Stability analysis of switched linear systems defined by graphs

Stability Analysis of Switched Linear Systems Defined by Graphs

Nikolaos Athanasopoulos* and Mircea Lazar

Abstract— **We present necessary and sufficient conditions for global exponential stability for switched discrete-time linear systems, under arbitrary switching, which is constrained within a set of admissible transitions. The class of systems studied includes the family of systems under arbitrary switching, periodic systems, and systems with minimum and maximum dwell time specifications. To reach the result, we describe the set of rules that define the admissible transitions with a weighted directed graph. This allows to express the system dynamics as a time invariant difference inclusion. In turn, a modified version of the forward reachability set mapping is utilized to analyze global exponential stability. The developed framework leads to the establishment of an iterative stability verification algorithm.**

I. Introduction

Many systems in control engineering can be naturally modeled by switched discrete-time linear systems [1]–[5]. In the typical setting, the admissible transitions from one subsystem to another are determined by a set of rules. These rules constitute the switching constraints which can be time-dependent, state-dependent or they can be defined according to a discrete-event logic.

A few recent works have utilized graph-theoretic methods for stability analysis of switched systems [6]–[12]. For the continuous-time case, [6], [7] focus on the computation of mode-dependent upper bounds on dwell and average dwell time, while [8], [9] deal with the definition and characterization of the input-to-output stability properties of general switched systems defined by directed graphs. For the discrete-time case and in the context of analyzing stability via Lyapunov theory, directed graphs have been used to introduce finite path-dependent Lyapunov functions [10] and path-complete Lyapunov functions [11], which can capture well known approaches, e.g., maximum and minimum of quadratic functions as Lyapunov functions. Recently, in [12], directed graphs were used to capture rule-based constraints and to provide sufficient conditions of existence of stabilizing switching signals, when the switching frequency is larger than a constant.

In this article, we model switched discrete-time linear systems under constrained switching with weighted directed graphs (digraphs). By expressing the admissible transitions with a weighted digraph, we are able to model the dynamics of the system as a time invariant difference inclusion. Moreover, we allow more nodes than subsystems (or modes) in the digraph, incorporating rules with memory. For example, rules concerning the minimum or maximum dwell time can be taken into account. In the proposed setting, each node corresponds to one of the finite discrete states, generated by the switching rules, that the switched system can be in at each time instant.

To analyze stability, we utilize the forward reachability mapping of the obtained difference inclusion, applied on convex compact sets that contain the origin in their interior. The idea behind this approach, followed also in [13] for systems under arbitrary and unconstrained switching, is to establish stability by requiring confinement of the k-step forward reachability set in a scaled version of a starting set that enjoys the aforementioned properties. Consequently, necessary and sufficient conditions of global exponential stability are established in the form of set inclusions.

The practical contribution of the article concerns the establishment of a simple iterative stability verification algorithm. The computations involved at each step of the algorithm are convex unions and linear operations on convex sets. Thus, although it might appear counter intuitive, the stability analysis problem of the studied class of systems can be treated in a similar manner as the stability analysis problem of switched systems under arbitrary switching.

In Section II, the modeling of the family of systems under study is presented. The main theoretical stability result is established in Section III. In Section IV, the practical results concerning the efficient verification of stability are presented, along with an illustrative example. The conclusions are drawn in Section V. For clarity of exposition, the proofs are in the Appendix.

II. System description

Let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ denote the field of real numbers, the set of non-negative reals and the set of nonnegative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ we define the sets $\Pi_{\geq c} := \{k \in \Pi : k \geq c\}$, and similarly $\Pi_{\leq c}$, $\mathbb{R}_\Pi := \Pi$ and $\mathbb{N}_\Pi := \mathbb{N} \cap \Pi$. A $\mathcal{C}$-set $\mathcal{S} \subset \mathbb{R}^n$ is a closed, bounded and convex set which contains the origin while a proper $\mathcal{C}$-set $\mathcal{S} \subset \mathbb{R}^n$ is a $\mathcal{C}$-set that contains the origin in its interior. Given a set $\mathcal{S} \subset \mathbb{R}^n$ and a real matrix $A \in \mathbb{R}^{n\times n}$, the set $A\mathcal{S}$ is defined by $A\mathcal{S} := \{x \in \mathbb{R}^n : (\exists y \in \mathcal{S} : x = Ay)\}$. The convex hull of the sets $\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,N]}}$, $\mathcal{S}_i \subset \mathbb{R}^n$, $i \in \mathbb{N}_{[1,N]}$, will be denoted by $\mathrm{conv}(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,N]}})$. An arbitrary norm in $\mathbb{R}^n$ is denoted by $\|\cdot\|$. The unit ball of an arbitrary norm is denoted by $\mathcal{B} := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. For a set $\mathcal{S} \subset \mathbb{R}^n$ with a finite number of elements, the number of its elements is denoted by $\mathrm{card}(\mathcal{S})$. The boundary and the interior of a set $\mathcal{S} \subset \mathbb{R}^n$ are denoted by $\partial\mathcal{S}$ and $\mathrm{interior}(\mathcal{S})$ respectively.

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N. Athanasopoulos and M. Lazar are with the Department of Electrical Engineering, Eindhoven University of Technology, The Netherlands n.athanasopoulos@tue.nl, m.lazar@tue.nl

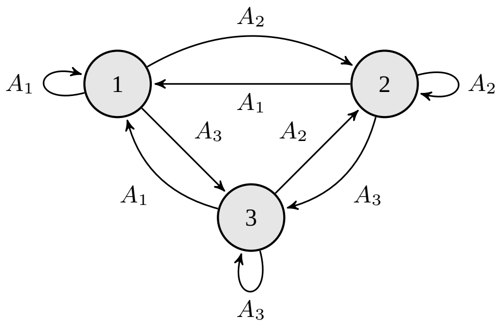

Fig. 1. The graph of a switched discrete time linear system under arbitrary switching. All transitions are admissible.

The empty set is denoted by $\emptyset$. The set of all subsets of $\mathbb{R}^n$, including the empty set, is denoted by $\mathcal{R}^n$.

A. Switching rules described by digraphs

To describe the switching rules that define the admissible transitions in the family of systems under study, we consider the following:

(i) A finite set of matrices

$$\mathcal{A} := \{A_l\}_{l\in\mathbb{N}_{[1,N]}}, \tag{1}$$

$A_l \in \mathbb{R}^{n\times n}$, $l \in \mathbb{N}_{[1,N]}$. The matrices induce $N \in \mathbb{N}_{\geq 1}$ systems of the form $x_{t+1} = A_l x_t$, $l \in \mathbb{N}_{[1,N]}$. We shall call these systems as subsystems of the system under study.

