MODELS OF MORTALITY RATES - ANALYSING THE RESIDUALS

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ABSTRACT. The area of mortality modelling has received significant attention over the last 25 years owing to the need to quantify and forecast improving mortality rates. This need is driven primarily by the concern of governments, insurance and actuarial professionals and individuals to be able to fund their old age. In particular, to quantify the costs of increasing longevity we need suitable model of mortality rates that capture the dynamics of the data and forecast them with sufficient accuracy to make them useful. In this paper we test several of the leading time series models by considering the fitting quality and in particular, testing the residuals of those models for normality properties. In a wide ranging study considering 30 countries we find that almost exclusively the residuals do not demonstrate normality. Further, in Hurst tests of the residuals we find evidence that structure remains that is not captured by the models.

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1. INTRODUCTION

The rapid growth in the development of models of mortality designed to capture patterns in mortality data and accurately forecast and quantify future mortality rates has been dramatic. Over the recent decades, life expectancy in developed countries has risen to historically unprecedented levels and there is clearly a need from a demographic, financial, social and actuarial perspective to understand and predict these improvements for the future. The prospects of future reductions in mortality rates are of fundamental importance in various areas such as public health and old age care planning, social insurance planning, welfare benefit forecasting and economic policy. Over recent years, significant progress has been made in mortality forecasting (for reviews see Booth and Tickle, 2008; Plat, 2009; O'Hare and Li, 2012) with the most popular approaches to long-term forecasting being based on the Lee and Carter (1992) model. A time series model, it describes the movement of age-specific mortality as a function of a latent level of mortality, also known as the overall mortality index, which can be forecasted using simple time-series methods. The method was initially used to forecast mortality in the US, but since then has been applied to many other countries (amongst others see Tuljapurkar and Boe, 1998; Carter and Prskawetz, 2001; Lee and Miller, 2001; Booth *et al.*, 2002; Brouhns *et al.*, 2002; Renshaw and Haberman, 2003 and Koissi *et al.*, 2005).

The success of the Lee Carter model can be seen in the number and variety of mortality models that extend the Lee Carter approach (see O'Hare and Li, 2012) for examples of these extensions. One thread of extensions to the Lee Carter model involves including additional latent age and period effects with the objective of better fitting the data, producing a less simplistic correlation structure between ages and capturing the nonlinear profile of mortality data. This has led for example to the models of Renshaw and Haberman (2003), Cairns, Blake and Dowd (2006, 2008, 2009), Plat (2009) and O'Hare and Li (2012) for example. These models extend the Lee Carter approach by including additional period effects and in some cases cohort effects and improve upon each other by producing better fits to the data and in the main better forecasts. In the literature however, there has been limited attempts to test the fitting of such models. The majority of papers calculate point estimates of the average errors produced between the fitted and actual rates using one of several measures (for example root mean square

errors, mean average percentage errors etc). There has been little work looking at the patterns of such errors.

One such paper that considered the shape of the residuals in a range of mortality models is that of Dowd *et al.* (2010) where the authors assess the residuals for normality carrying out several tests of the mean, variance and skewness of the residuals. Dowd *et al.* (2010) fitted a range of models, primarily the Lee Carter (1992) model and a selection of CBD models to data and then after calculating the in sample forecasts they derived standardised residuals from the forecasts and tested these for normality. Their paper concluded that none of the models considered performed well under these tests. In this paper we extend and modify this work in three ways. Firstly, rather than forecasting and testing the derived residuals we calculate the residuals directly from the fitted models. This will enable us to test the model to ensure that all of the structure of mortality is being captured prior to forecasting. Secondly, we extend the work by considering several multi factorial models, namely Plat (2009) and O'Hare and Li (2012), that were not considered in the previous study. Finally, in addition to the normality tests we calculate Hurst exponents for each of the residual time series for each country and gender to test for the presence of autocorrelations within the period or age dimensions of the residuals.

The paper is organized as follows. Section 2 presents a brief review of extrapolative models such as the Lee-Carter model and its extensions. Section 3 discusses the data we have used in this study. In section 4 we discuss the methodology we use to test the residuals for normality and in section 5 we present the results of our analysis and discuss some of the implications for pricing, forecasting and the making of policy. Finally, Section 6 concludes with some ideas for further research.

2. LEE-CARTER AND ITS VARIANTS

The current leading method for forecasting mortality rates is the stochastic extrapolation approach. In this method data is first transformed (by taking natural logarithms) and then analysed using statistical methods to identify and extract patterns. These patterns are then forecast using well known time series approaches. The resulting forecasts are then used to predict future mortality rates. The first and most well known stochastic mortality model of this type is the Lee and Carter (1992) model. Based on US data the model uses a stochastic, time series framework to identify a single period effect pattern in the natural logarithm of mortality rates. This linear

trend over time is extracted and using Box-Jenkins an appropriate ARIMA processes is fitted to the data (a random walk with drift in each case). The random walk with drift is forecast and resulting future mortality rates predicted. Also known as a one factor or one principle component approach the model became a benchmark and underlined a new approach to modelling mortality rates for several reasons: the model has an extremely simple structure and so is very easy to communicate; and the use of the random walk with drift enabled the authors not only to predict the expected future mortality rates but also to visualise the uncertainty associated with the predictions. The **Lee-Carter** model, outlined below includes two age dependent parameters a_x and b_x which respectively represent the intercept and gradient for the log mortality rate at each age and the time or period trend κ_t which is forecast using a random walk with drift:

$$\ln(m_{x,t}) = a_x + b_x\kappa_t + \epsilon_{x,t}, \tag{1}$$

where a_x and b_x are age effects and κ_t is a random period effect.

The model is known to be over parameterised and applying the necessary constraints as in the original Lee and Carter (1992) paper leads to an estimate of a_x of

$$a_x = \frac{1}{N}\sum_{t=1}^{N} \ln m_{x,t}.$$

In the original paper the bilinear part $b_x\kappa_t$ of the model specification was determined as the first singular component of a singular value decomposition (SVD), with the remaining information from the SVD considered to be part of the error structure. The κ_t were then estimated and refitted to ensure the model mapped onto historic data. Finally the subsequent time series κ_t was used to forecast mortality rates.

Despite the attractiveness of the models simplicity it has several weaknesses. Among many discussions of the Lee-Carter model, Cairns *et al.* (2006, 2009, and 2011) summarized the main disadvantage of the model as having only one factor, resulting in mortality improvements at all ages being perfectly correlated (trivial correlation structure). They also note that for countries where a cohort effect is observed in the past, the model gives a poor fit to historical data. The uncertainty in future death rates is proportional to the average improvement rate b_x which for high ages can lead to this uncertainty being too low, since historical improvement

rates have often been lower at high ages. Also, the model can result in a lack of smoothness in the estimated age effect b_x.

Despite the weaknesses of the Lee-Carter model it's simplicity has led to it being taken as a benchmark against which other stochastic mortality models can be assessed. There has been a significant amount of literature developing additions to, or modifications of, the Lee-Carter model. For example Booth *et al.* (2002), Brouhns *et al.* (2002), Lee and Miller (2001), Girosi and King (2005), De Jong and Tickle (2006), Delwarde *et al.* (2007) and Renshaw and Haberman (2003, 2006). In this paper we consider four models from the time series mortality modelling literature. The Lee Carter (1992) model, the CBD (2006) model, the Plat (2009) model and the O'Hare and Li (2013) model. The structure of these models is outlined in table 1. We have selected a range of simple factor models and larger multifactorial models to see if the addition of latent factors affects the residuals in any way.

