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Blanco, A. (2017). On the regularity of homomorphisms between Riesz subalgebras of L^r(X), II. *Indagationes Mathematicae*, *28*(4), 863-883. https://doi.org/10.1016/j.indag.2017.06.009

Published in:
Indagationes Mathematicae

Document Version:
Peer reviewed version

ON THE REGULARITY OF HOMOMORPHISMS BETWEEN RIESZ SUBALGEBRAS OF $\mathcal{L}^r(X)$, II.

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Abstract. We extend and improve our earlier results on automatic regularity of continuous algebra homomorphisms between Riesz algebras of regular operators.

1. Introduction.

Many important examples of algebras in Analysis carry order structures compatible with that of the algebra, in the sense that positivity is preserved by the product. It is also well known that continuity of an algebra homomorphism can follow from mere algebraic assumptions on the algebras involved, and possibly assumptions on the homomorphism itself, such as surjectivity. Thus, it is not unreasonable to wonder whether, in the presence of order structures compatible with those of the algebras, one could go further and conclude not just the continuity but also the regularity of an algebra homomorphism (where by regularity we mean that can be represented as a linear combination of positive linear maps). It is known, for instance, that if the domain and codomain of an algebra homomorphism Θ are both Archimedean semiprime f-algebras, and in addition, the domain is relatively uniformly complete, then Θ must be a Riesz homomorphism [11, Theorem 5.1]. The few results of this kind we are aware of, though, are for Archimedean f-algebras, and therefore confined to the commutative setting.

In [2], we initiated the study of automatic regularity of algebra homomorphisms between Riesz algebras of regular operators on Banach lattices. One of the aims of [2] was to give conditions on the underlying Banach lattices, forcing the automatic regularity of certain algebra homomorphisms (e.g., bounded and injective) between the corresponding algebras of regular operators. We were able to establish, for instance, that if X is a purely atomic Banach lattice whose atoms form a subsymmetric basis for X, and Y is a purely atomic reflexive Banach lattice, then every bounded (and hence, any) homomorphism $\Theta : \mathcal{A}^r(X) \to \mathcal{A}^r(Y)$ is necessarily regular (see below for definitions). Furthermore, we showed that, under the same hypotheses on X and Y, also any injective algebra homomorphism $\Theta : \mathcal{L}^r(X) \to \mathcal{L}^r(Y)$, such that $\operatorname{im}\Theta \cap \mathcal{F}(Y) \neq \{0\}$, must be regular.

In this note we extend the results of [2] to Riesz algebras of regular operators acting on non-atomic Banach lattices that we shall (provisionally) describe as completions of 'direct

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2000 *Mathematics Subject Classification.* Primary 47B48, 47B65; Secondary 46B42.

Key words and phrases. Automatic regularity, Banach algebra, Banach lattice, regular operator.

limits' of atomic Banach lattices of the kind considered in [2]. The resulting class of Banach lattices will be seen to contain important examples such as non-atomic L^p-spaces and rearrangement invariant spaces of functions on the real line (not covered in any way by our previous results). It will follow from our results, in particular from Corollary 3.3, that if X is an L^p-space ($1 \leq p < \infty$) or a rearrangement invariant space on $\mathbb{R}$, and Y is reflexive and also an L^p-space or a rearrangement invariant space on $\mathbb{R}$, then any injective continuous algebra homomorphism $\Theta : \mathcal{L}^r(X) \to \mathcal{L}^r(Y)$, whose image contains elements of rank-one and such that the weak* closure of $\Theta(\mathcal{F}(X))$ (in $\mathcal{L}^r(Y)$) contains an identity for $\operatorname{im}\Theta$, must be regular.

The paper is mostly self-contained, however, arguments that were provided in [2] will be typically omitted. The organization is as follows. In Section 2, we introduce the terminology and notation we shall use throughout the paper and recall some results we will need from [2]. In Section 3, we state and prove our main results on automatic regularity, Corollaries 3.2 and 3.3. Lastly, in Section 4, we discuss some ideas in connection with the problem of finding continuous non-regular algebra homomorphisms between Riesz algebras of regular operators, raised in [2].

2. Some background and notations.

The notation and terminology we shall use throughout the paper is mostly standard and entirely consistent with that of [2]. For the reader's convenience we have collected most of it in this section.

The topological dual of a normed space X shall be denoted by X', and given a subset $\{x_i : i \in I\}$ of X, we shall write $[x_i : i \in I]$ for the closure of its linear span. If the index set I is clear from context we shall write $\{x_i\}$ for $\{x_i : i \in I\}$. We shall also use the notation $\overline{S}^\tau$ for the closure of a subset S of a topological vector space (X, τ). Given a sequence (x_i) and an infinite subset $S = \{n_1 < n_2 < \ldots\}$ of $\mathbb{N}$, we shall often write $(x_i)_{i\in S}$ for the subsequence (x_{n_i}).

If X is an ordered vector space, we denote by X_+ its positive cone. Recall if X is a real Riesz space then the module of an element $x \in X$ is defined by $|x| = x \vee (-x)$. A complex Riesz space is defined as the complexification $X_\mathbb{R} + iX_\mathbb{R}$ of a real Riesz space $X_\mathbb{R}$ with the property that for every sequence (x_n) in X_+, satisfying $x_n \leq \lambda_n x$ ($n \in \mathbb{N}$) for some $x \in X_+$ and $(\lambda_n) \in \ell_1$, the series $\sum_i x_i$ order converges. In this case, $|x| := \sup_{\theta\in[0,2\pi)} \operatorname{Re}(e^{i\theta}x)$ ($x \in X$). All Riesz spaces in the note will be assumed to be Archimedean.

By a **Riesz subspace** of a Riesz space X, we shall mean, as usual, a linear subspace of X closed under the map $x \mapsto |x|$, $X \to X$; and in the complex case, also closed under the map $x + iy \mapsto x - iy$, $X_\mathbb{R} + iX_\mathbb{R} \to X_\mathbb{R} + iX_\mathbb{R}$, so the Riesz subspaces of a complex Riesz space, X, are precisely those linear subspaces of the form $E_\mathbb{R} + iE_\mathbb{R}$, with $E_\mathbb{R}$ a (real) Riesz subspace of $X_\mathbb{R}$.

Recall a complex Banach lattice is defined to be the complexification of a real Banach lattice $(X_\mathbb{R}, \|\cdot\|)$, endowed with the norm $\|x\|_X := \||x|\|$ ($x \in X$). As in [2], we shall say that a separable purely atomic Banach lattice X satisfies ($\star$) if

for some constant $\mu \geq 0$ there exist a sequence $(\mu_n) \subset [1, \mu]$ and a sequential arrangement of its normalized atoms, $\{x_i : i \in \mathbb{N}\}$ say, such that, for every $n \in \mathbb{N}$, *if* $l_1 < l_2 < \cdots < l_n$ *are such that* $(x_i)_{i=1}^n \overset{\mu_n}{\sim} (x_{l_i})_{i=1}^n$ *then for every* $k \in \mathbb{N}$ *there exists* $l_{n+1} \geq l_n + k$ *such that* $(x_i)_{i=1}^{n+1} \overset{\mu_{n+1}}{\sim} (x_{l_i})_{i=1}^{n+1}$

(where the notation $(y_i)_{i=1}^m \overset{\kappa}{\sim} (z_i)_{i=1}^m$ stands for $\kappa^{-1}\big\|\sum_i \alpha_i y_i\big\| \leq \big\|\sum_i \alpha_i z_i\big\| \leq \kappa\big\|\sum_i \alpha_i y_i\big\|$ for every $(\alpha_i) \in c_{00}$). Any Banach lattice whose atoms, in a certain order, form a subsymmetric basis, satisfies ($\star$) almost trivially, and so does any 1-unconditional sum $(\bigoplus_i X_i)_{\mathbf{e}}$ (see paragraph below for definitions) of a sequence (X_i) of Banach lattices satisfying ($\star$) for the same constant μ.

Given a sequence of Banach lattices $(X_i, \|\cdot\|_i)_{i=1}^{\mathfrak{m}}$, where $\mathfrak{m}$ stands for either a natural number or ∞, and a 1-unconditional basis $\mathbf{e} = (e_i)_{i=1}^{\mathfrak{m}}$ for some Banach space E, we shall write $(\bigoplus_i X_i)_{\mathbf{e}}$ for the space of all sequences $(x_i) \in \prod_i X_i$ such that $\sum_i \|x_i\|_i e_i$ converges. It is easy to see that, endowed with the norm $\|(x_i)\| := \big\|\sum_i \|x_i\|_i e_i\big\|_E$ and the positive cone $(\bigoplus_i X_i)_{\mathbf{e}} \bigcap (\prod_i X_i^+)$, the latter becomes a Banach lattice. As customary, by an **order-continuous** Banach lattice we shall mean a Banach lattice whose norm is order-continuous.

By a **topological algebra** we shall mean an associative algebra $\mathcal{A}$, endowed with a topology τ, such that $(\mathcal{A}, \tau)$ is a Hausdorff locally convex topological vector space and multiplication on $\mathcal{A}$ is separately continuous. A bounded net (e_α) in $\mathcal{A}$ such that $\lim_\alpha a e_\alpha = \lim_\alpha e_\alpha a = a$ $(a \in \mathcal{A})$ shall be said to be a **bounded approximate identity** (b.a.i. in short). Two idempotents e, f in an algebra $\mathcal{B}$ shall be said to be **equivalent** if there are u and v in $\mathcal{B}$ such that $e = uv$ and $f = vu$. An associative algebra $\mathcal{A}$ shall be called **semiprime** if $\mathcal{I} = \{0\}$ is the only two-sided ideal of $\mathcal{A}$ with the property that $\mathcal{I}^2 = \{0\}$.

Recall a **Riesz algebra** is an associative algebra $\mathcal{A}$, which is also a Riesz space, where $a, b \in \mathcal{A}_+ \Rightarrow ab \in \mathcal{A}_+$. A Riesz algebra endowed with a lattice and algebra norm will be called a **normed Riesz algebra**. A normed Riesz algebra shall be said to be **Levi** if every normed bounded increasing net in it has a supremum. A complete normed Riesz algebra shall be called a **Banach lattice algebra**, and shall be said to be **dual** if it is a dual Banach space. An algebra ideal and Riesz subspace of a Riesz algebra $\mathcal{A}$ will be said to be a **Riesz algebra ideal**. Elements a, b from a Riesz algebra $\mathcal{A}$ shall be said to be **orthogonal** if $ab = 0$, and **disjoint** if $|a| \wedge |b| = 0$.

Given a linear map $T : X \to Y$ and a subspace E of X (resp. F of Y), we shall write $T|_E$ (resp. $T|^F$) for the restriction to E (resp. corestriction to F) of T. The image of T shall be denoted by $T(X)$.

As customary, a linear map T between Riesz spaces X and Y shall be called **positive** if $T(X_+) \subseteq Y_+$, and **regular** if it is a linear combination of positive maps. We shall write $\mathcal{L}^r(X, Y)$ for the space of all regular maps from X to Y. We note that, in the complex case, $\mathcal{L}^r(X, Y)$ can be naturally identified with the complexification of $\mathcal{L}^r(X_{\mathbb{R}}, Y_{\mathbb{R}})$. If X and Y are Banach lattices, the map $\|\cdot\|_r : \mathcal{L}^r(X, Y) \to \mathbb{R}_+$ defined by $\|T\|_r := \inf\{\|S\| : S \in \mathcal{B}(X, Y) \text{ and } |Tx| \leq S|x| \ (x \in X)\}$ $(T \in \mathcal{L}^r(X, Y))$ defines a norm on $\mathcal{L}^r(X, Y)$ which turns the latter into a Banach space (here $\mathcal{B}(X, Y)$ stands for the space of all bounded linear operators from X to Y with the operator norm $\|\cdot\|$). If X and Y are Banach lattices, $\mathcal{L}^r(X, Y)$ should be understood as $(\mathcal{L}^r(X, Y), \|\cdot\|_r)$.

In general, given Riesz spaces X and Y, $\mathcal{L}^r(X, Y)$ need not be a lattice. Indeed, for $T \in \mathcal{L}^r(X, Y)$, $|T|$ exists if and only if $\sup_{|\xi| \leq x} |T\xi|$ exists for every $x \in X_+$, in which case

$|T|(x) = \sup_{|\xi| \le x} |T\xi|$ $(x \in X_+)$. This result shall provide the main criterion for deciding on the regularity of a given linear map. As an immediate consequence of it, if Y happens to be Dedekind complete, then $\mathcal{L}^r(X,Y)$ is a lattice, actually, a Dedekind complete Riesz space. If X and Y are Banach lattices, with Y Dedekind complete, then $\|\cdot\|_r$ is, in addition, a Banach lattice norm and $\|T\|_r = \||T|\|$ $(T \in \mathcal{L}^r(X,Y))$.

Given Banach lattices X and Y, we shall write $\mathcal{F}(X,Y)$ for the operator ideal of all continuous finite-rank operators from X to Y, and $\mathcal{A}^r(X,Y)$ for its closure in $\mathcal{L}^r(X,Y)$ with respect to the regular norm. As customary, we write $\mathcal{A}^r(X)$ and $\mathcal{L}^r(X)$ if $X = Y$. In this case, $\mathcal{L}^r(X)$ and $\mathcal{A}^r(X)$ are Banach algebras. If X is Dedekind complete then $\mathcal{L}^r(X)$ is a Banach lattice algebra and $\mathcal{A}^r(X)$ is a Riesz algebra ideal of $\mathcal{L}^r(X)$. It is also well known that $\mathcal{A}^r(X)$ is always a Banach lattice algebra.

We shall write $X \otimes_{|\pi|} Y$ for the **positive projective tensor product** of two Banach lattices X and Y (for its definition in the real case see [8, Section 1]; in the complex case, $X \otimes_{|\pi|} Y$ is just the complexification of $X_{\mathbb{R}} \otimes_{|\pi|} Y_{\mathbb{R}}$). The positive projective tensor product has the (universal) property that for every regular bilinear map $\phi : X \times Y \to Z$, where Z is another Banach lattice, there is a unique regular linear map $\varphi : X \otimes_{|\pi|} Y \to Z$ such that $\phi(x,y) = \varphi(x \otimes y)$ $(x \in X,\ y \in Y)$ (here by a regular bilinear map we mean a bilinear map that can be represented as a linear combination of positive bilinear maps). The positive projective tensor product $X \otimes_{|\pi|} Y$ contains $X \otimes Y$ as a norm-dense subspace. Furthermore, its topological dual can be isometrically identified as a Riesz space with $\mathcal{L}^r(X, Y')$. Precisely, the map $\psi \mapsto \big[T_\psi : X \to Y,\ x \mapsto \psi\big(x \otimes (\cdot)\big)\big]$, $(X \otimes_{|\pi|} Y)' \to \mathcal{L}^r(X, Y')$, is an isometric Riesz isomorphism.

