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On Constrained Stabilization of Discrete-time Linear Systems

Nikolaos Athanasopoulos*, Alina I. Doban and Mircea Lazar

Abstract—The concepts of controlled (k, λ) -contractive sets and set-induced finite-time control Lyapunov functions are introduced in this paper. These tools are then employed to derive new synthesis methods for constrained stabilization of linear systems. Two classes of state-feedback control strategies are proposed, namely, periodic conewise linear control laws and periodic vertex-interpolation control laws. The benefits of these synthesis methods are demonstrated for the constrained stabilization of a DC-DC buck converter.

I. INTRODUCTION

Existing methods for constrained stabilization of linear discrete-time systems stem from a combination of tools from set theory and Lyapunov theory [1], [2]. More specifically, these methods are based on the existence of controlled λ -contractive sets, which in turn yield set-induced control Lyapunov functions [3]. The concept of controlled λ -contractive sets was utilized in several methods for synthesis of stabilizing control laws for constrained linear systems. Most of these methods generate either conewise linear control laws [4], [5], which require the solution of a point location problem at every time instant, or vertex-interpolation control laws, which require the solution of a linear program at every time instant [6]–[9].

However, the impact of these methods in real-world applications is limited by some inherent characteristics. For example, most of the existing iterative algorithms for constructing controlled λ -contractive sets, see e.g. [2], [3], [10]–[14], are computationally expensive, lack scalability with respect to the dimension of the state-space and, they typically generate complex polytopes. Unfortunately, the complexity of the resulting polytopes is further inherited by the corresponding vertex-interpolation control laws. Few alternative algorithms for computing controlled λ -contractive sets, see, e.g., [15], [16], are able to produce sets with a specific shape, and thus, they can provide a trade-off between generality and tractability. Still, all of the above approaches cannot cover the entire basin of attraction, i.e., the maximal controlled invariant set, as invariant sets do not induce control Lyapunov functions.

It is worth to point out that results for constrained stabilization of linear systems that do not make use of controlled λ -contractive sets can be found in the early work [17]. Therein, it was shown that, given a predefined polytopic

set of initial conditions, feasibility of the corresponding stabilization problem is equivalent with existence of finite sequences of control actions that steer each vertex in the strict interior of the set. Furthermore, given the finite sequence of control actions for each vertex, it was indicated how a stabilizing control action can be computed online by linear programming.

Recently, the concepts of (k, λ) -contractive sets and set-induced finite-time Lyapunov functions were introduced in [18]. Therein it was shown that the construction of (k, λ) -contractive sets is easier compared to the construction of λ -contractive sets and, moreover, that any invariant set is a (k, λ) -contractive set for some integer k strictly larger than one. As such, it would be of interest to exploit these new concepts for constrained stabilization of linear systems.

To this aim, in this paper we define the corresponding concepts of controlled (k, λ) -contractive sets and set-induced finite-time control Lyapunov functions (CLFs). Then, via the results in [18], we establish equivalence between existence of (i) controlled (k, λ) -contractive sets, (ii) finite-time CLFs and (iii) stabilizing state-feedback control laws for homogeneous non-autonomous dynamics. It is worth to point out that the equivalence (i) \Leftrightarrow (iii) recovers the result in [17] for the particular setting of linear dynamics and polytopic sets. Moreover, we establish that any proper \mathcal{C} -subset of the maximal controlled invariant set (including the maximal controlled invariant set, whenever it is a proper \mathcal{C} -set), is a controlled (k, λ) -contractive set for some integer k strictly larger than one. This result enables a trade-off between the complexity of the controlled (k, λ) -contractive set and capturing the whole basin of attraction.

The main contribution of this paper is then to derive novel controller synthesis algorithms based on controlled (k, λ) -contractive polytopic sets for linear systems. More specifically, we propose two classes of control laws, namely periodic conewise linear control laws and periodic vertex-interpolation control laws. The computation and implementation of both types of control laws is based on linear programming. For the case when k equals one, these methods recover the existing methods based on controlled λ -contractive polytopic sets.