TABLE 1. The stochastic mortality models

Name	Model and Name
M1	Lee-Carter (1992)
	$\ln(m_{x,t}) = a_x + b_x\kappa_t + \epsilon_{x,t}$
M5	CBD (2006)
	$\text{logit}(q_{x,t}) = \kappa_t^1 + \kappa_t^2(x - \bar{x}) + \epsilon_{x,t}$
M9	Plat (2009)
	$\ln(m_{x,t}) = a_x + \kappa_t^1 + \kappa_t^2(\bar{x} - x) + \kappa_t^3(\bar{x} - x)^+ + \gamma_{t-x} + \epsilon_{x,t}$
M10	O'Hare and Li (2012)
	$\ln(m_{x,t}) = a_x + \kappa_t^1 + \kappa_t^2(\bar{x} - x) + \kappa_t^3\big((\bar{x} - x)^+ + [(\bar{x} - x)^+]^2\big) + \gamma_{t-x} + \epsilon_{x,t}$

Note: The models selected form a sample of the existing time series models in the literature and represent models with both small and large numbers of factors.

For a review of the main extensions and modifications of the Lee Carter model the interested reader is directed to O'Hare and Li (2012).

3. DATA

The data that we use in this paper comes from the Human Mortality Database.[1] The data available for each country includes number of deaths $D_{x,t}$ and exposure to death $E_{x,t}$ for lives aged x last birthday during year t. We can use this to gain a proxy for the central mortality rate

[1]This can be found at http://www.mortality.org/. The database is maintained in the Department of Demography at the University of California, Berkeley, USA, and at the Max Planck Institute for Demographic Research in Rostock, Germany.

TABLE 2. Countries considered in this study along with HMD codes

HMD Code	Country	HMD Code	Country
ast	Austria	nor	Norway
bel	Belgium	nth	Netherlands
blr	Belarus	nzd	New Zealand
bul	Bulgaria	pol	Poland
can	Canada	por	Portugal
czr	Czechoslovakia	rus	Russia
den	Denmark	spa	Spain
est	Estonia	svk	Slovakia
fin	Finland	swe	Sweeden
fra	France	swi	Switzerland
hun	Hungary	uke	England
ity	Italy	ukr	Ukraine
jap	Japan	uks	Scotland
lat	Latvia	ukt	United Kingdom
lit	Lithuania	usa	United States of America

for lives aged x during year t as:

$$m_{x,t} = \frac{D_{x,t}}{E_{x,t}}. \tag{2}$$

The data provides an estimate of the true mortality due to issues with the recording of data. Death data tends to be recorded accurately, with death certificates in most cases. However, exposure data is taken from census data which may only be accurately recorded every 5 or 10 years adjusting these figures for migration, deaths and births etc. The resulting mortality estimates are therefore quite noisy, particularly at the older ages where there is less data available. Data is available going back to the mid nineteenth century in some cases but we have restricted this study to data from 1960-2009 in order to have a consistent period across all countries. This has resulted in the 30 countries we have considered in this paper. The countries along with their 3 letter codes are outlined in table 2.

The wide range of countries give a good spread of populations both geographically and in terms of economic development. The inclusion of Male and Female data also enables gender differences to be considered. We focus on the age range 20-89 for several reasons. Firstly, the models upon which we have based our comparisons are also fitted to this age range. Secondly, and as identified by Currie (2011), data at the older ages provide additional problems in terms of the reliability. Indeed in several cases mortality rates determined using older data appear to fall sharply beyond age 95.

4. METHODOLOGY

We begin by fitting each of the models considered to the data above for the 30 countries considered and for both males and females. In this paper we will consider the four models Lee Carter (1992), CBD(2006), Plat (2009) and O'Hare and Li(2012). We fit the models using a maximum likelihood approach using code developed in R and publicly available for several of the models.[2]. The results of fitting are assessed and presented using three point measures of fit quality outlined below.

The average error, $E1$ – this equals the average of the standardized errors,

$$E1 = \frac{1}{X_1 - X_2 + 1} \sum_{x=X_1}^{X_2} \sum_{t=1}^{T} \frac{(\hat{m}_{x,t} - m_{x,t})}{\hat{m}_{x,t}}, \tag{3}$$

this is a measure of the overall bias in the projections. The average absolute error, $E2$ – this equals the average of absolute value of the standardized errors,

$$E2 = \frac{1}{X_1 - X_2 + 1} \sum_{x=X_1}^{X_2} \sum_{t=1}^{T} \left| \frac{\hat{m}_{x,t} - m_{x,t}}{\hat{m}_{x,t}} \right|, \tag{4}$$

this is a measure of the magnitude of the differences between the actual and projected rates. The standard deviation of the error, $E3$ – this equals the square root of the average of the squared errors,

$$E3 = \sqrt{\frac{1}{X_1 - X_2 + 1} \sum_{x=X_1}^{X_2} \sum_{t=1}^{T} \left(\frac{\hat{m}_{x,t} - m_{x,t}}{\hat{m}_{x,t}} \right)^2}. \tag{5}$$

where X_1 and X_2 and the age limits of our sample $X_1 = 20$ and $X_2 = 89$, and $T = 60$ is the number of years of data we have in our sample.

The models are fitted by assuming that death rates are drawn from a poisson distribution with parameter given by $E_{x,t} m_{x,t}$. We then calculate the corresponding fitted mortality rates $\hat{m}_{x,t}$ and calculate the standardised residuals using the following formula

$$\frac{\hat{m}_{x,t} - m_{x,t}}{\sqrt{m_{x,t}/E_{x,t}}} \tag{6}$$

This approach to calculating the residuals is consistent with that of the Dowd *et al.* (2010) paper and should represent samples drawn from a standard normal distribution if indeed the residuals

[2]The open source coding used can be found at `http://www.macs.hw.ac.uk/~andrewc/lifemetrics/`

are reflecting no more than random noise. The tests used in this section aim to identify whether the mortality residuals described above are consistent with *i.i.d.* $N(0,1)$. We carry out the following tests on the matrix of mortality residuals:

- A t-test of the prediction that their mean should be 0.
- A variance ratio (VR) test of the prediction that the variance should be 1 (see Cochrane, 1988; Lo and MacKinley, 1988, 1989), and
- A Jarque Bera normality test based on the skewness and kurtosis predictions (see Jarque and Bera, 1980).

In addition, we calculate Hurst exponent, H, for each of the time series extracted from the residuals. The Hurst exponent is referred to as the "index of dependence" or "index of long-range dependence". It quantifies the relative tendency of a time series either to regress strongly to the mean or to cluster in a direction. A value of H in the range $0.5 < H < 1$ indicates a time series with long-term positive autocorrelation, meaning both that a high value in the series will probably be followed by another high value and that the values a long time into the future will also tend to be high. A value in the range $0 < H < 0.5$ indicates a time series with long-term switching between high and low values in adjacent pairs, meaning that a single high value will probably be followed by a low value and that the value after that will tend to be high, with this tendency to switch between high and low values lasting a long time into the future. A value of $H = 0.5$ can indicate a completely uncorrelated series, but in fact it is the value applicable to series for which the autocorrelations at small time lags can be positive or negative but where the absolute values of the autocorrelations decay exponentially quickly to zero. Given that we are expecting the residuals to be samples for a $N(0,1)$ distribution we should not expect any correlations between residuals. In other words a Hurst exponent of 0,5 would be ideal.