Recall a linear projection P on a Riesz space X is a **band** (or **order**) **projection** if and only if $0 \le P \le \mathrm{id}_X$ in the real case; and if and only if $P(X_{\mathbb{R}}) \subseteq X_{\mathbb{R}}$ and $P|_{X_{\mathbb{R}}}^{X_{\mathbb{R}}}$ is a (real) band projection on $X_{\mathbb{R}}$ in the complex case. Recall also that if $\mathcal{A}$ is an algebra, then a linear map $T : \mathcal{A} \to \mathcal{A}$ such that $T(ab) = T(a)b$ (resp. $T(ab) = aT(b)$) $(a, b \in \mathcal{A})$ is a **left** (resp. **right**) **multiplier of** $\mathcal{A}$. The algebra of left (resp. right) multipliers of $\mathcal{A}$ will be denoted by $\mathcal{M}_l(\mathcal{A})$ (resp. $\mathcal{M}_r(\mathcal{A})$).

Lastly, we collect in a theorem some facts established in [2] that will be required in the next section. First recall from [2, § 3] that an idempotent e in a Riesz algebra $\mathcal{A}$ is **o-minimal** if it is positive and $e\mathcal{A}e = \mathbb{K}e$, where $\mathbb{K}$ stands for the underlying field; if $\mathcal{A}$ is a topological algebra, a sequence $(a_n) \in \mathcal{A}$ is **convergence preserving** if for every convergent sequence $(\xi_n) \subset \mathcal{A}$, the sequences $(a_n\xi_n)$ and $(\xi_n a_n)$ converge; furthermore, (a_n) is said to **factor through** a sequence $(b_n) \subset \mathcal{A}$ if there are bounded convergence preserving sequences (u_n) and (v_n) in $\mathcal{A}_+$ such that $a_n = u_n b_{k_n} v_n$ $(n \in \mathbb{N})$ for some increasing sequence (k_n) in $\mathbb{N}$. Recall also a Riesz space is Dedekind σ-complete if every non-empty countable subset of it which is bounded above has a supremum.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Riesz and topological algebras and let $\Theta : \mathcal{A} \to \mathcal{B}$ be a sequentially continuous algebra homomorphism. Suppose $\mathcal{A}_+$ is closed, $\mathcal{A}$ is semiprime, $\mathcal{B}$ embeds continuously as a Riesz algebra ideal into a Dedekind σ-complete Riesz and topological algebra $(\widetilde{\mathcal{B}}, \omega)$, and there are*

- *a b.a.i. $(e_i)_{i\in\mathbb{N}}$ for $\mathcal{A}$;*
- *a sequence (p_i) of mutually orthogonal o-minimal idempotents in $\mathcal{A}$; and*

– *sequences* $(P_i) \subset \mathcal{M}_l(\widetilde{\mathcal{B}})$ *and* $(Q_i) \subset \mathcal{M}_r(\widetilde{\mathcal{B}})$ *of continuous disjoint band projections;*

such that

i) *the sequence* $\pi_n := \sum_{i=1}^n p_i$ $(n \in \mathbb{N})$ *is bounded, convergence preserving and* (e_i) *factors through it;*
ii) $\sup_i P_i$ *and* $\sup_j Q_j$ *are sequentially continuous;*
iii) $\sum_i (P_i(\Theta(p_i)) - \Theta(p_i))$ *and* $\sum_i (Q_i(\Theta(p_i)) - \Theta(p_i))$ *exist in* $\mathcal{B}$*; and*
iv) $\sum_i |P_i(\Theta(p_i)) - \Theta(p_i)||\Theta(p_i)|$ *and* $\sum_i |\Theta(p_i)||Q_i(\Theta(p_i)) - \Theta(p_i)|$ *exist in* $\mathcal{B}$.

Then

$$\Theta(a) = \widetilde{b}_u \Psi\Big(\lim_i \lim_j v_i a u_j\Big)\widetilde{b}_v \quad (a \in \mathcal{A}),$$

where (u_i) *and* (v_i) *are bounded convergence preserving sequences in* $\mathcal{A}_+$ *such that* $e_i = u_i \pi_{k_i} v_i$ $(i \in \mathbb{N})$ *for some increasing sequence* (k_i) *in* $\mathbb{N}$, $\widetilde{b}_u$ *and* $\widetilde{b}_v$ *are limit points of* $\{\Theta(u_i) : i \in \mathbb{N}\}$ *and* $\{\Theta(v_i) : i \in \mathbb{N}\}$*, respectively, and* $\Psi : \mathcal{A} \to \mathcal{B}$ *is the regular map defined by*

$$\Psi(a) = \Theta\Big(\lim_m \lim_n \pi_m a \pi_n\Big) \quad (a \in \mathcal{A}).$$

Furthermore, if $\Phi : \mathcal{A} \to \mathcal{B}$, $a \mapsto \sum_i \sum_j P_i(Q_j(\Theta(p_i a p_j)))$, $\Phi_{Q,k} : \mathcal{A} \to \mathcal{B}$, $a \mapsto \sum_j Q_j(\Theta(p_k a p_j))$, $\Phi_{P,l} : \mathcal{A} \to \mathcal{B}$, $a \mapsto \sum_i P_i(\Theta(p_i a p_l))$ *and* $\Theta_{kl} : \mathcal{A} \to \mathcal{B}$, $a \mapsto \Theta(p_k a p_l)$, $(k, l \in \mathbb{N})$*, then, for every* $a \in \mathcal{A}_+$, $|\Phi|(a) = |\Phi(a)|$, $|\Phi_{Q,k}|(a) = |\Phi_{Q,k}(a)|$, $|\Phi_{P,l}|(a) = |\Phi_{P,l}(a)|$, $|\Theta_{kl}|(a) = |\Theta_{kl}(a)|$ $(k, l \in \mathbb{N})$*, and*

$$\begin{aligned}\Psi(\cdot) = \Phi(\cdot) &+ \sum_i (\Theta(p_i) - P_i(\Theta(p_i)))\Phi_{Q,i}(\cdot) + \sum_j \Phi_{P,j}(\cdot)(\Theta(p_j) - Q_j(\Theta(p_j))) \\ &+ \sum_i \sum_j (\Theta(p_i) - P_i(\Theta(p_i)))\Theta_{ij}(\cdot)(\Theta(p_j) - Q_j(\Theta(p_j))).\end{aligned}$$

Lastly, if the map $a \mapsto \Theta(a)$, $\mathcal{A} \to (\widetilde{\mathcal{B}}, \omega)$*, is compact, i.e., maps bounded sets to precompact ones, then* Θ *is regular.*

All the above was established in [2], although not everything was explicitly stated there. We should notice, though, that in Theorem 3.2 from [2], we assumed the embedding of $\mathcal{B}$ into $\widetilde{\mathcal{B}}$ to be compact, and therefore, the statement of Theorem 2.1, above, is slightly stronger than that of [2, Theorem 3.2], for if $\mathcal{B}$ embeds compactly in $(\widetilde{\mathcal{B}}, \omega)$ and $\Theta : \mathcal{A} \to \mathcal{B}$ is sequentially continuous then $a \mapsto \Theta(a)$, $(\mathcal{A}, \tau) \to (\widetilde{\mathcal{B}}, \omega)$, is necessarily compact. The proof of Theorem 2.1, on the other hand, remains the same as that of Theorem 3.2 of [2], for the compactness of the embedding was used only at the end of the proof of [2, Theorem 3.2] to conclude the existence of limit points for the sets $\{\Theta(u_i) : i \in \mathbb{N}\}$ and $\{\Theta(v_i) : i \in \mathbb{N}\}$; clearly, the same conclusion can be achieved under the weaker assumption that $a \mapsto \Theta(a)$, $(\mathcal{A}, \tau) \to (\widetilde{\mathcal{B}}, \omega)$, is compact.

We should also mention here that it was the stronger result above that we had in mind while proving Theorem 4.1 of [2]. Indeed, in proving Theorem 4.1 of [2], we replaced the norm topology on $\mathcal{B}$ by that induced by ω, and clearly, the embedding of $(\mathcal{B}, \omega)$ into $(\widetilde{\mathcal{B}}, \omega)$ need not be compact, though it is still continuous. On the other hand, the topology τ from the proof of [2, Theorem 4.1] is such that τ-boundedness and norm-boundedness on $\mathcal{A}$ coincide; taking this into account, compactness of the map $a \mapsto \Theta(a)$, $(\mathcal{A}, \tau) \to$

$(\widetilde{\mathcal{B}}, \omega)$, follows easily from the norm-continuity of Θ together with the compactness of the embedding of $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ into $(\widetilde{\mathcal{B}}, \omega)$. We apologize to the reader for any confusion this oversight on our part may have caused.

Remark 2.2. Although it will not be needed here, it seems worth pointing out that the assumption of $\mathcal{A}$ being a Riesz algebra, in Theorem 2.1, can be weaken to $\mathcal{A}$ being an algebra in which multiplication preserves positivity and also an ordered vector space with the decomposition property (recall an ordered vector space X is said to have the decomposition property if whenever $x, u, v \in X_+$ and $x \leq u + v$, there are $y, z \in X_+$ such that $y \leq u$, $z \leq v$ and $y + z = x$). Indeed, the only reason for assuming $\mathcal{A}$ is a lattice (and not just an ordered vector space) is because it is amongst the hypotheses on the domains of the maps of Theorems 1.10, 1.14 and 1.19 of [1] (which we used in the proof of Theorem 4.1 of [2]). However, it is known and apparent from their proofs, that all these results still hold under the weaker assumptions just mentioned.

Further concepts, together with any material relevant to their definitions, shall be presented as they are needed. Moreover, as in [2], we shall deal (whenever possible) with the real and complex cases simultaneously. For any unexplained material on Banach lattices we refer the reader to [1] and [15].

3. Main results.

As stated in the introduction, it is the main aim of the note to extend the results of [2] to 'direct limits' of atomic Banach lattices of the kind considered in [2]. Let us start this section by making this statement more precise.

First of all, given a Banach lattice X, we shall call a (bounded) family of positive projections $\Pi \subset \mathcal{L}^r(X)$ such that $\pi(X)$ is a Riesz subspace of X for every $\pi \in \Pi$ and $\overline{\bigcup_{\pi\in\Pi} \pi(X)} = X$, a (**bounded**) **generating system for** X. Furthermore, if the set $\{\pi(X) : \pi \in \Pi\}$, partially ordered by inclusion, is a directed set, we shall say that Π is **directed**. If X and Π are clear from context we shall write p_π for the corestriction $\pi|^{\pi(X)}$ and $\imath_\pi$ for the natural inclusion of $\pi(X)$ into X $(\pi \in \Pi)$.

Also, for a Banach lattice X, we shall denote by $\mathfrak{F}_X^{\star,\mu}$ the collection of all its vector sublattices, isometrically lattice isomorphic to Banach lattices of the form $(\bigoplus_i F_i)_{\mathbf{e}}$, where $(F_i)_{i=1}^{\mathfrak{m}}$ is a sequence of infinite dimensional separable purely atomic Banach lattices satisfying $(\star)$ for the same constant μ and $\mathbf{e}$ is a 1-unconditional basis with $|\mathbf{e}| = \mathfrak{m}$. Our concern here will be, precisely, with Banach lattices for which there is a bounded generating system Π, as above, such that $\pi(X) \in \mathfrak{F}_X^{\star,\mu}$ $(\pi \in \Pi)$ for some fixed μ. Clearly, we can think of one such X (in the obvious way), as a direct limit of atomic Banach lattices of the kind considered in [2].

Before going any further, let us point out that, apart from those cases in which $X \in \mathfrak{F}_X^{\star,\mu}$ (such cases were considered in [2]), the above class of Banach lattices includes:

i) All infinite dimensional L^p-spaces $(1 \leq p < \infty)$, for if $X = L^p(\Omega, \Sigma, \nu)$ then one can take as Π, the collection of all averaging projections $\pi_{(\Omega_i)} : X \to X$, $f \mapsto \sum_i \big(\nu(\Omega_i)^{-1} \int_{\Omega_i} f \, d\nu\big) \chi_{\Omega_i}$, with $(\Omega_i) \subset \Sigma \setminus \{A \in \Sigma : \nu(A) = 0\}$ a disjoint sequence.

ii) Any order-continuous rearrangement invariant space on $[0,\infty)$. Indeed, if X is one such space then one can take as Π the sequence of averaging projections $\pi_n : X \to X$, $f \mapsto \sum_i 2^n\big(\int_{\Delta_{i,n}} f\,d\lambda\big)\chi_{\Delta_{i,n}}$ $(n \in \mathbb{N})$, where $\Delta_{i,n} := [(i-1)/2^n, i/2^n]$ $(i,n \in \mathbb{N})$ and λ stands for the Lebesgue measure (to see Π is generating note that since X is order continuous, the simple integrable functions are dense on it; combining this fact with the regularity of λ, it is then easy to see that the linear span of the characteristic functions of dyadic intervals is dense in X).

