Invariant Sets Analysis for Constrained Switching Systems

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Invariant Sets Analysis for Constrained Switching Systems

N. Athanasopoulos, K. Smpoukis and R. M. Jungers

***Abstract*— We study discrete time linear constrained switching systems with additive disturbances, in the general setting where the switching acts on the system matrices, the disturbance sets and the state constraint sets. Our primary goal is to extend the existing invariant set constructions when the switching signal is constrained by a given automaton. We achieve it by working with a relaxation of invariance, namely the *multi–set invariance*. By exploiting recent results on computing the stability metrics for these systems, we establish explicit bounds on the number of iterations required for each construction. Last, as an application, we develop new maximal invariant set constructions for the case of linear systems in far fewer iterations compared to the state-of-the-art.**

I. INTRODUCTION

Switching systems pose major theoretical challenges, provide an accurate modeling framework for many processes and are good approximations of complex dynamics [1]–[5]. In interesting cases, switching is not arbitrary, see, e.g., dwell time and fault-detection settings [6]–[9]. Constrained switching can be described by labeled directed graphs [10]–[14]. A switching sequence is *admissible* if it can be realized by the labels of the edges appearing in a walk in the graph.

Although the stability and stabilizability problems are addressed in the literature, a systematic approach dealing with the safety analysis, as in e.g. [15], [16], of these systems is missing. Related studies include works on Markov Jump Linear Systems [17], dwell-time [6], [18], [19], periodic [20] and cyclic [21] invariance. In this article, we focus on how invariance[1] generalizes to constrained switching and develop the notion of *invariant multi-sets*. A multi-set is an M–tuple of sets, one per node in the switching constraints graph (see for an example Figure 2): at each time instant, the state is required to be in only one of these sets. We define forward and backward reachability multi-set sequences and characterize, to the best of our knowledge for the first time, the maximal invariant multi-set and the minimal invariant multi-set and its approximations. The contributions are:

• We show existence and uniqueness of the minimal invariant multi-set. We construct ϵ–approximations of the minimal invariant multi-set, starting from the approaches concerning unconstrained systems in [24] (inner approximation) and [8], [22], [25] (outer approximations).

• We show existence and uniqueness of the maximal invariant multi-set following a similar path as in [26].

• We provide new constructions of the maximal invariant set for linear time-invariant systems, with a bound on the number of iterations proportional to the square root of the iterations in the standard algorithm.

Most importantly, by utilizing recent, practicable, results concerning approximations of the exponential stability metrics [11], [12], [27], we establish *a priori upper bounds on the maximum number of iterations required for computing the multi-sets*. Preliminary results are in [28].

Notation: The ball of radius α of an arbitrary norm is $\mathbb{B}(\alpha)$. The Minkowski sum of two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is $\mathcal{S}_1 \oplus \mathcal{S}_2$. The interior and the convex hull of a set $\mathcal{S}$ is $\mathrm{int}\,(\mathcal{S})$ and $\mathrm{conv}(\mathcal{S})$ respectively. A C-set $\mathcal{S} \subset \mathbb{R}^n$ is a convex compact set for which a $\delta > 0$ exists such that $\mathbb{B}(\delta) \subseteq \mathcal{S}$ [15].

II. PRELIMINARIES

We consider a set of matrices $\mathcal{A} := \{A_1, ..., A_N\} \subset \mathbb{R}^{n\times n}$ and disturbance sets $\mathbb{W} = \{\mathcal{W}_1, ..., \mathcal{W}_N\}$, $\mathcal{W}_i \subset \mathbb{R}^n$. We consider a set of nodes $\mathcal{V} := \{1, 2, ..., M\}$ and a set of edges $\mathcal{E} = \{(s, d, \sigma) : s \in \mathcal{V}, d \in \mathcal{V}, \sigma \in \{1, ..., N\}\}$, where s is the source node, d is the destination node and σ is the label of the edge. We denote the corresponding graph by $\mathcal{G}(\mathcal{V}, \mathcal{E})$, or, $\mathcal{G}$. The set of outgoing nodes of a node $s \in \mathcal{V}$ is $\mathrm{Outgoing}(s, \mathcal{G}) := \{d \in \mathcal{V} : (\exists \sigma \in \{1, ..., N\} : (s, d, \sigma) \in \mathcal{E})\}$. Finally, we consider constraint sets $\mathcal{X}_i \subset \mathbb{R}^n$, $i \in \{1, ..., M\}$. The System we study is

$$x(t+1) = A_{\sigma(t)}x(t) + w(t), \tag{1}$$
$$z(t+1) \in \mathrm{Outgoing}(z(t), \mathcal{G}), \tag{2}$$
$$w(t) \in \mathcal{W}_{\sigma(t)}, \tag{3}$$
$$(x(0), z(0)) \in \mathbb{R}^n \times \mathcal{V}, \tag{4}$$

subject to the constraints

$$\sigma(t) \in \{\sigma : (z(t), z(t+1), \sigma) \in \mathcal{E}\}, \tag{5}$$
$$x(t) \in \mathcal{X}_{z(t)}, \tag{6}$$

for all $t \geq 0$. We call *nominal* the disturbance-free system, i.e., the system $x(t+1) = A_{\sigma(t)}x(t)$ together with (2), (4)–(6). The stability of the nominal system is characterized by the *constrained joint spectral radius* [10] $\check{\rho}(\mathcal{A}, \mathcal{G}) := \lim_{k\to\infty} \check{\rho}_k(\mathcal{A}, \mathcal{G})$, where $\check{\rho}_k(\mathcal{A}, \mathcal{G}) := \max\{\| \prod_{j=1}^{k} A_{\sigma_j} \|^{1/k} : \sigma_k...\sigma_1$ *is an admissible switching sequence*$\}$ is the maximum growth rate up to time k. The nominal system is asymptotically stable if and only if $\check{\rho}(\mathcal{A}, \mathcal{G}) < 1$ [10, Corollary 2.8].

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[1]By stability we mean asymptotic stability and by invariance we mean robust positive invariance [15], [22], also referred to as forward invariance [23] or d-invariance [24].

Assumption 1 (State constraints) *The constraint sets $\mathcal{X}_i \subset \mathbb{R}^n$, $i = 1, ..., M$, are C-sets.*

Assumption 2 (Disturbances) *The disturbance sets $\mathcal{W}_i \subset \mathbb{R}^n$, $i = 1, ..., N$, are C-sets.*

Assumption 3 (Stability) $\check{\rho}(\mathcal{A}, \mathcal{G}) < 1$.