The Hurst exponent, H, is defined in terms of the asymptotic behaviour of the rescaled range as a function of the time span of a time series as follows

$$E\Big[\frac{R(n)}{S(n)}\Big] = Cn^H \text{ as } n \to \infty \tag{7}$$

where;

- $R(n)$ is the range of the first n values, and $S(n)$ is their standard deviation
- $E[\_]$ is the expected value

- n is the time span of the observation (number of data points in a time series)
- C is a constant

In order to consider the Hurst exponent analysis we must apply it to a time series of residuals not a matrix of residuals. We therefore consider both the age dimension and the period dimension separately. We should not expect any correlations between residuals across age nor should we expect any across the period dimension. In the empirical section following we present the analysis in both dimensions and comment accordingly.

5. Empirical analysis

In this action we present and discuss our findings. We firstly show the fitting results measured using the standard E1, E2 and E3 measures of fitting quality. These are calculated as shown in the methodology section and in the main confirm the reported findings of each of the previous papers proposing the models. We follow this with a discussion of some of the residuals calculated for each of the countries in the study. We present some of the residual plots and comment on some common characteristics we find. Finally, we empirically test the residuals using a range of tests as discussed above.

5.1. **Fitting the models and assessing with point estimates.** We consider each of the 30 countries covered in the paper for both male and female data, fitting the models to data from 1960 - 2009. We present results below in tables 3 - 5 using the three measures of error E1, E2 and E3 outlines earlier.

TABLE 3. Fitting results (expressed as percentages) measured using the mean absolute percentage error (E1) for Males and Females for the Lee-Carter, CBD, Plat, and OL models.

Country	Male				Female			
	LC(1992)	CBD(2006)	Plat(2009)	OL(2013)	LC(1992)	CBD(2006)	Plat(2009)	OL(2013)
Austria	6.39	14.11	5.83	5.91	7.96	23.1	7.9	8.23
Belarus	6.51	8.55	5.04	4.96	8.33	15.58	7.24	7.45
Belgium	6.27	14.87	4.93	4.94	7.52	22.24	6.92	7.26
Bulgaria	6.75	11.58	4.88	5.42	7.77	19.45	6.92	7.1
Canada	4.83	14.81	3.98	4.14	5	12.52	4.44	4.48
Czechoslovakia	6.46	11.28	4.81	4.82	7.15	15.48	6.6	6.93
Denmark	8.02	12.52	7.13	6.98	12.92	14.13	9.93	10.04
England	4.94	15.05	3.9	4.2	4.98	12.37	3.62	3.86
Estonia	11.66	13.21	9.76	9.86	18.55	20.93	16.83	16.91
Finland	7.48	11.63	6.4	6.34	10.9	22.56	10.36	10.73
France	5.3	12.38	3.06	3.33	5.04	26.54	3.76	4.19
Hungary	10.59	13.97	5.93	5.24	8.46	15.65	6.38	6.21
Italy	6.31	14.06	3.78	3.93	5.3	21.83	4.06	4.46
Japan	4.86	14.09	3.45	3.78	6.84	25.02	2.94	3.57
Latvia	9.54	11.4	6.98	7.16	12.1	19.04	10.2	10.07
Lithuania	8.67	11.27	6.93	7.08	11.41	19.27	10	10.17
Netherlands	5.69	13.74	4.38	4.47	5.81	18.85	5.6	6.01
New Zealand	9.52	19.66	9.35	9.52	12.14	16.41	12.2	12.09
Norway	7.18	15.41	6.62	6.79	9.54	20.24	9.56	9.77
Poland	6.93	9.45	3.1	3.37	5.13	15.1	3.98	4.68
Portugal	8.51	17.69	6.1	5.83	7.05	25.99	7.17	7.28
Russia	5.03	8.71	3.7	3.72	6.48	17.53	4.16	4.42
Scotland	7.88	15.06	7.45	7.92	9.98	11.48	8.76	8.84
Slovakia	10	11.83	6.19	6.13	10.61	15.51	9.89	9.85
Spain	7.65	14.4	4.49	4.68	6.7	26.24	5.45	5.67
Sweeden	6.3	15.86	6.17	6.04	8.55	21.86	8.39	8.6
Switzerland	8.73	18.31	6.71	6.69	10.46	24.33	9.41	9.5
Ukraine	5.09	9.01	4.3	4.18	7.84	17.63	5.08	5.42
United Kingdom	4.74	14.87	3.76	4.1	4.88	11.78	3.45	3.7
USA	4.46	13.05	2.65	2.67	3.85	11.81	3.01	3

TABLE 4. Fitting results (expressed as percentages) measured using the mean average percentage error (E2) for Males and Females for the Lee-Carter, CBD, Plat, and OL models.

Country	Male				Female			
	LC(1992)	CBD(2006)	Plat(2009)	OL(2013)	LC(1992)	CBD(2006)	Plat(2009)	OL(2013)
Austria	0.9	-4.45	0.52	0.57	1.47	-14.51	1.09	1.16
Belarus	0.85	-1.17	0.49	0.46	1.16	-7.84	1.08	1.09
Belgium	0.9	-4.25	0.57	0.6	1.38	-13.37	1.02	0.99
Bulgaria	0.91	-2.81	0.57	0.47	1.39	-10.99	1.16	1.08
Canada	0.48	-2.68	0.48	0.49	0.43	-6.86	0.41	0.45
Czechoslovakia	0.92	-0.69	0.46	0.37	1.23	-9.92	0.92	0.89
Denmark	1.36	-3.02	1.11	1.02	3.31	-4.26	2.21	2.18
England	0.5	-1.76	0.57	0.65	0.37	-8.2	0.34	0.41
Estonia	2.05	-0.76	1.82	1.88	6.95	-7.53	6.18	6.06
Finland	1.13	-3.3	0.74	0.79	2.65	-12.41	2.27	2.27
France	1.07	-4.7	0.3	0.3	0.87	-13.97	0.39	0.24
Hungary	2.16	2.51	0.93	0.25	1.63	-7.04	1.23	0.59
Italy	1.05	-4.84	0.47	0.51	0.7	-14.31	0.39	0.35
Japan	0.42	-7.02	0.3	0.47	0.32	-13.99	0.28	0.33
Latvia	1.99	-1.27	0.9	0.81	3.03	-9.35	2.52	2.19
Lithuania	1.27	-1.37	0.85	0.79	2.58	-8.32	2.16	2.01
Netherlands	0.22	-2.43	0.45	0.44	0.97	-11.07	0.68	0.54
New Zealand	1.89	-1.05	1.73	1.73	2.94	-5.26	2.97	2.95
Norway	0.98	-5.22	0.87	0.95	2.04	-12.73	2.04	2.04
Poland	1.03	-1.7	0.31	0.16	0.56	-8.35	0.4	0.28
Portugal	2.25	-6.58	1.08	0.76	1.13	-14.08	1.14	1.01
Russia	0.59	-2.08	0.39	0.41	0.7	-9.18	0.35	0.37
Scotland	1.27	0.11	1.38	1.52	1.97	-2.89	1.89	2
Slovakia	1.27	-0.25	0.76	0.54	2.4	-7.09	2.22	2.09
Spain	2.15	-5.92	0.62	0.45	1.33	-15.05	0.89	0.65
Sweeden	0.73	-8.47	0.69	0.76	1.72	-14.29	1.24	1.39
Switzerland	2.09	-5.24	1.09	1.02	2.93	-13.92	1.98	1.91
Ukraine	0.63	-2.3	0.56	0.45	1.19	-9.91	0.68	0.68
United Kingdom	0.46	-1.68	0.55	0.64	0.33	-7.75	0.3	0.37
USA	0.69	-2.67	0.31	0.33	0.23	-6.45	0.26	0.25

TABLE 5. Fitting results (expressed as percentages) measured using the root mean square percentage error (E3) for Males and Females for the Lee-Carter, CBD, Plat, and OL models.