(ii) A weighted digraph

$$\mathcal{G} := (\mathcal{V}, \mathcal{E}), \tag{2}$$

which corresponds to the set of admissible switching rules. In (2), $\mathcal{V} := \{i\}_{i\in\mathbb{N}_{[1,V]}}$ denotes the set of nodes of $\mathcal{G}$ of cardinality $V \in \mathbb{N}_{\geq N}$, while $\mathcal{E} \subseteq \mathbb{N}_{[1,V]} \times \mathbb{N}_{[1,V]}$ denotes the set of the directed edges. The directed graph $\mathcal{G}$ is allowed to have self-loops.

(iii) The weight matrices

$$W_{i,j} := A_l, \quad l \in \mathbb{N}_{[1,N]}, \tag{3}$$

defined for each directed edge $(i, j) \in \mathcal{E}$.

Remark 1 *The nodes $i \in \mathcal{V}$ of the directed graph $\mathcal{G}$ do not need to correspond to the extreme subsystems induced by the matrices A_l, $l \in \mathbb{N}_{[1,N]}$ of the set $\mathcal{A}$. Such cases are illustrated in Example 1, for the systems described by the graphs in Figures 2 and 3.*

The in-neighbor set of a node $j \in \mathcal{V}$ of the digraph $\mathcal{G}$ is

$$\mathcal{I}_j := \{i \in \mathbb{N}_{[1,V]} : (i, j) \in \mathcal{E}\}, \tag{4}$$

The indegree of each node $j \in \mathcal{V}$ is equal to the cardinality of the set $\mathcal{I}_j$. The out-neighbor set and the outdegree of a node are defined in a similar manner. A variety of switching rules can be described by a weighted directed graph, as illustrated in the following example.

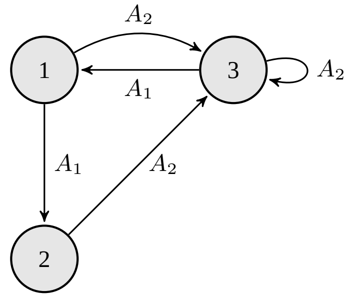

Fig. 2. The graph of a switched discrete time linear system that consists of two subsystems and obeys a maximum dwell time rule. In specific, the system leaves the subsystem induced by matrix A_1 at most after two time instants. Node two was introduced to realize this specification.

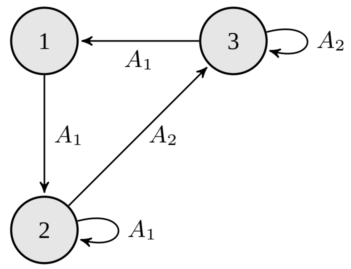

Fig. 3. The graph of a switched discrete time linear system that consists of two subsystems and obeys a minimum dwell time rule. In specific, the system stays at the subsystem induced by matrix A_1 for at least two time instants. Node one was introduced to realize this specification.

Example 1 *In Figure 1, the induced graph of a switched linear system under arbitrary switching is depicted. Each node $i \in \mathcal{V}$ corresponds to a subsystem induced by the matrix A_l, $l \in \mathbb{N}_{[1,3]}$. Thus, $\mathcal{A} := \{A_l\}_{l\in\mathbb{N}_{[1,3]}}$, $\mathcal{V} := \{i\}_{i\in\mathbb{N}_{[1,3]}}$, $\mathcal{E} := \mathbb{N}_{[1,3]} \times \mathbb{N}_{[1,3]}$, and $W_{i,j} = A_j$, for all $(i, j) \in \mathcal{E}$.*

In Figure 2, the graph of a switched system consisting of two subsystems, i.e., $\mathcal{A} := \{A_l\}_{l\in\mathbb{N}_{[1,2]}}$, is shown. For this case, $\mathcal{V} := \{i\}_{i\in\mathbb{N}_{[1,3]}}$, $\mathcal{E} := \{(1,3),(1,2),(2,3),(3,1),(3,3)\}$ and $W_{1,3} = A_2$, $W_{1,2} = A_1$, $W_{2,3} = A_2$, $W_{3,1} = A_1$, $W_{3,3} = A_3$. The switching rule posed is the system to leave the subsystem induced by A_1 at most after two time instants. This is a rule with memory, which leads to the introduction of the additional node 2 in the corresponding graph. This constraint can be considered as a maximum dwell time specification.

In Figure 3, the graph of a switched system consisting of two subsystems, i.e., $\mathcal{A} := \{A_l\}_{l\in\mathbb{N}_{[1,2]}}$, is shown. For this case, $\mathcal{V} := \{i\}_{i\in\mathbb{N}_{[1,3]}}$, $\mathcal{E} := \{(1,2),(2,2),(2,3),(3,1),(3,3)\}$, and $W_{1,2} = A_1$, $W_{2,2} = A_1$, $W_{2,3} = A_2$, $W_{3,3} = A_2$, $W_{3,1} = A_1$. The switching rule posed is the system to remain in the subsystem induced by A_1 for at least two consecutive time instants. The node 1 has been added to realize this switching constraint. This constraint can be considered as a minimum dwell time specification.

B. Switched system as a difference inclusion

Given (1)–(3), we consider the set-valued mappings $\Phi_j(\cdot) : \mathcal{R}^{n^{\mathrm{card}(\mathcal{I}_j)}} \to \mathcal{R}^n$, $j \in \mathbb{N}_{[1,N]}$, where

$$\Phi_j(\{\mathcal{X}_i\}_{i\in\mathcal{I}_j}) := \bigcup_{i\in\mathcal{I}_j} W_{i,j}\mathcal{X}_i, \tag{5}$$

with $\mathcal{X}_i \in \mathcal{R}^n$, $i \in \mathbb{N}_{[1,V]}$. Consequently, the switched discrete-time linear system under arbitrary switching, subject to the constraints expressed by (2) and with initial condition $x_0 \in \mathbb{R}^n$, can be described by the difference inclusions

$$x_{t+1} \in \bigcup_{j=1}^{V} \Phi_j(\{\mathcal{X}_{t,l}\}_{l\in\mathcal{I}_j}), \tag{6}$$