Assumption 4 (Connectedness) *$\mathcal{G}(\mathcal{V}, \mathcal{E})$ is strongly connected.*

These assumptions are standard, or necessary for our purpose, see e.g., [16] for Assumptions 1 and 2. Assumption 3 is standard since $\check{\rho}(\mathcal{A}, \mathcal{G}) > 1$ excludes non-trivial invariant multi-sets or safe sets[2]. Assumption 4 guarantees the completeness of solutions and can be alleviated, at the price of further technicalities. Below, we generalize the notion of invariance, which concerns the confinement of the system trajectories to a single set, see e.g. [15], to *multi-sets*.

Definition 1 (Multi–set invariance) *The collection of sets $\{\mathcal{S}^i\}_{i\in\mathcal{V}}$ is an* invariant multi-set *with respect to the System* (1)–(5) *if $x(0) \in \mathcal{S}^{z(0)}$ implies $x(t) \in \mathcal{S}^{z(t)}$, for all $t \geq 0$, $z(0) \in \mathcal{V}$ and $\sigma(t), t \geq 0$, satisfying* (5). *If also $\mathcal{S}^i \subseteq \mathcal{X}_i$, $i \in \mathcal{V}$, then $\{\mathcal{S}^i\}_{i\in\mathcal{V}}$ is called an* admissible invariant multi-set *with respect to* (1)–(6). *The multi-set $\{\mathcal{S}^i_M\}_{i\in\mathcal{V}}$ is the* maximal admissible invariant multi-set *if for any admissible invariant multi-set $\{\mathcal{S}^i\}_{i\in\mathcal{V}}$, $i \in \mathcal{V}$, we have $\mathcal{S}^i \subseteq \mathcal{S}^i_M$. The invariant multi-set $\{\mathcal{S}^i_m\}_{i\in\mathcal{V}}$ is the* minimal invariant multi-set *if for any invariant multi-set $\{\mathcal{S}^i\}_{i\in\mathcal{V}}$, $i \in \mathcal{V}$, $\mathcal{S}^i_m \subseteq \mathcal{S}^i$.*

Definition 2 (Safety) *A set $\mathcal{S}_\mathcal{Y} \subset \mathbb{R}^n$ is* safe *with respect to the System* (1)–(6) *and a set of nodes $\mathcal{Y} \subseteq \mathcal{V}$ if $(x(0), z(0)) \in \mathcal{S}_\mathcal{Y} \times \mathcal{Y}$, implies $x(t) \in \mathcal{X}_{z(t)}$, $t \geq 0$.*

The Assumptions do not suffice for System (1)–(6) to possess a non-trivial invariant set[3]. The connection to invariance can be made via the Ω-*lift* and the *Kronecker lift* [29], [13]. Although it is tempting to work in that lifted space, the computations are significantly harder since they involve set operations in higher dimensions. Invariant multi-sets can also be connected to a particular case of *hybrid invariant sets*, e.g., [23], defined in an extended state space $[x^\top \quad z]^\top \in \mathbb{R}^n \times \{1, ..., M\}$. See [30] for new results on invariance and ω-limit sets in the framework of *set dynamical systems*.

Definition 3 (Reachability) *Consider the System* (1)–(4) *and a switching sequence $\sigma_1 ... \sigma_p$, $\sigma_i \in \{1, ..., N\}$, $p \geq 1$. The p-step* forward reachability map *is* $\mathcal{R}(\sigma_1...\sigma_p, \mathcal{S}) := (\prod_{i=1}^p A_{\sigma_{p+1-i}}\mathcal{S}) \oplus (\bigoplus_{j=1}^{p}\prod_{i=1}^{p-j} A_{\sigma_{p+1-i}}\mathcal{W}_{\sigma_j})$. *The* p-step backward reachability map *is* $\mathcal{C}(\sigma_1...\sigma_p, \mathcal{S}) := \{x : (\prod_{i=1}^p A_{\sigma_{p+1-i}}\{x\}) \oplus (\bigoplus_{j=1}^{p}\prod_{i=1}^{p-j} A_{\sigma_{p+1-i}}\mathcal{W}_{\sigma_j}) \in \mathcal{S}\}$ (Note that we use the convention $\prod_{i=1}^0 A_{\sigma(i)} = 1$).

We write $\mathcal{R}_\mathrm{N}(\sigma_1...\sigma_p, \mathcal{S}) := \{\prod_{i=1}^p A_{\sigma_{p+1-i}}x : x \in \mathcal{S}\}$. Moreover, we define the convex versions of the forward mappings as $\mathcal{R}_\mathrm{C}(\sigma_1...\sigma_p, \mathcal{S}) := \mathrm{conv}(\mathcal{R}(\sigma_1...\sigma_p, \mathcal{S}))$, $\mathcal{R}_\mathrm{CN}(\sigma_1...\sigma_p, \mathcal{S}) := \mathrm{conv}(\mathcal{R}_\mathrm{N}(\sigma_1...\sigma_p, \mathcal{S}))$.

Example 1 *By considering a sequence $\sigma_1\sigma_2$ and a set $\mathcal{S} \subset \mathbb{R}^n$, we have $\mathcal{R}(\sigma_1\sigma_2, \mathcal{S}) = A_{\sigma_2}A_{\sigma_1}\mathcal{S} \oplus A_{\sigma_2}\mathcal{W}_{\sigma_1} \oplus \mathcal{W}_{\sigma_2}$ and $\mathcal{C}(\sigma_1\sigma_2, \mathcal{S}) = \{x : A_{\sigma_2}A_{\sigma_1}\{x\} \oplus A_{\sigma_2}\mathcal{W}_{\sigma_1} \oplus \mathcal{W}_{\sigma_2} \in \mathcal{S}\}$. See Figure 1 for an illustration.*

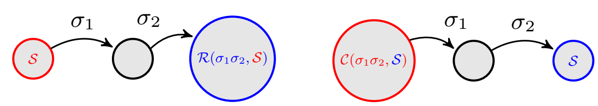

Fig. 1. Example 1, illustration of the Definition 3 for a two-step forward and backward reachability map.