Country	Male				Female			
	LC(1992)	CBD(2006)	Plat(2009)	OL(2013)	LC(1992)	CBD(2006)	Plat(2009)	OL(2013)
Austria	1.01	1.86	1.36	1.17	0.71	1.9	1.79	1.35
Belarus	1.44	1.95	1.17	1.18	1.28	1.87	1.22	1.15
Belgium	0.94	1.63	1.01	0.97	0.62	1.93	1.87	1.46
Bulgaria	1.62	1.82	1.8	1.39	1.41	1.78	1.74	1.48
Canada	0.78	1.35	0.59	0.66	0.52	1.37	0.58	0.55
Czechoslovakia	1.1	1.37	1.12	1.02	0.72	1.34	1.21	0.96
Denmark	1.14	1.23	1.01	0.97	1.02	1.55	0.85	0.78
England	0.67	1.46	0.83	1	0.5	1.01	0.65	0.59
Estonia	2.8	2.86	2.18	2.13	1.57	2.19	1.72	1.53
Finland	1.37	1.77	1.27	1.29	0.97	1.87	2.76	2.19
France	0.67	2.15	1.39	1.15	0.49	2.37	1.8	1.37
Hungary	2.64	1.88	1.57	1.07	0.92	1.82	1.83	1.34
Italy	0.79	1.41	0.49	0.5	0.53	1.67	1.26	0.99
Japan	0.73	1.86	1.3	1.02	0.91	2.23	1.83	1.42
Latvia	1.86	2.61	1.53	1.53	1.22	2.18	1.38	1.18
Lithuania	1.76	2.47	1.54	1.46	1.39	2.31	1.97	1.57
Netherlands	0.92	1.16	0.49	0.5	0.48	1.72	1.52	1.17
New Zealand	1.47	2.07	1.45	1.51	1.19	1.8	1.25	1.22
Norway	0.88	1.47	0.72	0.75	0.68	1.58	1.33	1.1
Poland	1.27	1.59	1.2	0.89	0.67	1.69	1.48	1.02
Portugal	0.99	2.59	1.6	1.39	0.69	2.08	1.79	1.51
Russia	1.38	2.49	1.17	1.16	1.16	2.23	1.22	1.08
Scotland	1.07	1.68	0.96	1.07	0.89	1.04	0.8	0.8
Slovakia	2.17	2.13	1.17	1.15	1.16	1.72	1.2	1.07
Spain	0.72	1.81	1.27	1.08	0.55	1.95	2.22	1.82
Sweeden	0.74	1.44	0.86	0.81	0.61	1.66	1.3	1.02
Switzerland	0.97	1.95	1.15	1.1	0.74	2.1	1.87	1.54
Ukraine	1.29	2.4	1.23	1.13	1.09	2.02	1.13	1.03
United Kingdom	0.65	1.44	0.77	0.95	0.5	0.98	0.63	0.57
USA	0.64	1.68	0.72	0.77	0.44	1.49	0.46	0.45

The results in tables 3 to 5 show the fitting results across the 30 countries and the four models considered do vary significantly. Some noticeable comments include;

- The results for the multifactorial models, Plat (2009) and O'Hare and Li (2012), are markedly better than those for the smaller models of Lee and Carter (1992) and Cairns, Blake and Dowd (2006). This is to be expected given the additional parameters in the larger models.
- Between the Plat (2009) and the O'Hare and Li (2012) models there is very little to separate them apart on the point estimate measures. It should be noted that the O'Hare and Li (2012) model outperformed the Plat (2009) model more noticeably on the wider age range of 5-89.
- As has been well written before, the CBD(2006) fairs poorly on the wide age range.

The results show that if the purpose of the exercise is purely to find the best fitting model then from these results the multi factor models do outperform. One of the questions of this research however, is do the additional factors and additional structure lead to models which capture more of the structure of mortality. In other words do they results in residuals that conform more to the random noise we should expect.

5.2. **Analysing the residuals.** To test for the normality of the residuals we follow an approach similar to that of Dowd *et al.* (2010). We carry out three statistical tests of the residuals. Firstly we test the mean of the residuals. If the residuals do represent random noise then their mean should be zero. We carry out a t-test to test for this. Similarly the variance of the residuals should be 1, and we use the variance ratio test to test this. Finally we test for skewness using the Jarque Bera test. Below we plot the results of the sample mean, variance and skewness before presenting the test results.

Figure 1 shows the plots of the sample means and variances for the male and female residuals after the models have been fitted. The models represented in these plots are the Lee Carter model (blue), the Cairns, Blake and Dowd model (red), the Plat model (green) and the O'Hare and Li model (black). As can be seen the mean figures are above zero and the variances are significantly different from 1. Figures 2 through to 4 show the t-test results, the variance ratio test results and the Jarque Bera test results.

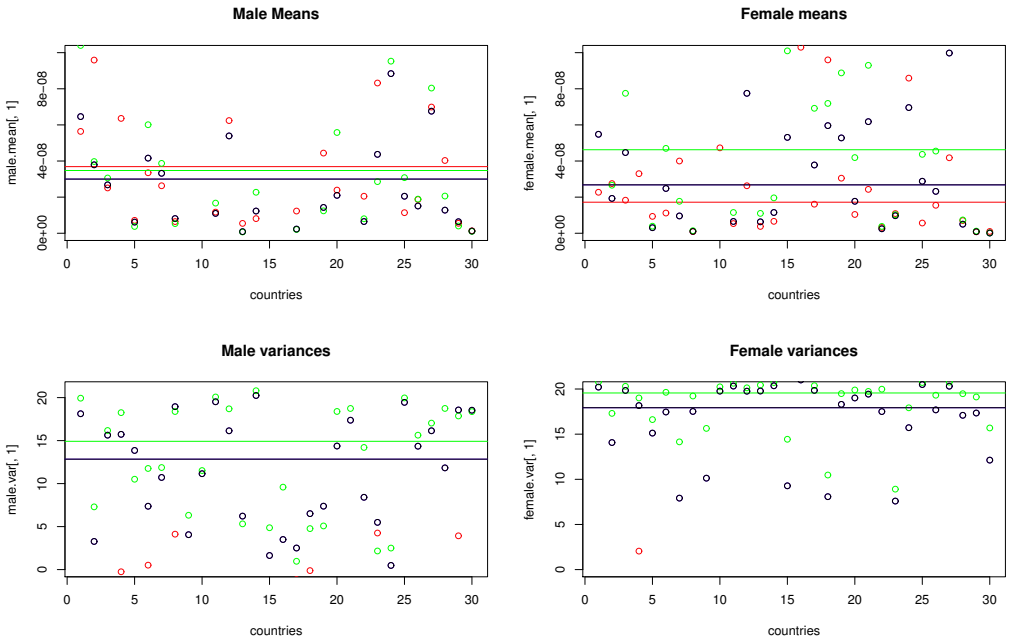

FIGURE 1. Sample means and variances for both males and females for the Lee Carter (blue), CBD (red), Plat (green) and O'Hare and Li (black) models