where the sets $\mathcal{X}_{t,j}$ satisfy the difference equations

$$\mathcal{X}_{t+1,j} := \Phi_j(\{\mathcal{X}_{t,l}\}_{l\in\mathcal{I}_j}), \tag{7}$$

for all $(t,j) \in \mathbb{N} \times \mathbb{N}_{[1,V]}$, where $\mathcal{X}_{0,j} := \{x_0\}$, $j \in \mathbb{N}_{[1,V]}$. Given an integer $k \in \mathbb{N}$, the k–th iterated mapping $\Phi_j^k(\cdot) : \mathcal{R}^{n^V} \to \mathcal{R}^n$, $j \in \mathbb{N}_{[1,V]}$ is defined as follows. For $k = 0$, for all $j \in \mathbb{N}_{[1,V]}$, $\Phi_j^0(\{\mathcal{X}_i\}_{i\in\mathbb{N}_{[1,V]}}) := \mathcal{X}_j$ holds by convention. For $k = 1$, it holds that $\Phi_j^1(\{\mathcal{X}_i\}_{i\in\mathbb{N}_{[1,V]}}) := \Phi_j(\{\mathcal{X}_i\}_{i\in\mathcal{I}_j})$. For $k \in \mathbb{N}_{\geq 2}$, it holds that $\Phi_j^k(\{\mathcal{X}_i\}_{i\in\mathbb{N}_{[1,V]}}) := \Phi_j(\{\Phi_i^{k-1}(\{\mathcal{X}_p\}_{p\in\mathbb{N}_{[1,V]}})\}_{i\in\mathcal{I}_j})$.

Remark 2 *When switching rules with memory are present, the initial conditions $\mathcal{X}_{0,j}$ for the nodes $j \in \mathbb{N}_{[1,V]}$ that realize these rules have to be taken with care. To this end, if there exists a path in the corresponding digraph such that the switching rule is not satisfied when starting from node j, the initial condition is set equal to $\mathcal{X}_{0,j} := \emptyset$ to guarantee satisfaction of the switching constraints. This is illustrated in the remaining part of Example 1. Nevertheless, for clarity of exposition and without loss of generality, the theoretical results are obtained by considering $\mathcal{X}_{0,j} = \{x_0\}$, for all $j \in \mathbb{N}_{[1,V]}$.*

Example 1 (continued) *Consider the system described by the weighted digraph in Figure 3. The corresponding in-neighbor sets, defined by* (4)*, are $\mathcal{I}_1 = \{3\}$, $\mathcal{I}_2 = \{1,2\}$ and $\mathcal{I}_3 = \{2,3\}$. We compute the sets $\mathcal{X}_{t,j}$, $(t,j) \in \mathbb{N}_{[1,3]} \times \mathbb{N}_{[1,3]}$. Taking into account Remark 2, given an initial condition $x_0 \in \mathbb{R}^n$, it follows that $\mathcal{X}_{0,1} = \mathcal{X}_{0,2} = \emptyset$, $\mathcal{X}_{0,3} = \{x_0\}$. For $t = 1$, it follows that $\mathcal{X}_{1,1} = \Phi_1(\{\mathcal{X}_{0,i}\}_{i\in\{3\}}) = \{A_1x_0\}$, $\mathcal{X}_{1,2} = \Phi_2(\{\mathcal{X}_{0,i}\}_{i\in\{1,2\}}) = \emptyset$ and $\mathcal{X}_{1,3} = \Phi_3(\{\mathcal{X}_{0,i}\}_{i\in\{2,3\}}) = \{A_2x_0\}$. For $t = 2$, we calculate $\mathcal{X}_{2,1} = \{A_1A_2x_0\}$, $\mathcal{X}_{2,2} = \{A_1^2x_0\}$, $\mathcal{X}_{2,3} = \{A_2^2x_0\}$, while for $t = 3$, it follows that $\mathcal{X}_{3,1} = \{A_1A_2^2x_0\}$, $\mathcal{X}_{3,2} = \{A_1^3x_0, A_1^2A_2x_0\}$ and $\mathcal{X}_{3,3} = \{A_2A_1^2x_0, A_2^3x_0\}$.*

III. Stability analysis

In this section, we derive necessary and sufficient conditions for global exponential stability for the family of switched discrete time linear systems described by (6).

Definition 1 *The system* (6) *is called globally exponentially stable if and only if there exists a pair $(\Gamma, \varepsilon) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[0,1)}$ such that for all $x_0 \in \mathbb{R}^n$ it holds that*

$$\|x_t\| \leq \Gamma\varepsilon^t\|x_0\|, \ \forall t \in \mathbb{N}, \tag{8}$$

for all x_t satisfying (6)*.*

To establish the main theoretical result, a few technical results are required first. To this end, the following result establishes that the set valued mappings $\Phi_j^t(\cdot)$ are order preserving with respect to set inclusion, for all $(t,j) \in \mathbb{N} \times \mathbb{N}_{[1,V]}$.

Lemma 1 *Consider the sets $\{\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}}$, $\mathcal{Y}_i \subset \mathbb{R}^n$, $\{\mathcal{Z}_i\}_{i\in\mathbb{N}_{[1,V]}}$, $\mathcal{Z}_i \subset \mathbb{R}^n$, such that $\mathcal{Y}_i \subseteq \mathcal{Z}_i$, $i \in \mathbb{N}_{[1,V]}$. Then, relation*

$$\Phi_j^t(\{\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}}) \subseteq \Phi_j^t(\{\mathcal{Z}_i\}_{i\in\mathbb{N}_{[1,V]}}) \tag{9}$$

holds, for all $(t,j) \in \mathbb{N} \times \mathbb{N}_{[1,V]}$.

Definition 2 *A set-valued mapping $\Phi(\cdot) : \mathcal{R}^{n^V} \to \mathcal{R}^n$ is called a positively homogeneous mapping of order one, or simply a homogeneous mapping, if for any set sequence $\{\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}}$, $\mathcal{Y}_i \in \mathcal{R}^n$, and any nonnegative scalar $\alpha \in \mathbb{R}_+$ it holds that $\Phi(\{a\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}}) = a\Phi(\{\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}})$.*

Lemma 2 *The mappings $\Phi_j^t(\cdot)$, $(t,j) \in \mathbb{N} \times \mathbb{N}_{[1,V]}$, where $\Phi_j(\cdot)$, $j \in \mathbb{N}_{[1,V]}$ are defined in* (5)*, are homogeneous, i.e.,*

$$\Phi_j^t(\{a\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}}) = a\Phi_j^t(\{\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}}) \tag{10}$$

holds, for all $(t,j) \in \mathbb{N} \times \mathbb{N}_{[1,V]}$.