iii) Certain Banach lattices of vector-valued functions (examples (i) and (ii) can be seen as particular cases of this one). Precisely, let X be a Banach lattice with a directed bounded generating system $\Pi_1 \subset \mathcal{F}(X)$, and let Y be a Banach lattice with a directed bounded generating system Π_2 such that $\kappa(Y) \in \mathfrak{F}_Y^{\star,\mu}$ $(\kappa \in \Pi_2)$. For each pair $(\pi,\kappa) \in \Pi_1 \times \Pi_2$, let $X_\pi := \pi(X)$, let $Y_\kappa := \kappa(Y)$, let $(e_{i,\pi})_{i=1}^{n_\pi}$ be a basis of atoms for X_π, and let $X_\pi(Y_\kappa)$ be the vector space $X_\pi \otimes Y_\kappa$, endowed with the norm $\big\|\sum_{i=1}^{n_\pi} e_{i,\pi}\otimes y_i\big\| := \big\|\sum_{i=1}^{n_\pi}\|y_i\|e_{i,\pi}\big\|$ $(\sum_{i=1}^{n_\pi} e_{i,\pi}\otimes y_i \in X_\pi \otimes Y_\kappa)$ and the positive cone $\{\sum_{i=1}^{n_\pi} e_{i,\pi}\otimes y_i : y_i \in Y_+,\ 1 \le i \le n_\pi\}$. It is easy to see that if $(\pi',\kappa') \in \Pi_1 \times \Pi_2$ is such that $\pi(X) \subseteq \pi'(X)$ and $\kappa(Y) \subseteq \kappa'(Y)$, then the natural embedding of $X_\pi(Y_\kappa)$ into $X_{\pi'}(Y_{\kappa'})$ is an isometric Riesz isomorphism, while the map $\pi\otimes\kappa : X_{\pi'}(Y_{\kappa'}) \to X_\pi(Y_\kappa)$, $\sum_{i=1}^{n_{\pi'}} e_{i,\pi'}\otimes y_i \mapsto \sum_{i=1}^{n_{\pi'}} \pi(e_{i,\pi'})\otimes\kappa(y_i)$, is a projection of norm $\le \|\pi\|\|\kappa\|$. To see the latter, let $\sum_i e_{i,\pi'}\otimes y_i \in X_{\pi'}(Y_{\kappa'})$ arbitrary, and for every $1 \le j \le n_\pi$, let $I_j := \big\{i : \pi(e_{i,\pi'}) \in [e_{j,\pi}]\big\}$ and let $\pi(e_{i,\pi'}) = \gamma_i e_{j,\pi}$ $(i \in I_j)$. Then

$$\begin{aligned}\Big\|\pi\otimes\kappa\Big(\sum\nolimits_i e_{i,\pi'}\otimes y_i\Big)\Big\| &= \Big\|\sum\nolimits_i \pi(e_{i,\pi'})\otimes\kappa(y_i)\Big\| = \Big\|\sum\nolimits_j e_{j,\pi}\otimes\Big(\sum\nolimits_{i\in I_j}\gamma_i\kappa(y_i)\Big)\Big\|\\ &\le \|\kappa\|\Big\|\sum\nolimits_j\Big(\sum\nolimits_{i\in I_j}\gamma_i\|y_i\|\Big)e_{j,\pi}\Big\| = \|\kappa\|\Big\|\sum\nolimits_i \|y_i\|\pi(e_{i,\pi'})\Big\|\\ &\le \|\kappa\|\|\pi\|\Big\|\sum\nolimits_i e_{i,\pi'}\otimes y_i\Big\|.\end{aligned}$$

Let $X(Y)$ be the norm closure of the direct limit of the system $\{X_\pi(Y_\kappa) : \pi \in \Pi_1,\ \kappa \in \Pi_2\}$ (with the natural inclusion maps taken as the morphisms). An straightforward consequence of the preceding discussion is that for every $(\pi,\kappa) \in \Pi_1 \times \Pi_2$ there is a positive projection $P_{\pi\otimes\kappa} : X(Y) \to X(Y)$ with range $X_\pi(Y_\kappa)$ and norm $\le \|\pi\|\|\kappa\|$. It is then easy to verify that the set $\Pi := \{P_{\pi\otimes\kappa} : \pi \in \Pi_1,\ \kappa \in \Pi_2\}$ is a directed bounded generating system for $X(Y)$ such that $P_{\pi\otimes\kappa}(X(Y)) = X_\pi(Y_\kappa) \in \mathfrak{F}_{X(Y)}^{\star,\mu}$ $((\pi,\kappa) \in \Pi_1 \times \Pi_2)$ (to see this last note that $X_\pi(Y_\kappa) = \big(\bigoplus_{i=1}^{n_\pi} Y_i\big)_{\mathbf{e}_\pi}$, where $\mathbf{e}_\pi := (e_{i,\pi})_{i=1}^{n_\pi}$ and $Y_i = Y_\kappa$, $1 \le i \le n_\pi$). Spaces of Bochner p-integrable functions with values in a Banach lattice Y with a generating system Π_2 as above, are probably the most obvious examples one can obtain in this way. However, the construction just outlined allows for much more general Banach lattices in the role of X. Indeed, the above assumptions on X could be seen as some strong form of positive approximation property, but nonetheless, one that is satisfied by most known examples of Banach lattices.

As in [2], given a Banach lattice X, by a **Riesz operator subalgebra of** $\mathcal{L}^r(X)$, we shall mean a subalgebra of $\mathcal{L}^r(X)$, which is a Riesz subspace and contains the ideal $\mathcal{F}(X)$. Naturally, if the lattice structure of X and its generating system Π are going to play any role in our arguments there should be a special interplay between them and the Riesz operator subalgebras of $\mathcal{L}^r(X)$ that we plan to consider. With this in mind, we introduce one more notion (at this stage, mainly for the sake of presenting our results in slightly more generality). Given a Banach lattice X with a generating system Π and an operator subalgebra $\mathcal{A}$ of $\mathcal{L}^r(X)$, we shall say that $\mathcal{A}$ is a Π **hereditary Riesz operator subalgebra of** $\mathcal{L}^r(X)$ if for every $\pi \in \Pi$, $\pi\mathcal{A}\pi \subseteq \mathcal{A}$ and the algebra

$$\mathcal{A}_\pi := \{T \in \mathcal{L}^r(\pi(X)) : \imath_\pi T p_\pi \in \mathcal{A}\},$$

is a Riesz operator subalgebra of $\mathcal{L}^r(\pi(X))$.

At first glance, the definition of a Π hereditary Riesz operator subalgebra may seem rather restrictive, however, it is still weak enough to accommodate important examples of Riesz operator subalgebras. For instance, any order and algebra ideal of $\mathcal{L}^r(X)$ is a Π hereditary Riesz operator subalgebra of $\mathcal{L}^r(X)$ with respect to any generating system Π of X. Indeed, if $\mathcal{A}$ is an order and algebra ideal of $\mathcal{L}^r(X)$ then for any positive projection $\pi \in \mathcal{L}^r(X)$ and every $T \in \mathcal{A}_\pi$, $T \in \mathcal{A}_\pi \Rightarrow \imath_\pi T p_\pi \in \mathcal{A} \Rightarrow |\imath_\pi T p_\pi| \in \mathcal{A} \Rightarrow \pi|\imath_\pi T p_\pi|\pi \in \mathcal{A}$ $\Rightarrow \imath_\pi |T| p_\pi \in \mathcal{A}$ (because $\imath_\pi |T| p_\pi = \imath_\pi |p_\pi \imath_\pi T p_\pi \imath_\pi| p_\pi \leq \pi|\imath_\pi T p_\pi|\pi$) $\Rightarrow |T| \in \mathcal{A}_\pi$. We should notice though that not every Π hereditary Riesz operator subalgebra of $\mathcal{L}^r(X)$ is a ring ideal. For instance, if X is order continuous (so $\mathcal{A}^r(X)$ is an order and algebra ideal of $\mathcal{L}^r(X)$) and $\mathcal{D}$ is the solid hull of the subalgebra generated by the π's, then $\mathcal{A}^r(X) + \mathcal{D}$ is a Π hereditary Riesz operator subalgebra. It is not difficult to find examples of order continuous Banach lattices and generating systems for which $\mathcal{A}^r(X) + \mathcal{D}$ is not an algebra ideal of $\mathcal{L}^r(X)$.

We are at last almost ready to state and prove the first result of the note, but before, to simplify, we introduce some more notation. We shall write $\mathfrak{P}_\infty(\mathbb{N})$ for the family of all infinite subsets of $\mathbb{N}$ and $\mathcal{L}_{w^*}(X)$ for the subspace of all weak*-continuous maps in $\mathcal{L}^r(X)$, where X is a Banach lattice. Furthermore, given $S \subset X$, we shall write $S^\wedge$ (resp. $S^\vee$) for the set $\big\{x \in X : x = \sup_n x_n$ for some sequence (x_n) in $S\big\}$ (resp. $\big\{x \in X :$ $x = \inf_n x_n$ for some sequence (x_n) in $S\big\}$).

Theorem 3.1. *Let X be an order continuous Banach lattice with a directed bounded generating system Π such that $\pi(X) \in \mathfrak{F}_X^{\star,\mu}$ $(\pi \in \Pi)$ for some $\mu \geq 1$, and let $\mathcal{A}$ be a Π hereditary Riesz operator subalgebra of $\mathcal{L}^r(X)$. Let $\widetilde{\mathcal{B}}$ be a dual Banach lattice algebra and let $\mathcal{B}$ be a Levi Riesz algebra ideal of $\widetilde{\mathcal{B}}$. Furthermore, suppose $\mathcal{B}$ contains a closed Riesz algebra ideal $\mathcal{B}_0$ such that whenever $(b_i) \subset \mathcal{B}_0$ is a sequence of mutually orthogonal equivalent idempotents, equivalent to the unit vector basis of c_0, there are sets $\{P_{n,S} : n \in \mathbb{N}$ & $S \in \mathfrak{P}_\infty(\mathbb{N})\}$ and $\{Q_{n,S} : n \in \mathbb{N}$ & $S \in \mathfrak{P}_\infty(\mathbb{N})\}$ of band projections satisfying:*

- *$(\{P_{n,S}\}^\wedge)^\vee \subset \mathcal{M}_l(\widetilde{\mathcal{B}}) \cap \mathcal{L}_{w^*}(\widetilde{\mathcal{B}})$ and $(\{Q_{n,S}\}^\wedge)^\vee \subset \mathcal{M}_r(\widetilde{\mathcal{B}}) \cap \mathcal{L}_{w^*}(\widetilde{\mathcal{B}})$,*
- *$P_{n,S}(\mathcal{B}_0) \subseteq \mathcal{B}$ and $Q_{n,S}(\mathcal{B}_0) \subseteq \mathcal{B}$ $(n \in \mathbb{N},\ S \in \mathfrak{P}_\infty(\mathbb{N}))$,*
- *$\lim_n P_{n,S}(b_i) = b_i = \lim_n Q_{n,S}(b_i)$ $(i \in \mathbb{N},\ S \in \mathfrak{P}_\infty(\mathbb{N}))$, and*
- *$\inf_{i\in S} \|P_{n,S}(b_i)\| = 0 = \inf_{i \in S} \|Q_{n,S}(b_i)\|$ $(n \in \mathbb{N},\ S \in \mathfrak{P}_\infty(\mathbb{N}))$.*

Then any continuous injective algebra homomorphism $\Theta : \mathcal{A} \to \mathcal{B}$, such that $\overline{\Theta(\mathcal{A})} \cap \mathcal{B}_0 \neq \{0\}$ and $w^*\text{-}\lim_\pi \Theta(\pi T \pi) = \Theta(T)$ $(T \in \mathcal{A})$, is necessarily regular.

If Π can be taken to be $\{\mathrm{id}_X\}$, i.e., if $X \in \mathfrak{F}_X^{\star,\mu}$, then the Levi assumption on $\mathcal{B}$ can be dropped and instead of $\|\cdot\|_{\mathcal{B}} := \|\cdot\|_{\widetilde{\mathcal{B}}}$ one can simply assume that $\|\cdot\|_{\mathcal{B}}$ is such that $\mathcal{B}$ embeds continuously into $\widetilde{\mathcal{B}}$.

In proving Theorem 3.1, we shall make use of the fact, proven in [2, Lemma 4.5], that under the above hypotheses, $\Theta(\mathcal{F}(X)) \subseteq \mathcal{B}_0$. In this regard, we take the opportunity to make a correction to the statements of Lemma 4.5 and Theorem 4.1 of [2]. The assumption of $\mathcal{B}_0$ being closed in $\mathcal{B}$ is missing in the statement of [2, Lemma 4.5], although it is implicitly used in its proof. In turn, the assumption should have been made also in the statement of [2, Theorem 4.1]. We note, though, that in the special case where $\mathcal{F}(Y) \subseteq \mathcal{B}_0 \subseteq \mathcal{A}^r(Y)$ for some Banach lattice Y, $\Theta(\mathcal{F}(X)) \subset \mathcal{B}_0$ is still true, for $\Theta(\mathcal{F}(X)) \subset \overline{\mathcal{F}(Y)} \Rightarrow \Theta(\mathcal{F}(X)) \subset \mathcal{F}(Y)$ (simply note that every idempotent in $\mathcal{A}^r(Y)$ is a fortiori in $\mathcal{F}(Y)$). It is easy to see that no other result from [2] is affected by this change (although, for consistency, one should let $\mathcal{B}_0 = \mathcal{B} \cap \mathcal{A}^r(Y)$ in the proof of [2, Corollary 4.7]). We also note in passing that the assumption of $\mathcal{B}_0$ being closed in $\mathcal{B}$ is not needed if $\mathcal{B}$ is complete.

Proof of Theorem 3.1. Let $\Theta : \mathcal{A} \to \mathcal{B}$ be an algebra homomorphism as in the hypotheses. Fix $\pi \in \Pi$ and set $F := \pi(X)$, so we can assume $F = (\bigoplus_i F_i)_{\mathbf{e}}$ for some sequence (F_i) of infinite-dimensional separable purely atomic Banach lattices satisfying $(\star)$ for the same constant μ, and some 1-unconditional basis $\mathbf{e}$. Let $\Theta_\pi : \mathcal{A}_\pi \to \mathcal{B}$, $T \mapsto \Theta(\imath_\pi T p_\pi)$. It is then easy to see that Θ_π is a continuous injective algebra homomorphism such that $\Theta_\pi(\mathcal{F}(F)) \subset \mathcal{B}_0$.

Let $I := \{i \in \mathbb{N} : i \leq |\mathbf{e}|\}$, and for each $i \in I$, let $(f_{i,j})_{j\in\mathbb{N}}$ and $(\mu_{i,j})_{j\in\mathbb{N}}$ be a sequential arrangement of the atoms in F_i and a sequence in $[1,\mu]$, respectively, as in the definition of $(\star)$. Let $\phi : I \times \mathbb{N} \to \mathbb{N}$ be a bijective map, order preserving on subsets of the form $\{i\} \times \mathbb{N}$ $(i \in I)$, and let (f_i) be the basis of F defined by $f_{\phi(i,j)} := f_{i,j}$ $(i \in I,\ j \in \mathbb{N})$. Set $t_i := f_i^* \otimes f_i \in \mathcal{F}(F)$ $(i \in \mathbb{N})$.