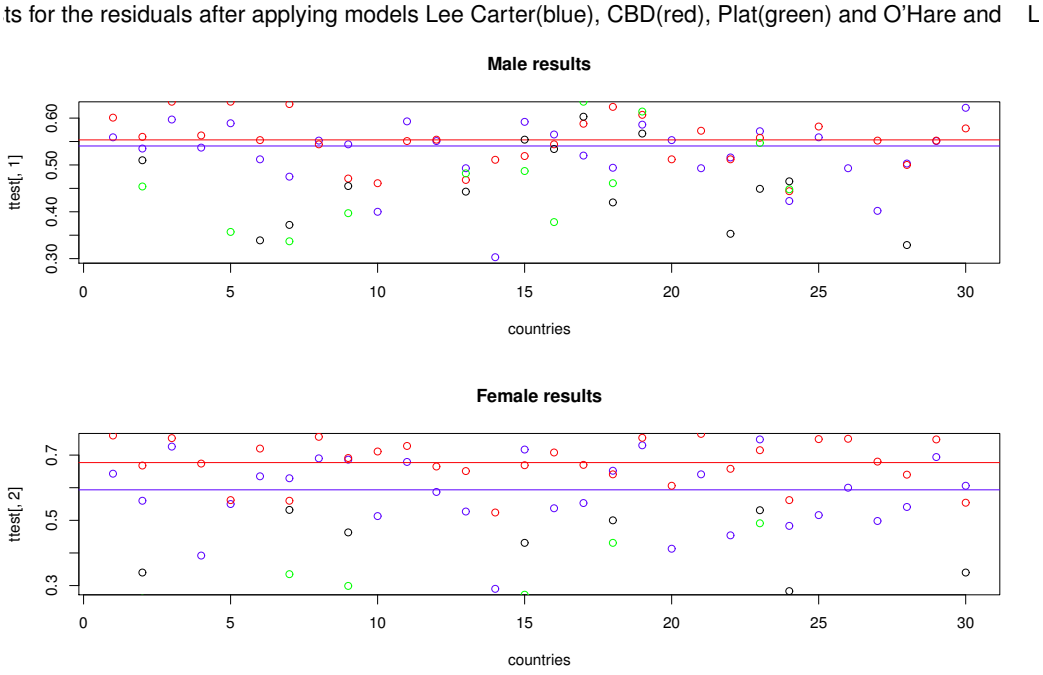

FIGURE 2. t-test results of the mean of the residuals for both males and females for the Lee Carter (blue), CBD (red), Plat (green) and O'Hare and Li (black) models

As can be seen for the test results in every case the fitted models fail the basic normality tests suggesting that the residuals mean and variances do not conform to those of the standard normal distribution, nor do the higher moments. In addition, the Hurst exponent calculations show long

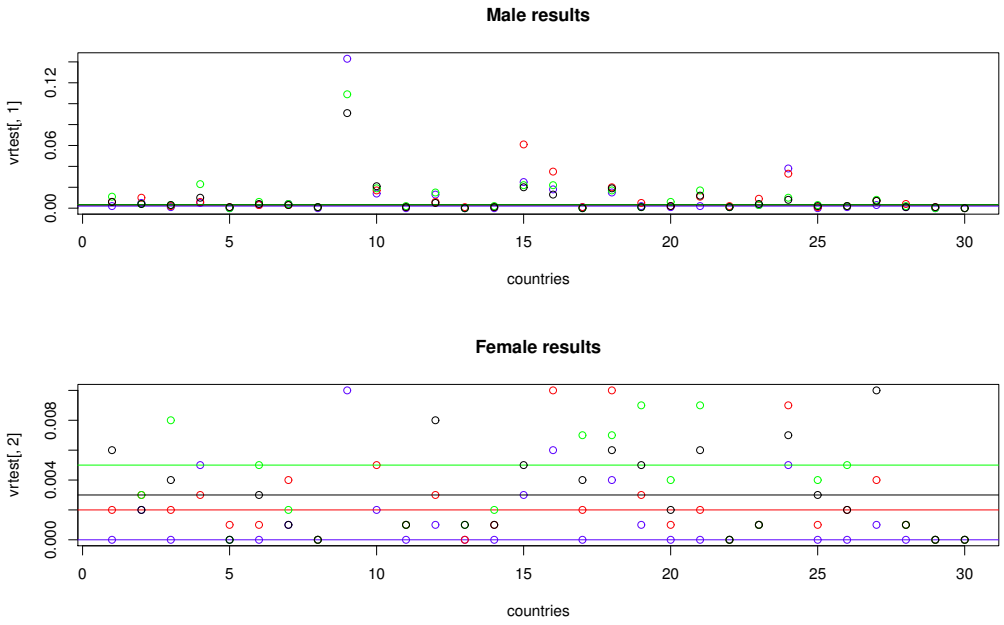

FIGURE 3. Variance ratio test results of the variance of the residuals for both males and females for the Lee Carter (blue), CBD (red), Plat (green) and O'Hare and Li (black) models

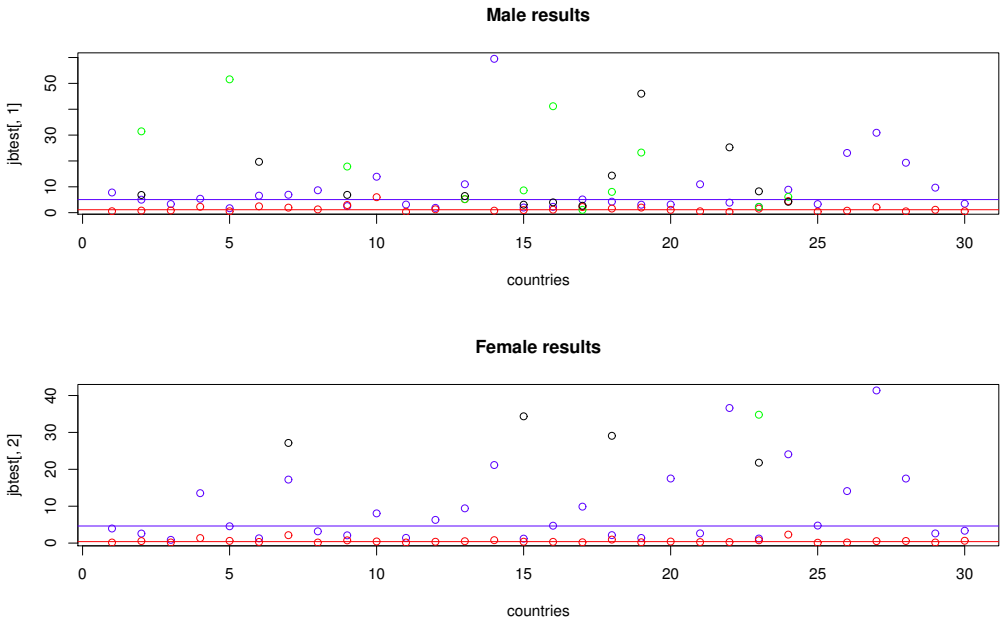

FIGURE 4. Jarque Bera test results of the kurtosis of the residuals for both males and females for the Lee Carter (blue), CBD (red), Plat (green) and O'Hare and Li (black) models

term positive correlation in the residuals both in the age dimension and the period dimension. This suggests that perhaps there is still some structure in the residuals that might be identified. In particular the inclusion of additional period effects (as in the Plat, 2009; and O'Hare and Li, 2013) models does not compensate for this. This is an area of further research.

