the bottom edge e_b . By convention, we have $[\emptyset : v] = 1$ for every vertex v .

For each face F we define the ring A_F to be the monoid R -algebra

$$A_F = R[T_F \cap \mathbb{Z}^2]$$

where T_F is the tangent cone $T_F = \text{span}_{\geq 0}\{s - f \mid s \in S, f \in F\}$ of S at F . For example, the ring A_S is the LAURENT polynomial ring in two indeterminates $R[x, x^{-1}, y, y^{-1}]$, while $A_{v_{br}} = R[x^{-1}, y]$ is a polynomial ring. In general we have $A_F \subseteq A_G$ if $F \subseteq G$.

For a face F of S we let $v_F \in \mathbb{Z}^2$ denote its barycentre, or centre of mass. For a pair of faces $F \subset G$, the difference $v_G - v_F$ represents a monomial m_{FG} in the centre of A_F ; for example, $m_{v_{bl}e_b} = x$, $m_{v_{tl}S} = xy^{-1}$ and $m_{e_rS} = x^{-1}$. The algebra A_G is the localisation of A_F by m_{FG} :

$$\text{For every element } u \in A_G \text{ there is } k_0 \geq 0 \text{ such that} \quad (12)$$

$$m_{FG}^k u \in A_F \text{ for all } k \geq k_0.$$

We also choose $m_{\emptyset F}$ to be the monomial represented by the lattice point v_F , for every face F of S , so that $m_{\emptyset e_l} = x^{-1}$ and $m_{\emptyset v_{tl}} = x^{-1}y$. Clearly,

$$m_{FH} = m_{FG} \cdot m_{GH} \text{ for all triples of (possibly empty) faces} \quad (13)$$

$$F \subseteq G \subseteq H \text{ of } S.$$

When F is a codimension-1 face of G , the monomial m_{FG} is the indeterminate $z \in \{x, x^{-1}, y, y^{-1}\}$ such that $\{z, z^{-1}\} \subset A_G$ and $z^{-1} \notin A_F$.

We denote by $\mathcal{F}(S)$ the join semi-lattice of non-empty faces of S ordered by inclusion, and by $\text{Ch}(R\text{-Mod})$ the category of cochain complexes of right R -modules.

- Definition IV.1.1.** (1) An $\mathcal{F}(S)$ -diagram X of cochain complexes is a functor $X: \mathcal{F}(S) \longrightarrow \text{Ch}(R\text{-Mod})$, $F \mapsto X_F$ equipped with extra data giving each X_F the structure of a cochain complex of (right) A_F -modules, such that for each pair of non-empty faces $F \subseteq G$ of S the corresponding structure map $s_{FG}: X_F \longrightarrow X_G$ is in fact A_F -linear.
- (2) The $\mathcal{F}(S)$ -diagram X is called *bounded* if the cochain complex X_F is bounded for every face F of S .
- (3) An $\mathcal{F}(S)$ -diagram X is said to be *quasi-coherent* if for each pair of non-empty faces $F \subset G$ of S the adjoint map

$$X_F \otimes_{A_F} A_G \longrightarrow X_G$$

of s_{FG} is an isomorphism of A_G -module cochain complexes.

As a matter of notation, given a functor $X: \mathcal{F}(S) \longrightarrow \text{Ch}(R\text{-Mod})$ we denote the differential of the cochain complex X_F by d_F .

An example of a quasi-coherent $\mathcal{F}(S)$ -diagram (concentrated in a single cochain degree) is the diagram $D(k)$ shown in Fig. 1. We have $D(k)_F = A_F$, and the structure maps s_{FG} are the inclusions followed by multiplication with the monomial m_{FG}^{-k} .

Construction IV.1.2. Let C be a bounded cochain complex consisting of finitely generated free $R[x, x^{-1}, y, y^{-1}]$ -modules, with a specified finite basis $B_a \subset C^a$ so that we have an identification $C_a = \bigoplus_{B_a} R[x, x^{-1}, y, y^{-1}]$.

$$\begin{array}{ccccc}
R[x, y^{-1}] & \xrightarrow{x^{-k}} & R[x, x^{-1}, y^{-1}] & \xleftarrow{x^k} & R[x^{-1}, y^{-1}] \\
\downarrow y^k & & \downarrow y^k & & \downarrow y^k \\
R[x, y, y^{-1}] & \xrightarrow{x^{-k}} & R[x, x^{-1}, y, y^{-1}] & \xleftarrow{x^k} & R[x^{-1}, y, y^{-1}] \\
\uparrow y^{-k} & & \uparrow y^{-k} & & \uparrow y^{-k} \\
R[x, y] & \xrightarrow{x^{-k}} & R[x, x^{-1}, y] & \xleftarrow{x^k} & R[x^{-1}, y]
\end{array}$$

FIGURE 1. The diagram $D(k)$

We will show how to construct a sequence k_i of non-negative integers and a bounded $\mathcal{F}(S)$ -diagram Y with $Y_S = C$ such that $Y^j \cong \bigoplus_{B_j} D(k_j)$ for every $j \in \mathbb{Z}$.

We start by making the default convention that all numbers k_j which are not explicitly defined are taken to be 0, and similarly that all diagrams Y^j and all boundary maps that are not explicitly defined are taken to be trivial.

If C is the 0-complex we can choose the trivial diagram with all values 0. Otherwise, C^i is trivial for $i < a$, say, and $C^a \neq 0$. For a face F of S we define $Y_F^a = A_F[B_a]$ to be the free A_F -module on the basis B_a , and for a pair of faces $F \subseteq G$ we define the structure map $s_{FG}^a: Y_F^a \rightarrow Y_G^a$ as the map induced by the inclusion $A_F \subseteq A_G$ and the identity on basis elements. In other words, we define $Y^a = \bigoplus_{B_a} D(0)$. We set $k_a = 0$.

If C^a is the only non-trivial module in C the construction process terminates. Otherwise, we define Y^{a+1} by setting $Y_F^{a+1} = A_F[B_{a+1}]$, the free A_F -module with basis B_{a+1} . The structure maps will be of the form “inclusion followed by multiplication with the monomial $m_{FG}^{k_{a+1}}$ ”, for some suitable k_{a+1} to be determined presently. In other words, we will have $Y^{a+1} = \bigoplus_{B_{a+1}} D(k_{a+1})$.

To find a suitable value of k_{a+1} recall that we also need to construct differentials $d_F^{a+1}: Y_F^a \rightarrow Y_F^{a+1}$, for all the faces F of S , which are compatible with the structure maps. We choose $d_S^a = d_C^a$ to be the given differential of the cochain complex C . For a proper face F and fixed basis element $b \in B_a$ the image $w_{F,b}$ of b under the composition

$$\bigoplus_{B_a} A_F = Y_F^a \xrightarrow{s_{FS}^a} Y_S^a = C^a \xrightarrow{d_C^a} C^{a+1} = Y_S^{a+1} = \bigoplus_{B_{a+1}} A_S$$

is such that $m_{FS}^k w_{F,b} \in \bigoplus_{B_{a+1}} A_F = Y_F^{a+1}$ for large k , by (12). We choose a number $k_{a+1} = k$ large enough to work for all $b \in B_a$, and all proper faces F of S .