Combining the fact that Θ_π is an injective algebra homomorphism with the fact that $\|\Theta_\pi(\sum_i \alpha_i t_i)\| \leq \|\Theta_\pi\| \|\sum_i \alpha_i t_i\|_r = \|\Theta_\pi\| \sup_i |\alpha_i|$ for every $(\alpha_i) \in c_{00}$, it is readily seen that $(\Theta_\pi(t_i))$ is a sequence of mutually orthogonal equivalent idempotents in $\mathcal{B}_0$, equivalent to the unit vector basis of c_0. To simplify notations set $b_i := \Theta_\pi(t_i)$ $(i \in \mathbb{N})$.

Let $\varepsilon > 0$ arbitrary. We construct next an increasing sequence $(l_i) \subset \mathbb{N}$ and sequences $(\mathcal{P}_i) \subset \mathcal{M}_l(\widetilde{\mathcal{B}})$ and $(\mathcal{Q}_i) \subset \mathcal{M}_r(\widetilde{\mathcal{B}})$ of disjoint band projections such that

$$\max\Big\{\big\|\mathcal{P}_i(b_{l_i}) - b_{l_i}\big\|, \big\|\mathcal{Q}_i(b_{l_i}) - b_{l_i}\big\|\Big\} \leq \frac{\varepsilon}{2^i} \quad (i \in \mathbb{N}), \tag{1}$$

and

$$\mu^{-1}\Big\|\sum_i a_i f_i\Big\| \leq \Big\|\sum_i a_i f_{l_i}\Big\| \leq \mu\Big\|\sum_i a_i f_i\Big\| \quad ((a_i) \in c_{00}). \tag{2}$$

Let $\{P_{i,S} : i \in \mathbb{N}\ \&\ S \in \mathfrak{P}_\infty(\mathbb{N})\}$ and $\{Q_{i,S} : i \in \mathbb{N}\ \&\ S \in \mathfrak{P}_\infty(\mathbb{N})\}$ be sets as in the hypotheses and let $\varepsilon > 0$ arbitrary. Set $l_1 := 1$, suppose $\phi^{-1}(2) = (i,j)$ and let S_1 be any infinite subset of $\phi(\{i\} \times \mathbb{N}) \cap \{l \in \mathbb{N} : l > l_1\}$ with the property that $(f_{\phi(i,k)})_{k=1}^j \overset{\mu_{i,j}}{\sim} (f_{l_{\phi(i,k)}})_{k=1}^j$ whenever $l_2 \in S_1$ (which exists by our definition of property $\star$). Choose $\kappa \in \mathbb{N}$ so that

$$\max\Big\{\|P_{\kappa,S_1}(b_{l_1}) - b_{l_1}\|, \|Q_{\kappa,S_1}(b_{l_1}) - b_{l_1}\|\Big\} \leq \frac{\varepsilon}{8},$$

and then choose $l_2 \in S_1$ so that

$$\max\Big\{\|P_{\kappa,S_1}(b_{l_2})\|, \|Q_{\kappa,S_1}(b_{l_2})\|\Big\} \leq \frac{\varepsilon}{8}.$$

Set $P_1 := P_{\kappa,S_1}$ and $Q_1 := Q_{\kappa,S_1}$. In general, if $l_1,\dots,l_n$, $P_1,\dots,P_{n-1}$ and $Q_1,\dots,Q_{n-1}$ have been defined and $\phi^{-1}(n+1) = (i,j)$, let S_n be any infinite subset of $\phi(\{i\}\times\mathbb{N}) \cap \{l \in$

$\mathbb{N} : l > l_n\}$ such that $(f_{\phi(i,k)})_{k=1}^{j} \overset{\mu_{i,j}}{\sim} (f_{l_{\phi(i,k)}})_{k=1}^{j}$ whenever $l_{n+1} \in S_n$, choose $\kappa \in \mathbb{N}$ big enough so that

$$\max\Big\{\|P_{\kappa,S_n}(b_{l_i}) - b_{l_i}\|, \|Q_{\kappa,S_n}(b_{l_i}) - b_{l_i}\|\Big\} \leq \frac{\varepsilon}{2^{n+2}} \quad (1 \leq i \leq n),$$

and then choose $l_{n+1} \in S_n$ so that

$$\max\Big\{\|P_{\kappa,S_n}(b_{l_{n+1}})\|, \|Q_{\kappa,S_n}(b_{l_{n+1}})\|\Big\} \leq \frac{\varepsilon}{2^{n+2}}.$$

Set $P_n := P_{\kappa,S_n}$ and $Q_n := Q_{\kappa,S_n}$.

Define

$$\mathcal{P}_n := \bigwedge_{n\leq i} P_i - \bigwedge_{n-1\leq i} P_i \quad \text{and} \quad \mathcal{Q}_n := \bigwedge_{n\leq i} Q_i - \bigwedge_{n-1\leq i} Q_i \quad (n \in \mathbb{N}),$$

where $P_0 := 0 =: Q_0$. The sequences $(\mathcal{P}_i)$ and $(\mathcal{Q}_i)$ are disjoint sequences of weak*-continuous band projections in $\mathcal{M}_l(\widetilde{B})$ and $\mathcal{M}_r(\widetilde{B})$, respectively (by the first condition on the sets $\{P_{n,S}\}$ and $\{Q_{n,S}\}$). Furthermore, it is easy to see that

$$\mathrm{id}_{\widetilde{\mathcal{B}}} - \mathcal{P}_n = \bigvee_{n\leq k} \Big(\bigwedge_{n\leq i\leq k-1} P_i \Big)(\mathrm{id}_{\widetilde{\mathcal{B}}} - P_k) + \bigwedge_{n-1\leq i} P_i \quad (n \in \mathbb{N}),$$

and since $\big((\mathrm{id}_{\widetilde{\mathcal{B}}} - P_k)\bigwedge_{n\leq i\leq k-1} P_i\big)_{k\in\mathbb{N}}$ is a disjoint sequence and $\sum_{n\leq k}(b_{l_n} - P_k(b_{l_n}))$ is absolutely convergent,

$$b_{l_n} - \mathcal{P}_n(b_{l_n}) = \sum_{k=n}^{\infty} \Big(\bigwedge_{n\leq i\leq k-1} P_i \Big)(b_{l_n} - P_k(b_{l_n})) + \Big(\bigwedge_{n-1\leq i} P_i \Big)(b_{l_n}).$$

It follows that

$$\begin{aligned}\|\mathcal{P}_n(b_{l_n}) - b_{l_n}\| &\leq \sum_{k=n}^{\infty} \|P_k(b_{l_n}) - b_{l_n}\| + \|P_{n-1}(b_{l_n})\| \\ &\leq \sum_{k=n}^{\infty} \frac{\varepsilon}{2^{k+2}} + \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2^n}.\end{aligned}$$

That $\|\mathcal{Q}_n(b_{l_n}) - b_{l_n}\| \leq \varepsilon 2^{-n}$ $(n \in \mathbb{N})$ is shown in the same way. As for (2), it follows easily from the definition of the norm on F, combined with the facts that $(f_{\phi(i,k)})_{k=1}^{j} \overset{\mu_{i,j}}{\sim} (f_{l_{\phi(i,k)}})_{k=1}^{j}$ $(i, j \in \mathbb{N})$ and $\sup_{i,j} \mu_{i,j} \leq \mu$.

Note also that $\bigvee_i \mathcal{P}_i$ and $\bigvee_i \mathcal{Q}_i$ are weak*-continuous, for $\bigvee_i \mathcal{P}_i = \bigvee_i(\sum_{j=1}^{i} \mathcal{P}_j) = \bigvee_i(\bigwedge_{i\leq j} P_j) \in \mathcal{L}_{w^*}(\widetilde{\mathcal{B}})$ and $\bigvee_i \mathcal{Q}_i = \bigvee_i(\bigwedge_{i\leq j} Q_j) \in \mathcal{L}_{w^*}(\widetilde{\mathcal{B}})$ (again by our first condition on the sets $\{P_{n,S}\}$ and $\{Q_{n,S}\}$).

Endow $\mathcal{B}$ with the topology ω induced by the weak*-topology on $\widetilde{\mathcal{B}}$, endow $\mathcal{A}_\pi$ with the topology τ generated by the system of seminorms $\{\tau_f : f \in F\}$, where $\tau_f(T) := \||T|(|f|)\|$ $(T \in \mathcal{A}_\pi)$, and define

$$p_i := t_{l_i} \quad \text{and} \quad e_i := \sum_{j=1}^{i} t_j \quad (i \in \mathbb{N}).$$

If one considers the map $a \mapsto \Theta_\pi(a)$, $(\mathcal{A}_\pi, \tau) \to (\mathcal{B}, \omega)$, then all hypotheses of Theorem 2.1 are satisfied. Indeed, for the proof that the latter map is sequentially continuous and that $(\mathcal{A}_\pi, \tau)$ is a semiprime Riesz and topological algebra with a closed positive cone, having (e_i) as a b.a.i., see the first part of the proof of [2, Theorem 4.1] (on page 203). Condition

(i) is verified as in the final part of the proof of [2, Theorem 4.1] (on page 205), with the linear maps

$$u_i := \sum_{j=1}^{i} f_{l_j}^* \otimes f_j \quad \text{and} \quad v_i := \sum_{j=1}^{i} f_j^* \otimes f_{l_j} \quad (i \in \mathbb{N}),$$

providing the required factorization of (e_i); condition (ii) was established earlier; as for (iii) and (iv), they follow readily from (1). Lastly, that $a \mapsto \Theta_\pi(a)$, $(\mathcal{A}_\pi, \tau) \to (\widetilde{\mathcal{B}}, w^*)$, is compact, follows on noting that a subset of $\mathcal{A}_\pi$ is τ-bounded if and only if it is norm-bounded.

We thus have that

$$\Theta_\pi(\cdot) = \widetilde{b}_{u,\pi} \Psi_\pi\Big(\tau\text{-}\lim_i \, \tau\text{-}\lim_j \, v_i(\cdot) u_j\Big) \widetilde{b}_{v,\pi}, \tag{3}$$

where $\Psi_\pi : \mathcal{A}_\pi \to \mathcal{B}$, $T \mapsto \Theta_\pi\big(\tau\text{-}\lim_m \tau\text{-}\lim_n \pi_m T \pi_n\big)$ and $\widetilde{b}_{u,\pi}, \widetilde{b}_{v,\pi} \in \widetilde{\mathcal{B}}$ have norms not greater than $\mu\|\Theta_\pi\|$. Moreover, if $\xi_i = b_{l_i} - \mathcal{P}_i(b_{l_i})$, $\eta_j = b_{l_j} - \mathcal{Q}_j(b_{l_j})$, $\Phi_\pi : \mathcal{A}_\pi \to \mathcal{B}$, $T \mapsto \sum_i \sum_j \mathcal{P}_i(\mathcal{Q}_j(\Theta_\pi(p_i T p_j)))$, $\Phi^\pi_{Q,i} : \mathcal{A}_\pi \to \mathcal{B}$, $T \mapsto \sum_j \mathcal{Q}_j(\Theta_\pi(p_i T p_j))$, $\Phi^\pi_{P,j} : \mathcal{A}_\pi \to \mathcal{B}$, $T \mapsto \sum_i \mathcal{P}_i(\Theta_\pi(p_i T p_j))$, and $\Theta^\pi_{ij} : \mathcal{A}_\pi \to \mathcal{B}$, $T \mapsto \Theta_\pi(p_i T p_j)$, $(i, j \in \mathbb{N})$, then Ψ_π satisfies

$$\Psi_\pi(\cdot) = \Phi_\pi(\cdot) + \sum_i \xi_i \Phi^\pi_{Q,i}(\cdot) + \sum_j \Phi^\pi_{P,j}(\cdot)\eta_j + \sum_{i,j} \xi_i \Theta^\pi_{ij}(\cdot)\eta_j, \tag{4}$$

and for every $S \in (\mathcal{A}_\pi)_+$,

$$|\Phi_\pi|(S) = |\Phi_\pi(S)|, \quad |\Phi^\pi_{Q,i}|(S) = |\Phi^\pi_{Q,i}(S)|,$$

$$|\Phi^\pi_{P,j}|(S) = |\Phi^\pi_{P,j}(S)| \quad \text{and} \quad |\Theta^\pi_{ij}|(S) = |\Theta^\pi_{ij}(S)| \quad (i, j \in \mathbb{N}).$$

One readily sees that, for every $S \in (\mathcal{A}_\pi)_+$,

$$\big|\Psi_\pi(S)\big| \le \big|\Phi_\pi\big|(S) + \sum_i |\xi_i||\Phi^\pi_{Q,i}|(S) + \sum_j |\Phi^\pi_{P,j}|(S)|\eta_j| + \sum_{i,j} |\xi_i||\Theta^\pi_{ij}|(S)|\eta_j|. \tag{5}$$

Our next task will be to find suitable estimates for $\|\Phi_\pi\|$, $\|\Phi^\pi_{P,j}\|$, $\|\Phi^\pi_{Q,i}\|$ and $\|\Theta^\pi_{i,j}\|$. We start with $\|\Phi^\pi_{Q,i}\|$. For every $T \in \mathcal{A}_\pi$,

$$\begin{aligned}
\Phi^\pi_{Q,i}(T) &= \Theta_\pi(p_i) \sum_j \mathcal{Q}_j(\Theta_\pi(T p_j)) \\
&= \Theta_\pi(p_i) \sum_j \big(\Theta_\pi(T p_j) - \Theta_\pi(T)\eta_j\big) \\
&= \Theta_\pi(p_i) \Big(\omega\text{-}\lim_n \Theta_\pi(T\pi_n) - \Theta_\pi(T) \sum_j \eta_j\Big) \\
&= \Theta_\pi\Big(\tau\text{-}\lim_n p_i T \pi_n\Big) - \Theta_\pi(p_i T) \sum_j \eta_j.
\end{aligned} \tag{6}$$

In order to estimate the norm of the latter, note first that if $(S_i) \subset \mathcal{A}_\pi$ is bounded and convergence preserving then, for every $T \in \mathcal{A}_\pi$,

$$\begin{aligned}
\Big\|\Big|\tau\text{-}\lim_j S_j T\Big|(f)\Big\| &= \tau_f\Big(\tau\text{-}\lim_j S_j T\Big) \\
&\le \tau_f\Big(S_i T - \tau\text{-}\lim_j S_j T\Big) + \tau_f(S_i T) \quad (f \in F_+,\, i \in \mathbb{N}).
\end{aligned}$$

Letting $i \to \infty$ in the last inequality, one obtains that

$$\begin{aligned}\Big\|\Big|\tau\text{-}\lim_j S_jT\Big|(f)\Big\| &\le \sup_i \tau_f(S_iT)\\ &\le \Big(\sup_i \|S_i\|_r\Big)\|T\|_r\|f\| \quad (f \in F_+),\end{aligned}$$

and hence that $\|\tau\text{-}\lim_i S_iT\|_r \le \big(\sup_i \|S_i\|_r\big)\|T\|_r$. Almost the same argument gives that $\|\tau\text{-}\lim_i TS_i\|_r \le \big(\sup_i \|S_i\|_r\big)\|T\|_r$ $(T \in \mathcal{A}_\pi)$. It follows then from (6), taking into account the latter discussion and (1), that

$$\|\Phi^\pi_{Q,i}\| \le \Big(\sup_n \|\pi_n\|\Big)\|\Theta_\pi\| + \varepsilon\|\Theta_\pi\| \le (1+\varepsilon)\varkappa\|\Theta\| \quad (i \in \mathbb{N}),$$

where $\varkappa := \sup_{\pi\in\Pi}\|\pi\|$.