To illustrate the developed methodology and to demonstrate its effectiveness and benefits we consider the problem of constrained stabilization of a Buck DC-DC converter. This frequently used power electronics circuit [19] is subject to hard physical constraints, i.e., constraints on the output voltage and the current of the inductor. In addition, the state that corresponds to zero voltage and zero current, i.e., when the converter is turned off, is a significant initial condition

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which does not lie in any controlled λ -contractive set. We show that there exists a controlled (k, λ) -contractive set that contains the zero voltage and zero current state and, as such, the developed synthesis methods can be used to obtain a stabilizing and admissible control law.

The remainder of the article is structured as follows. In Section II, notation and basic definitions are presented. The theoretical results are established in Section III. Two novel classes of stabilizing control laws based on the proposed concepts are described in Section IV. The results obtained for the Buck DC-DC converter case study are reported in Section V and conclusions are drawn in Section VI.

II. NOTATION AND BASIC DEFINITIONS

Let \mathbb{R} , \mathbb{R}_+ and \mathbb{N} denote the field of real numbers, the set of non-negative reals and the set of nonnegative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ we define $\Pi_{\geq c} := \{k \in \Pi \mid k \geq c\}$, and similarly $\Pi_{\leq c}$, $\mathbb{R}_\Pi := \Pi$ and $\mathbb{Z}_\Pi := \mathbb{Z} \cap \Pi$. For a matrix $A \in \mathbb{R}^{n \times m}$, $[A]_{ij}$ denotes the element in the i -th row and j -th column, $[A]_{i:} \in \mathbb{R}^m$ denotes the i -th row and $[A]_{:j} \in \mathbb{R}^n$ denotes the j -th column. The identity matrix is denoted by $I_n \in \mathbb{R}^{n \times n}$. The vector with all its elements equal to one is denoted by $\mathbf{1}_n \in \mathbb{R}^n$. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes an arbitrary Hölder norm. A set $\mathcal{S} \subset \mathbb{R}^n$ is a *proper C-set* if it is compact, convex, and contains the origin in its interior. The collection of proper C-sets in \mathbb{R}^n is denoted by $\text{PC}(\mathbb{R}^n)$. The interior of a set $\mathcal{S} \subset \mathbb{R}^n$ is denoted by $\text{int}(\mathcal{S})$. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces and a *polytope* is a closed and bounded polyhedron. Given a proper C-set $\mathcal{X} \subset \mathbb{R}^n$ the function $\text{gauge}(\mathcal{X}, \cdot)$ given by $\text{gauge}(\mathcal{X}, x) := \inf_{\mu} \{\mu : x \in \mu\mathcal{X}, \mu \geq 0\}$ for $x \in \mathbb{R}^n$, is called the Minkowski, or gauge, function. A function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class \mathcal{K} if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class \mathcal{K}_∞ if $\varphi \in \mathcal{K}$ and $\lim_{s \rightarrow \infty} \varphi(s) = \infty$. A map $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be positively homogeneous of the first degree, or simply, homogeneous, if $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{R}_+$ and all $x \in \mathbb{R}^n$. A map $f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^l \rightarrow \mathbb{R}^m$ is said to be positively homogeneous of the first degree with respect to both arguments if $f(\alpha x, u) = \alpha f(x, \frac{1}{\alpha}u)$ and $f(x, \alpha u) = \alpha f(\frac{1}{\alpha}x, u)$, for all $\alpha \in \mathbb{R}_+$, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^l$. For the standard definitions of regional asymptotic stability in the Lyapunov sense in $\mathbb{X} \subseteq \mathbb{R}^n$, exponential stability, the corresponding global variants of these properties and the definition of a standard Lyapunov function, we refer the interested reader to, for example, [20].