Now we can define our A_F -linear differential d_F^a uniquely by the requirement that it sends $b \in B_a$ to $m_{FS}^{k_{a+1}} w_{F,b} \in Y_F^{a+1}$ for a proper face F of C .

We claim that this yields a map of $\mathcal{F}(S)$ -diagrams $d^{a-1}: Y^{a-1} \rightarrow Y^a$. We need to verify $s_{FG}^{a+1} \circ d_F^a = d_G^a \circ s_{FG}^a$ for faces $F \subset G$. In fact, this

equality holds for $G = S$ by construction: the basis element $b \in B_a$ is sent to $w_{F,b}$ by the composition on the right, while on the left we have $s_{FS}^{a+1} \circ d_F^a(b) = s_{FS}^{a+1}(m_{FS}^{k_{a+1}} w_{F,b}) = m_{FS}^{-k_{a+1}} m_{FS}^{k_{a+1}} w_{F,b} = w_{F,b}$ as required. If G is a proper face of S we note that, by construction and the argument just given, we have

$$s_{GS}^{a+1} \circ s_{FG}^{a+1} \circ d_F^a = s_{FS}^{a+1} \circ d_F^a = d_S^a \circ s_{FS}^a = d_S^a \circ s_{GS}^a \circ s_{FG}^a = s_{GS}^{a+1} \circ d_G^a \circ s_{FG}^a ;$$

since s_{GS}^{a+1} is injective, the claim follows.

If there are no further non-trivial entries in C the construction terminates. Otherwise, we extend from Y^j to Y^{j+1} by repeating the process above: choose a sufficiently large integer $k_{j+1} \geq 0$ so that, for each proper face F of S and each basis element $b \in B_j$, we have $m_{FS}^{k_{j+1}} w_{F,b} \in \bigoplus_{B_{j+1}} A_F = Y_F^{j+1}$, where $w_{F,b}$ is the image of b under the composition

$$\bigoplus_{B_j} A_F = Y_F^j \xrightarrow{s_{FS}^j} Y_S^j = C^j \xrightarrow{d_C^j} C^{j+1} = Y_S^{j+1} = \bigoplus_{B_{j+1}} A_S .$$

We let $Y^{j+1} = \bigoplus_{B_{j+1}} D(k_{j+1})$, and define differentials d_F^j by the requirement that they send the basis element $b \in B_j$ to $m_{FS}^{k_{j+1}} w_{F,b}$.

Since C is bounded this process terminates, and results in a quasi-coherent $\mathcal{F}(S)$ -diagram Y with $Y_S = C$. It might be worth pointing out that the composition of two differentials is the zero map as required. Indeed, for every i the structure map s_{FS}^{i+2} is injective, and we have

$$s_{FS}^{i+2} \circ d_F^{i+1} \circ d_F^i = d_C^{i+1} \circ d_C^i \circ s_{FS}^i = 0$$

(since d_C is a differential) so that $d_F^{i+1} \circ d_F^i = 0$.

IV.2. DOUBLE COMPLEXES FROM DIAGRAMS. ČECH COMPLEXES

Let P be a poset. We suppose that P is equipped with a strictly increasing *degree function* $\deg: P \rightarrow \mathbb{N}$, and an *incidence function*

$$[- : -]: P \times P \rightarrow \mathbb{Z}$$

satisfying the following properties:

- (DI1) $[x : y] = 0$ unless $x < y$ and $\deg(y) = 1 + \deg(x)$;
- (DI2) for all $x < z$ with $\deg(z) = 2 + \deg(x)$, the open interval $I(x : z) = \{y \in P \mid x < y < z\}$ is finite, and we have

$$\sum_{y \in I(x:z)} [x : y] \cdot [y : z] = 0 ;$$

- (DI3) for $z \in P$ with $\deg(z) = 1$ the set $I(< z) = \{y \in P \mid y < z\}$ is finite, and we have

$$\sum_{y \in I(<z)} [y : z] = 0 .$$

Now let $X: P \longrightarrow R\text{-Mod}$, $x \mapsto X_x$ be a diagram of R -modules. We define the ČECH complex of X , denoted $\check{\Gamma}(X)$, to be the cochain complex with

$$(\check{\Gamma}(X))^n = \bigoplus_{\substack{x \in P \\ \deg(x)=n}} X_x$$

and differential induced by the structure maps of X modified by incidence numbers $[x : y]$. It follows from (DI2) that this defines indeed a cochain complex.

More generally, for a P -indexed diagram X of cochain complexes of R -modules we define a double complex $D^{*,*}$, and define the ČECH complex of X , denoted $\check{\Gamma}(X)$, to be the totalisation:

$$\check{\Gamma}(X) = \text{Tot } D^{*,*}$$

The double complex $D^{*,*}$ is defined by saying that

- the q th row $D^{*,q}$ is the ČECH complex $\check{\Gamma}(X^q)$ of X^q , the P -indexed diagram of R -modules consisting of all the cochain level q terms of entries in X , and
- the vertical differential in column p is induced by the differentials of the cochain complexes X_x with $\deg(x) = p$, modified by the sign $(-1)^p$.

By construction, vertical and horizontal differentials anti-commute.

We will apply this construction in specific cases only. For example, for $P = \mathcal{F}(S)$ we can choose \deg to be the dimension function, and use the incidence numbers introduced at the beginning of §IV.1. This data satisfies the conditions (DI1–3) above. In particular, every $\mathcal{F}(S)$ -diagram X in the sense of definition IV.1.1 gives rise to a double complex $D^{*,*}$ of R -modules, and a cochain complex $\check{\Gamma}(X) = \text{Tot } D^{*,*}$ of R -modules. Explicitly, we have

$$D^{i,j} := \bigoplus_{\dim F=i} X_F^j \quad (14)$$

with horizontal and vertical differentials induced by

$$d_h = [F : G]_{\mathcal{S}_{FG}}, \quad d_v = (-1)^i \cdot \bigoplus_{\dim F=i} d_F$$

where $[F : G]$ are our chosen incidence numbers, and

$$\check{\Gamma}(X)^n = (\text{Tot } D^{*,*})^n = \bigoplus_{i+j=n} D^{i,j}$$

equipped with differential $d = d_h + d_v$.

IV.3. ČECH COHOMOLOGY. COMPUTATIONS

Let C and Y be as in Construction IV.1.2, and let $D^{*,*}$ be the double complex (14) used to define $\check{\Gamma}(Y)$. In particular the module $C^t = D^{2,t}$ is a free $R[x, x^{-1}, y, y^{-1}]$ -module with basis B_t . By construction, $D^{*,t}$ is the ČECH complex of a B_t -indexed direct sum of diagrams $D(k_t)$, where $k_t \geq 0$ depends on t according to Construction IV.1.2. (As before we use the symbol $D(k)$ to denote the diagram of Fig. 1. We will never refer to an object ‘ D ’ alone, so our notation should not lead to any confusion.) To compute the

cohomology of $D^{*,t}$ it is thus enough to compute the cohomology of $\check{\Gamma}(D(k))$ for $k \geq 0$.

Recall that $S = [-1, 1]^2$. We denote by kS the k -th dilate of S which is $[-k, k]^2$. By $R[kS \cap \mathbb{Z}^2]$ we mean the free R -module with basis $kS \cap \mathbb{Z}^2 = \{(i, j) \in \mathbb{Z}^2 \mid -k \leq i, j \leq k\}$, the set of lattice points in kS .