Similarly, using the identities $\mathcal{P}_i(\Theta_\pi(p_iT)) = \Theta_\pi(p_iT) - \xi_i\Theta_\pi(T)$ $(T \in \mathcal{A}_\pi)$ and the sequential continuity of the map $a \mapsto \Theta_\pi(a)$, $(\mathcal{A}_\pi, \tau) \to (\mathcal{B}, \omega)$, one obtains that $\|\Phi^\pi_{P,j}\| \le (1+\varepsilon)\varkappa\|\Theta\|$ $(j \in \mathbb{N})$. Clearly, $\|\Theta^\pi_{ij}\| \le \varkappa\|\Theta\|$ $(i, j \in \mathbb{N})$. As for $\|\Phi_\pi\|$, it follows from (4) and our findings so far, that

$$\begin{aligned}\|\Phi_\pi\| &\le \|\Psi_\pi\| + \varepsilon\sup_i\|\Phi^\pi_{Q,i}\| + \varepsilon\sup_j\|\Phi^\pi_{P,j}\| + \varepsilon^2\sup_{i,j}\|\Theta^\pi_{ij}\|\\ &\le \Big(\sup_n\|\pi_n\|\Big)^2\varkappa\|\Theta\| + \varepsilon(2+3\varepsilon)\varkappa\|\Theta\|\\ &\le (1+2\varepsilon+3\varepsilon^2)\varkappa\|\Theta\|.\end{aligned}$$

Combining the definition of $\|\cdot\|_r$ with (5) and the estimates found for $\|\Phi_\pi\|$, $\|\Phi^\pi_{Q,i}\|$, $\|\Phi^\pi_{P,j}\|$ and $\|\Theta^\pi_{ij}\|$, we obtain that

$$\begin{aligned}\|\Psi_\pi\|_r &\le \|\Phi_\pi\| + \varepsilon\sup_i\|\Phi^\pi_{Q,i}\| + \varepsilon\sup_j\|\Phi^\pi_{P,j}\| + \varepsilon^2\sup_{i,j}\|\Theta^\pi_{ij}\|\\ &\le (1+4\varepsilon+6\varepsilon^2)\varkappa\|\Theta\|,\end{aligned}$$

and since ε and π above were arbitrary,

$$\|\Psi_\pi\|_r \le \varkappa\|\Theta\| \quad (\pi \in \Pi).$$

Now let $\widetilde{\mathcal{B}}_*$ be a predual of $\widetilde{\mathcal{B}}$. For every $\pi \in \Pi$ let

$$\Lambda_\pi : \mathcal{A} \to \mathcal{B},\ T \mapsto \widetilde{b}_{u,\pi}\Psi_\pi\Big(\tau\text{-}\lim_i\ \tau\text{-}\lim_j\ v_ip_\pi T\imath_\pi u_j\Big)\widetilde{b}_{v,\pi},$$

and let $\phi_\pi : \mathcal{A}\otimes\widetilde{\mathcal{B}}_* \to \mathbb{K}$ (where $\mathbb{K}$ denotes, as usual, the underlying field) be the linear map whose values on elementary tensors are given by the formula $\phi_\pi(T\otimes\beta) := (\Lambda_\pi(T))(\beta)$. Furthermore, let $\phi : \mathcal{A}\otimes\widetilde{\mathcal{B}}_* \to \mathbb{K}$ be the linear map defined by $\phi(T\otimes\beta) := (\Theta(T))(\beta)$ $(T \in \mathcal{A},\ \beta \in \widetilde{\mathcal{B}}_*)$. Then, for every $T \in \mathcal{A}$ and $\beta \in \widetilde{\mathcal{B}}_*$, taking into account that w^*-$\lim_\pi \Theta(\pi T\pi) = \Theta(T)$ $(T \in \mathcal{A})$ and (3), one finds that

$$\begin{aligned}\phi(T\otimes\beta) = (\Theta(T))(\beta) &= \lim_\pi\,(\Theta(\pi T\pi))(\beta) = \lim_\pi\,(\Theta_\pi(p_\pi T\imath_\pi))(\beta)\\ &= \lim_\pi\Big(\widetilde{b}_{u,\pi}\Psi_\pi\Big(\tau\text{-}\lim_i\ \tau\text{-}\lim_j\ v_ip_\pi T\imath_\pi u_j\Big)\widetilde{b}_{v,\pi}\Big)(\beta)\\ &= \lim_\pi\,(\Lambda_\pi(T))(\beta) = \lim_\pi\ \phi_\pi(T\otimes\beta).\end{aligned}$$

Now let $\imath:\mathcal{B}\to\widetilde{\mathcal{B}}$ be the natural inclusion map. Since $\psi\mapsto\big[L_\psi:\mathcal{A}\to\widetilde{\mathcal{B}},\, a\mapsto\psi(a\otimes(\cdot))\big]$, $(\mathcal{A}\otimes_{|\pi|}\widetilde{\mathcal{B}}_*)'\to\mathcal{L}^r(\mathcal{A},\widetilde{\mathcal{B}})$, is an isometric Riesz isomorphism,

$$\begin{aligned}\|\phi_\pi\|=\|\imath\circ\Lambda_\pi\|_r\le\|\Lambda_\pi\|_r&\le\big\|\widetilde{b}_{u,\pi}\big\|\Big\|\Psi_\pi\Big(\tau\text{-}\lim_i\,\tau\text{-}\lim_j\,v_ip_\pi(\cdot)\imath_\pi u_j\Big)\Big\|_r\big\|\widetilde{b}_{v,\pi}\big\|\\&\le\mu^2\varkappa^2\|\Theta\|^2\|\Psi_\pi\|_r\Big(\sup_i\|u_i\|\Big)\Big(\sup_i\|v_i\|\Big)\big\|p_\pi(\cdot)\imath_\pi\big\|\le\mu^4\varkappa^4\|\Theta\|^3,\end{aligned}$$

for every $\pi\in\Pi$. Thus, for every $u\in(\mathcal{A}\otimes\widetilde{\mathcal{B}}_*,\|\cdot\|_{|\pi|})$,

$$|\phi(u)|=\lim_\pi\big|\phi_\pi(u)\big|\le\Big(\sup_\pi\big\|\phi_\pi\big\|\Big)\|u\|_{|\pi|}\le\mu^4\varkappa^4\|\Theta\|^3\|u\|_{|\pi|},$$

and therefore, $\imath\circ\Theta\in\mathcal{L}^r(\mathcal{A},\widetilde{\mathcal{B}})$.

Lastly, we show $\Theta\in\mathcal{L}^r(\mathcal{A},\mathcal{B})$. For this, fix $T\in\mathcal{A}_+$ and note that, since $\imath$ is a Riesz homomorphism, for every finite subset $\mathfrak{S}$ of $\mathfrak{A}_T:=\{S\in\mathcal{A}:|S|\le T\}$,

$$\imath\Big(\bigvee\nolimits_{S\in\mathfrak{S}}\big|\Theta(S)\big|\Big)=\bigvee\nolimits_{S\in\mathfrak{S}}\big|\imath(\Theta(S))\big|\le|\imath\circ\Theta|(T).$$

So, $\sup\big\{\|\bigvee_{S\in\mathfrak{S}}|\Theta(S)|\|:\mathfrak{S}\subset\mathfrak{A}_T\text{ finite}\big\}<\infty$ (recall $\imath$ is also an isometry). But $\mathcal{B}$ is Levi and $\sup_{|S|\le T}|\Theta(S)|=\sup\big\{\bigvee_{S\in\mathfrak{S}}|\Theta(S)|:\mathfrak{S}\subset\mathfrak{A}_T\text{ finite}\big\}$, so $|\Theta|(T)$ exists in $\mathcal{B}$. Since $T\in\mathcal{A}_+$ was arbitrary, $\Theta\in\mathcal{L}^r(\mathcal{A},\mathcal{B})$.

We now turn our attention to the last claim of the theorem. In this case, the same argument gives $\Theta(\cdot)=\widetilde{b}_u\Psi\big(\tau\text{-}\lim_i\,\tau\text{-}\lim_j\,v_i(\cdot)u_j\big)\widetilde{b}_v$, where $\widetilde{b}_u,\widetilde{b}_v\in\widetilde{\mathcal{B}}$ have norms $\le\mu\|\Theta\|$ and $\Psi:\mathcal{A}\to\mathcal{B}$, $T\mapsto\Theta\big(\lim_m\lim_n\pi_mT\pi_n\big)$, is regular with $\|\Psi\|_r\le\|\Theta\|$. As for dropping the Levi assumption on $\mathcal{B}$, simply recall that $\mathcal{B}$ is an algebra ideal of $\widetilde{\mathcal{B}}$. The rest is clear. □

Theorem 3.1 is mostly a revised version of [2, Theorem 4.1]. It is not an extension of the latter because of the stronger hypotheses on the algebra $\widetilde{\mathcal{B}}$.

As a relatively straightforward consequence of Theorem 3.1, we present next a generalization of [2, Corollary 4.7]. To simplify our statements, we shall call a seminormalised sequence (x_n) in a Banach lattice X, with the property that for some disjoint sequence (ξ_n) in X, $\lim_n\|x_n-\xi_n\|=0$, **asymptotically disjoint** (a.d. in short).

Corollary 3.2. *Let X be a Banach lattice as in Theorem 3.1, and let Y be a reflexive Banach lattice with the property that any complemented seminormalised unconditional basic sequence, either in Y or in Y', contains an a.d. subsequence. Then every continuous one-to-one algebra homomorphism Θ from a Π hereditary Riesz operator subalgebra $\mathcal{A}$ of $\mathcal{L}^r(X)$ into a Levi Riesz algebra ideal $\mathcal{B}$ of $\mathcal{L}^r(Y)$, such that $\overline{\Theta(\mathcal{A})}\cap\mathcal{F}(Y)\ne\{0\}$ and $w^*\text{-}\lim_\pi\Theta(\pi T\pi)=\Theta(T)$ $(T\in\mathcal{A})$ is automatically regular. Furthermore, if $X\in\mathfrak{F}_X^{\star,\mu}$ and $\Pi=\{\mathrm{id}_X\}$ then the Levi assumption on $\mathcal{B}$ can be dropped.*

Before embarking on the proof of the corollary, we recall a few facts about unconditional structures. First, a Banach space X is said to have **local unconditional structure** (l.u.s. in short) if for every finite-dimensional subspace E of X there is a Banach lattice Z and linear operators $R:Z\to X$ and $S:E\to Z$ such that $\imath_E=RS$, where $\imath_E:E\to X$ stands for the natural inclusion map. It is known that a Banach space X has l.u.s. if and only if X'' is isomorphic to a complemented subspace of a Banach lattice (see, for instance, [13, Theorem 8.11]). In particular, every complemented subspace of a reflexive Banach lattice has l.u.s.

Recall also that a basic sequence (x_i) in a Banach space X is unconditional if and only if

$$\Big\| \sum_i \xi_i \alpha_i x_i \Big\| \leq 2M \Big(\sup_i |\xi_i| \Big) \Big\| \sum_i \alpha_i x_i \Big\| \quad ((\xi_i), (\alpha_i) \in c_{00}),$$

where M is the unconditionality constant of (x_i). Moreover, every subsequence of an unconditional basic sequence is itself unconditional and if a basic sequence (x_i) is unconditional and shrinking, then the sequence (x_i^*) of associated biorthogonal functionals on $[x_i : i \in \mathbb{N}]$, is an unconditional basis for the dual $[x_i : i \in \mathbb{N}]'$.

Following common practice, we shall call a sequence (X_i) of k-dimensional subspaces of a Banach space X, such that every $x \in X$ can be represented in a unique way as an unconditional sum $\sum_i x_i$ with $x_i \in X_i$ $(i \in \mathbb{N})$, a **k-unconditional finite-dimensional Schauder decomposition** (k-UFDD in short) of X.

Proof of Corollary 3.2. Let $\widetilde{\mathcal{B}}$ be the dual Banach lattice algebra $\mathcal{L}^r(Y)$, let $\mathcal{B}$ be a Levi Riesz algebra ideal of $\mathcal{L}^r(Y)$ (or just a Riesz algebra ideal of $\mathcal{L}^r(Y)$ if $X \in \mathfrak{F}_X^{\star,\mu}$ and $\Pi = \{\mathrm{id}_X\}$), and let $\mathcal{B}_0 := \mathcal{B} \cap \mathcal{A}^r(Y)$ (i.e., the smallest closed non-zero Riesz algebra ideal of $\mathcal{B}$). The weak*-topology on $\mathcal{L}^r(Y)$ will be the one it inherits via the isomorphism $\mathcal{L}^r(Y) \cong (Y \otimes_{|\pi|} Y')'$.