III. THEORETICAL FOUNDATION

We consider discrete-time, time invariant non-autonomous dynamical systems of the form

$$x_{t+1} = \Phi(x_t, u_t), \quad (1)$$

where $x_t \in \mathbb{R}^n$ is the state vector and $u_t \in \mathbb{R}^m$ is the input vector at time instant $t \in \mathbb{N}$, and $\Phi(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is

the map describing the dynamics of the system. The system state and input variables are subject to hard constraints, i.e.,

$$x_t \in \mathbb{X}, \quad u_t \in \mathbb{U}, \quad \forall t \in \mathbb{N}, \quad (2)$$

with $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$. The following assumption is imposed.

Assumption 1 *The state and input constraints sets $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ are proper C-sets.*

We consider the class of admissible state-feedback control laws¹ $u := g(x)$, $g : \mathbb{X} \rightarrow \mathbb{U}$, such that $g(0) = 0$, that satisfy the following assumption.

Assumption 2 *The map $g(\cdot) : \mathbb{X} \rightarrow \mathbb{U}$ is \mathcal{K} -bounded at zero, i.e., for all $x \in \mathbb{R}^n$ there exists a $\varphi \in \mathcal{K}$ such that $\|g(x)\| \leq \varphi(\|x\|)$.*

Definition 1 *The map $\Phi(\cdot, \cdot)$ is called controlled \mathcal{K} -bounded at zero if and only if for any map $g(\cdot)$ satisfying Assumption 2 and for all $x \in \mathbb{R}^n$ there exists a $\kappa \in \mathcal{K}$ such that $\|\Phi(x, g(x))\| \leq \kappa(\|x\|)$.*

The following assumptions concern the dynamics of system (1).

Assumption 3 *The map $\Phi(\cdot, \cdot)$ is positively homogeneous of the first degree with respect to both arguments.*

Assumption 4 *The map $\Phi(\cdot, \cdot)$ is controlled \mathcal{K} -bounded at zero.*

Given any $k \in \mathbb{N}$ and any control law $u := g(x)$ satisfying Assumption 2, the k -th iterated map $\Phi^k(x, g(x))$, $\Phi^k(\cdot, g(\cdot)) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by $\Phi^k(x, g(x)) := \Phi(\Phi^{k-1}(x, g(x)), g(\Phi^{k-1}(x, g(x))))$, for any $k \in \mathbb{N}_{\geq 1}$. By convention, for all $x \in \mathbb{X}$, $\Phi^0(x, g(x)) = x$.

In what follows, the notions of controlled (k, λ) -contractive sets are given. These notions are suitably adapted from the analogous concepts established in [18], where the case of autonomous and homogeneous dynamics was considered.

Definition 2 *Given a real scalar $\lambda \in [0, 1]$ and an integer $k \in \mathbb{N}_+$, the set $\mathcal{S} \in \text{PC}(\mathbb{R}^n)$ is called a controlled (k, λ) -contractive set with respect to system (1) and with respect to the state and input constraint sets \mathbb{X} and \mathbb{U} , if and only if $\mathcal{S} \subseteq \mathbb{X}$ and there exists a state-feedback control law $g(\cdot) : \mathbb{X} \rightarrow \mathbb{U}$ such that, for all $x \in \mathcal{S}$ and for all $i \in \mathbb{N}_{[1, k-1]}$, $\Phi^i(x, g(x)) \in \mathbb{X}$, and $\Phi^k(x, g(x)) \in \lambda\mathcal{S}$.*

Remark 1 *In light of Definition 2, and similar to observations made in [18] for the autonomous case, controlled $(1, \lambda)$ -contractive sets and controlled $(1, 1)$ -contractive sets recover the standard notions of controlled λ -contractive*

¹Note that all results that follow can be extended to the case of set-valued control laws.

and controlled invariant sets respectively, see e.g., [2]. Also, controlled $(k, 1)$ -contractive sets are periodically controlled invariant sets with period $k \in \mathbb{N}_+$.

Similar to the established relationship between controlled λ -contractive sets and control Lyapunov functions [3] for linear systems affected by parametric uncertainties, controlled (k, λ) -contractive sets induce *finite-time control Lyapunov functions*, for systems of the form (1) which satisfy Assumptions 3 and 4. The concept of a finite-time control Lyapunov function is presented below.