Let us consider the sequence $\{\mathcal{N}^j_l\}_{j\in\mathcal{V}}$, $l \geq 0$, generated by

$$\mathcal{N}^j_0 := \cup_{(s,j,\sigma)\in\mathcal{E}}\mathcal{W}_\sigma, \quad j \in \mathcal{V}, \tag{7}$$

$$\mathcal{N}^j_{l+1} := \cup_{(s,j,\sigma)\in\mathcal{E}}\mathcal{R}_\mathrm{N}(\sigma, \mathcal{N}^s_l), \quad j \in \mathcal{V}. \tag{8}$$

We can express the exponential shrinking of the elements of the multi-set sequence (7), (8) with the set inclusion

$$\mathcal{N}^j_t \subseteq \Gamma\rho^t\mathcal{N}^j_0, \quad \forall j \in \mathcal{V}, \quad \forall t \geq 0. \tag{9}$$

Several methods exist for computing the scalars $\Gamma \geq 1$, $\rho \in (0, 1)$ in (9), see, e.g., [11], [12], [27].

III. The minimal invariant multi-set

Let us consider the forward reachability multi-set sequence $\{\mathcal{F}^j_l\}_{j\in\mathcal{V}}$, $l \geq 0$, with

$$\mathcal{F}^j_0 := \{0\}, \quad j \in \mathcal{V}, \tag{10}$$

$$\mathcal{F}^j_{l+1} := \cup_{(s,j,\sigma)\in\mathcal{E}}\mathcal{R}(\sigma, \mathcal{F}^s_l), \quad j \in \mathcal{V}. \tag{11}$$

First, we characterize the minimal invariant multi–set.

Theorem 1 *The minimal invariant multi–set $\{\mathcal{S}^j_m\}_{j\in\mathcal{V}}$ with respect to the System* (1)–(5) *is unique and equal to $\mathcal{S}^j_m = \lim_{l\to\infty}\mathcal{F}^j_l$, $j \in \mathcal{V}$.*

Proof Following a reasoning similar to [24, Section 4], we can show that $\mathcal{F}^j_l \subseteq \mathcal{F}^j_{l+1} \subseteq \mathcal{F}^j_l \oplus (\Gamma\rho^l \cup_{i=1}^N \mathcal{W}_i)$, $l \geq 0$, where (Γ, ρ) satisfy (9). Consequently, the multi-set sequence is convergent to a compact multi-set, in the space of compact multi-sets paired with the Hausdorff metric. To show uniqueness, we work as in [31, Lemma 3.1] and assume there is a compact invariant multi-set $\{\mathcal{S}^j\}_{j\in\mathcal{V}}$ and $j^\star \in \{1, ..., M\}$ such that $\mathcal{F}^{j^\star}_\infty \nsubseteq \mathcal{S}^{j^\star}$. We pick a $x(0) \in \mathbb{R}^n$, $z(0) \in \mathcal{V}$ and set $w(t) = 0$, for all $t \geq 0$. By Assumption 4, we choose a solution $(x(t), z(t))$, $t \geq 0$, for which $z(t_i) = j^\star$, $i \geq 0$, for a sequence $\{t_i\}_{i\geq 0}$. We have $x(t) \to 0$ as $t \to \infty$. Since $\mathcal{S}^{j^\star}$ is compact, $x(t_i) \in \mathcal{S}^{j^\star}$ and $\{x(t_i)\}_{i\geq 0}$ converges to 0, it necessarily holds that

[2] The study of the limiting case $\check{\rho}(\mathcal{A}, \mathcal{G}) = 1$ is outside the scope of this study: in this case the problem is undecidable [3].

[3] Consider $x(t+1) = a_{\sigma(t)}x(t)$, $a_1 = -2$, $a_2 = 0.25$ and $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, 2\}$, $\mathcal{E} = \{(1, 1, 2), (1, 2, 1), (2, 1, 2)\}$. The system does not admit a non–trivial invariant set but has the invariant multi-set $\{\mathcal{S}^1, \mathcal{S}^2\} = \{[-0.5, 0.5], [-1, 1]\}$.

$0 \in \mathcal{S}^{j^\star}$. By invariance of $\{\mathcal{S}^j\}_{j\in\mathcal{V}}$, $\mathcal{F}_\infty^{j^\star} \subseteq \mathcal{S}^{j^\star}$, leading to a contradiction. ■

Since the multi-set sequence (10), (11) does not necessarily converge in finite time, we provide ways to approximate it, by the following theorem.

Theorem 2 *Consider the System* (1)–(5)*, a pair* (Γ, ρ) *satisfying* (9)*, the multi-set sequence* (10)*,* (11) *and the minimal invariant multi-set* $\{\mathcal{S}_m^j\}_{j\in\mathcal{V}}$*. Let* $\alpha_1 := \min\{a : \cup_{i=1}^N \mathcal{W}_i \subseteq \mathbb{B}(a)\}$*. Given a desired accuracy* $\epsilon > 0$*, the following hold.*
(i) *For any* $l \geq \lceil \log_\rho(\frac{\epsilon(1-\rho)}{\alpha_1 \Gamma})\rceil$*, it holds that*

$$\mathcal{F}_l^j \subseteq \mathcal{S}_m^j \subseteq \mathcal{F}_l^j \oplus \mathbb{B}(\epsilon), \quad j \in \mathcal{V}.$$

(ii) *Let* $\alpha_2 = \min\{\alpha : \cup_{i=1}^N \mathcal{W}_i \subseteq \alpha \cap_{i=1}^N \mathcal{W}_i\}$ *and consider a pair* (k, λ) *that satisfies the inequalities*

$$\alpha_2 \Gamma \rho^k \leq \lambda, \tag{12}$$

$$\tfrac{\Gamma(1-\rho^{k-1})}{1-\rho} \leq \tfrac{\epsilon(1-\lambda)}{\alpha_1 \lambda}. \tag{13}$$

Then, the multi-set $\{\frac{1}{1-\lambda}\mathcal{F}_{k-1}^j\}_{j\in\mathcal{V}}$ *is invariant, and furthermore,*

$$\mathcal{S}_m^j \subseteq \tfrac{1}{1-\lambda}\mathcal{F}_{k-1}^j \subseteq \mathcal{S}_m^j \oplus \mathbb{B}(\epsilon), \quad j \in \mathcal{V}.$$