We now want to consider the following cochain complex of R -modules, an augmented version of the complex $\check{\Gamma}(Y)$:

$$0 \longrightarrow R[kS \cap \mathbb{Z}^2] \xrightarrow{d^{-1}} \check{\Gamma}(D(k))^0 \xrightarrow{d^0} \check{\Gamma}(D(k))^1 \xrightarrow{d^1} \check{\Gamma}(D(k))^2 \longrightarrow 0 \quad (15)$$

The map d^{-1} is given by multiplication with the monomials $m_{\emptyset v}^{-k}$ for the various vertices v of S , followed by an inclusion map. More explicitly, bar the zero terms this is the complex

$$R[kS \cap \mathbb{Z}^2] \xrightarrow{d^{-1}} \begin{array}{c} R[x, y] \\ \oplus \\ R[x^{-1}, y] \\ \oplus \\ R[x, y^{-1}] \\ \oplus \\ R[x^{-1}, y^{-1}] \end{array} \xrightarrow{d^0} \begin{array}{c} R[x, x^{-1}, y] \\ \oplus \\ R[x, y, y^{-1}] \\ \oplus \\ R[x^{-1}, y, y^{-1}] \\ \oplus \\ R[x, x^{-1}, y^{-1}] \end{array} \xrightarrow{d^1} R[x, x^{-1}, y, y^{-1}]$$

where d^{-1} , d^0 and d^1 are given by matrices

$$d^{-1} = \begin{pmatrix} x^k y^k \\ x^{-k} y^k \\ x^k y^{-k} \\ x^{-k} y^{-k} \end{pmatrix}, \quad d^0 = \begin{pmatrix} -x^{-k} & x^k & 0 & 0 \\ y^{-k} & 0 & -y^k & 0 \\ 0 & -y^{-k} & 0 & y^k \\ 0 & 0 & x^{-k} & -x^k \end{pmatrix}$$

$$\text{and } d^1 = (y^{-k}, x^{-k}, x^k, y^k);$$

the entries of these matrices are all of the form $[F : G] \cdot m_{FG}^{-k}$, for (possibly empty) faces $F \subset G$.

Lemma IV.3.1. *The complex (15) is exact for $k \geq 0$.*

Proof. It is easy to see that d^1 is surjective; for example, we can write any element $p \in R[x, x^{-1}, y, y^{-1}]$ as a sum $p = p_+ + p_-$ where $p_+ \in R[x, x^{-1}][y]$ and $p_- \in y^{-1}R[x, x^{-1}][y^{-1}]$, and note that d^1 maps the quadruple $(y^k p_+, 0, 0, y^{-k} p_-)$ to p .

To show that the complex is exact at $\check{\Gamma}(D(k))^1$ for a given cocycle $e_1 \in \ker(d^1)$ we will construct an explicit cochain $e_0 \in \check{\Gamma}(D(k))^0$ for which $d^0(e_0) = e_1$.

Fix exponents i and j . On the one hand, the coefficient of the monomial $x^i y^j$ in $d^1(e_1) = 0$ is zero. On the other hand, we can systematically work out which terms of the components of e_1 contribute to the coefficient of $x^i y^j$ in $d^1(e_1)$, as follows.

Case 1: $|i| > k$ and $|j| > k$. The coefficient of $x^i y^j$ receives contribution from exactly two of the four components of e_1 . By symmetry we may assume

that $i, j > k$ (the other cases being similar); then the first and second components of e_1 must contain terms of the form $ax^i y^{j+k}$ and $bx^{i+k} y^j$, respectively, with $a + b = 0$. We set $z_{i,j} = (-ax^{i+k} y^{j+k}, 0, 0, 0)$ and note that $d^0(z_{i,j}) = (ax^i y^{j+k}, bx^{i+k} y^j, 0, 0)$.

Case 2: $|i| \leq k$ and $|j| > k$. The coefficient of $x^i y^j$ receives contributions from three of the components of e_1 . Let us again assume that both i and j are non-negative. Then the first three components of e_1 must contain terms of the form $ax^i y^{j+k}$, $bx^{i+k} y^j$ and $cx^{i-k} y^j$, respectively, with $a + b + c = 0$. We set $z_{i,j} = (bx^{i+k} y^{j+k}, -cx^{i-k} y^{j+k}, 0, 0)$ and note that $d^0(z_{i,j}) = (ax^i y^{j+k}, bx^{i+k} y^j, cx^{i-k} y^j, 0)$.

Case 3: $|i| > k$ and $|j| \leq k$. This is dealt with in a manner similar to the previous case.

Case 4: $|i| \leq k$ and $|j| \leq k$. The coefficient of $x^i y^j$ receives contributions from all four components of e_1 , which must contain terms of the form $ax^i y^{j+k}$, $bx^{i+k} y^j$, $cx^{i-k} y^j$ and $dx^i y^{j-k}$, respectively, with $a + b + c + d = 0$. We choose

$$z_{i,j} = ((b+d)x^{i+k} y^{j+k}, -cx^{i-k} y^{j+k}, dx^{i+k} y^{j-k}, 0)$$

and note that $d^0(z_{i,j}) = (ax^i y^{j+k}, bx^{i+k} y^j, cx^{i-k} y^j, dx^i y^{j-k})$. (Unlike before, the definition of $z_{i,j}$ does involve a choice between many alternatives. The one given will do.)

Now define $e_0 = \sum_{i,j \in \mathbb{Z}} z_{i,j}$; this is a finite sum as only finitely many of the $z_{i,j}$ can be non-zero. By construction we have $d^0(e_0) = e_1$ so that $\text{im}(d^0) = \ker(d^1)$ as required.

We now show that $\ker(d^0)$ coincides with the image of d^{-1} . An element e_0 of $\check{\Gamma}(D(k))^0$ is of the form

$$e_0 = \left(\sum_{i,j \geq 0} a_{ij} x^i y^j, \sum_{i \leq 0 \leq j} b_{ij} x^i y^j, \sum_{j \leq 0 \leq i} c_{ij} x^i y^j, \sum_{i,j \leq 0} d_{ij} x^i y^j \right),$$

with all sums being finite. If $d^0(e_0) = 0$, that is, if $e_0 \in \ker d^0$, then in particular (by considering the first component)

$$-x^{-k} \cdot \sum_{i,j \geq 0} a_{ij} x^i y^j + x^k \cdot \sum_{i \leq 0 \leq j} b_{ij} x^i y^j = 0 \in R[x, x^{-1}, y].$$

This implies for all $j \geq 0$ that

$$\begin{aligned} a_{ij} &= b_{i-2k,j} && \text{for } 0 \leq i \leq 2k, \\ a_{ij} &= 0 && \text{for } i > 2k, \\ b_{ij} &= 0 && \text{for } i < -2k. \end{aligned}$$

Considering the other components of $d^0(e_0)$ in a similar manner shows that all coefficients with subscripts $|i| > 2k$ or $|j| > 2k$ vanish, and that

$$\begin{aligned} a_{ij} &= c_{i,j-2k} && \text{for } 0 \leq j \leq 2k, i \geq 0, \\ b_{ij} &= d_{i,j-2k} && \text{for } 0 \leq j \leq 2k, i \leq 0, \\ c_{ij} &= d_{i,j-2k} && \text{for } 0 \leq j \leq 2k, j \leq 0. \end{aligned}$$

Now we choose

$$e_{-1} = \sum_{-k \leq i, j \leq k} a_{i+k, j+k} x^i y^j \in R[kS \cap \mathbb{Z}^2],$$

and our characterisation of the coefficients of e_0 above immediately gives that $d^{-1}(e_{-1}) = e_0$.