Consider the system (1), the state and input constraints sets \mathbb{X} and \mathbb{U} , and let \mathcal{S} be a controlled $(k, 1)$ -contractive proper \mathcal{C} -set with respect to system (1) and constraint sets \mathbb{X} and \mathbb{U} . Suppose there exists a function $V : \mathbb{X} \rightarrow \mathbb{R}_+$, functions $\kappa_1, \kappa_2 \in \mathcal{K}$, a real scalar $\rho \in (0, 1)$, an integer $k \in \mathbb{N}_+$, and a state-feedback control law $u = g(x)$, $g(\cdot) : \mathbb{X} \rightarrow \mathbb{U}$, such that the following inequalities hold:

$$\kappa_1(\|x\|) \leq V(x) \leq \kappa_2(\|x\|), \quad \forall x \in \mathbb{X}, \quad (3a)$$

$$V(\Phi^k(x, g(x))) \leq \rho V(x), \quad \forall x \in \mathcal{S}. \quad (3b)$$

Definition 3 A function $V(\cdot)$ that satisfies (3) is called a *finite-time control Lyapunov function associated with the $(k, 1)$ -contractive proper \mathcal{C} -set \mathcal{S} , relative to \mathcal{S} and with respect to \mathbb{X} and \mathbb{U}* . A function $V(\cdot)$ that satisfies (3) for $\mathcal{S} = \mathbb{X} = \mathbb{R}^n$ and $\mathbb{U} = \mathbb{R}^m$ is called a *global finite-time control Lyapunov function*.

The next result, which is a direct consequence of the results established for the autonomous case in [18], demonstrates the equivalence between controlled (k, λ) -contractive proper \mathcal{C} -sets and finite-time control Lyapunov functions for the class of systems under study.

Proposition 1 Suppose that Assumptions 1–4 hold. Consider the dynamical system (1), systems constraints (2) and a proper \mathcal{C} -set $\mathcal{S} \subseteq \mathbb{X}$. Then, the following statements are equivalent.

(i) The set \mathcal{S} is a controlled (k, λ) -contractive set with respect to system (1) and constraints (2).

(ii) The function $V(x) := \text{gauge}(\mathcal{S}, x)$ is a finite-time control Lyapunov function in \mathcal{S} , with respect to \mathbb{X} and \mathbb{U} .

(iii) There exists a stabilizing state feedback control law $u := g(x)$, $g(\cdot) : \mathbb{X} \rightarrow \mathbb{U}$ such that the closed-loop system $x_{t+1} = \Phi(x_t, g(x_t))$ is \mathcal{KL} -stable in \mathcal{S} , with respect to \mathbb{X} and \mathbb{U} .

Proof: (i) \Leftrightarrow (ii). By Definition 2, for all $x \in \mathcal{S}$ there exists a control law $u = g(x)$ such that $\Phi^k(x, g(x)) \in \lambda\mathcal{S}$. Since Assumption 3 holds, from [18, Theorem V.1], equivalence between (i) and (ii) is established.

(ii) \Rightarrow (iii). From (ii) it follows that there exists a state-feedback control law $u = g(x)$ such that $V(x)$ is a finite-time Lyapunov function for the closed-loop system $x_{t+1} = \Phi(x_t, g(x_t))$. Then, the result is a direct consequence of [18, Theorem IV.5].

(iii) \Rightarrow (i). Direct consequence of Assumption 4 and [18, Theorem IV.3]. \blacksquare

From Proposition 1, it is evident that controlled (k, λ) -contractive sets and the corresponding set-induced finite-time control Lyapunov functions are non-conservative synthesis tools for the class of homogeneous (with respect to both arguments) non-autonomous discrete-time systems, which are controlled \mathcal{K} -bounded at zero.