(iii) *Let* $\{\mathcal{S}_0^j\}_{j\in\mathcal{V}}$ *be an invariant multi-set and let* $\alpha_3 = \min\{a : \mathcal{S}_0^j \subseteq a\mathcal{N}_0^j, j \in \mathcal{V}\}$*. Let* $\{\mathcal{S}_l^j\}_{j\in\mathcal{V}}$*,* $l \geq 0$ *be the multi-set sequence generated by the update* (11)*, initialized by* $\{\mathcal{S}_0^j\}_{j\in\mathcal{V}}$*. Then, for any* $l \geq \lceil \log_\rho(\frac{\epsilon}{\alpha_1 \alpha_3 \Gamma})\rceil$*, the multi-set* $\{\mathcal{S}_{l+1}^j\}_{j\in\mathcal{V}}$ *is invariant, and furthermore,*

$$\mathcal{S}_m^j \subseteq \mathcal{S}_{l+1}^j \subseteq \mathcal{S}_m^j \oplus \mathbb{B}(\epsilon), \quad j \in \mathcal{V}.$$

Proof **(i)** The left inclusion holds by construction. To prove the right, we have for any $l \geq \lceil \log_\rho(\frac{\epsilon(1-\rho)}{\alpha_1 \Gamma})\rceil$, $j \in \mathcal{V}$, $\mathcal{S}_m^j = \lim_{i\to\infty} \mathcal{F}_i^j = \lim_{k\to\infty} \mathcal{F}_{l+k}^j \subseteq \lim_{k\to\infty} \mathcal{F}_{l+k-1}^j \oplus \mathbb{B}(\Gamma\alpha_1\rho^{l+k}) \subseteq \ldots \subseteq \lim_{k\to\infty} \mathbb{B}(\Gamma\alpha_1\rho^l \frac{1-\rho^k}{1-\rho}) \oplus \mathcal{F}_l^j = \mathbb{B}(\Gamma\alpha_1\rho^l\frac{1}{1-\rho}) \oplus \mathcal{F}_l^j \subseteq \mathbb{B}(\epsilon) \oplus \mathcal{F}_l^j$.
(ii) From (13) we have that $\frac{\lambda}{1-\lambda}\mathcal{F}_{k-1}^j \subseteq \frac{\lambda}{1-\lambda}\mathbb{B}(\Gamma\alpha_1\frac{1-\rho^{k-1}}{1-\rho}) \oplus \mathcal{F}_0^j \subseteq \mathbb{B}(\epsilon)$. Consequently, to show the right inclusion we have $\frac{1}{1-\lambda}\mathcal{F}_{k-1}^j = (1+\frac{\lambda}{1-\lambda})\mathcal{F}_{k-1}^j \subseteq \mathcal{F}_{k-1}^j \oplus \mathbb{B}(\epsilon) \subseteq \mathcal{S}_m^j \oplus \mathbb{B}(\epsilon)$. From (12) it follows $\mathcal{N}_k^j \subseteq \Gamma\rho^k\mathcal{N}_0^j \subseteq \lambda \cap_{i=1}^N \mathcal{W}_i$ and using a similar reasoning as in [22], [25] we can prove invariance of $\{\frac{1}{1-\lambda}\mathcal{F}_{k-1}^j\}_{j\in\mathcal{V}}$. Consequently, the left inclusion holds by definition.
(iii) For any admissible switching sequence $\sigma_1 \ldots \sigma_l$ such that there is $d \in \mathcal{V}$ so that $(j, d, \sigma_1) \in \mathcal{E}$, we have $\mathcal{R}_N(\sigma_1\ldots\sigma_l, \mathcal{S}_0^j) \subseteq \mathcal{R}_N(\sigma_1\ldots\sigma_l, \alpha_3\mathcal{N}_0^j) \subseteq \alpha_3\Gamma\rho^l\mathcal{N}_0^j \subseteq \mathbb{B}(\alpha_3\alpha_1\Gamma\rho^l) \subseteq \mathbb{B}(\epsilon)$. Consequently, we have for any edge $(i,j,\sigma) \in \mathcal{E}$ that $A_\sigma\mathcal{S}_l^i \oplus \mathcal{W}_\sigma \subseteq \mathbb{B}(\epsilon) \oplus \mathcal{F}_{l+1}^j \subseteq \mathbb{B}(\epsilon) \oplus \mathcal{S}_m^j$, thus, $\mathcal{S}_{l+1}^j \subseteq \mathcal{S}_m^j \oplus \mathbb{B}(\epsilon)$. Invariance of $\{\mathcal{S}_{l+1}^j\}_{j\in\mathcal{V}}$ can be shown using a reasoning similar to [8], thus, the left inclusion holds by definition. ■

It is worth comparing the different approximation schemes of Theorem 2: The inner approximation (i) is not invariant[4] whereas the outer approximations (ii), (iii) are. On the other hand, for any number l of iterations of the multi-set sequence (10), (11), we always obtain an $\epsilon(l)$–approximation. In comparison, the number of iterations in (ii) has to be larger than a threshold, implied by (12), (13). In this sense, approximation (iii) might seem favorable, the challenge there however is to compute an initial invariant multi-set.

A. Convexifications

It may be difficult in practice to compute the sequences generated by (8) and (11) since their members are typically non-convex[5]. As it is the case with unconstrained systems [8], [25], [6], [16], we may establish convex approximations of the *minimal convex invariant multi-set*. We consider the multi-set sequences, $\{\overline{\mathcal{N}}_l^j\}_{j\in\mathcal{V}}$, $\{\overline{\mathcal{F}}_l^j\}_{j\in\mathcal{V}}$, updated by

$$\overline{\mathcal{N}}_{l+1}^j := \cup_{(s,j,\sigma)\in\mathcal{E}} \mathcal{R}_{\mathrm{CN}}(\sigma, \overline{\mathcal{N}}_l^s) \tag{14}$$

and

$$\overline{\mathcal{F}}_{l+1}^j := \cup_{(s,j,\sigma)\in\mathcal{E}} \mathcal{R}_{\mathrm{C}}(\sigma, \overline{\mathcal{F}}_l^s) \tag{15}$$

respectively, with $\overline{\mathcal{N}}_0^j = \mathcal{N}_0^j$, $\overline{\mathcal{F}}_0^j = \mathcal{F}_0^j$, $j \in \mathcal{V}$. We can show that the minimal convex invariant multi-set $\{\overline{\mathcal{S}}_m^j\}_{j\in\mathcal{V}}$ is $\overline{\mathcal{S}}_m^j = \mathrm{conv}(\mathcal{S}_m^j)$, $j \in \mathcal{V}$ and that similar approximation schemes as in Theorem 2 can be made, using the technical facts summarized below.