Finally, it remains to observe that d^{-1} is injective; in fact, each of the four components of d^{-1} is injective by itself. \square

Corollary IV.3.2. *The complex $D^{*,t}$ is quasi-isomorphic to the finitely generated free R -module (considered as a cochain complex concentrated in degree 0)*

$$\bigoplus_{B_t} R[k_t S \cap \mathbb{Z}^2]$$

via the map d^{-1} from the preceding lemma. \square

Proposition IV.3.3. *Let C and Y be as in Construction IV.1.2. There is a quasi-isomorphism $\chi: B' \longrightarrow \check{\Gamma}(Y)$ from a bounded cochain complex B' of finitely generated free R -modules with $B'^i \cong \bigoplus_{B_i} R[k_i S \cap \mathbb{Z}^2]$.*

Proof. Let $D^{*,*}$ denote the right half-plane double complex (14) associated to Y as at the beginning of this subsection. We define a new double complex $E^{*,*}$ which agrees with $D^{*,*}$ everywhere except that in column -1 we put the free R -modules $E^{-1,t} = \bigoplus_{B_t} R[k_t S \cap \mathbb{Z}^2]$, with vertical differential induced by the negative of the differential of Y and horizontal differential d^{-1} as above. The resulting double complex has exact rows, by Corollary IV.3.2, hence the induced map $\chi: E^{-1,*} \longrightarrow \check{\Gamma}(Y) = \text{Tot } D^{*,*}$ is a quasi-isomorphism by Lemma II.2.3. (To apply the Lemma we may need to re-index C and Y temporarily to make sure that all non-zero entries live in non-negative cochain degrees.) \square

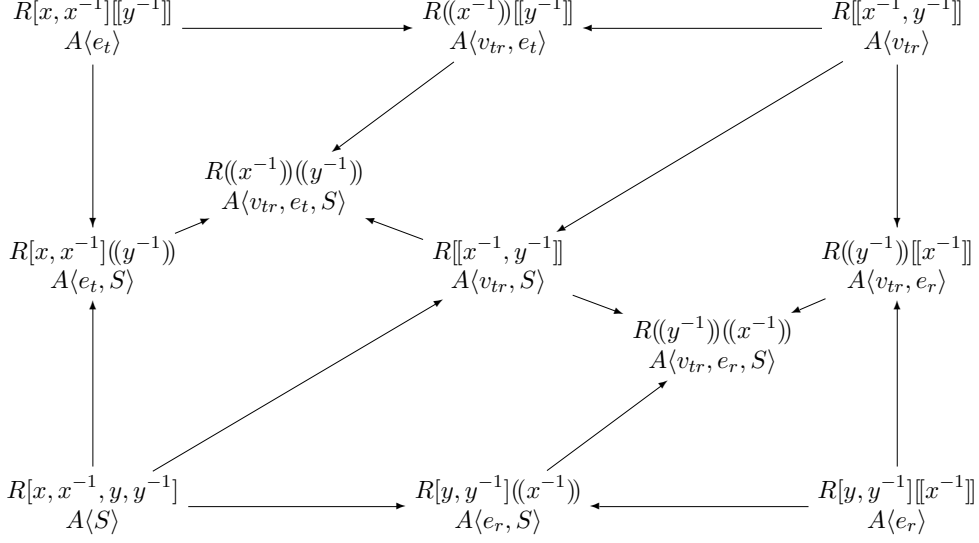
IV.4. THE NERVE OF THE SQUARE. MORE NOVIKOV RINGS

In this section we will describe diagrams involving both the NOVIKOV rings from Part III along with other rings we will describe shortly. For each face F we will construct a diagram E_F , the ČECH complex of which will be quasi-isomorphic to A_F considered as a cochain complex concentrated in degree zero.

Definition IV.4.1. For a non-empty face F of the square S we make the following definitions.

- (1) The *star of F* , denoted $\text{st}(F)$, is the set of faces of S which contain F .
- (2) We will denote by \mathcal{N}_F the *nerve of $\text{st}(F)$* , the simplicial complex on the vertex set $\text{st}(F)$ in which the faces are given by the sequences of strict inclusions of elements of $\text{st}(F)$. The faces of \mathcal{N}_F will also referred to as *flags* of faces of S .

For each face τ of \mathcal{N}_F we will define a ring $A\langle\tau\rangle$. Note that τ can occur in \mathcal{N}_F for many faces F ; the definition of $A\langle\tau\rangle$ does not depend on F , however, which is reflected by the notation.

FIGURE 2. The diagram E_v for $v = v_{tr} = (1, 1)$

Definition IV.4.2. By our previous definitions we have $A_{v_{bl}} = R[x, y]$, $A_{e_b} = R[x, x^{-1}, y]$ and $A_S = R[x, x^{-1}, y, y^{-1}]$. We now define the following rings:

$$\begin{aligned} A\langle v_{bl} \rangle &= R[[x, y]] & A\langle v_{bl}, S \rangle &= R((x, y)) \\ A\langle e_b \rangle &= R[x, x^{-1}][[y]] & A\langle e_b, S \rangle &= R[x, x^{-1}]((y)) \\ A\langle v_{bl}, e_b \rangle &= R((x))[[y]] & A\langle v_{bl}, e_b, S \rangle &= R((x))((y)) \end{aligned}$$

Also, we define $A\langle S \rangle = A_S$. If e_b is replaced by e_l throughout, we have the same definitions with x and y swapped. To replace the subscript “ l ” by “ r ” everywhere we replace x by x^{-1} , and finally to replace the subscript “ b ” by “ t ” everywhere we replace y by y^{-1} .

This defines a ring $A\langle \tau \rangle$ for any non-empty flag τ of faces of S . For example, we have

$$\begin{aligned} A\langle v_{br} \rangle &= R[[x^{-1}, y]] \text{ and} \\ A\langle v_{tr}, e_r \rangle &= R((y^{-1})][[x^{-1}]] . \end{aligned}$$

For each face F of the square we have a diagram of R -modules

$$E_F: \mathcal{N}_F \longrightarrow R\text{-Mod}, \quad \tau \mapsto A\langle \tau \rangle$$

with maps given by inclusion of rings. Here and elsewhere we consider \mathcal{N}_F as a poset ordered by inclusion of flags.

We first exhibit the case when $F = \{v\}$ is a vertex. The star of v is the set $\{v, e_x, e_y, S\}$ where e_x and e_y are the two edges incident to v . Specifically we take $v = v_{tr} = (1, 1)$, $e_x = e_r$ to be the right vertical edge of S , and $e_y = e_t$ to be the top horizontal edge (the situation for the other vertices will be similar). The diagram E_v is given in Fig. 2. The entries are specified using both the abstract notation $A\langle \tau \rangle$ as well as the concrete NOVIKOV rings.