It will suffice to show that $\mathcal{B}_0$ satisfies the hypotheses of Theorem 3.1. To this end, let $(b_i) \subset \mathcal{B}_0$ be a sequence of mutually orthogonal equivalent idempotents, equivalent to the unit vector basis of c_0. Let P be a weak*-limit point of $\{\sum_{i=1}^n b_i : n \in \mathbb{N}\}$ and let $E := P(Y)$. It is easy to see, using the separate weak*-continuity of the product in $\mathcal{L}^r(Y)$ (see for instance the proof of [2, Corollary 4.7]) that P is a projection. Furthermore, E coincides with the norm closure of $\sum_i b_i(Y)$. To see this last, set $P_n := \sum_{i=1}^n b_i$ $(n \in \mathbb{N})$, and let (P_{n_α}) be a subnet of (P_n) such that $P = w^*\text{-}\lim_\alpha P_{n_\alpha}$. Then, for every $y \in E$ and every $y' \in Y'$, we have that $y'(y) = y'(Py) = \lim_\alpha y'(P_{n_\alpha} y)$, i.e., $y = w\text{-}\lim_\alpha P_{n_\alpha} y$, so $E \subseteq \overline{\sum_i b_i(Y)}^{w} = \overline{\sum_i b_i(Y)}^{\|\cdot\|}$ and the opposite inclusion is obvious.

Next note that $(b_i(Y))$ is a k-UFDD for E, where k is the common rank of the b_i's, for if $(z_i) \in \prod_i b_i(Y)$ is a finitely nonzero sequence, $(\varepsilon_i) \in \{\pm 1\}^{\mathbb{N}}$ and $m \in \mathbb{N}$, then we have that

$$\sum_{i=1}^m \varepsilon_i z_i = \Big(\sum_{i=1}^m \varepsilon_i b_i \Big) \Big(\sum_i z_i \Big).$$

Since E is a complemented subspace of a reflexive Banach lattice it has l.u.s. (see the discussion preceding the corollary), and so, by [3, Theorem 3.8], E has an unconditional basis (y_n) such that $(y_{ki+j})_{j=1}^k$ is a basis for $b_{i+1}(Y)$ $(i \in \mathbb{N} \cup \{0\})$. We shall assume, as we can, that the basis (y_n) is, in addition, seminormalized. Let (y_n^*) be the corresponding sequence of biorthogonal functionals on E, and define $\phi_n := y_n^* \circ P$ $(n \in \mathbb{N})$. Note that $b_{i+1} = \sum_{j=1}^k \phi_{ki+j} \otimes y_{ki+j}$ $(i \in \mathbb{N} \cup \{0\})$.

Set $\xi_{i+1} := \sum_{j=1}^k |y_{ki+j}|$ and $\eta_{i+1} := \sum_{j=1}^k |\phi_{ki+j}|$ $(i \in \mathbb{N} \cup \{0\})$. Then, for every $n \in \mathbb{N}$,

$$\lim_l \Big\| \Big(\sum_{i \leq n} \xi_i \Big) \wedge \xi_l \Big\| = 0 = \lim_l \Big\| \Big(\sum_{i \leq n} \eta_i \Big) \wedge \eta_l \Big\|. \tag{7}$$

We give the proof of the first equality – the proof of the second is completely analogous. Since $\xi \wedge \xi_{i+1} \leq \sum_{j=1}^{k} \xi \wedge |y_{ik+j}|$ $(\xi \in Y_+)$, it will suffice to show that $\lim_l \|(\sum_{i\leq n} \xi_i) \wedge |y_l|\| = 0$ $(n \in \mathbb{N})$. So fix n and suppose towards a contradiction there exist $\delta > 0$ and $S \in \mathfrak{P}_\infty(\mathbb{N})$ such that $\|(\sum_{i\leq n} \xi_i) \wedge |y_l|\| \geq \delta$ for every $l \in S$. By our assumptions, the subsequence $(y_l)_{l\in S}$ of (y_l) must contain an a.d. subsequence, $(y_l)_{l\in S_1}$ say, where $S_1 \subseteq S$. Let $(\gamma_l)_{l\in S_1}$ be a disjoint sequence in Y such that $\lim_{l\in S_1} \|\gamma_l - y_l\| = 0$ and set $y := \sum_{i\leq n} \xi_i$. The sequence $(y \wedge |\gamma_l|)_{l\in S_1}$ is order bounded, and since Y is order continuous, $\lim_{l\in S_1} \|y \wedge |\gamma_l|\| = 0$ (by a classical result of Dodds and Fremlin, see for instance [1, Theorem 4.14]). In turn, we would have that $\lim_{l\in S_1} \|y \wedge |y_l|\| = 0$ (because $\|y \wedge |y_l| - y \wedge |\gamma_l|\| \leq \||y_l| - |\gamma_l|\|$ $(l \in \mathbb{N})$), and hence, that $\inf_{l\in S} \|(\sum_{i\leq n} \xi_i) \wedge |y_l|\| = 0$, which is a contradiction.

Now let $S \in \mathfrak{P}_\infty(\mathbb{N})$ fixed. Set $l_1 := \min S$ and choose $l_2 \in S$ so that $l_2 > l_1$ and $\max\{\|\xi_1 \wedge \xi_{l_2}\|, \|\eta_1 \wedge \eta_{l_2}\|\} \leq 1/4$ (which is possible by (7)). In general, if $l_1, l_2, \ldots, l_n \in S$ have been chosen, choose $l_{n+1} \in S$ so that $l_{n+1} > l_n$ and

$$\max\left\{\left\|\left(\sum_{i\leq n} \xi_i\right) \wedge \xi_{l_{n+1}}\right\|, \left\|\left(\sum_{i\leq n} \eta_i\right) \wedge \eta_{l_{n+1}}\right\|\right\} \leq \frac{1}{2^{n+1}},$$

(the latter again possible by (7)). For each $n \in \mathbb{N}$, let $P_{n,S}$ and $Q_{n,S}$ be the band projections onto the projection bands in Y and Y', generated by the vectors

$$\widetilde{\xi}_n := \sum_{i\leq n} \xi_i - \left(\sum_{i\leq n} \xi_i\right) \wedge \left(\sum_{n<i} \xi_{l_i}\right) \quad \text{and} \quad \widetilde{\eta}_n := \sum_{i\leq n} \eta_i - \left(\sum_{i\leq n} \eta_i\right) \wedge \left(\sum_{n<i} \eta_{l_i}\right),$$

respectively.

For every pair $i, n \in \mathbb{N}$, with $i \leq n$,

$$\begin{aligned}
&|P_{n,S} \circ b_i - b_i| \\
&\qquad \leq \sum_{j=1}^{k} |\phi_{k(i-1)+j}| \otimes (\mathrm{id}_Y - P_{n,S})|y_{k(i-1)+j}| \\
&\qquad \leq \eta_i \otimes (\mathrm{id}_Y - P_{n,S})\left(\sum_{i\leq n} \xi_i\right) = \eta_i \otimes (\mathrm{id}_Y - P_{n,S})\left(\sum_{i\leq n} \xi_i - \widetilde{\xi}_n\right) \\
&\qquad \leq \eta_i \otimes \left(\sum_{i\leq n} \xi_i\right) \wedge \left(\sum_{n<i} \xi_{l_i}\right),
\end{aligned}$$

and hence,

$$\|P_{n,S} \circ b_i - b_i\| \leq \|\eta_i\| \sum_{n<j} \left\|\left(\sum_{i\leq n} \xi_i\right) \wedge \xi_{l_j}\right\| \leq \|\eta_i\| \sum_{n\leq j} \frac{1}{2^{j+1}} \leq \frac{\|\eta_i\|}{2^n},$$

so $\lim_n P_{n,S} \circ b_i = b_i$ $(i \in \mathbb{N})$. Likewise, $\|b_i \circ Q'_{n,S} - b_i\| \leq 2^{-n}\|\xi_i\|$ $(n \in \mathbb{N})$, and hence, $\lim_n b_i \circ Q'_{n,S} = b_i$ $(i \in \mathbb{N})$.

Furthermore, it follows from the definition of $\widetilde{\xi}_n$ that $\widetilde{\xi}_n \wedge (\xi_{l_i} - \xi_{l_i} \wedge \sum_{j\le n} \xi_j) = 0$ for every pair $i, n \in \mathbb{N}$, with $i > n$, so $P_{n,S}(\xi_{l_i}) = P_{n,S}(\xi_{l_i} \wedge \sum_{j\le n} \xi_j)$ and

$$\begin{aligned}\|P_{n,S} \circ b_{l_i}\| &\le \|\eta_{l_i} \otimes P_{n,S}(\xi_{l_i})\| \\ &\le \|\eta_{l_i}\| \left\| \left(\sum_{j\le n} \xi_j \right) \wedge \xi_{l_i} \right\| \le \frac{2kM \sup_j \|b_j\|}{2^i \inf_j \|y_j\|},\end{aligned}$$

where M is the unconditionality constant of (y_i). Thus, $\inf_{i\in S} \|P_{n,S} \circ b_i\| = 0$ $(n \in \mathbb{N})$, and a completely analogous argument shows that $\inf_{i\in S} \|b_i \circ Q'_{n,S}\| = 0$ $(n \in \mathbb{N})$.

It is clear that repeating the above construction for every $S \in \mathfrak{P}_\infty(\mathbb{N})$, we obtain sets $\{P_{n,S}\}$ and $\{Q'_{n,S}\}$ satisfying the last three conditions from the list of Theorem 3.1 (where we are identifying the elements of these sets with the corresponding multiplication operators on $\widetilde{\mathcal{B}}$). As for the first condition, it suffices to note that $\{P_{n,S}\} \subset \mathcal{L}^r(Y) \subseteq \mathcal{M}_l(\widetilde{\mathcal{B}}) \cap \mathcal{L}_{w^*}(\widetilde{\mathcal{B}})$ and $\{Q'_{n,S}\} \subset \mathcal{L}^r(Y) \subseteq \mathcal{M}_r(\widetilde{\mathcal{B}}) \cap \mathcal{L}_{w^*}(\widetilde{\mathcal{B}})$, where elements of $\mathcal{L}^r(Y)$ are being thought of as left multiplication operators on $\widetilde{\mathcal{B}}$, in the first chain of inclusions, and as right multiplication operators on $\widetilde{\mathcal{B}}$, in the second chain. Clearly, only the inclusion $\mathcal{L}^r(Y) \subseteq \mathcal{L}_{w^*}(\widetilde{\mathcal{B}})$ needs a proof. But this follows easily from the fact that multiplication on $\mathcal{L}^r(Y)$ is separately weak*-continuous. □

A few comments in connection with Corollary 3.2 seem in order:

– The condition $w^*\text{-}\lim_\pi \Theta(\pi T \pi) = \Theta(T)$ $(T \in \mathcal{A})$ holds trivially if $X \in \mathfrak{F}_X^{\star,\mu}$ and $\Pi = \{\mathrm{id}_X\}$, or if $\lim_\pi \pi T \pi = T$ $(T \in \mathcal{A})$. Furthermore (and more relevant to the non-atomic situation, which is central to this note), the condition holds also whenever the weak*-closure of $\Theta(\mathcal{F}(X))$ in $\mathcal{L}^r(Y)$ contains a unit for $\Theta(\mathcal{A})$. To see this last, let $Q \in \overline{\Theta(\mathcal{F}(X))}^{w^*}$ be such that $Q\Theta(T) = \Theta(T) = \Theta(T)Q$ $(T \in \mathcal{A})$ and let $(S_\alpha) \subset \mathcal{F}(X)$ be a net such that $Q = w^*\text{-}\lim_\alpha \Theta(S_\alpha)$. Then Q is a unit for $\overline{\Theta(\mathcal{A})}^{w^*}$ (recall multiplication on $\mathcal{L}^r(Y)$ is separately weak*-continuous), and if $T \in \mathcal{A}$ and $R = w^*\text{-}\lim_\gamma \Theta(\pi_\gamma T \pi_\gamma)$ for some net $(\pi_\gamma)_{\gamma\in\Gamma} \subset \Pi$, with Γ a cofinal subset of Π, then

$$\begin{aligned}\Theta(T) = Q\Theta(T)Q &= w^*\text{-}\lim_\alpha\, w^*\text{-}\lim_\beta\, \Theta(S_\alpha)\Theta(T)\Theta(S_\beta) \\ &= w^*\text{-}\lim_\alpha\, w^*\text{-}\lim_\beta\, \lim_\gamma \Theta(S_\alpha \pi_\gamma T \pi_\gamma S_\beta) \\ &= w^*\text{-}\lim_\alpha\, w^*\text{-}\lim_\beta\, \Theta(S_\alpha)\Big(w^*\text{-}\lim_\gamma \Theta(\pi_\gamma T \pi_\gamma)\Theta(S_\beta)\Big) \\ &= w^*\text{-}\lim_\alpha\, w^*\text{-}\lim_\beta\, \Theta(S_\alpha) R \Theta(S_\beta) = QRQ = R.\end{aligned}$$

A special case of the above occurs when $\Theta(\mathcal{F}(X))$ contains a bounded left approximate identity for $\mathcal{F}(Y)$.

– The Levi condition on $\mathcal{B}$ might seem like a very restrictive assumption since, for instance, in the case of a purely atomic Banach lattice Y, it implies that $\mathcal{B} = \mathcal{L}^r(Y)$ (recall $\mathcal{B}$ is, in particular, an algebra ideal, so $\mathcal{F}(Y) \subset \mathcal{B}$). However, it seems to us that if X has no atoms and Y is purely atomic then one-to-one algebra homomorphisms from $\mathcal{A}^r(X)$ into $\mathcal{A}^r(Y)$ (recall that $\overline{\Theta(\mathcal{A})} \cap \mathcal{F}(Y) \neq \{0\} \Rightarrow \Theta(\mathcal{F}(X)) \subseteq \mathcal{F}(Y)$), if at all possible, should be rare. In addition, we should notice that, in the non-atomic case, $\mathcal{L}^r(Y)$ may contain non-trivial proper Levi Riesz algebra ideals. For instance, if Y' coincides with the order continuous dual of Y (such is the case for any space $L^p(\mu)$ with μ a σ-finite measure and $1 < p < \infty$) then the band in $\mathcal{L}^r(Y)$, generated by $\mathcal{F}(Y)$, is a proper algebra ideal, which in the case of an L^p-space over a σ-finite measure space consists of the so-called kernel operators, and clearly has the Levi property.