Lemma 3 *For any pair $(\rho, k) \in \mathbb{R}_{[0,1)} \times \mathbb{N}_{\geq 1}$, there exists a pair $(M, \lambda) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[\rho,1)}$ such that*

$$\rho^{\lfloor \frac{i}{k} \rfloor} \leq M\lambda^i, \quad \forall i \in \mathbb{N}. \tag{11}$$

Lemma 4 *For any proper $\mathcal{C}$-set $\mathcal{S} \subset \mathbb{R}^n$, there exists a pair $(c_1, c_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{(0,c_1]}$ such that*

$$c_2\mathcal{B} \subseteq \mathcal{S} \subseteq c_1\mathcal{B}. \tag{12}$$

Consider the set–valued mappings $\mathcal{F}_k(\cdot) : \mathcal{R}^{n^V} \to \mathcal{R}^n$,

$$\mathcal{F}_k(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}}) := \bigcup_{j=1}^{V} \Phi_j^k(\{\mathcal{S}_i\}_{i\in\mathcal{I}_j}), \tag{13}$$

with $k \in \mathbb{N}_{\geq 1}$. The main theoretical result follows.

Theorem 1 *The system* (6) *is globally exponentially stable if and only if for any proper $\mathcal{C}$–set $\mathcal{S} \subset \mathbb{R}^n$, there exists a pair $(k, \rho) \in \mathbb{N}_{\geq 1} \times \mathbb{R}_{[0,1)}$ such that*

$$\mathcal{F}_k(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}}) \subseteq \rho\mathcal{S}, \tag{14}$$

where $\mathcal{S}_i := \mathcal{S}$, $i \in \mathbb{N}_{[1,V]}$.

IV. Verification of stability

Theorem 1 establishes necessary and sufficient conditions for system (6) to be exponentially stable. However, the result cannot be used directly since it requires verification of condition (14) for all proper $\mathcal{C}$–sets $\mathcal{S} \subset \mathbb{R}^n$. To this end, in what follows we show that it is enough for condition (14) of Theorem 1 to be satisfied for a single, arbitrary proper $\mathcal{C}$-set $\mathcal{S} \subset \mathbb{R}^n$ for the system (6) to be globally exponentially stable.

Lemma 5 *Consider two proper $\mathcal{C}$-sets $\mathcal{S} \subset \mathbb{R}^n$ and $\mathcal{M} \subset \mathbb{R}^n$. Then, there exist positive numbers $\alpha_i \in \mathbb{R}_+$, $i \in \mathbb{N}_{[1,2]}$, such that*

$$\alpha_2 \mathcal{M} \subseteq \alpha_1 \mathcal{S} \subseteq \mathcal{M}, \tag{15}$$

Theorem 2 *Consider a single, arbitrary, proper $\mathcal{C}$-set $\mathcal{S} \subset \mathbb{R}^n$. Then, the system* (6) *is globally exponentially stable if and only if there exists a pair $(k, \rho) \in \mathbb{N}_{\geq 1} \times \mathbb{R}_{[0,1)}$ such that* (14) *holds, with $\mathcal{S}_i := \mathcal{S}$, $i \in \mathbb{N}_{[1,V]}$.*

The value of Theorem 2 lies in the observation that relation (14) needs to be verified for a single arbitrary proper $\mathcal{C}$–set $\mathcal{S} \subset \mathbb{R}^n$ for stability to hold. Nevertheless, computing the sets $\mathcal{F}_t(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}})$, $t \in \mathbb{N}$, is difficult, since these are radially convex sets [14], which are nonconvex and have an increasingly complex description as t grows. To this end, in what follows it is proven that the condition (14) is equivalent to a simpler one which can be computed efficiently.

Lemma 6 *Let $\mathcal{X} \subset \mathbb{R}^n$ be a proper $\mathcal{C}$-set and $\mathcal{Y} \subset \mathbb{R}^n$ be a set. Then, relation $\mathcal{Y} \subseteq \mathcal{X}$ holds if and only if $\operatorname{conv}(\mathcal{Y}) \subseteq \mathcal{X}$.*

We define the set–valued mappings $\widehat{\Phi}_j(\cdot) : \mathcal{R}^{n^{\operatorname{card}(\mathcal{I}_j)}} \to \mathcal{R}^n$, $j \in \mathbb{N}_{[1,V]}$, induced from (5),

$$\widehat{\Phi}_j(\{\mathcal{S}_i\}_{i\in\mathcal{I}_j}) := \operatorname{conv}(\Phi_j(\{\mathcal{S}_i\}_{i\in\mathcal{I}_j})). \tag{16}$$

The k–th iterations of the mappings (16), $\widehat{\Phi}_j^k(\cdot) : \mathcal{R}^{n^V} \to \mathcal{R}^n$, $k \in \mathbb{N}$, are defined in a similar manner.

Proposition 1 *Given a set of matrices $\mathcal{A} \subset \mathbb{R}^{n\times n}$* (1), *a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$* (2) *the weights $W_{i,j}$, $(i,j) \in \mathcal{E}$* (3) *and a sequence of sets $\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}}$, $\mathcal{S}_i \subset \mathbb{R}^n$, $i \in \mathbb{N}_{[1,V]}$, the relation*

$$\widehat{\Phi}_j^t(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}}) = \operatorname{conv}(\Phi_j^t(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}})) \tag{17}$$

holds, for all $(t, j) \in \mathbb{N} \times \mathbb{N}_{[1,V]}$.

The next result is a direct consequence of Theorem 2, Lemma 6 and Proposition 1.

Corollary 1 *Consider a single, arbitrary proper $\mathcal{C}$-set $\mathcal{S} \subset \mathbb{R}^n$. Then, the system* (6) *is globally exponentially stable if and only if there exists a pair $(k, \rho) \in \mathbb{N}_{\geq 1} \times \mathbb{R}_{[0,1)}$ such that*

$$\operatorname{conv}\Big(\bigcup_{j=1}^{V} \widehat{\Phi}_j^k(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}})\Big) \subseteq \rho\mathcal{S}, \tag{18}$$

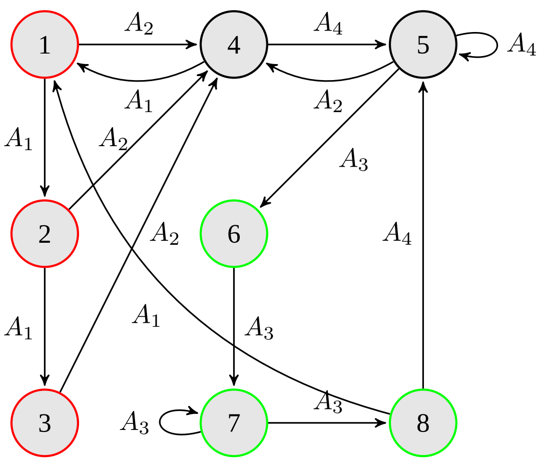

Fig. 4. The weighted digraph that describes the admissible switching rules for the system in Example 2.