For the case when F is an edge we take $F = e_x$ as defined above (the other cases being similar), and note that the diagram E_{e_x} appears as a restriction

of E_v since \mathcal{N}_{e_x} is an order ideal of \mathcal{N}_v . Explicitly, E_{e_r} looks like this:

$$\begin{array}{ccccc} R[x, x^{-1}, y, y^{-1}] & \longrightarrow & R[y, y^{-1}](x^{-1}) & \longleftarrow & R[y, y^{-1}][x^{-1}] \\ A\langle S \rangle & & A\langle e_r, S \rangle & & A\langle e_r \rangle \end{array} \quad (16)$$

Finally, if F is the square S itself, then the nerve \mathcal{N}_S of $\text{st}(S)$ is just $\{S\}$ and the diagram E_S consists of only $A_S = A\langle S \rangle$.

Back to general F , we equip \mathcal{N}_F with degree and incidence functions in the sense of §IV.2: the degree function is given by the (simplicial) dimension, which assigns to a flag with $k + 1$ entries dimension k , and the standard simplicial incidence numbers. To explain the latter, note that a flag τ is totally ordered, so we can let $d_i(\tau)$ denote the flag obtained by omitting the i th entry ($0 \leq i \leq \dim \tau$) and set $[d_i(\tau) : \tau] = (-1)^i$; all other incidence numbers vanish. — The diagram E_F then has an associated ČECH complex $\check{\Gamma}(E_F)$. Explicitly,

$$\check{\Gamma}(E_F)^t = \bigoplus_{\substack{\tau \in \mathcal{N}_F \\ \dim \tau = t}} A\langle \tau \rangle$$

with differential induced by

$$A\langle \tau \rangle \xrightarrow{[\tau : \mu]} A\langle \mu \rangle .$$

IV.5. DECOMPOSING DIAGRAMS. ČECH COHOMOLOGY CALCULATIONS

We keep the notation from §IV.4. For a fixed face F of S , the R -module diagram E_F is in fact a diagram of A_F -modules: all its entries contain A_F as a subring, and all structure maps are A_F -linear. In particular, its ČECH complex $\check{\Gamma}(E_F)$ is an A_F -module complex. Moreover, by the property (DI3) of incidence numbers, the inclusion maps of subrings assemble to a map of A_F -module cochain complexes

$$\sigma_F : A_F \longrightarrow \check{\Gamma}(E_F) , \quad (17)$$

where we consider the left hand side as a cochain complex concentrated in degree 0 as usual.

Lemma IV.5.1. *The map σ_F is a quasi-isomorphism.*

Proof. We note that there is nothing to prove in case $F = S$ as $\check{\Gamma}(E_S) = A\langle S \rangle = A_S$ and $\sigma_S = \text{id}_{A_S}$.

Next suppose that F is an edge of S . We will treat the case $F = e_r = \{1\} \times [-1, 1]$ only, the other cases are identical apart from possible coordinate changes replacing an indeterminate x or y by its inverse, or swapping their roles. — The diagram E_{e_r} is depicted in (16). Considered as a diagram of R -modules it decomposes as the direct sum of two diagrams of the same shape, corresponding to non-positive and positive powers of x respectively: $E_{e_r} = \mathcal{D}_+ \oplus \mathcal{D}_-$ where

$$\mathcal{D}_- = \left(R[x^{-1}, y, y^{-1}] \longrightarrow R[y, y^{-1}][x^{-1}] \longleftarrow R[y, y^{-1}][x^{-1}] \right)$$

and

$$\mathcal{D}_+ = \left(xR[x, y, y^{-1}] \longrightarrow xR[x, y, y^{-1}] \longleftarrow 0 \right) .$$

Upon application of $\check{\Gamma}$, the short exact sequence of diagrams

$$0 \longrightarrow \mathcal{D}_+ \longrightarrow E_{e_r} \longrightarrow \mathcal{D}_- \longrightarrow 0$$

translates into a short exact sequence of ČECH complexes

$$0 \longrightarrow \check{\Gamma}(\mathcal{D}_+) \longrightarrow \check{\Gamma}(E_{e_r}) \xrightarrow{\beta} \check{\Gamma}(\mathcal{D}_-) \longrightarrow 0 .$$

Now $\check{\Gamma}(\mathcal{D}_+)$ is acyclic (it is a two-step complex with only non-trivial differential being an isomorphism) so that β is a quasi-isomorphism. Introducing the diagrams

$$\mathcal{E}_0 = (0 \longrightarrow R[y, y^{-1}][[x^{-1}]] \longleftarrow R[y, y^{-1}][[x^{-1}]])$$

and

$$\mathcal{E}_1 = (R[x^{-1}, y, y^{-1}] \longrightarrow 0 \longleftarrow 0)$$

we have a short exact sequence

$$0 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{D}_- \longrightarrow \mathcal{E}_1 \longrightarrow 0$$

and consequently a short exact sequence

$$0 \longrightarrow \check{\Gamma}(\mathcal{E}_0) \longrightarrow \check{\Gamma}(\mathcal{D}_-) \xrightarrow{\gamma} \check{\Gamma}(\mathcal{E}_1) \longrightarrow 0 .$$

Now $\check{\Gamma}(\mathcal{E}_0)$ is acyclic by the same reasoning as before so that γ is a quasi-isomorphism to $\check{\Gamma}(\mathcal{E}_1) = A_{e_r}$. As clearly $\gamma \circ \beta \circ \sigma_{e_r} = \text{id}$ it follows that σ_{e_r} is a quasi-isomorphism.

We finally deal with the case $F = v$ a vertex. We will treat $v = v_{tr} = (1, 1)$ explicitly, the other cases are similar (and follow formally by change of variables). — Informally speaking, we decompose the diagram $E_{v_{tr}}$ as a direct sum of R -module diagrams by restriction to the sets

$$\begin{aligned} \bullet Q &= \{x^i y^j \mid i \leq 0, j > 0\} , & Q^\bullet &= \{x^i y^j \mid i > 0, j > 0\} , \\ \bullet Q &= \{x^i y^j \mid i \leq 0, j \leq 0\} , & Q_\bullet &= \{x^i y^j \mid i > 0, j \leq 0\} \end{aligned}$$

which correspond to the four quadrants (with or without various boundary components included). We denote the restricted diagrams by E^\bullet , $\bullet E$, $\bullet E$ and E_\bullet respectively. The four resulting summands are shown in Fig. 3. The diagram appearing in the top right is E^\bullet . The top left diagram is $\bullet E$. Similarly, $\bullet E$ is the bottom left diagram and E_\bullet is the bottom right diagram.

Note that the splitting $E_{v_{tr}} = \bullet E \oplus \bullet E \oplus E_\bullet \oplus E^\bullet$ yields a corresponding splitting of ČECH complexes

$$\check{\Gamma}(E_{v_{tr}}) = \check{\Gamma}(\bullet E) \oplus \check{\Gamma}(\bullet E) \oplus \check{\Gamma}(E_\bullet) \oplus \check{\Gamma}(E^\bullet) .$$

We will show

- (i) that the last three summands are acyclic so that the projection map $\pi: \check{\Gamma}(E_{v_{tr}}) \longrightarrow \check{\Gamma}(\bullet E)$ is a quasi-isomorphism, and
- (ii) that there is a quasi-isomorphism $\beta: \check{\Gamma}(\bullet E) \longrightarrow A_{v_{tr}}$ such that the composite map

$$A_{v_{tr}} \xrightarrow{\sigma_{v_{tr}}} \check{\Gamma}(E_{v_{tr}}) \xrightarrow{\pi} \check{\Gamma}(\bullet E) \xrightarrow{\beta} A_{v_{tr}}$$

is the identity.