Lemma 1 *Consider the multi-set sequences generated by the updates rules* (8)*,* (14) *and* (11)*,* (15)*. The following hold.*
(i) $\mathrm{conv}(\mathcal{N}_l^j) = \mathrm{conv}(\overline{\mathcal{N}}_l^j)$, $\quad \forall l \geq 0, \quad \forall j \in \mathcal{V}$.
(ii) $\mathrm{conv}(\mathcal{F}_l^j) = \mathrm{conv}(\overline{\mathcal{F}}_l^j)$, $\quad \forall l \geq 0, \quad \forall j \in \mathcal{V}$.
(iii) Consider two sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^n$ *and let* $\mathcal{S}_2$ *be convex. Then,* $\mathcal{S}_1 \subseteq \mathcal{S}_2$ *if and only if* $\mathrm{conv}(\mathcal{S}_1) \subseteq \mathcal{S}_2$*.*
(iv) For any two sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^n$*, it holds that* $\mathrm{conv}(\mathcal{S}_1 \oplus \mathcal{S}_2) = \mathrm{conv}(\mathcal{S}_1) \oplus \mathrm{conv}(\mathcal{S}_2)$*.*

Proof To show (ii), we can use a similar reasoning to [25, Section 3], [11, Proposition 1]. In specific, the relation holds for $l = 0$ and assuming that the relation holds for $l = k$, we have $\mathrm{conv}(\mathcal{F}_{l+1}^j) = \mathrm{conv}(\cup_{(s,j,\sigma)\in\mathcal{E}}\mathcal{R}(\sigma, \mathcal{F}_l^s)) = \mathrm{conv}(\cup_{(s,j,\sigma)\in\mathcal{E}} \mathrm{conv}(\mathcal{R}(\sigma, \mathrm{conv}(\mathcal{F}_l^s)))) = \mathrm{conv}(\cup_{(s,j,\sigma)\in\mathcal{E}}\mathcal{R}_{\mathrm{C}}(\sigma, \overline{\mathcal{F}}_l^s)) = \mathrm{conv}(\overline{\mathcal{F}}_{l+1}^j)$. The proof of (i) is identical. Items (iii) and (iv) follow from standard convexity arguments. ■

Taking into account Lemma 1, we can establish the corollary convex versions of Theorems 1 and 2.

Corollary 1 (Theorem 2, Lemma 1) *Consider the System* (1)–(5)*, a pair* (Γ, ρ) *satisfying* (9)*, the multi-set sequence generated by* (15)*,* $\overline{\mathcal{F}}_0^j = \{0\}$ *and the minimal convex invariant multi-set* $\{\overline{\mathcal{S}}_m^j\}_{j\in\mathcal{V}}$*. Let* $\alpha_1 := \min\{a : \cup_{i=1}^N \mathcal{W}_i \subseteq \mathbb{B}(a)\}$*. Given a desired accuracy* $\epsilon > 0$*, the following hold.*
(i) *For any* $l \geq \lceil \log_\rho(\frac{\epsilon(1-\rho)}{\alpha_1\Gamma})\rceil$*, it holds that*

$$\mathrm{conv}(\overline{\mathcal{F}}_l^j) \subseteq \overline{\mathcal{S}}_m^j \subseteq \mathrm{conv}(\overline{\mathcal{F}}_l^j) \oplus \mathbb{B}(\epsilon), \quad j \in \mathcal{V}.$$

[4]unless equal to the minimal invariant multi-set.

[5]However, they have some structure since they are *radially convex sets*.

(ii) *Let* $\alpha_2 = \min\{\alpha : \cup_{i=1}^N \mathcal{W}_i \subseteq \alpha \cap_{i=1}^N \mathcal{W}_i\}$ *and consider a pair* (k, λ) *that satisfies* (12), (13). *Then, the multi-set* $\{\mathrm{conv}(\frac{1}{1-\lambda}\overline{\mathcal{F}}_{k-1}^j)\}_{j\in\mathcal{V}}$ *is invariant, and furthermore,*

$$\overline{\mathcal{S}}_m^j \subseteq \mathrm{conv}(\tfrac{1}{1-\lambda}\overline{\mathcal{F}}_{k-1}^j) \subseteq \overline{\mathcal{S}}_m^j \oplus \mathbb{B}(\epsilon), \quad j \in \mathcal{V}.$$

(iii) *Let* $\{\overline{\mathcal{S}}_0^j\}_{j\in\mathcal{V}}$ *be an invariant multi-set and let* $\alpha_3 = \min\{a : \overline{\mathcal{S}}_0^j \subseteq a\mathcal{N}_0^j, j \in \mathcal{V}\}$. *Let* $\{\overline{\mathcal{S}}_l^j\}_{j\in\mathcal{V}}$, $l \geq 0$ *be the multi-set sequence generated by the update* (15), *initialized by* $\{\overline{\mathcal{S}}_0^j\}_{j\in\mathcal{V}}$. *Then, for any* $l \geq \lceil \log_\rho(\frac{\epsilon}{\alpha_1\alpha_3\Gamma})\rceil$, *the multi-set* $\{\mathrm{conv}(\overline{\mathcal{S}}_{l+1}^j)\}_{j\in\mathcal{V}}$ *is invariant, and furthermore,*

$$\overline{\mathcal{S}}_m^j \subseteq \mathrm{conv}(\overline{\mathcal{S}}_{l+1}^j) \subseteq \overline{\mathcal{S}}_m^j \oplus \mathbb{B}(\epsilon), \quad j \in \mathcal{V}.$$

IV. The maximal invariant multi-set

Let us consider the backward reachability multi-set sequence $\{\mathcal{B}_l^j\}_{j\in\mathcal{V}}$, $l \geq 0$, where

$$\mathcal{B}_0^j = \mathcal{X}_j, \quad j \in \mathcal{V}, \tag{16}$$

$$\mathcal{B}_{l+1}^j = (\cap_{(j,d,\sigma)\in\mathcal{E}}\mathcal{C}(\sigma, \mathcal{B}_l^d)) \cap \mathcal{B}_0^j, \quad j \in \mathcal{V}. \tag{17}$$

The l-th term of the multi-set sequence (16), (17) contains the initial conditions $(x(0), z(0)) \in \mathcal{X}_{z(0)} \times \mathcal{V}$ which satisfy the state constraints for at least l time instants.