A. Problem formulation

In what follows we exploit the concept of controlled (k, λ) -contractive sets to construct a prototype problem for synthesis of stabilizing and admissible (i.e., with respect to the constraint sets \mathbb{X} and \mathbb{U}) control laws.

Problem 1 Suppose that Assumptions 1–4 hold. Consider the dynamical system (1), i.e., $x_{t+1} = \Phi(x_t, u_t)$, and the state and input constraint sets \mathbb{X} and \mathbb{U} . Given a controlled (k, λ) -contractive proper \mathcal{C} -set $\mathcal{S} \subseteq \mathbb{X}$, find a sequence of state-feedback control laws $\{g_i(\cdot)\}_{i \in \mathbb{N}_{[0, k-1]}}$, $g_i : \mathbb{X} \rightarrow \mathbb{U}$, $i \in \mathbb{N}_{[0, k-1]}$, such that for all $x_0 \in \mathcal{S}$, it holds that

$$x_{i+1} = \Phi(x_i, g_i(x_i)), \quad i \in \mathbb{N}_{[0, k-1]}, \quad (4a)$$

$$x_i \in \mathbb{X}, \quad i \in \mathbb{N}_{[1, k-1]}, \quad (4b)$$

$$g_i(x_i) \in \mathbb{U}, \quad i \in \mathbb{N}_{[0, k-1]}, \quad (4c)$$

$$x_k \in \lambda\mathcal{S}. \quad (4d)$$

Next, it is formally shown that the control laws obtained by solving Problem 1 result in a stable closed-loop system.

Proposition 2 Suppose that Assumptions 1, 2, and 4 hold. Consider the dynamical system (1), i.e., $x_{t+1} = \Phi(x_t, u_t)$, the state and input constraint sets \mathbb{X} and \mathbb{U} , the controlled (k, λ) -contractive set $\mathcal{S} \subseteq \mathbb{X}$, and let the sequence $\{g_i(\cdot)\}_{i \in \mathbb{N}_{[0, k-1]}}$, $g_i(\cdot) : \mathbb{X} \rightarrow \mathbb{U}$, $i \in \mathbb{N}_{[0, k-1]}$, be a feasible solution to Problem 1. Consider the control law $u = \pi(x)$, $\pi(\cdot) : \mathbb{X} \rightarrow \mathbb{U}$, where

$$\pi(x_t) := g_i(x_t) \quad \text{if } t = kN + i, \quad N \in \mathbb{N}. \quad (5)$$

Then, the closed-loop system

$$x_{t+1} = \Phi(x_t, \pi(x_t)) \quad (6)$$

is \mathcal{KL} -stable in \mathcal{S} , with respect to \mathbb{X} and \mathbb{U} .

Proof: It suffices to observe that $V(x) := \text{gauge}(\mathcal{S}, x)$ is a finite-time Lyapunov function for the closed-loop system (6). Then, from [18, Theorem IV.5], it follows that system (6) is \mathcal{KL} -stable in \mathcal{S} with respect to \mathbb{X} . \blacksquare

Remark 2 The resulting autonomous closed-loop dynamics (6) is not required to be homogeneous. Therefore, the feasible space of Problem 1 is not restricted to homogeneous functions $\{g_i(\cdot)\}_{i \in \mathbb{N}_{[0, k-1]}}$, but to functions $\{g_i(\cdot)\}_{i \in \mathbb{N}_{[0, k-1]}}$ that are \mathcal{K} -bounded at zero. This class of functions is rather general and allows for discontinuous (except at $x = 0$) control laws.

IV. SYNTHESIS ALGORITHMS FOR LINEAR SYSTEMS

In the remainder of the paper we focus on discrete-time invariant non-autonomous linear systems of the form

$$x_{t+1} = Ax_t + Bu_t, \quad \forall t \in \mathbb{N}, \quad (7)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ are the system matrices. Due to linearity, Assumptions 3 and 4 are naturally satisfied.