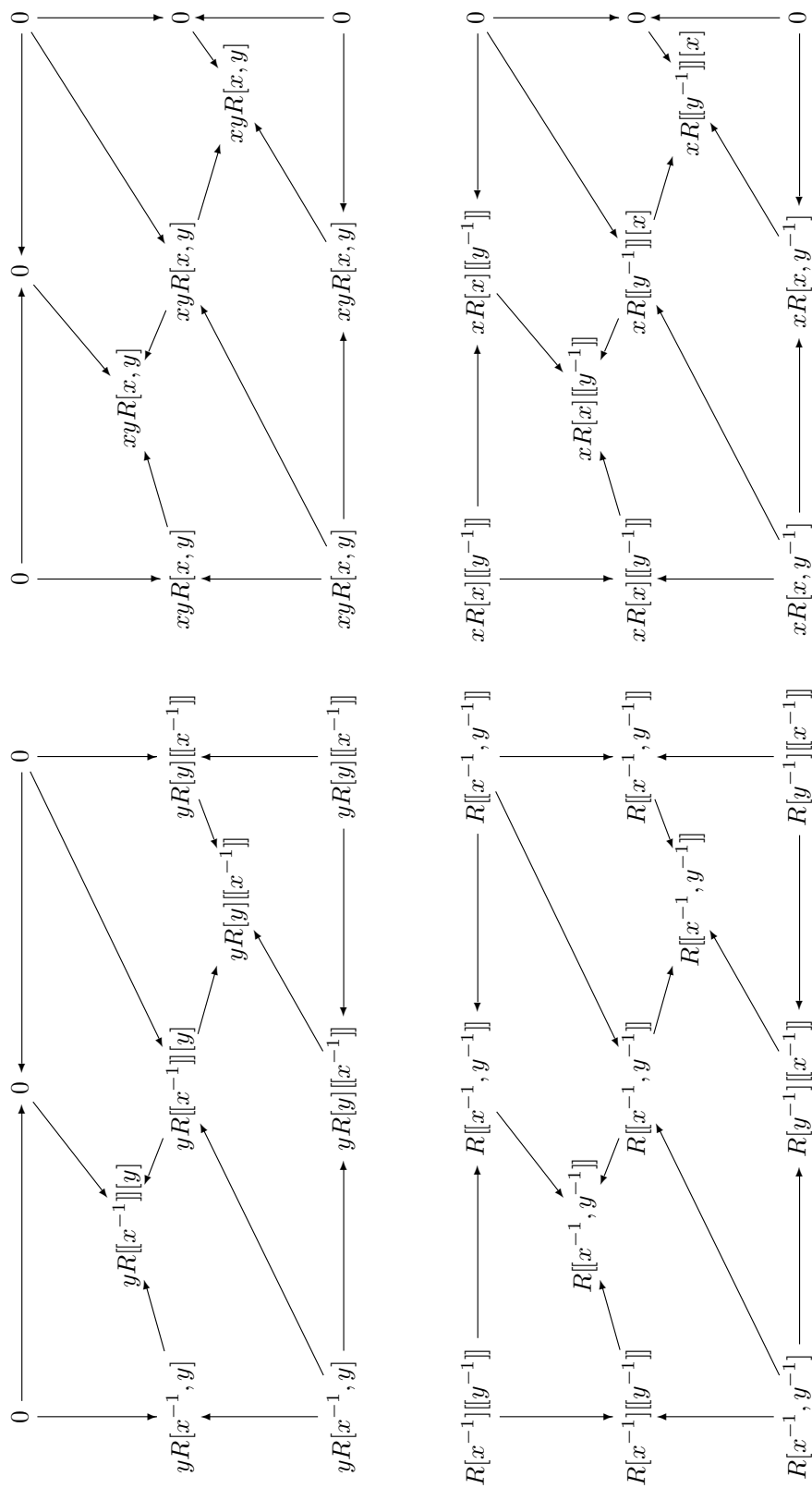


FIGURE 3. Decomposition of $E_{v_{tr}}$

It then follows that $\sigma_{v_{tr}}$ is a quasi-isomorphism as claimed.

The main idea in both cases is to use a suitable filtration of diagrams, as was done implicitly in the case of an edge above. More precisely, we will look at a chain of epimorphisms of R -module diagrams indexed by $\mathcal{N}_{v_{tr}}$

$$X_0 \xrightarrow{\kappa_1} X_1 \xrightarrow{\kappa_2} \dots \xrightarrow{\kappa_k} X_k$$

such that $\check{\Gamma}(\ker \kappa_j)$ is acyclic for $1 \leq j \leq k$. From the short exact sequence

$$0 \longrightarrow \check{\Gamma}(\ker \kappa_j) \longrightarrow \check{\Gamma}(X_{j-1}) \xrightarrow{\check{\Gamma}(\kappa_j)} \check{\Gamma}(X_j) \longrightarrow 0$$

we then infer that the map $\check{\Gamma}(\kappa_j)$ is a quasi-isomorphism.

Let us consider specifically the diagram $X_0 = \bullet E$ (top left in Fig. 3). We let X_1 have the same entries and structure maps as X_0 except at the flags $\{v_{tr}, e_r\}$ and $\{v_{tr}, e_r, S\}$ where X_1 is trivial (see Fig. 2 for a reminder on the indexing); the incident structure maps are forced to be zero maps, of course. The map κ_1 is the identity where possible, or else the zero map. The ČECH complex $\check{\Gamma}(\ker \kappa_1)$ is a two-step complex with an isomorphism as differential and is thus acyclic. — We construct further diagrams X_j in a similar manner from X_{j-1} , by prescribing two flags τ_1 and τ_2 on which the former differs from the latter in taking the zero module as value, and by declaring κ_j to be the identity where possible. In detail, we choose

$$\begin{aligned} j = 2: & \tau_1 = \{e_r\} \text{ and } \tau_2 = \{e_r, S\}; \\ j = 3: & \tau_1 = \{v_{tr}, S\} \text{ and } \tau_2 = \{v_{tr}, e_t, S\}; \\ j = 4: & \tau_1 = \{S\} \text{ and } \tau_2 = \{e_t, S\}. \end{aligned}$$

This makes X_4 the trivial all-zero diagram so that $\check{\Gamma}(\bullet E)$ is quasi-isomorphic to the zero complex via $\check{\Gamma}(\kappa_4 \kappa_3 \kappa_2 \kappa_1)$. — The diagrams E^\bullet and E_\bullet can be dealt with in a similar manner. This proves (i).