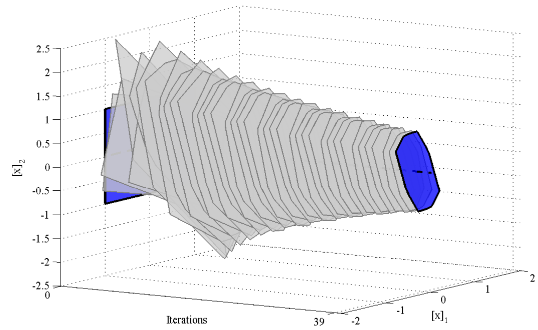

Fig. 5. The sets $\{\widehat{\Phi}_2^i(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,8]}})\}_{i\in\mathbb{N}_{[1,38]}}$, shown in grey color and the sets $\mathcal{S}_2$ and $\widehat{\Phi}_2^{39}(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,8]}})$, shown in blue color.

where $\mathcal{S}_i := \mathcal{S}$, $i \in \mathbb{N}_{[1,V]}$.

The importance of Corollary 1 lies in the fact that the mappings defined by (16) and their iterations, are easier to compute compared to the set mappings (5). Indeed, the set mappings $\widehat{\Phi}_j^t(\cdot)$ can be computed by calculating convex hulls of a finite number of $\mathcal{C}$-sets. Moreover, it is worth observing that the conditions for global exponential stability for the arbitrary switching setting [13] (see Example 1, Figure 1 for the corresponding directed graph) are recovered directly from Corollary 1 as a particular case.

Example 2 *We consider a switched discrete time linear system that consists of four subsystems, i.e, $\mathcal{A} := \{A_i\}_{i\in\mathbb{N}_{[1,4]}}$, where $A_1 = \left[\begin{smallmatrix} 1.10 & 0.11 \\ -0.33 & 1.00 \end{smallmatrix}\right]$, $A_2 = \left[\begin{smallmatrix} -0.55 & 0.21 \\ -0.11 & -0.35 \end{smallmatrix}\right]$, $A_3 = \left[\begin{smallmatrix} 0.99 & 0.38 \\ -0.58 & -1.09 \end{smallmatrix}\right]$, $A_4 = \left[\begin{smallmatrix} 0.60 & 0.00 \\ 1.00 & 0.50 \end{smallmatrix}\right]$. It is worth to observe that the matrix A_1 is unstable. The switching constraints are realized on the weighted digraph shown in Figure 4, where $\mathcal{V} := \{i\}_{i\in\mathbb{N}_{[1,8]}}$ and $\mathcal{E} := \{(1,2),(2,3),(2,4),(3,4),(4,1),(4,5),(5,4),(5,5),(5,6),(6,7),(7,7),(7,8),(8,5)\}$. Apart from the rules that correspond to the admissible transitions, two additional rules* *are present, namely the system must stay at the subsystem induced by the matrix* A_3 *for at least three consecutive instants and must leave the subsystem induced by* A_1 *at most after three consecutive instants. To realize these two rules, nodes* $6, 7, 8$ *and* $1, 2, 3$ *have been added respectively in the digraph and are shown in Figure 4 in green and red color. The in-neighbor sets* (4) $\mathcal{I}_i$, $i \in \mathbb{N}_{[1,8]}$ *are* $\mathcal{I}_1 = \{4, 8\}$, $\mathcal{I}_2 = \{1\}$, $\mathcal{I}_3 = \{2\}$, $\mathcal{I}_4 = \{1, 2, 3, 5\}$, $\mathcal{I}_5 = \{4, 5, 8\}$, $\mathcal{I}_6 = \{5\}$, $\mathcal{I}_7 = \{6, 7\}$, $\mathcal{I}_8 = \{7\}$.

Corollary 1 was utilized to verify global exponential stability. For this example, the unit sublevel set of the infinity norm was chosen to be the proper $\mathcal{C}$*-set* $\mathcal{S}$*, i.e.,* $\mathcal{S} := \mathcal{B}_\infty = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\}$*. The necessary and sufficient condition for global exponential stability* (18) *was satisfied for the pair* $k = 39$, $\rho = 0.903$.

The set $\mathcal{B}_\infty$ *is a simple proper* $\mathcal{C}$*–polytopic set, consisting of four vertices. All subsequent sets* $\widehat{\Phi}_j^t(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,8]}})$, $(t, j) \in \mathbb{N}_{[1,39]} \times \mathbb{N}_{[1,8]}$*, that were computed in order to verify relation* (18)*, are polytopes generated by the convex union of linear mappings on polytopes. For this example, the vertex representation (see, e.g.,* [15, Chapter 1]*) of the polytope was utilized, using* MATLAB 2011b *in an up-to-date standard desktop computer. In detail, the polytopic sets* $\widehat{\Phi}_j^t(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,8]}})$, $(t, j) \in \mathbb{N}_{[1,39]} \times \mathbb{N}_{[1,8]}$ *were computed by collecting products of matrices with vectors, while the removal of redundant vertices of the sets and verification of the set relation* (18) *for* $k \in \mathbb{N}_{[1,39]}$ *was performed using the* Multi-Parametric Toolbox [16]*. Under this setting, the computation time for generating the sets* $\widehat{\Phi}_j^t(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,8]}})$, $(t, j) \in \mathbb{N}_{[1,39]} \times \mathbb{N}_{[1,8]}$ *and verifying relation* (18) *of Corollary 1 was* 0.95 *seconds. In Figure 5, the evolution of the set sequences* $\{\widehat{\Phi}_2^i(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,8]}})\}_{i\in\mathbb{N}_{[1,38]}}$ *which correspond to node* 2 *of the graph is shown in grey color. The sets* $\mathcal{B}_\infty$ *and* $\widehat{\Phi}_2^{39}(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,8]}})$ *are shown in blue color.*

V. Conclusions

A necessary and sufficient condition for global exponential stability for switched discrete-time linear systems under arbitrary, yet constrained, switching was established. The set of rules that define the admissible switching sequences was described by a weighted directed graph that allows self-loops. Several families of relevant switched systems can be described in this setting, such as systems under arbitrary unconstrained switching, periodic systems and systems with dwell time specifications. A modified convex version of the forward reachability set mapping was used to construct a systematic iterative algorithmic procedure for stability verification, which is based on convex operations on convex sets.