Finally, figures 5 and 6 show point plots and averages of the Hurst exponent calculations across the age dimension and the period dimension respectively. As can be seen from the plots, and from the tables in the appendix the hurst exponents primarily lie between 0.5 and 1 showing that there is some autocorrelation present in both the age and period dimensions of the residuals. The hurst calculation being larger on average across the period dimension than the age one. This suggests that far from being samples of random noise the residuals are still showing some patterns or structure in both age and period. Note also that the inclusion of more period effects (as in the Plat (2009) and O'Hare and Li (2012) models does not diminish this effect. This is an area for further research.

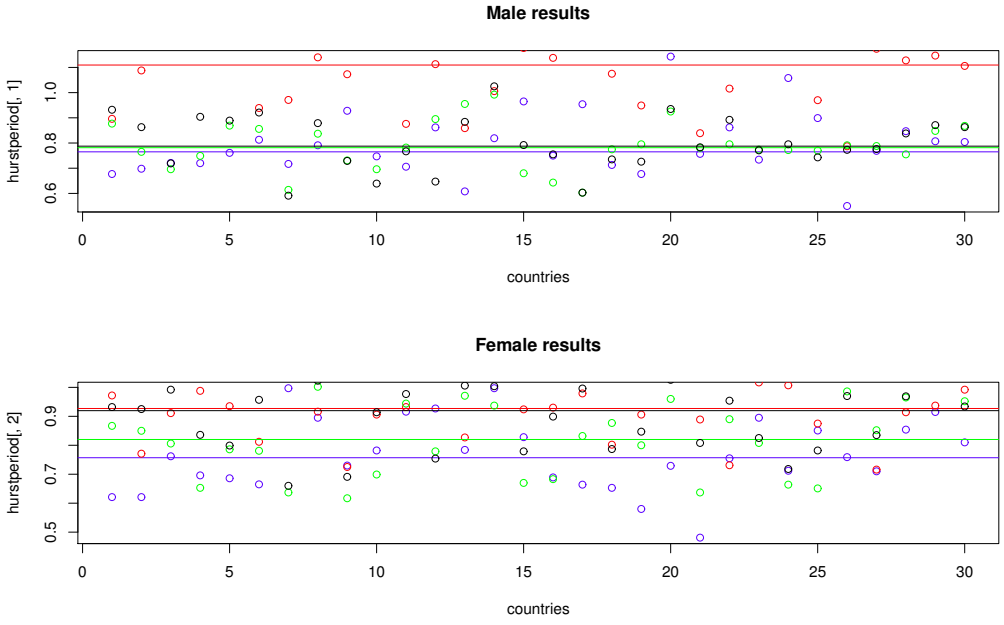

FIGURE 5. Average Hurst calculations across the period dimension of the residuals for both males and females for the Lee Carter (blue), CBD (red), Plat (green) and O'Hare and Li (black) models

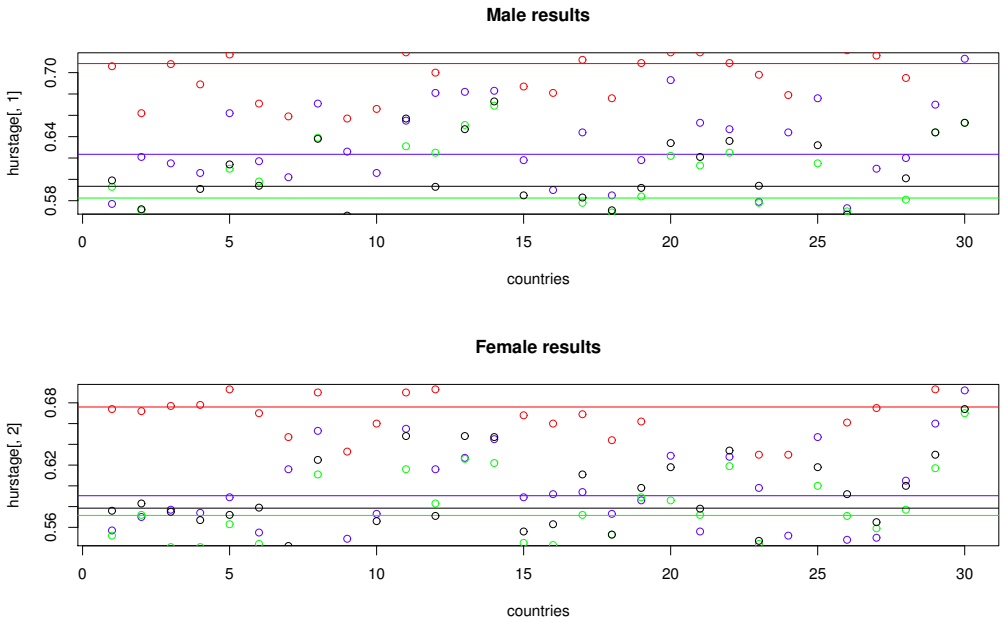

FIGURE 6. Average Hurst calculations across the age dimension of the residuals for both males and females for the Lee Carter (blue), CBD (red), Plat (green) and O'Hare and Li (black) models

5.3. **Implications for pricing, forecasting and policy.** The analysis in this paper has shown that within the extrapolative family of models of mortality considered there can still be found some structure within the residuals after fitting to the data. In particular, for all the models considered, and the vast majority of the counties considered, the models failed the tests on the normality of those residuals. For comparative purposes the lack of normality of the residuals may be considered a minor issue, particularly when the aims is to compare the ability of various models to fit to the same mortality data. However, when using such models to forecast mortality rates, particularly long term forecasting for welfare purposes, for pension pricing purposes or for government longer term planning there remains a significant risk that the forecasts will be inappropriate. In particular, the prevailing trend of mortality improvement determined through the modelling process could be mis specified. In addition, the assumption of independence in the residuals could result in forecasting confidence intervals being narrower than they should be, and hence providing false confidence in the forecasts.

In some cases these concerns may be overcome by repeatedly refitting and forecasting at periodic intervals, for example as actuaries do when creating updated life tables, In many other

cases however, long terms forecasts are used to determine funding and resource allocations. In these cases it would be important to appreciate the limitations of the modelling and forecasting process identified here.

6. CONCLUSIONS

In this paper we have considered several of the leading extrapolative models of mortality rates and have applied normality tests and Hurst calculations to the fitted residuals. More specifically we have fitted the models of Lee and Carter (1992), Cairns, Blake and Dowd (2006), Plat (2009) and O'Hare and Li (2012) to the data for 30 countries for both males and females and tests the resulting residuals using t-tests, variance ratio tests and Jaque bear tests. We have also calculated age and prior Hurst exponents for each of the countries and genders and note that exclusively these Hurst exponents lie in the region $0.5 < 1$. This suggests some positive correlations between residuals.