To prove (ii) we employ a suitable filtration of $X_0 = \bullet E$: we let X_j and κ_j be determined in the manner described above by the choices

$$\begin{aligned} j = 1: & \tau_1 = \{v_{tr}, e_r\} \text{ and } \tau_2 = \{v_{tr}, e_r, S\}; \\ j = 2: & \tau_1 = \{e_r\} \text{ and } \tau_2 = \{e_r, S\}; \\ j = 3: & \tau_1 = \{v_{tr}, e_t\} \text{ and } \tau_2 = \{v_{tr}, e_t, S\}; \\ j = 4: & \tau_1 = \{e_t\} \text{ and } \tau_2 = \{e_t, S\}; \\ j = 5: & \tau_1 = \{v_{tr}\} \text{ and } \tau_2 = \{v_{tr}, S\}. \end{aligned}$$

The diagram X_5 has a single non-trivial entry, *viz.*, the entry $A_{v_{tr}}$ at position S so that $\check{\Gamma}(X_5) = A_{v_{tr}}$. The map $\beta = \check{\Gamma}(\kappa_5 \kappa_4 \kappa_3 \kappa_2 \kappa_1)$ satisfies all the required properties. \square

We also need to record naturality properties of the maps σ_F . Let $F \subseteq G$ be faces of S . Since every $\tau \in \mathcal{N}_G$ is also an element of \mathcal{N}_F , we can define a “projection” map

$$\lambda_{FG}: \check{\Gamma}(E_F) \longrightarrow \check{\Gamma}(E_G) \tag{18}$$

which maps summands occurring in both complexes by the identity, and maps all other summands of the source to 0. (Note that $\check{\Gamma}(E_G)$ is, in general, not a direct summand of $\check{\Gamma}(E_F)$.) It is a matter of straightforward checking that $\sigma_G|_{A_F} = \lambda_{FG} \circ \sigma_F$, and that we have

$$\lambda_{FH} = \lambda_{GH} \circ \lambda_{FG} \tag{19}$$

for a triple of faces $F \subseteq G \subseteq H$ of S .

IV.6. PARTIAL TOTALISATIONS OF TRIPLE COMPLEXES. APPLICATIONS

For this last section of the paper we assume throughout that C and Y are as in Construction IV.1.2; that is, we assume that C is a bounded cochain complex of finitely generated free $R[x, x^{-1}, y, y^{-1}]$ -modules, with C^n having basis B_n , and that Y is a bounded $\mathcal{F}(S)$ -diagram in the sense of Definition IV.1.1 with $Y_S = C$, with each Y^n isomorphic to a B_n -indexed direct sum of diagrams of the form $D(k_n)$. We write $s_{FG}: Y_F \longrightarrow Y_G$ for the structure map associated to the inclusion of faces $F \subseteq G$.

We will introduce the following complexes and maps between them:

$$B' \xrightarrow{\simeq} \text{Tet } S^{*,*,*} \xrightarrow{\simeq} \text{Tet } T^{*,*,*} \xrightarrow{\simeq} \text{Tet } U^{*,*,*}$$

and

$$\text{Tet } V^{*,*,*} \cong \text{Tet } W^{*,*} \xleftarrow{\simeq} C ,$$

and a splitting of complexes

$$\text{Tet } U^{*,*,*} \cong \text{Tet } V^{*,*,*} \oplus ? .$$

Provided all the cochain complexes listed in (1a) and (1b) are acyclic, the maps marked “ \simeq ” above are quasi-isomorphisms; hence we obtain, in the derived category of R , morphisms $C \xrightarrow{\bar{s}} B' \xrightarrow{\bar{r}} C$ with $\bar{r} \circ \bar{s} = \text{id}_C$. Since both C and B' are bounded complexes of free R -modules we can lift these morphisms to R -linear maps $C \xrightarrow{s} B' \xrightarrow{r} C$ with $r \circ s \simeq \text{id}_C$. Since B' is finitely generated this shows C to be R -finitely dominated [Ran85, Proposition 3.2], thereby finishing the proof of the implication (b) \Rightarrow (a) of the Main Theorem.

We start with the triple complex

$$S^{u,s,t} = \begin{cases} 0 & \text{for } t \neq 0 , \\ \bigoplus_{\substack{F \subseteq S \\ \dim F = u}} Y_F^s & \text{for } t = 0 . \end{cases}$$

Differentials are necessarily trivial in z -direction; for fixed s , the complex $S^{*,s,0}$ is the ČECH complex of the $\mathcal{F}(S)$ -indexed diagram Y^s , and for fixed u the complex $S^{u,*,0}$ is a direct sum of complexes Y_F with differential changed by the sign $(-1)^u$.

We note that $S^{*,*,*}$ is actually a double complex in disguise, and that $\text{Tet } S^{*,*,*} = \text{Tet } S^{*,*,0} = \check{\Gamma}(Y)$. Hence by Proposition IV.3.3:

Lemma IV.6.1. *There exists a bounded cochain complex B' of finitely generated free R -modules, together with a quasi-isomorphism*

$$\chi: B' \longrightarrow \text{Tet } S^{*,*,*} . \quad \square$$

If the face F is contained in the minimal element of the flag τ then $A_F \subseteq A\langle\tau\rangle$. Consequently, we can define a triple complex $T^{*,*,*}$ by

$$T^{u,s,t} = \bigoplus_{\substack{F \subseteq S \\ \dim F = u}} \left(Y_F^s \otimes_{A_F} \bigoplus_{\substack{\tau \in \mathcal{N}_F \\ \dim \tau = t}} A\langle\tau\rangle \right) \quad (20)$$

with differentials induced by

$$\begin{aligned} d_x &= [F : G] \cdot (s_{FG} \otimes \lambda_{FG}) , \\ d_y &= (-1)^u \cdot (d_F \otimes 1) , \text{ and} \\ d_z &= (-1)^{u+s} \cdot (1 \otimes d_{\mathcal{N}_F}) , \end{aligned}$$

with λ_{FG} is as in (18), where $d_{\mathcal{N}_F}$ denotes the differential of the cochain complex $\check{\Gamma}(E_F)$ and d_F denotes the differential of the cochain complex Y_F . (Note that for fixed t the maps d_x are induced by the cochain complex maps

$$[F : G] \cdot (s_{FG} \otimes \lambda_{FG}): Y_F \otimes_{A_F} A\langle\tau\rangle \longrightarrow Y_G \otimes_{A_G} A\langle\tau\rangle ,$$

where τ is a t -dimensional flag in \mathcal{N}_G ; this implies, in view of (19), that $d_x \circ d_x = 0$.)