– Every complemented seminormalized unconditional basic sequence in a reflexive purely atomic Banach lattice Y contains an a.d. subsequence (this is just a particular instance of the well-known Bessaga-Pelczynski Selection Principle). Thus, every reflexive purely atomic Banach lattice Y satisfies the condition of Corollary 3.2. Unfortunately, we do not know exactly which Banach lattices have this property and it seems difficult to decide, in the non-atomic case, whether a given Banach lattice satisfies the property or not. We will show next that if Θ happens to preserve ranks in a certain sense, then this requirement on Y can be dropped.

Corollary 3.3. *Let X be a Banach lattice as in Theorem 3.1, and let Y be a reflexive Banach lattice. Then every continuous one-to-one algebra homomorphism Θ from a Π hereditary Riesz operator subalgebra $\mathcal{A}$ of $\mathcal{L}^r(X)$ into a Levi Riesz algebra ideal $\mathcal{B}$ of $\mathcal{L}^r(Y)$ such that $\overline{\Theta(\mathcal{A})} \cap \mathcal{F}(Y)$ contains elements of rank-one and $w^*\text{-}\lim_\pi \Theta(\pi T \pi) = \Theta(T)$ $(T \in \mathcal{A})$ is automatically regular. As before, if $X \in \mathfrak{F}_X^{\star,\mu}$ and $\Pi = \{\mathrm{id}_X\}$, the Levi assumption on $\mathcal{B}$ can be dropped.*

Proof. First note that Θ must map rank-one elements to rank-one elements. Indeed, let $a = \lambda \otimes x \in \mathcal{A} \setminus \{0\}$ arbitrary. Let $b \in \overline{\Theta(\mathcal{A})} \cap \mathcal{F}(Y)$ be a rank-one element and let $(a_n) \subset \mathcal{A} \setminus \{0\}$ be such that $b = \lim_n \Theta(a_n)$. Clearly, we can assume, without loss of generality, $\|\Theta(a_n)\| = \|b\| = 1$ $(n \in \mathbb{N})$. For each a_n choose $x_n \in X_{[1]}$ and $x'_n \in X'_{[\|\Theta\|]}$ so that $x'_n(a_n(x_n)) = 1$. Then $a = (x'_n \otimes x) \circ a_n \circ (\lambda \otimes x_n)$ $(n \in \mathbb{N})$, and if R and S are weak*-limit points of $\{\Theta(x'_n \otimes x) : n \in \mathbb{N}\}$ and $\{\Theta(\lambda \otimes x_n) : n \in \mathbb{N}\}$, respectively, then $\Theta(a) = w^*\text{-}\lim_n \Theta(x'_n \otimes x) b \Theta(\lambda \otimes x_n) = RbS$.

Let $\mathcal{B}_0$ and $\widetilde{\mathcal{B}}$ be as in the proof of Corollary 3.2. It follows from the previous paragraph that for any $\pi \in \Pi$ the sequence (b_i), defined as in the proof of Theorem 3.1, is a sequence of rank-one idempotents. Thus, to proof the corollary, it will suffice to show that for every sequence $(b_i) \subset \mathcal{L}^r(Y)$ of rank-one mutually orthogonal idempotents, equivalent to the unit vector basis of c_0, there are sets $\{P_{n,S}\}$ and $\{Q_{n,S}\}$ satisfying the conditions of Theorem 3.1. So let (b_i) be one such sequence. As in the proof of Corollary 3.2, there are complemented seminormalised unconditional basic sequences $(y_i) \subset Y$ and $(\phi_i) \subset Y'$ so that $b_i = \phi_i \otimes y_i$ $(i \in \mathbb{N})$. If every subsequence of (y_i) has an a.d. subsequence, and every subsequence of (ϕ_i) has an a.d. subsequence, then one could argue as in the proof of Corollary 3.2 to produce sets of band projections as required by Theorem 3.1. Accordingly, to prove the corollary, it will suffice to show that every subsequence of (y_i) has an a.d. subsequence, and similarly for every subsequence of (ϕ_i). We shall argue by contradiction.

Suppose first (y_i) has a subsequence without a.d. subsequences. To simplify notations, we shall continue to denote this subsequence by (y_i). We show first that, in this situation, (ϕ_i) must contain an a.d. subsequence. Indeed, the sequence $b_i := \phi_i \otimes y_i$ $(i \in \mathbb{N})$ of rank-one operators in $\mathcal{A}^r(Y)$ is an unconditional basic sequence equivalent to the unit vector basis of c_0, and hence, according to [12, Proposition 1.c.10], it must contain and a.d. subsequence $(b_i)_{i \in S}$ (note that since Y is reflexive, $\mathcal{A}^r(Y)$ is order continuous [4, Theorem 2.8]). Since

$$\begin{aligned}(|\phi_i| \wedge |\phi_j|)&\otimes(|y_i| \wedge |y_j|)\\ &\le (|\phi_i| \otimes |y_i|) \wedge (|\phi_j| \otimes |y_j|) = |b_i| \wedge |b_j| \quad (i, j \in S),\end{aligned}$$

there is (by Ramsey Theorem) an infinite subset $I \subseteq S$ such that either $\||\phi_i| \wedge |\phi_j|\|^2 \le \||b_i| \wedge |b_j|\|$ for every pair $i, j \in I$, or $\||y_i| \wedge |y_j|\|^2 \le \||b_i| \wedge |b_j|\|$ for every pair $i, j \in I$. Suppose first $\||\phi_i| \wedge |\phi_j|\|^2 \le \||b_i| \wedge |b_j|\|$ for every pair $i, j \in I$. Set $i_1 := \min I$ and choose $i_2 \in I$ so that $i_2 > i_1$ and $\||b_{i_1}| \wedge |b_{i_2}|\| \le 4^{-1}$. In general, if $i_1, \dots, i_n \in I$ have been chosen, choose $i_{n+1} \in I$ so that $i_{n+1} > i_n$ and $\||b_{i_k}| \wedge |b_{i_{n+1}}|\| \le n^{-2}4^{-n}$ $(1 \le k \le n)$ (note that since $\mathcal{A}^r(Y)$ is order continuous we must have $\lim_{i \in S} \|b \wedge |b_i|\| = 0$ for every $b \in \mathcal{A}^r(Y)_+$, so the latter is certainly possible). Then (ϕ_{i_n}) is an a.d. subsequence. Indeed, suppose first we are in the real situation, and define (φ_n) in Y' by

$$\varphi_n := \left(\phi^+_{i_n} - \phi^+_{i_n} \wedge \sum_{k:k\ne n} |\phi_{i_k}|\right) - \left(\phi^-_{i_n} - \phi^-_{i_n} \wedge \sum_{k:k\ne n} |\phi_{i_k}|\right) \quad (n \in \mathbb{N}).$$

It is easy to see that (φ_n) is disjoint, and for every $n \in \mathbb{N}$, we have that

$$\begin{aligned}\|\varphi_n - \phi_{i_n}\| \le 2\left\||\phi_{i_n}| \wedge \sum_{k:k\ne n} |\phi_{i_k}|\right\| &\le 2\sum_{k:k<n} \||\phi_{i_k}| \wedge |\phi_{i_n}|\| + 2\sum_{k:k>n} \||\phi_{i_n}| \wedge |\phi_{i_k}|\|\\ &\le 2\sum_{k:k<n} \sqrt{\||b_{i_k}| \wedge |b_{i_n}|\|} + 2\sum_{k:k>n} \sqrt{\||b_{i_n}| \wedge |b_{i_k}|\|} \le \frac{1}{2^{n-3}},\end{aligned}$$

so (ϕ_{i_n}) is a.d. in this case. In the complex case, define (φ_n) by

$$\Re\varphi_n := \left((\Re\phi_{i_n})^+ - (\Re\phi_{i_n})^+ \wedge \sum_{k:k\ne n} |\phi_{i_k}|\right) - \left((\Re\phi_{i_n})^- - (\Re\phi_{i_n})^- \wedge \sum_{k:k\ne n} |\phi_{i_k}|\right)$$

and

$$\Im\varphi_n := \left((\Im\phi_{i_n})^+ - (\Im\phi_{i_n})^+ \wedge \sum_{k:k\ne n} |\phi_{i_k}|\right) - \left((\Im\phi_{i_n})^- - (\Im\phi_{i_n})^- \wedge \sum_{k:k\ne n} |\phi_{i_k}|\right) \quad (n \in \mathbb{N}),$$

where $\Re$ and $\Im$ stand for the real and imaginary parts, respectively, of the elements indicated. Then, again, it is easy to see that (φ_n) is disjoint, and almost the same argument as in the real case, shows that (ϕ_{i_n}) is a.d.

It is also clear now that we cannot have $\||y_i| \wedge |y_j|\|^2 \le \||b_i| \wedge |b_j|\|$ for every pair i, j in an infinite subset I of S, for the same argument as above would produce an a.d. subsequence of (y_i), contrary to our assumptions.

By passing to a subsequence if necessary, assume $\sum_k \|\varphi_k - \phi_{i_k}\| < \infty$. Since (y_i) has no a.d. subsequences and Y is order continuous, there is $y' \in Y'_+$ such that $\inf_i y'(|y_i|) =: \delta > 0$. Indeed, by [12, Proposition 1.a.9], Y contains a projection band Y_0 with a weak unit and such that $(y_i) \subset Y_0$. We can then assume there is a probability measure space (Ω, Σ, μ) such that Y_0 embeds continuously as an order ideal into $L^1(\Omega, \Sigma, \mu)$ (by [12, Theorem 1.b.14]). Since (y_i) contains no a.d. subsequences, there is $t > 0$ such that $(y_i) \subset M(t) = \{y \in Y_0 : \mu(\{\omega \in \Omega : |y(\omega)| \ge t\|y\|_Y\}) \ge t\}$ (see the proofs of [12, Propositions 1.c.8 and 1.c.10]), and hence, $\int_\Omega |y_i(\omega)|\, d\mu \ge t^2\|y_i\|_Y$ $(i \in \mathbb{N})$. Define $y'_0 \in Y'_0$ by $y'_0(y) := \int_\Omega y(\omega)\, d\mu$ $(y \in Y_0)$, and let $y' := y'_0 \circ P_0$, where $P_0 : Y \to Y$ stands for the band projection onto Y_0. We readily see that $y'(|y_i|) = y'_0(|y_i|) \geq t^2\|y_i\|_Y$ $(i \in \mathbb{N})$, so y' meets the requirement with $\delta \geq t^2 \inf_i \|y_i\|_Y$. Clearly, we can assume $\|y'\| = 1$. Then, for every $n \in \mathbb{N}$,

$$\begin{aligned}
\Big\| \sum_{i_k \leq n} \phi_{i_k} \otimes y_{i_k} \Big\|_r &\geq \Big\| \sum_{i_k \leq n} |\varphi_k| \otimes |y_{i_k}| \Big\| - \sum_{i_k \leq n} \|\phi_{i_k} - \varphi_k\| \|y_{i_k}\| \\
&\geq \Big\| \sum_{i_k \leq n} y'(|y_{i_k}|)\varphi_k \Big\| - \sup_k \|y_k\| \sum_k \|\phi_{i_k} - \varphi_k\| \\
&\geq \Big\| \sum_{i_k \leq n} y'(|y_{i_k}|)\phi_{i_k} \Big\| - 2\sup_k \|y_k\| \sum_k \|\phi_{i_k} - \varphi_k\| \\
&\geq \frac{\delta}{M'} \Big\| \sum_{i_k \leq n} \phi_{i_k} \Big\| - 2\sup_k \|y_k\| \sum_k \|\phi_{i_k} - \varphi_k\|,
\end{aligned}$$

where M' stands for the unconditionality constant of (ϕ_i). Since (b_i) is equivalent to the unit vector basis of c_0, $\sup_n \big\| \sum_{i_k \leq n} \phi_{i_k} \otimes y_{i_k} \big\|_r < \infty$. But Y' is reflexive, and so, we must also have $\big\| \sum_{i_k \leq n} \phi_{i_k} \big\| \to \infty$ as $n \to \infty$, since otherwise (ϕ_{i_k}) would be equivalent to the unit vector basis of c_0. It follows from this contradiction that (y_i) without a.d. subsequences is just impossible.

If instead we assume it is (ϕ_i) the one that does not have a.d. subsequences, then essentially the same argument as above would give that (y_i) contains an a.d. subsequence (y_{i_k}) such that $\sup_n \big\| \sum_{i_k \leq n} y_{i_k} \big\| < \infty$, contradicting the reflexivity of Y. □

Remark 3.4. The main difficulty with extending the above proof to the case in which the b_i's are not rank-one, arises at the end of the argument – we simply do not know the exact conditions on Y, under which a sequence $(\phi_{ki+j} \otimes y_{ki+j})_{1 \leq j \leq k,\, i \in \mathbb{N}}$, like the one used in the proof of Corollary 3.2, remains unconditional when seen as a sequence in $\mathcal{L}^r(Y)$.

As an application of Corollary 3.3, we are now able to extend the conclusion of [2, Theorem 5.1] to a much larger class of Banach lattices. Precisely, we have the following.