References

[1] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Systems Magazine*, vol. 19, pp. 59–70, 1999.

[2] R. N. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability Criteria for Switched and Hybrid Systems," *SIAM Review*, vol. 49, no. 4, pp. 545–592, 2007.

[3] F. Blanchini and S. Miani, *Set-theoretic methods in control*, ser. Systems & Control: Foundations & Applications. Boston, MA: Birkhäuser, 2008.

[4] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: A survey of recent results," *IEEE Transactions on Automatic Control*, vol. 54, pp. 308–322, 2009.

[5] Z. Sun, "Stability of piecewise linear systems revisited," *Annual Reviews in Control*, vol. 34, pp. 221–231, 2010.

[6] O. Karabacak, "Dwell time and average dwell time methods based on the cycle ratio of the switching graph," *Systems and Control Letters*, vol. 62, pp. 1032–1037, 2013.

[7] F. Blanchini, D. Casagrande, and S. Miani, "Modal and transition dwell time computation in switching systems: A set-theoretic approach," *Automatica*, vol. 46, pp. 1477–1482, 2010.

[8] J. L. Mancilla-Aguilar, R. Garcia, E. Sontag, and Y. Wang, "Uniform stability properties of switched systems with switchings governed by digraphs," *Nonlinear Analysis*, vol. 63, pp. 472–490, 2005.

[9] J. L. Mancilla-Aguilar and R. A. Garcia, "Invariance Results for Constrained Switched Systems," in *American Control Conference*, Baltimore, USA, 2010, pp. 2493–2498.

[10] J. W. Lee and G. E. Dullerud, "Uniform stabilization of discrete-time switched and Markovian jump linear systems," *Automatica*, vol. 42, pp. 205–218, 2006.

[11] A. A. Ahmadi, R. M. Jungers, P. A. Parrilo, and M. Roozbehani, "Analysis of the Joint Spectral Radius via Lyapunov Functions on Path-Complete Graphs," in *International Conference on Hybrid Systems: Computation and Control*, Chicago, USA, 2011, pp. 13–22.

[12] A. Kundu and D. Chatterjee, "Stabilizing discrete-time switched linear systems," in *17th International Conference on Hybrid Systems: Computation and Control*, Berlin, Germany, 2014, pp. 11–20.

[13] N. Athanasopoulos and M. Lazar, "Alternative Stability Conditions for Switched Discrete Time Linear Systems," in *IFAC World Congress*, Cape Town, South Africa, 2014, pp. 6007–6012.

[14] A. M. Rubinov and A. A. Yagubov, "The space of star–shaped sets and its applications in nonsmooth optimization," *Quasidifferential Calculus, Mathematical Programming Studies*, vol. 28, pp. 176–202, 1986.

[15] G. M. Ziegler, *Lectures on Polytopes, Updated Seventh Printing*. Springer, 2007.

[16] M. Kvasnica, P. Grieder, and M. Baotic, M. amd Morari, "Multi–Parametric Toolbox," in *7th International Workshop on Hybrid Systems: Computation and Control*, Philadelphia, USA, 2004, pp. 448–462.

[17] B. Grunbaum, *Convex Polytopes*. Springer, 2nd edition, 2003.

Appendix

A. Proof of Lemma 1

For $t = 0$, it holds that $\Phi_j^0(\{\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}}) = \mathcal{Y}_j \subseteq \mathcal{Z}_j = \Phi_j^0(\{\mathcal{Z}_i\}_{i\in\mathbb{N}_{[1,V]}})$. Suppose that relation (9) holds for iteration t, for all $j \in \mathbb{N}_{[1,V]}$. Let $\mathcal{Y}_j^\star := \Phi_j^t(\{\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}})$, $\mathcal{Z}_j^\star := \Phi_j^t(\{\mathcal{Z}_i\}_{i\in\mathbb{N}_{[1,V]}})$, $j \in \mathbb{N}_{[1,V]}$. Then, for any $j \in \mathbb{N}_{[1,V]}$, it holds that $\Phi_j^{t+1}(\{\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}}) = \Phi_j(\{\mathcal{Y}_i^\star\}_{i\in\mathcal{I}_j}) = \bigcup_{i\in\mathcal{I}_j} W_{i,j}\mathcal{Y}_i^\star \subseteq \bigcup_{i\in\mathcal{I}_j} W_{i,j}\mathcal{Z}_i^\star = \Phi_j^{t+1}(\{\mathcal{Z}_i\}_{i\in\mathbb{N}_{[1,V]}})$. Thus, relation (9) holds for all $(t, j) \in \mathbb{N} \times \mathbb{N}_{[1,V]}$. $\blacksquare$

B. Proof of Lemma 2

For $t = 0$, it holds that $\Phi_j^0(\{\alpha\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}}) = \alpha\mathcal{Y}_j = \alpha\Phi_j^0(\{\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}})$. Suppose that relation (10) holds for iteration t, for all $j \in \mathbb{N}_{[1,V]}$, and let $\mathcal{Y}_j^\star := \Phi_j^t(\{\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}})$, $j \in \mathbb{N}_{[1,V]}$. Then, for any $j \in \mathbb{N}_{[1,V]}$, it holds that $\Phi_j^{t+1}(\{\alpha\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}}) = \Phi_j(\{\alpha\mathcal{Y}_i^\star\}_{i\in\mathcal{I}_j}) = \bigcup_{i\in\mathcal{I}_j} W_{i,j}\alpha\mathcal{Y}_i^\star = \alpha\bigcup_{i\in\mathcal{I}_j} W_{i,j}\mathcal{Y}_i^\star = \alpha\Phi_j^{t+1}(\{\mathcal{Y}_i\}_{i\in\mathbb{N}_{[1,V]}})$. Thus, relation (10) holds for all $(t, j) \in \mathbb{N} \times \mathbb{N}_{[1,V]}$. $\blacksquare$

C. Proof of Lemma 3

Relation (11) is satisfied with $M := \rho^{\frac{-k+1}{k}}$, $\lambda := \rho^{\frac{1}{k}}$. $\blacksquare$