The triple complex just defined is such that $T^{u,s,*}$, for fixed indices u and s , is a direct sum of complexes of the form $Y_F^s \otimes_{A_F} \check{\Gamma}(E_F)$, with differential changed by a sign $(-1)^{u+s}$. But as $\check{\Gamma}(E_F)$ is quasi-isomorphic to A_F via the map σ_F defined in (17), and as Y_F^s is a free A_F -module we have a quasi-isomorphism $Y_F^s \otimes_{A_F} \check{\Gamma}(E_F) \simeq Y_F^s$. In fact, the compositions

$$Y_F^s \cong Y_F^s \otimes_{A_F} A_F \xrightarrow{\text{id} \otimes \sigma_F} Y_F^s \otimes_{A_F} \check{\Gamma}(E_F)$$

assemble to a map of triple complexes

$$v: S^{*,*,*} \longrightarrow T^{*,*,*}$$

which is a quasi-isomorphism on complexes in z -direction in the sense of Lemma II.3.3. We thus have:

Lemma IV.6.2. *The map $\text{Tet}(v): \text{Tet} S^{*,*,*} \longrightarrow \text{Tet} T^{*,*,*}$ is a quasi-isomorphism. \square*

Next, we define a triple complex $U^{*,*,*}$ which is, informally speaking, the restriction of $T^{*,*,*}$ to those flags which are either zero-dimensional, or do not involve S . Explicitly,

$$U^{u,s,t} = \begin{cases} 0 & \text{for } t \neq 0, 1 , \\ \bigoplus_{\substack{F \subset S \\ \dim F = u}} \left(Y_F^s \otimes_{A_F} \bigoplus_{\substack{\tau \in \mathcal{N}_F \\ \dim \tau = 1 \\ S \not\subset \tau}} A\langle\tau\rangle \right) & \text{for } t = 1 , \\ \bigoplus_{\substack{F \subset S \\ \dim F = u}} \left(Y_F^s \otimes_{A_F} \bigoplus_{G \supseteq F} A\langle G\rangle \right) & \text{for } t = 0 ; \end{cases} \quad (21)$$

note that $U^{u,s,1} = 0$ if $u \neq 0$, and that the second direct sum in the last line is taken over all 0-dimensional flags in \mathcal{N}_F , *i.e.*, over all faces G containing F . The differentials are either trivial by necessity, or the restrictions of the corresponding differentials of $T^{*,*,*}$ where possible.

There is an obvious “projection” map of triple complexes

$$\omega: T^{*,*,*} \longrightarrow U^{*,*,*} ;$$

it is given by sending the summand $Y_F^s \otimes_{A_F} A\langle\tau\rangle$ to itself via the identity map if the target contains the same summand, and by sending it to 0 otherwise. (Note that $U^{*,*,*}$ is not a direct summand of $T^{*,*,*}$ due to the presence of too many non-trivial differentials in z -direction in the latter.)

Lemma IV.6.3. *If all the cochain complexes listed in (1a) and (1b) are acyclic, the map $\text{Tet}(\omega)$ is a quasi-isomorphism.*

Proof. First note that the complexes listed in (1a) are of the type $C \otimes A\langle e, S \rangle$ for an edge e of S (all unmarked tensor products are over the ring $A_S = R[x, x^{-1}, y, y^{-1}]$ in this proof). Similarly, the complexes listed in (1b) are of the form $C \otimes A\langle v, S \rangle$ for v a vertex of S .

Suppose now that these complexes are acyclic. Since C is a bounded complex of free modules, they are then in fact contractible, *i.e.*, homotopy equivalent to the trivial complex. Since tensor products preserve homotopies, it follows that for each 2-dimensional flag $\tau = (v \subset e \subset S)$ the cochain complex

$$Y_S \otimes_{A_S} A\langle \tau \rangle \cong Y_S \otimes_{A_S} A\langle v, S \rangle \otimes_{A\langle v, S \rangle} A\langle \tau \rangle$$

is contractible and hence acyclic. That is, acyclicity of the complexes (1a) and (1b) implies acyclicity of the additional eight complexes

$$\begin{array}{ll} C \otimes R((x))((y)) , & C \otimes R((x))((y^{-1})) , \\ C \otimes R((x^{-1}))((y)) , & C \otimes R((x^{-1}))((y^{-1})) , \\ C \otimes R((y))((x)) , & C \otimes R((y))((x^{-1})) , \\ C \otimes R((y^{-1}))((x)) , & C \otimes R((y^{-1}))((x^{-1})) . \end{array}$$

Let F be a face of S . If $\tau \in \mathcal{N}_F$ denotes a positive-dimensional flag ending in S , we know that $A_S \subset A\langle \tau \rangle$ and thus

$$Y_F \otimes_{A_F} A\langle \tau \rangle \cong Y_F \otimes_{A_F} A_S \otimes_{A_S} A\langle \tau \rangle \cong C \otimes_{A_S} A\langle \tau \rangle \simeq 0$$

where we made use of the fact that $Y_F \otimes_{A_F} A_S \cong Y_S = C$ according to Construction IV.1.2.

But this means that ω is a quasi-isomorphism of cochain complexes

$$T^{u,*,t} \longrightarrow U^{u,*,t}$$

for all $u, t \in \mathbb{Z}$. Indeed, for $t = 0$ it is an identity map, for $t = 1$ it is a direct sum of identity maps (corresponding to summands indexed by flags of the form $v \subset e$) and maps from acyclic to trivial complexes (all other summands), for $t = 2$ it is a map from an acyclic to a trivial one, using the results of the previous two paragraphs. From Lemma II.3.3 we conclude that $\text{T\oet}(\omega)$ is a quasi-isomorphism as claimed. \square

Now $U^{*,*,*}$ has a direct summand consisting, informally speaking, of the summands indexed by $G = S$ at height $t = 0$ only:

$$V^{u,s,t} = \begin{cases} 0 & \text{for } t \neq 0 , \\ \bigoplus_{\substack{F \subset S \\ \dim F = u}} (Y_F^s \otimes_{A_F} A\langle S \rangle) & \text{for } t = 0 ; \end{cases} \quad (22)$$

differentials are obtained by restricting the corresponding ones of $U^{*,*,*}$. Clearly then $\text{T\oet} V^{*,*,*}$ is a direct summand of $\text{T\oet} U^{*,*,*}$.

The triple-complex totalisation of $V^{*,*,*}$ agrees with the double complex totalisation $\text{T\oet} V^{*,*,0}$ (due to the absence of non-trivial terms for $z \neq 0$). Also, we have $A\langle S \rangle = A_S$ and $Y_F^s \otimes_{A_F} A_S \cong C^s$, by construction of Y ; it follows that the double complex $V^{*,*,0}$ is isomorphic to the double complex $W^{*,*}$ which, in column p , has a direct sum of copies of C indexed by the p -dimensional faces of S , with differential changed by the sign $(-1)^p$, and

has in row q the ČECH complex of the constant $\mathcal{F}(S)$ -indexed diagram with value C^s .

Lemma IV.6.4. *There is a quasi-isomorphism $C \longrightarrow \mathrm{T\oet}(W^{*,*})$.*

Proof. Since C is bounded we may, by simple re-indexing, assume that C is concentrated in non-negative degrees and that consequently $W^{*,*}$ is concentrated in the first quadrant. For $q \geq 0$ let h_q denote the diagonal inclusion $C^q \longrightarrow W^{0,q} = \bigoplus_v C^q$, modified by a sign $(-1)^q$ (the direct sum taken over all vertices of S). By construction the two composites

$$C^q \longrightarrow W^{0,q} \longrightarrow W^{0,q+1} \quad \text{and} \quad C^q \longrightarrow C^{q+1} \longrightarrow W^{0,q+1}$$

agree up to sign. The complexes

$$0 \longrightarrow C^q \longrightarrow W^{0,q} \longrightarrow W^{1,q} \longrightarrow \dots$$

are exact; this follows, for example, from the observation that they can be obtained by tensoring the dual of the augmented cellular chain complex of S (which computes $\tilde{H}_*(S; R) = 0$) with the free R -module C^q . — We can now apply Proposition II.2.3 to conclude that $C \simeq \mathrm{T\oet} W^{*,*}$ as claimed. \square

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