Theorem 3 *Consider the System* (1)–(6) *and the sequence* (16), (17). *Let the pair* (Γ, ρ) *satisfy* (9) *and* $\{\mathcal{S}_m^j\}_{j\in\mathcal{V}}$ *be the minimal invariant multi-set. Assume* $\mathcal{S}_m^j \subseteq \mathrm{int}\,(\mathcal{X}_j)$, $j \in \mathcal{V}$ *and let* $R_j := \max\{R : \mathbb{B}(R) \subseteq \mathcal{X}_j\}$, $r_j := \min\{r : \mathcal{S}_m^j \subseteq \mathbb{B}(r)\}$, $\alpha_1 := \min\{a : \cup_{i=1}^N \mathcal{W}_i \subseteq \mathbb{B}(a)\}$, $c := \min\{c : \mathcal{X}_j \subseteq c\mathcal{N}_0^j, j \in \mathcal{V}\}$, $\mathcal{N}_0^j$ *given in* (7). *Then, there is an integer* $\overline{k}$ *such that* $\mathcal{B}_{\overline{k}+1}^j = \mathcal{B}_{\overline{k}}^j$, $j \in \mathcal{V}$, *with*

$$\overline{k} \leq \log_\rho\left(\frac{\min_{j\in\mathcal{V}}(R_j - r_j)}{\alpha_1\Gamma c}\right). \tag{18}$$

Moreover, the multi-set $\{\mathcal{B}_{\overline{k}}^j\}_{j\in\mathcal{V}}$ *is the maximal admissible invariant multi-set.*

Proof For any initial condition $(x(0), z(0)) \in \mathcal{X}_{z(0)} \times \mathcal{V}$ we have $x(t) = x_1(t) + x_2(t)$, with $x_1(t) := \prod_{i=0}^{t-1} A_{\sigma(t-1-i)}x(0)$ and $x_2(t) \in \mathcal{F}_t^{z(t)}$, where $\{\mathcal{F}_l^j\}_{j\in\mathcal{V}}$ is generated by (10), (11). Consequently, we have $\|x(t)\| \leq \alpha_1\Gamma\rho^t c + r_{z(t)}$, $t \geq 0$. Thus, $\|x(t)\| \leq R_{z(t)}$, or, $x(t) \in \mathcal{X}_{z(t)}$, for all $t \geq \overline{k}$, where $\overline{k}$ is given in (18). Let us assume that $x(0) \in \mathcal{B}_{\overline{k}}^{z(0)}$ but $x(0) \notin \mathcal{B}_{\overline{k}+1}^{z(0)}$. Then, $x(\overline{k}+1) \notin \mathcal{X}_{z(\overline{k}+1)}$ which is a contradiction. Thus, $\mathcal{B}_{\overline{k}+1}^{z(0)} \supseteq \mathcal{B}_{\overline{k}}^{z(0)}$. Since $\mathcal{B}_{l+1}^j \subseteq \mathcal{B}_l^j$ holds by construction, the result follows. To show maximality, we work similarly to [24]: Multi-set invariance follows from the fact that $\mathcal{B}_{\overline{k}+1}^j = \mathcal{B}_{\overline{k}}^j$, $j \in \mathcal{V}$. Suppose there is an admissible invariant multi-set $\{\mathcal{M}^j\}_{j\in\mathcal{V}}$ and a $j^\star$ for which $\mathcal{M}^{j^\star} \not\subseteq \mathcal{B}_{\overline{k}}^{j^\star}$. Then, for all $x(0) \in \mathcal{M}^{j^\star} \setminus \mathcal{B}_{\overline{k}}^{j^\star}$, $z(0) = j^\star$, we have $x(\overline{k}) \notin \mathcal{X}_{z(\overline{k})}$ and $\{\mathcal{M}^{j^\star}\}_{j\in\mathcal{V}}$ is not admissible, which is a contradiction. ∎

The upper bound (18) in Theorem 3 can be computed a priori: The pair (Γ, ρ) can be computed by [27], the scalars R_j, c, α_1 depend on the problem data and can be computed, e.g., for polyhedral or ellipsoidal disturbance and state constraint sets. By applying the results of Section III, we can compute approximations of r_j, $j \in \mathcal{V}$ and use them in (18). The relation with the maximal safe set is stated in Corollary 2 which is a consequence of Theorem 3 and the definition of the maximal invariant multi-set. We note that we can deduce even more refined types of invariance, such as, e.g., returnability [32] and recurrence [33].

Corollary 2 *Consider the System* (1)–(6) *and let* $\{\mathcal{S}_M\}_{j\in\mathcal{V}}$ *be the maximal invariant multi-set. Let* $\mathcal{Y} \subseteq \mathcal{V}$ *be a set of nodes in* $\mathcal{G}(\mathcal{V}, \mathcal{E})$. *The maximal safe set* $\mathcal{S}_\mathcal{Y}$ *with respect to the System* (1)–(6) *and the node set* $\mathcal{Y} \subseteq \mathcal{V}$ *is* $\mathcal{S}_\mathcal{Y} = \cap_{j\in\mathcal{Y}}\mathcal{S}_M^j$.