D. Proof of Lemma 4

Relation (12) is satisfied with

$$c_1 := \max_{x\in\mathcal{S}} \|x\|, \quad c_2 := \min_{x\in\partial\mathcal{S}} \|x\|. \blacksquare$$

E. Proof of Theorem 1

Suppose that (14) holds. From (13), relation (14) implies $\Phi_j^k(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}}) \subseteq \rho\mathcal{S}$, for all $j \in \mathbb{N}_{[1,V]}$. Then, from Lemma 1 and Lemma 2, for any pair $(N,j) \in \mathbb{N}_{\geq 1}\times\mathbb{N}_{[1,V]}$, it holds that

$$\begin{aligned}&\Phi_j^{Nk}(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,N]}}) = \Phi_j^{(N-1)k}(\{\Phi_i^k(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,V]}})\}_{i\in\mathbb{N}_{[1,V]}})\\ &\subseteq \Phi_j^{(N-1)k}(\{\rho\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}}) = \rho\Phi_j^{(N-1)k}(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}})\\ &\subseteq \ldots \subseteq \rho^N\mathcal{S}.\end{aligned}$$

Consequently, it follows that $\mathcal{F}_{Nk}(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}}) = \bigcup_{j=1}^V \Phi_j^{Nk}(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}}) \subseteq \bigcup_{j=1}^V \rho^N\mathcal{S} = \rho^N\mathcal{S}$. For any $t \in \mathbb{N}$, there exists a pair $(N,j) \in \mathbb{N}\times\mathbb{N}_{[0,k-1]}$ such that $t := kN + j$. Then, $\mathcal{F}_t(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}}) \subseteq \rho^N \bigcup_{l=1}^V \Phi_l^j(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}}) = \rho^N\mathcal{F}_j(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}})$. The set $\mathcal{S}$ is a bounded set and moreover, since iteration (13) is a union of linear operations on sets, it is bounded in any bounded set in $\mathbb{R}^n$. Thus, $\mathcal{F}_j(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}})$ is bounded, for any $t\in\mathbb{N}_{\geq 1}$. Consequently, since $\mathcal{S}$ is a proper $\mathcal{C}$-set, there exists a number $c_3 \in \mathbb{R}_{\geq 1}$ such that $\mathcal{F}_j(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}}) \subseteq c_3\mathcal{S}$, for all $j \in \mathbb{N}_{[1,k-1]}$. Thus, $\mathcal{F}_t(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,V]}}) \subseteq \rho^N c_3\mathcal{S}$, and taking into account Lemma 3, there exists a pair $(M,\lambda) \in \mathbb{R}_{\geq 1}\times\mathbb{R}_{[\rho,1)}$ such that

$$\mathcal{F}_t(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,V]}}) \subseteq M\lambda^t c_3\mathcal{S}, \quad \forall i \in \mathbb{N}. \tag{19}$$

From Lemma 4, there exist positive numbers $c_1, c_2 \in \mathbb{R}_+$ such that (12) holds. For any $x_0 \in \partial\mathcal{S}$, $x_t \in \mathcal{F}^t(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,V]}})$, and from (19), it holds that $x_t \in M\lambda^t c_3 c_1\mathcal{B}$, or, equivalently,$\|x_t\| \leq M\lambda^t c_3 c_1$, $\forall t \in \mathbb{N}$. For any $x_0 \in \partial\mathcal{S}$, $\|x_0\| \geq c_2$, or, $1 \leq c_2^{-1}\|x_0\|$. Thus, $\|x_t\| \leq M\lambda^t c_3 c_1 c_2^{-1}\|x_0\|$, $\forall i \in \mathbb{N}$, for any $x_0 \in \partial\mathcal{S}$. Since $\mathcal{S}$ is a proper $\mathcal{C}$-set, for any $x_0 \in \mathbb{R}^n$, there exists a pair $(\alpha, x_0^\star) \in \mathbb{R}_{\geq 0}\times\partial\mathcal{S}$ such that $x_0 = \alpha x_0^\star$. Then, for all $t \in \mathbb{N}$ and for each solution $x_t \in \mathcal{F}_t(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,V]}})$ of (6), from Lemma 2 there exists a vector $x_t^\star \in \mathcal{F}_t(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,V]}})$ such that $x_t = \alpha x_t^\star$, because the map (13) is homogeneous. Thus, for any $x \in \mathbb{R}^n$, $\|x_t\| = \|\alpha x_t^\star\| \leq \alpha M\lambda^t c_3 c_1 c_2^{-1}\|x_0^\star\| = M\lambda^t c_3 c_1 c_2^{-1}\|\alpha x_0^\star\| = M\lambda^t c_3 c_1 c_2^{-1}\|x_0\|$, for all $t \in \mathbb{N}_{\geq 0}$. Thus, relation (8) is satisfied with $\Gamma := Mc_3c_1c_2^{-1}$, $\varepsilon := \lambda$, and the system (6) is globally exponentially stable.

Conversely, suppose that the system (6) is globally exponentially stable. Then, there exists a pair $(\Gamma, \varepsilon) \in \mathbb{R}_{\geq 1}\times\mathbb{R}_{[0,1)}$ such that $\|x_t\| \leq \Gamma\varepsilon^t\|x_0\|$, for all $x_0 \in \mathbb{R}^n$. From Lemma 4, for any proper $\mathcal{C}$-set $\mathcal{S} \subset \mathbb{R}^n$, for any $x_0 \in \mathcal{S}$, there exists a positive number $c_1 \in \mathbb{R}_{>0}$ such that $\|x_0\| \leq c_1$. Then, for all $x_0 \in \mathcal{S}$ it follows that $\|x_t\| \leq \Gamma\varepsilon^t\|x_0\| \leq \Gamma\varepsilon^t c_1$. If there exists a finite integer $k \in \mathbb{N}_{\geq 1}$, such that $x_0 \in \mathcal{S}$ implies $x_k \in \rho\mathcal{S}$, for some $\rho \in \mathbb{R}_{(0,1)}$, then relation (14) can be verified. This is true if $x_k \in \rho c_2\mathcal{B}$, or, $\|x_k\| \leq \rho c_2$. Thus, it has to be verified that there exists a k such that $\|x_k\| \leq \Gamma\varepsilon^k c_1 \leq \rho c_2$, or, $k\log\varepsilon \leq \log(\frac{\rho c_2}{\Gamma c_1})$, or, $k \geq \left\lceil \frac{\log\frac{\rho c_2}{\Gamma c_1}}{\log\varepsilon} \right\rceil$. Such a finite integer always exists for the quintuple $(\rho, c_2, \Gamma, c_1, \varepsilon) \in \mathbb{R}_{(0,1)}\times\mathbb{R}_{(0,c_1]}\times\mathbb{R}_{\geq 1}\times\mathbb{R}_+\times\mathbb{R}_{(0,1)}$. Thus, there exists a pair $(k,\rho) \in \mathbb{N}_{\geq 1}\times\mathbb{R}_{[0,1)}$ such that (14) holds. $\blacksquare$