Corollary 3.5. *Let X be a reflexive Banach lattice with generating system Π, as in Theorem 3.1, and let $\mathcal{A}$ be a closed Levi Π hereditary Riesz operator subalgebra of $\mathcal{L}^r(X)$ (in particular, any closed Levi order and algebra ideal). Then, for every algebra automorphism Θ of $\mathcal{A}$, there exists $U \in \mathcal{L}^r(X)$ invertible, such that $\Theta(T) = UTU^{-1}$ $(T \in \mathcal{A})$. As before, if Π can be taken to be $\{\mathrm{id}_X\}$, then the Levi assumption can be omitted.*

Proof. Let $\mathcal{A}$ be a closed Levi Π hereditary Riesz operator subalgebra of $\mathcal{L}^r(X)$ and let $\Theta : \mathcal{A} \to \mathcal{A}$ be an algebra automorphism. By [14, Theorem 2.5.19], there exists $U \in \mathcal{B}(X)$ invertible such that $\Theta(T) = UTU^{-1}$ $(T \in \mathcal{A})$. It follows readily that $\Theta : (\mathcal{A}, \|\cdot\|) \to (\mathcal{A}, \|\cdot\|)$ is continuous, and a straightforward application of the closed graph theorem shows that $\Theta : (\mathcal{A}, \|\cdot\|_r) \to (\mathcal{A}, \|\cdot\|_r)$ is continuous too. Clearly, $\Theta(\mathcal{F}(X)) \subseteq \mathcal{F}(X)$. Furthermore, $w^*\text{-}\lim_\pi \Theta(\pi T \pi) = \Theta(T)$ $(T \in \mathcal{A})$. To see this last fix $T \in \mathcal{A} \setminus \{0\}$ and note first that, for every $x \in X$ and $\lambda \in X'$, $\lim_\pi \lambda(U\pi T\pi U^{-1}x) = \lambda(UTU^{-1}x)$ (by our definition of a directed generating system). Next let $\phi \in X \otimes_{|\pi|} X'$ and $\varepsilon > 0$ arbitrary. Choose $\psi \in X \otimes X'$ so that $\|\phi - \psi\|_{|\pi|} < \varepsilon/(3\|\Theta\|\|T\|_r \sup_\pi \|\pi\|)$ and choose $\pi_0 \in \Pi$ so that $|\psi(U\pi T\pi U^{-1})-\psi(UTU^{-1})|<\varepsilon/3$ whenever π is 'greater' than π_0 (which is possible by the previous observation), then

$$\begin{aligned}&\big|\phi(U\pi T\pi U^{-1})-\phi(UTU^{-1})\big|\\&\qquad\le\|\phi-\psi\|_{|\pi|}\big(\|U\pi T\pi U^{-1}\|_r+\|UTU^{-1}\|_r\big)+\big|\psi(U\pi T\pi U^{-1})-\psi(UTU^{-1})\big|\le\varepsilon,\end{aligned}$$

so $\lim_\pi\phi(\Theta(\pi T\pi))=\phi(\Theta(T))$, as required. Since all hypotheses of Corollary 3.3 are satisfied, Θ must be regular. We will show next that U and U^{-1} must be regular too.

Let $x\in X_+$ and $x'\in X'_+\setminus\{0\}$ be arbitrary. Choose $x''\in X''_+$ so that $x''(|x'\circ U^{-1}|)=1$, and let $\phi:\mathcal{A}^r(X)\to X$ be the positive linear map defined on rank-one elements by $\phi(\lambda\otimes x):=x''(\lambda)x$ $(x\in X,\ \lambda\in X')$ (see the proof of [2, Theorem 5.1] for details). Next, let $\mathcal{A}^r(X)^{dd}$ be the band in $\mathcal{L}^r(X)$ generated by $\mathcal{A}^r(X)$, and for every $T\in(\mathcal{A}^r(X)^{dd})_+$, define $\phi(T):=\sup\big\{\phi(S):S\in[0,T]\cap\mathcal{A}^r(X)\big\}$. First of all note that ϕ is well-defined for $\big\{\phi(S):S\in[0,T]\cap\mathcal{A}^r(X)\big\}$ is a norm-bounded upwards directed set and X is reflexive (hence, Levi). Moreover, ϕ is additive on $(\mathcal{A}^r(X)^{dd})_+$. (To see it, let $S,T\in(\mathcal{A}^r(X)^{dd})_+$ arbitrary. If $R\in[0,S+T]\cap\mathcal{A}^r(X)$, then there are $R_1\in[0,S]\cap\mathcal{A}^r(X)$ and $R_2\in[0,T]\cap\mathcal{A}^r(X)$ such that $R_1+R_2=R$, so $\phi(R)=\phi(R_1)+\phi(R_2)\le\phi(S)+\phi(T)$, and $\phi(S+T)\le\phi(S)+\phi(T)$ follows. As for the opposite inequality, simply note that if $R_1\in[0,S]\cap\mathcal{A}^r(X)$ and $R_2\in[0,T]\cap\mathcal{A}^r(X)$, then $R_1+R_2\in[0,S+T]\cap\mathcal{A}^r(X)$ and $\phi(R_1)+\phi(R_2)=\phi(R_1+R_2)\le\phi(S+T)$.) It follows that ϕ has a unique extension, as a positive linear map, to the whole of $\mathcal{A}^r(X)^{dd}$, which we shall continue to denote by ϕ. Next note that, since X is reflexive, $\mathcal{A}^r(X)$ is an order ideal in $\mathcal{A}\subseteq\mathcal{L}^r(X)$ (this is immediate from [16, Theorem 4.1]), and therefore,

$$|\Theta|(x'\otimes x)=\sup_{S\in\mathcal{A}:|S|\le x'\otimes x}|\Theta(S)|=\sup_{S\in\mathcal{A}^r(X):|S|\le x'\otimes x}|\Theta(S)|\in\mathcal{A}^r(X)^{dd}.$$

In turn, for every $\xi\in X$ with $|\xi|\le x$, one has that

$$|U\xi|=\phi\big(|x'\circ U^{-1}|\otimes|U\xi|\big)\le\phi\Bigg(\sup_{S\in\mathcal{A}^r(X):|S|\le x'\otimes x}|USU^{-1}|\Bigg)=\phi\big(|\Theta|(x'\otimes x)\big),$$

and since X is Dedekind-complete, U must be regular. The regularity of U^{-1} follows on applying the same argument to the algebra automorphism Θ^{-1}.

As for the last claim of the corollary, simply note that if $X\in\mathfrak{F}_X^{\star,\mu}$ and $\Pi=\{\mathrm{id}_X\}$ then one is back to the situation of [2, Theorem 5.1]. □

4. Towards an example of a bounded non-regular homomorphism.

Once again, we have been unable to produce examples of non-regular continuous algebra homomorphisms between Banach lattice algebras of the kind considered in the note. Our results suggest the Levi assumption on the codomain could be, at least in some situations, a necessary condition but we have not been able to establish anything concrete in this direction.

At the end of Section 4 of [2], we mentioned without proof, that it is possible to construct continuous non-regular algebra homomorphisms between Banach algebras of regular operators. This involves adapting a well-known scheme for producing discontinuous algebra homomorphisms from algebras of bounded operators – first introduced in [6] – to the order setting. We do not know whether such scheme could produce also non-regular continuous homomorphisms. This would depend on the existence of homomorphisms with similar properties from the commutative Banach lattice algebra ℓ_∞/c_0. Precisely, the following holds:

If there is a non-regular bounded algebra homomorphism from the commutative Banach lattice algebra ℓ_∞/c_0 into some Dedekind complete Banach lattice algebra $\mathcal{B}$, then there are Banach lattices X and Y such that there exists a non-regular bounded algebra homomorphism $\Theta : \mathcal{L}^r(X) \to \mathcal{L}^r(Y)$.

For completeness, and since we do not have any reference for this, we provide the argument below (in any case, it will show how to produce discontinuous algebra homomorphisms in the present setting, for which we have no reference either).

Proof of the claim. Suppose first the underlying field is $\mathbb{R}$. Let X_G stand for the (real version of the) Banach space constructed by Gowers in [9]. The latter is a Banach space with a 1-unconditional basis, (x_i), and therefore, an order continuous, purely atomic Banach lattice in a natural way. A fundamental property of X_G, essential to the construction that follows, is the fact that every operator in $\mathcal{B}(X_G)$ is the sum of a strictly singular operator and a diagonal one (see [10, Section 5(5.1)]). Here, of course, we shall need a lattice version of this result. Precisely, let $\mathcal{SS}(X_G)$ be the ideal of strictly singular operators on $\mathcal{B}(X_G)$, and let $\mathcal{SS}^r(X_G) := \mathcal{L}^r(X_G) \cap \mathcal{SS}(X_G)$. Then $\mathcal{SS}^r(X_G)$ is a closed order and algebra ideal of $\mathcal{L}^r(X_G)$ and $\mathcal{L}^r(X_G) = \mathcal{D} + \mathcal{SS}^r(X_G)$, where $\mathcal{D}$ denotes the algebra of all diagonal operators on $\mathcal{B}(X_G)$. That $\mathcal{L}^r(X_G) = \mathcal{D} + \mathcal{SS}^r(X_G)$ follows easily on noting that $\mathcal{D} \subset \mathcal{L}^r(X_G)$. We verify next the other claims.

It is easy to see that $\mathcal{SS}^r(X_G)$ is an algebra ideal of $\mathcal{L}^r(X_G)$. As for it being closed, note that if (T_n) is a sequence in $\mathcal{SS}^r(X_G) \subset \mathcal{SS}(X_G)$ such that $\lim_n \|T_n - T\|_r = 0$ for some $T \in \mathcal{L}^r(X_G)$ then $\lim_n \|T_n - T\| = 0$ too, so $T \in \mathcal{SS}(X_G)$, and in turn, $T \in \mathcal{SS}^r(X_G)$, as required. To see that $\mathcal{SS}^r(X_G)$ is an order ideal, first note that if $S \in \mathcal{SS}^r(X_G)$ then $|S| \in \mathcal{SS}^r(X_G)$, for there is $S_1 \in \mathcal{SS}^r(X_G)$ such that $|S| - S_1 \in \mathcal{D}$, and since S and S_1 are strictly singular, $\lim_i x_i^*(|S|x_i) = \lim_i |x_i^*(Sx_i)| = 0$ and $\lim_i x_i^*(S_1 x_i) = 0$, so $|S| - S_1 \in \mathcal{A}^r(X_G) \subseteq \mathcal{SS}^r(X_G)$ (where as usual, x_i^* denotes the i-th biorthogonal functional associated with the basis (x_i)). Thus, $\mathcal{SS}^r(X_G)$ is indeed a Riesz subspace of $\mathcal{L}^r(X_G)$. Now let $S \in \mathcal{SS}^r(X_G)$ and let $T \in \mathcal{L}^r(X_G)$ be such that $0 \le |T| \le |S|$. Then, $0 \le T_+, T_- \le |S|$ and since X_G is atomic and order continuous, we must have $T_+, T_- \in \mathcal{SS}^r(X_G)$, by [7, Theorem 1.1]. In turn, $T \in \mathcal{SS}^r(X_G)$, so $\mathcal{SS}^r(X_G)$ is an order ideal, as claimed.

It is clear now that $\mathcal{L}^r(X_G)/\mathcal{SS}^r(X_G)$ is a Banach lattice algebra, and moreover, that it is isomorphic to the commutative Banach lattice algebra ℓ_∞/c_0. We shall write $\mathcal{Q}$ for the quotient map from $\mathcal{L}^r(X_G)$ onto $\mathcal{L}^r(X_G)/\mathcal{SS}^r(X_G)$, which in view of the previous discussion, is a Riesz and algebra homomorphism.

Now suppose there exists a continuous non-regular Banach algebra homomorphism Φ from ℓ_∞/c_0 into some order complete Banach lattice algebra $\mathcal{B}$. Let $\mathcal{B}^\#$ be the unitization of $\mathcal{B}$ (i.e., the vector space $\mathcal{B} \oplus \mathbb{R}$, with the product $(a,\lambda)(b,\mu) := (ab + \lambda b + \mu a, \lambda\mu)$ and the norm $\|(a,\lambda)\| := \|a\| + |\lambda|$), endowed with the obvious order structure, so $\mathcal{B}^\#$ becomes a Banach lattice algebra. Then let $\rho : \mathcal{B} \to \mathcal{L}^r(\mathcal{B}^\#)$, $b \mapsto \big[L_b : \mathcal{B}^\# \to \mathcal{B}^\#,\ (a,\lambda) \mapsto (ba + \lambda b, 0)\big]$ and let $\Theta := \rho \circ \Phi \circ \mathcal{Q}$. The continuity of Θ is clear, so we only need to check it cannot be regular. To this end, fix $T \in \mathcal{L}^r(X_G)_+$ and suppose Θ is regular. First note that,

$$(|\Theta|(T))(0,1) \geq \sup_{|S|\leq T} |\Theta(S)|(0,1) \geq \sup_{|S|\leq T} \big(|\Phi(\mathcal{Q}(S))|, 0\big). \tag{8}$$

Next note for $R, T \in \mathcal{L}^r(X_G)$, $|\mathcal{Q}(R)| \leq \mathcal{Q}(T) \iff \mathcal{Q}(|R|) \leq \mathcal{Q}(T) \iff$ there is $K_0 \in \mathcal{SS}^r(X_G)$ such that $|R| + K_0 \leq T \iff$ there is $K_1 \in \mathcal{SS}^r(X_G)$ such that $0 \leq |R| + K_1 \leq T$ (e.g., let $K_1 := -(T - |R|)_-$) $\Rightarrow$ there is $K_2 \in \mathcal{SS}^r(X_G)$ such that $-T \leq R + K_2 \leq T$ (e.g., define $K_2 \in \mathcal{L}^r(X_G)$ by $x_i^*(K_2 x_j) := \operatorname{sgn}(x_i^*(R x_j))\, x_i^*(K_1 x_j)$ $(i, j \in \mathbb{N})$), and so,

$$\sup_{|S|\leq T} |\Phi(\mathcal{Q}(S))| = \sup_{|\mathcal{Q}(R)|\leq \mathcal{Q}(T)} |\Phi(\mathcal{Q}(R))|.$$

Combining the last equality with (8), one finally arrives at

$$(|\Theta|(T))(0,1) \geq \left(\sup_{|W|\leq \mathcal{Q}(T)} |\Phi(W)|, 0 \right),$$

which contradicts the non-regularity of Φ.

In the complex case, the proof goes along the same lines, using instead the complex version of X_G. One simply needs to note that $\mathcal{SS}^r(X_G)$ can be identified with the complexification of $\mathcal{SS}^r((X_G)_{\mathbb{R}})$, where $(X_G)_{\mathbb{R}}$ stands for the real part of X_G (we leave the details to the reader). $\square$

By [5, Theorem 5.7.38], there are, in the complex case and assuming CH, discontinuous algebra homomorphisms from ℓ_∞/c_0 into the unitization of the weighted convolution algebra $L^1(\omega)$, for some continuous radical weight ω on $\mathbb{R}_+$. Such homomorphisms would give rise to discontinuous homomorphisms from $\mathcal{L}^r(X)$ into $\mathcal{L}^r(L^1(\omega)^\#)$, if used in the above argument. Also note that in view of the result of Huijsmans and de Pagter, mentioned in the introduction, if there was a continuous non-regular algebra homomorphism from the semiprime Banach lattice algebra ℓ_∞/c_0 into another Banach lattice algebra $\mathcal{B}$, the latter would not be a semiprime f-algebra. We do not know of any systematic study of automatic regularity of algebra homomorphisms in this situation.

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