Example 2 *We consider a constrained switching system modeling possible failures of a closed-loop linear system* [12, Section 4]. *We define* $\mathcal{A} = \{A_1, .., A_4\}$, *where* $A_i = A + BK_i$, *with* $A = \left[\begin{smallmatrix} 0.94 & 0.56 \\ 0.14 & 0.46 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $K_1 = \left[\begin{smallmatrix} -0.49 & 0.27 \end{smallmatrix}\right]$, $K_2 = \left[\begin{smallmatrix} 0 & 0.27 \end{smallmatrix}\right]$, $K_3 = \left[\begin{smallmatrix} -0.49 & 0 \end{smallmatrix}\right]$, $K_4 = \left[\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right]$. *In* [12] *it is shown that* $\check{\rho}(\mathcal{A}, \mathcal{G}) < 1$ *while in* [34] $\check{\rho}(\mathcal{A}, \mathcal{G})$ *is computed exactly. We consider different state constraint sets and disturbance sets* $\mathcal{X}_1 = \mathbb{B}_\infty(1.5)$, $\mathcal{X}_2 = \mathbb{B}_\infty(1.5) \oplus [0.5 \quad 0.5]^\top$, $\mathcal{X}_3 = \mathbb{B}_\infty(1.5) \oplus [-0.5 \quad -0.5]^\top$, $\mathcal{X}_4 = \mathbb{B}_\infty(2)$ *and* $\mathcal{W}_1 = \mathcal{W}_4 = \mathbb{B}_\infty(0.01)$, $\mathcal{W}_2 = T_1\mathbb{B}_\infty(0.01)$, $\mathcal{W}_3 = T_2\mathbb{B}_\infty(0.01)$, *where* $T_1 = \left[\begin{smallmatrix} 0.94 & -0.91 \\ 0.34 & 0.94 \end{smallmatrix}\right]$, $T_2 = \left[\begin{smallmatrix} 0.94 & 0.91 \\ -0.34 & 0.94 \end{smallmatrix}\right]$ *and* $\mathbb{B}_\infty(\alpha) := \{x \in \mathbb{R}^2 : |x|_\infty \leq \alpha\}$. *From Corollary 1(i), we compute a convex inner* ϵ*–approximation* $\{\mathcal{S}_\epsilon^j\}_{j\in\mathcal{V}}$ *of the minimal convex invariant multi-set for* $\epsilon = 10^{-2}$. *In specific, we calculate* $\mathcal{S}_\epsilon^j = \mathrm{conv}(\overline{\mathcal{F}}_{400}^j)$, $j \in \mathcal{V}$, *where* $\{\overline{\mathcal{F}}_l^j\}_{j\in\mathcal{V}}$, $l \geq 0$ *is generated by* (15). *The multi-set* $\{\mathcal{S}_\epsilon^j\}_{j\in\mathcal{V}}$ *is shown in Figure 3 with yellow color. From Corollary 1(ii) we can compute an invariant, convex outer* 10^{-2}*-approximation. We obtain the pair* (k, λ) *with* $k = 582$, $\lambda = 6.75 \cdot 10^{-4}$ *which satisfies the conditions* (12), (13) *with the smallest* k. *Thus, the convex outer* 10^{-2}*-approximation is* $\{\mathrm{conv}(\frac{1}{1-\lambda}\overline{\mathcal{F}}_{582}^j)\}_{j\in\mathcal{V}}$. *By utilizing Theorem 3, we compute the exact maximal invariant multi-set* $\{\mathcal{S}_M^j\}_{j\in\mathcal{V}}$ *in* $19 < \log_\rho(\frac{\min_{j\in\mathcal{V}}(R_j - r_j)}{\alpha_1\Gamma c})$ *iterations. The maximal invariant multi-set is shown in Figure 3 in light blue color. Last, from Corollary 1(iii), by considering the maximal invariant set as an initial invariant multi-set, i.e.,* $\overline{\mathcal{S}}_0^j = \mathcal{S}_M^j$, $j \in \mathcal{V}$ *we compute a second convex outer approximation* $\{\mathrm{conv}(\overline{\mathcal{S}}_{439}^j)\}_{j\in\mathcal{V}}$.

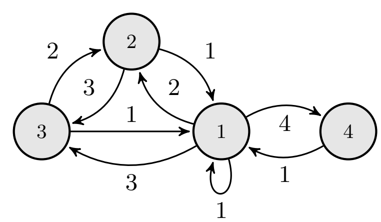

Fig. 2. The switching constraints graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, Example 2.

A. Maximal invariant sets for linear time-invariant systems

Relation (18) is an upper bound on the iterations required to retrieve the maximal invariant multi-set. For *linear systems*

$$x(t+1) = Ax(t), \tag{19}$$

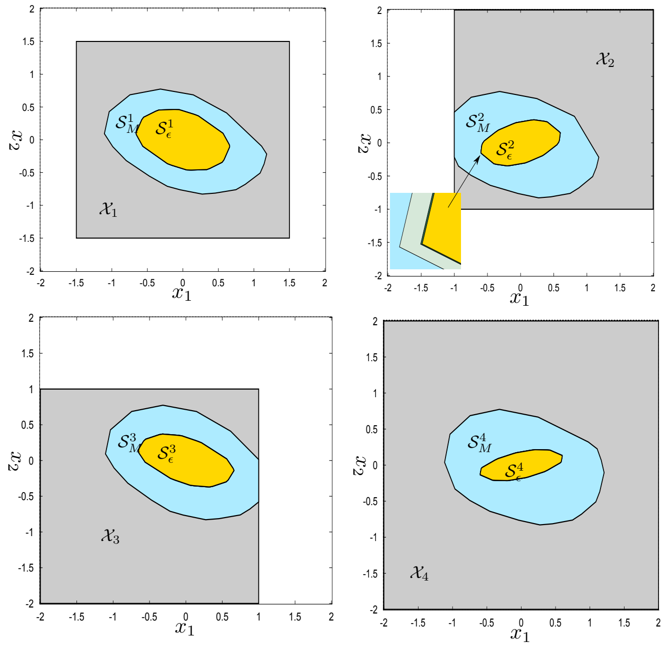

Fig. 3. Example 2, the convex inner 10^{-2}–approximation $\{\mathcal{S}_\epsilon^j\}_{j\in\mathcal{V}}$ (yellow) of the minimal convex invariant multi-set, the maximal invariant multi-set $\{\mathcal{S}_M^j\}_{j\in\mathcal{V}}$ (light blue) and the constraint multi-set $\{\mathcal{X}_i\}_{i\in\mathcal{V}}$ (grey). The outer approximations are not shown clearly since they almost coincide with the inner one. In the upper right part, a portion of the inner approximation, the outer approximation $\{\text{conv}(\frac{1}{1-\lambda}\overline{\mathcal{F}}^j_{582})\}_{j\in\mathcal{V}}$ (light green) and the outer approximation $\{\text{conv}(\overline{\mathcal{S}}^j_{439})\}_{j\in\mathcal{V}}$ (dark green) is shown.