F. Proof of Lemma 5

Relation (15) is satisfied with $\alpha_1 := \max_\alpha\{\alpha \in \mathbb{R}_+ : \alpha\mathcal{S} \subseteq \mathcal{M}\}$ and $\alpha_2 := \max_\alpha\{\alpha \in \mathbb{R}_+ : \alpha_2\mathcal{M} \subseteq \alpha\mathcal{S}\}$. $\blacksquare$

G. Proof of Theorem 2

Suppose there exists a proper $\mathcal{C}$-set $\mathcal{S} \subset \mathbb{R}^n$ and a pair $(k,\rho) \in \mathbb{N}_{\geq 1}\times\mathbb{R}_{[0,1)}$ such that (14) holds. We shall prove that for any proper $\mathcal{C}$-set $\mathcal{M} \subset \mathbb{R}^n$ there always exists a pair $(k^\star, \rho^\star) \in \mathbb{N}_{\geq 1}\times\mathbb{R}_{[0,1)}$ such that

$$\mathcal{F}_{k^\star}(\{\mathcal{M}_i\}_{i\in\mathbb{N}_{[1,V]}}) \subseteq \rho^\star\mathcal{M}, \tag{20}$$

where $\mathcal{M}_i := \mathcal{M}$, $i \in \mathbb{N}_{[1,V]}$. From Lemma 5, there exists a pair $(\alpha_1, \alpha_2) \in \mathbb{R}_+\times\mathbb{R}_{(0,1]}$ such that (15) holds. Then, for all $j \in \mathbb{N}_{[1,V]}$, from Lemma 1 and Lemma 2 it holds that $\Phi_j^k(\{\alpha_2\mathcal{M}_i\}_{i\in\mathbb{N}_{[1,V]}}) \subseteq \Phi_j^k(\{\alpha_1\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}}) = \alpha_1\Phi_j^k(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}}) \subseteq \alpha_1\rho\mathcal{S}$, where $\mathcal{S}_i := \mathcal{S}$, $i \in \mathbb{N}_{[1,V]}$. Applying the set map $\Phi_j(\cdot)$, $j \in \mathbb{N}_{[1,V]}$, kN times, for $N \in \mathbb{N}_{\geq 1}$, it follows that $\Phi_j^{Nk}(\{\alpha_2\mathcal{M}_i\}_{i\in\mathbb{N}_{[1,V]}}) \subseteq \rho^N\mathcal{M}$, or, $\Phi^{Nk}(\{\mathcal{M}_i\}_{i\in\mathbb{N}_{[1,V]}}) \subseteq \frac{\rho^N}{\alpha_2}\mathcal{M}$. There exists a pair $(k^\star, \rho^\star) \in \mathbb{N}_{\geq 1}\times\mathbb{R}_{[0,1)}$ such that (20) holds if and only if $\frac{\rho^N}{\alpha_2} \leq \rho^\star$, or, equivalently, if $k^\star \geq \left\lceil \frac{\log\rho^\star\alpha_2}{\log\rho} \right\rceil k$. Such an integer $k^\star$ always exists, thus (20) can be verified for any proper $\mathcal{C}$-set $\mathcal{M} \subset \mathbb{R}^n$, and according to Theorem 1, the system (6) is globally exponentially stable. $\blacksquare$

H. Proof of Lemma 6

If $\mathrm{conv}(\mathcal{Y}) \subseteq \mathcal{X}$, by definition of the convex hull it holds that $\mathcal{Y} \subseteq \mathrm{conv}(\mathcal{Y}) \subseteq \mathcal{X}$. Conversely, suppose that $\mathcal{Y} \subseteq \mathcal{X}$ and there exists a vector $y \in \mathrm{conv}(\mathcal{Y})$ such that $y \notin \mathcal{Y}$. By Caratheodory's theorem [17, Section 2.3, 5], there exist $n+1$ vectors $z_i \in \mathcal{Y}$, $i \in \mathbb{N}_{[1,n+1]}$ such that $y \in \mathrm{conv}(\{z_i\}_{i\in\mathbb{N}_{1,n+1}})$. However, this would imply that there exists at least one index $i^\star \in \mathbb{N}_{[1,n+1]}$ such that $z_{i^\star} \notin \mathcal{X}$, which is a contradiction. Thus, $\mathrm{conv}(\mathcal{Y}) \subseteq \mathcal{X}$. $\blacksquare$

I. Proof of Proposition 1

From definition (16), relation (17) holds for $t = 1$. Suppose that (17) holds for iteration t, for all $j \in \mathbb{N}_{[1,V]}$. Then, for any $j \in \mathbb{N}_{[1,V]}$, it follows that

$$\begin{aligned}&\mathrm{conv}(\Phi_j^{t+1}(\{\mathcal{S}_i\}_{i\in\mathbb{N}_{[1,V]}})) =\\ &\quad= \mathrm{conv}(\Phi_j(\{\Phi_i^t(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,V]}})\}_{i\in\mathbb{N}_{[1,V]}}))\\ &\quad= \mathrm{conv}(\bigcup_{i\in\mathcal{I}_j} W_{i,j}\Phi_i^t(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,V]}})\\ &\quad= \mathrm{conv}(\bigcup_{i\in\mathcal{I}_j} W_{i,j}\,\mathrm{conv}(\Phi_i^t(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,V]}}))\\ &\quad= \mathrm{conv}(\bigcup_{i\in\mathcal{I}_j} W_{i,j}\widehat{\Phi}_i^t(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,V]}})\\ &\quad= \widehat{\Phi}_j^{t+1}(\{\mathcal{S}_l\}_{l\in\mathbb{N}_{[1,V]}}\end{aligned}$$

Thus, relation (17) holds for all pairs $(t,j) \in \mathbb{N}\times\mathbb{N}_{[1,V]}$. $\blacksquare$