Further research will now focus further on the Hurst exponents analysing in more detail the patterns found within these exponents to try to identify what if any structure still remains in the data after fitting such time series model.

Appendix: Additional tables and figures

APPENDIX A. APPENDIX 1 - NORMALITY TESTS

TABLE 6. t-test statistic for the Lee-Carter, CBD, Plat, and OL models.

Country	Lee-Carter		CBD		Plat		OL	
	Male	Female	Male	Female	Male	Female	Male	Female
Austria	0.56	0.64	0.6	0.76	0.16	0.08	0.22	0.11
Belarus	0.54	0.56	0.56	0.67	0.45	0.26	0.51	0.34
Belgium	0.6	0.73	0.64	0.75	0.19	0.07	0.21	0.09
Bulgaria	0.54	0.39	0.56	0.67	0.11	0.09	0.18	0.12
Canada	0.59	0.55	0.64	0.56	0.36	0.21	0.27	0.26
Czechoslovakia	0.51	0.64	0.55	0.72	0.26	0.11	0.34	0.18
Denmark	0.48	0.63	0.63	0.56	0.34	0.34	0.37	0.53
England	0.55	0.69	0.54	0.76	0.14	0.14	0.12	0.19
Estonia	0.54	0.69	0.47	0.69	0.4	0.3	0.46	0.46
Finland	0.4	0.51	0.46	0.71	0.27	0.07	0.26	0.09
France	0.59	0.68	0.55	0.73	0.07	0.05	0.09	0.07
Hungary	0.55	0.59	0.55	0.67	0.13	0.07	0.21	0.09
Italy	0.49	0.53	0.47	0.65	0.48	0.06	0.44	0.09
Japan	0.3	0.29	0.51	0.52	0.06	0.04	0.09	0.06
Latvia	0.59	0.72	0.52	0.67	0.49	0.27	0.55	0.43
Lithuania	0.57	0.54	0.54	0.71	0.38	0.12	0.53	0.18
Netherlands	0.52	0.55	0.59	0.67	0.64	0.06	0.6	0.09
New Zealand	0.49	0.65	0.62	0.64	0.46	0.43	0.42	0.5
Norway	0.59	0.73	0.61	0.75	0.61	0.12	0.57	0.18
Poland	0.55	0.41	0.51	0.61	0.11	0.06	0.21	0.1
Portugal	0.49	0.64	0.57	0.77	0.12	0.09	0.16	0.11
Russia	0.52	0.45	0.51	0.66	0.26	0.14	0.35	0.2
Scotland	0.57	0.75	0.56	0.72	0.55	0.49	0.45	0.53
Slovakia	0.42	0.48	0.44	0.56	0.45	0.21	0.47	0.28
Spain	0.56	0.52	0.58	0.75	0.08	0.05	0.1	0.06
Sweden	0.49	0.6	0.65	0.75	0.23	0.11	0.28	0.17
Switzerland	0.4	0.5	0.55	0.68	0.18	0.08	0.2	0.1
Ukraine	0.5	0.54	0.5	0.64	0.21	0.17	0.33	0.24
United Kingdom Total	0.55	0.69	0.55	0.75	0.14	0.14	0.12	0.19
USA	0.62	0.61	0.58	0.55	0.12	0.25	0.11	0.34

TABLE 7. Variance ratio test results (x 10^5) for the Lee-Carter, CBD, Plat, and OL models.

Country	Lee-Carter		CBD		Plat		OL	
	Male	Female	Male	Female	Male	Female	Male	Female
Austria	0	0	0.01	0	0.01	0.01	0.01	0.01
Belarus	0.01	0	0.01	0	0	0	0	0
Belgium	0	0	0	0	0	0.01	0	0
Bulgaria	0.01	0.01	0.01	0	0.02	0.02	0.01	0.01
Canada	0	0	0	0	0	0	0	0
Czechoslovakia	0	0	0	0	0.01	0.01	0	0
Denmark	0	0	0	0	0	0	0	0
England	0	0	0	0	0	0	0	0
Estonia	0.14	0.01	0.16	0.02	0.11	0.03	0.09	0.01
Finland	0.01	0	0.02	0.01	0.02	0.07	0.02	0.05
France	0	0	0	0	0	0	0	0
Hungary	0.01	0	0.01	0	0.02	0.02	0.01	0.01
Italy	0	0	0	0	0	0	0	0
Japan	0	0	0	0	0	0	0	0
Latvia	0.03	0	0.06	0.01	0.02	0.01	0.02	0.01
Lithuania	0.02	0.01	0.04	0.01	0.02	0.03	0.01	0.02
Netherlands	0	0	0	0	0	0.01	0	0
New Zealand	0.02	0	0.02	0.01	0.02	0.01	0.02	0.01
Norway	0	0	0.01	0	0	0.01	0	0.01
Poland	0	0	0	0	0.01	0	0	0
Portugal	0	0	0.01	0	0.02	0.01	0.01	0.01
Russia	0	0	0	0	0	0	0	0
Scotland	0	0	0.01	0	0	0	0	0
Slovakia	0.04	0.01	0.03	0.01	0.01	0.01	0.01	0.01
Spain	0	0	0	0	0	0	0	0
Sweden	0	0	0	0	0	0.01	0	0
Switzerland	0	0	0.01	0	0.01	0.02	0.01	0.01
Ukraine	0	0	0	0	0	0	0	0
United Kingdom Total	0	0	0	0	0	0	0	0
USA	0	0	0	0	0	0	0	0

TABLE 8. Jaque Bera test results (x $10^{-}5$) for Males and Females for the Lee-Carter, CBD, Plat, and OL models.

Country	Lee-Carter		CBD		Plat		OL	
	Male	Female	Male	Female	Male	Female	Male	Female
Austria	7.8	3.95	0.56	0.15	377.33	328.14	315.07	309.82
Belarus	5.02	2.58	0.8	0.55	31.45	283.19	6.84	172.85
Belgium	3.33	0.88	0.85	0.17	160.93	291.78	147.88	273.41
Bulgaria	5.38	13.53	2.28	1.36	203.09	224.77	138.04	196.13
Canada	1.71	4.56	0.54	0.6	51.57	173.59	102.22	138.13
Czechoslovakia	6.58	1.29	2.45	0.31	74.48	276.87	19.67	200.02
Denmark	6.93	17.21	1.95	2.14	80.91	125.17	66.97	27.15
England	8.67	3.19	1.27	0.15	228.04	263.38	242.54	204.85
Estonia	2.52	2.1	2.96	0.74	17.84	180.93	6.88	57.61
Finland	13.9	8.05	5.99	0.44	106.46	285.74	95.23	267.86
France	3.13	1.41	0.35	0.16	274.54	305.68	254.13	291.96
Hungary	1.86	6.29	1.3	0.36	244.57	275.2	187.76	265.45
Italy	10.99	9.42	5.22	0.51	5.26	295.81	6.43	270.59
Japan	59.49	21.16	0.78	0.79	319.23	305.48	301.23	289.4
Latvia	1.99	1.21	1.01	0.38	8.61	112.95	3.08	34.36
Lithuania	2.26	4.71	1.17	0.33	41.16	551.08	3.98	469.41
Netherlands	5.11	9.88	2.18	0.22	1.09	290.96	2.67	271.96
New Zealand	4.25	2.16	1.59	0.95	8.04	60.7	14.34	29.08
Norway	3.09	1.39	1.97	0.22	23.25	263.23	46.01	227.96
Poland	3.16	17.51	1.1	0.42	215.97	257.18	109.65	225.64
Portugal	10.98	2.63	0.54	0.27	242.82	271.89	196.13	262.35
Russia	3.88	36.62	0.3	0.28	128.58	351.3	25.26	255.02
Scotland	2.16	1.23	1.54	0.76	2.25	34.79	8.25	21.8
Slovakia	8.88	24.07	4.48	2.3	6.11	337.76	4.21	280.69
Spain	3.31	4.75	0.38	0.1	278.68	317.27	263.26	307.53
Sweden	23.1	14.1	0.75	0.17	141.51	243.88	114.86	188.42
Switzerland	30.9	41.38	2.07	0.53	186.53	331.42	159.93	316.36
Ukraine	19.31	17.5	0.52	0.57	339.57	345.09	103.75	250.37
United Kingdom Total	9.68	2.62	1.11	0.16	206.39	258.31	224.6	197.56
USA	3.46	3.35	0.54	0.63	198.49	151.27	203.7	72.16