$A \in \mathbb{R}^{n\times n}$, $x(t) \in \mathcal{X} \subset \mathbb{R}^n$, Theorem 3 recovers the standard backward reachability approach for computing the maximal invariant set, see e.g., [16, Section 5.4]. In this case, the number of iterations[6] required is upper bounded by k, where

$$k = \lceil \log_\rho(\tfrac{R}{\Gamma c}) \rceil, \tag{20}$$

where ρ is the spectral radius of A, the constant Γ satisfies

$$\|x(t)\| \leq \Gamma \rho^t \|x(0)\| \tag{21}$$

and R, c are defined as in the statement of Theorem 3. In what follows, we leverage the results above and compute in the worst case the maximal invariant set for a number of iterations proportional to the square root of k. The method exploits the T-iterated dynamics of the system, which has been proven useful before for assessing stability, see e.g. [35], [12].

Theorem 4 *Consider a linear System* (19)*, a pair* (Γ, ρ) *satisfying* (21) *and* k *as in* (20)*. The maximal invariant set can be computed at worst after* $k^\star$ *iterations, where*

$$k^\star = \left\lceil 2\sqrt{k} - 1 \right\rceil. \tag{22}$$

Proof Given any $T \geq 1$, we observe that the pair (Γ_T, ρ_T) can be assigned in (21) for the T-iterated system $y(t+1) = A^T y(t)$, $y(t) \in \mathcal{X}$, with $\Gamma_T := \Gamma$, $\rho_T := \rho^T$. Consequently, we can compute the maximal invariant set $\check{\mathcal{S}}_M$ of the T-iterated system in at most $\check{k} = \lceil \log_{\rho_T}(\frac{R}{\Gamma c}) \rceil = \lceil \frac{k}{T} \rceil$ iterations. It is not difficult to show that the maximal invariant set of the true system (19) is $\mathcal{S}_M = \tilde{\mathcal{S}}_{T-1}$, where $\tilde{\mathcal{S}}_0 = \check{\mathcal{S}}_M$ and $\tilde{\mathcal{S}}_{l+1} = \{x : Ax \in \tilde{\mathcal{S}}_l\}$, $l = 0, ..., T-2$. Using this two-step approach, we can express the bound of the number of iterations required to compute the maximal invariant set as a function of T by

$$g(T) := \lceil \tfrac{k}{T} + T - 1 \rceil. \tag{23}$$

The optimal $T^\star$ that minimizes (23) is $T^\star = \sqrt{k}$. The result (22) is reached by computing $g(T^\star)$. ■

The reasoning may carry on in the setting of [36, Section V, Lemma 4], where we can obtain a similar amelioration on the number of linear inequalities in case the constraint set $\mathcal{X}$ is a polyhedral set. More generally, the same approach could be applied for switching systems with additive disturbances. Its exact formulation is left for further research.

Example 3 *We study the triple integrator* $\dot{x}(t) = Ax(t) + Bu(t)$*,* $A = \begin{bmatrix} 0&1&0\\0&0&1\\0&0&0 \end{bmatrix}$*,* $B = \begin{bmatrix} 0\\0\\1 \end{bmatrix}$*, by considering its discretized version* $x(k+1) = A_d x(k) + B_d u(k)$*, with* $A_d = I + \tau A$*,* $B_d = \tau B$*, with* $\tau = 0.3$*. We choose the LQR controller setting* $Q = 1$*,* $R = 10^7$*. Moreover, we consider the constraint set* $\mathcal{X} = \{x \in \mathbb{R}^3 : |x| \leq w\}$*, with* $w = [\,4\ 3\ 1\,]^\top$*. We compute the maximal invariant set* $\mathcal{S}_M$ *for the closed-loop system in two ways. First, from* (20)*, we compute* $k = 689$*, while* $\mathcal{S}_M$ *is computed in exactly* 164 *iterations. Second, following Theorem 4, we compute the T-iterated system setting* $T = \sqrt{k} = 27$*. The maximal invariant set of the T-iterated system* $\check{\mathcal{S}}_M$ *is computed in exactly* 7 *iterations. Next, we compute* $\mathcal{S}_M$ *after* $T - 1 = 26$ *additional basic iterations. Overall, the maximal invariant set is retrieved in* $7 + 26 = 33 < 164$ *iterations, while the theoretical upper bound is* $k^\star = 53 < 689$ *iterations. We illustrate in the lower and upper part of Figure 4 the sets* $\check{\mathcal{S}}_M$ *and* $\mathcal{S}_M$ *respectively.*

V. Conclusions

Invariance, safety and more refined set-theoretic notions are utilized more and more to tackle challenges appearing in safety-critical, resource-aware, embedded, Cyber-Physical systems. In this article, we focused on extending the available results to a family of systems with significant modeling power. We showed that it is necessary to generalize invariance to multi-set invariance and extended the available constructions for the minimal invariant multi-set (and its approximations) and the maximal invariant multi-set for systems under constrained switching.

We also characterized the computational complexity of the respective algorithms by providing in all cases *a priori* upper bounds on the maximum number of iterations. We believe this comprehensive outlook on invariant multi-set constructions may also provide new insights to well studied problems. For instance, in this article we established a new, provably much faster, way of computing the maximal invariant set for linear systems.

[6] Given a matrix $A \in \mathbb{R}^{n\times n}$ and a set $\mathcal{S} \subset \mathbb{R}^n$, we consider as one iteration the set mapping $f_-(\mathcal{S}) := \{x : Ax \in \mathcal{S}\}$.

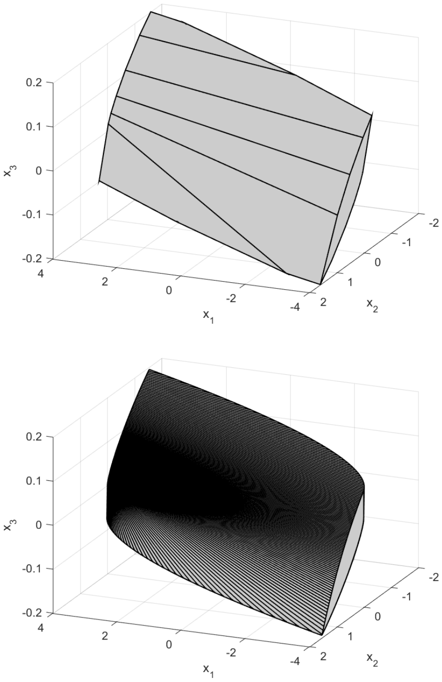

Fig. 4. Example 3, the maximal invariant set $\check{\mathcal{S}}_M$ for the T-iterated system, $T = 27$ (above) and the maximal invariant set $\mathcal{S}_M$ (below).

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