APPENDIX B. HURST TESTS

The Hurst exponent calculations are done by first splitting the matrix of residuals into time series of age specific residuals and period specific residuals. In other words by considering the columns and rows of the matrix separately. Of course we might also consider the cohort pattern (or the diagonals) of the matrix also but we defer this to further study. The results presented below should the Hurst exponents over period and over age.

TABLE 9. Hurst exponents for age specific residuals for the Lee-Carter, CBD, Plat, and OL models.

Country	Lee-Carter		CBD		Plat		OL	
	Male	Female	Male	Female	Male	Female	Male	Female
Austria	0.577	0.557	0.706	0.674	0.593	0.552	0.599	0.576
Belarus	0.621	0.57	0.662	0.672	0.571	0.572	0.572	0.583
Belgium	0.615	0.577	0.708	0.677	0.551	0.541	0.551	0.575
Bulgaria	0.606	0.574	0.689	0.678	0.543	0.541	0.591	0.567
Canada	0.662	0.589	0.717	0.693	0.61	0.563	0.614	0.572
Czechoslovakia	0.617	0.555	0.671	0.67	0.598	0.544	0.594	0.579
Denmark	0.602	0.616	0.659	0.647	0.547	0.526	0.54	0.542
England	0.671	0.653	0.723	0.69	0.639	0.611	0.638	0.625
Estonia	0.626	0.549	0.657	0.633	0.556	0.52	0.566	0.53
Finland	0.606	0.573	0.666	0.66	0.55	0.533	0.548	0.566
France	0.655	0.655	0.719	0.69	0.631	0.616	0.657	0.648
Hungary	0.681	0.616	0.7	0.693	0.625	0.583	0.593	0.571
Italy	0.682	0.627	0.723	0.709	0.651	0.626	0.647	0.648
Japan	0.683	0.645	0.731	0.708	0.669	0.622	0.673	0.647
Latvia	0.618	0.589	0.687	0.668	0.563	0.545	0.585	0.556
Lithuania	0.59	0.592	0.681	0.66	0.543	0.543	0.559	0.563
Netherlands	0.644	0.594	0.712	0.669	0.578	0.572	0.583	0.611
New Zealand	0.585	0.573	0.676	0.644	0.569	0.553	0.571	0.553
Norway	0.618	0.586	0.709	0.662	0.584	0.589	0.592	0.598
Poland	0.693	0.629	0.719	0.709	0.622	0.586	0.634	0.618
Portugal	0.653	0.556	0.719	0.704	0.613	0.572	0.621	0.578
Russia	0.647	0.628	0.709	0.722	0.625	0.619	0.636	0.634
Scotland	0.579	0.598	0.698	0.63	0.578	0.544	0.594	0.547
Slovakia	0.644	0.552	0.679	0.63	0.548	0.518	0.542	0.526
Spain	0.676	0.647	0.731	0.72	0.615	0.6	0.632	0.618
Sweden	0.573	0.548	0.721	0.661	0.57	0.571	0.567	0.592
Switzerland	0.61	0.55	0.716	0.675	0.558	0.559	0.558	0.565
Ukraine	0.62	0.605	0.695	0.709	0.581	0.577	0.601	0.6
United Kingdom	0.67	0.66	0.724	0.693	0.644	0.617	0.644	0.63
USA	0.713	0.692	0.73	0.706	0.653	0.67	0.653	0.674

TABLE 10. Hurst exponents for period specific residuals for the Lee-Carter, CBD, Plat, and OL models.

Country	Lee-Carter		CBD		Plat		OL	
	Male	Female	Male	Female	Male	Female	Male	Female
Austria	0.677	0.621	0.896	0.972	0.877	0.867	0.932	0.932
Belarus	0.698	0.621	1.088	0.771	0.765	0.85	0.863	0.925
Belgium	0.721	0.762	1.307	0.911	0.696	0.806	0.719	0.992
Bulgaria	0.72	0.696	1.195	0.988	0.749	0.653	0.904	0.836
Canada	0.761	0.686	1.223	0.935	0.869	0.786	0.889	0.799
Czechoslovakia	0.813	0.665	0.939	0.812	0.856	0.781	0.921	0.957
Denmark	0.717	0.997	0.971	1.106	0.614	0.637	0.591	0.66
England	0.791	0.895	1.14	0.916	0.837	1.002	0.879	1.023
Estonia	0.928	0.73	1.073	0.725	0.731	0.617	0.729	0.691
Finland	0.747	0.782	1.237	0.906	0.696	0.699	0.639	0.914
France	0.706	0.916	0.876	0.932	0.782	0.944	0.767	0.977
Hungary	0.862	0.927	1.113	1.077	0.895	0.779	0.647	0.754
Italy	0.608	0.784	0.859	0.827	0.955	0.971	0.884	1.006
Japan	0.819	0.997	1.006	1.089	0.992	0.937	1.025	1.004
Latvia	0.965	0.828	1.177	0.924	0.68	0.67	0.792	0.779
Lithuania	0.75	0.689	1.138	0.93	0.643	0.683	0.755	0.899
Netherlands	0.954	0.664	1.198	0.979	0.603	0.832	0.603	0.996
New Zealand	0.713	0.653	1.075	0.802	0.775	0.877	0.735	0.787
Norway	0.677	0.58	0.949	0.906	0.795	0.8	0.726	0.847
Poland	1.143	0.729	1.196	1.089	0.925	0.96	0.935	1.026
Portugal	0.757	0.481	0.839	0.889	0.781	0.637	0.783	0.808
Russia	0.862	0.755	1.016	0.731	0.795	0.89	0.892	0.954
Scotland	0.734	0.895	1.214	1.017	0.774	0.808	0.771	0.825
Slovakia	1.058	0.712	1.254	1.007	0.772	0.664	0.795	0.718
Spain	0.899	0.851	0.97	0.875	0.77	0.651	0.743	0.782
Sweden	0.55	0.759	0.787	1.061	0.791	0.986	0.773	0.97
Switzerland	0.769	0.71	1.174	0.716	0.788	0.852	0.776	0.835
Ukraine	0.847	0.854	1.128	0.914	0.755	0.965	0.838	0.969
United Kingdom Total	0.807	0.915	1.147	0.937	0.848	1.037	0.871	1.04
USA	0.804	0.81	1.106	0.992	0.868	0.952	0.863	0.934

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