In what follows we employ proper \mathcal{C} -polytopic sets, which can be defined either by half-space or vertex representations [21]. Generically, the half-space representation of an arbitrary proper \mathcal{C} -polytopic set corresponds to $\mathcal{S} := \{x \in \mathbb{R}^n : Hx \leq \mathbf{1}_p\}$, where $H \in \mathbb{R}^{p \times n}$ is a full column-rank matrix and $p \in \mathbb{N}_{\geq n+1}$. Generically, the vertex representation of \mathcal{S} corresponds to $\mathcal{S} := \text{convh}(\{v_i\}_{i \in \mathbb{N}_{[1,q]}})$, for some $q \in \mathbb{N}_{\geq n+1}$. Define $V := [v_1, v_2, \dots, v_q] \in \mathbb{R}^{n \times q}$ as the corresponding matrix that has as columns the vertices of \mathcal{S} . Note that the matrix V has full row-rank.

The state constraint set \mathbb{X} is of the form

$$\mathbb{X} := \{x \in \mathbb{R}^n : H_x x \leq \mathbf{1}_{p_x}\}, \quad (8)$$

where $H_x \in \mathbb{R}^{p_x \times n}$, $p_x \in \mathbb{R}_{\geq n+1}$. The input constraint set \mathbb{U} is of the form

$$\mathbb{U} := \{u \in \mathbb{R}^m : H_u u \leq \mathbf{1}_{p_u}\}, \quad (9)$$

where $H_u \in \mathbb{R}^{p_u \times m}$, $p_u \in \mathbb{R}_{\geq m+1}$. We consider controlled (k, λ) -contractive sets $\mathcal{S} \subset \mathbb{X}$ of the form

$$\mathcal{S} := \{x \in \mathbb{R}^n : H_0 x \leq \mathbf{1}_p\} = \text{convh}(\{v_0^j\}_{j \in \mathbb{N}_{[1,q]}}), \quad (10)$$

and with $V_0 := [v_0^1, v_0^2, \dots, v_0^q]$. We introduce the following problem.

Problem 2 Consider system (7) and the state and input constraint sets \mathbb{X} and \mathbb{U} as defined in (8) and (9), respectively. Given a controlled (k, λ) -contractive set \mathcal{S} as defined in (10), solve the following feasibility problem:

$$\min_{\{U_i\}_{i \in \mathbb{N}_{[0, k-1]}}, \{V_i\}_{i \in \mathbb{N}_{[1, k]}}} 0 \quad (11)$$

subject to

$$[V_0]_{:j} = v_0^j, \quad \forall j \in \mathbb{N}_{[1, q]}, \quad (12a)$$

$$[V_{i+1}]_{:j} = A[V_i]_{:j} + B[U_i]_{:j}, \quad \forall (i, j) \in \mathbb{N}_{[0, k-1]} \times \mathbb{N}_{[1, q]}, \quad (12b)$$

$$H_x[V_i]_{:j} \leq \mathbf{1}_{p_x}, \quad \forall (i, j) \in \mathbb{N}_{[1, k-1]} \times \mathbb{N}_{[1, q]}, \quad (12c)$$

$$H_u[U_i]_{:j} \leq \mathbf{1}_{p_u}, \quad (i, j) \in \mathbb{N}_{[0, k-1]} \times \mathbb{N}_{[1, q]}, \quad (12d)$$

$$H_0[V_k]_{:j} \leq \lambda \mathbf{1}_p, \quad \forall j \in \mathbb{N}_{[1, q]}. \quad (12e)$$

It is worth noting that Problem 2 is always feasible, since existence of the matrices $\{U_i\}_{i \in \mathbb{N}_{[0, k-1]}}$ is guaranteed by the assumption that \mathcal{S} is a controlled (k, λ) -contractive set. Moreover, it can be proven that it is necessary for conditions (12) to hold in order for the set \mathcal{S} to be controlled (k, λ) -contractive.

Next, consider a set of matrices $U_i \in \mathbb{R}^{m \times q}$ for $i \in \mathbb{N}_{[0, k-1]}$ and matrices $V_i \in \mathbb{R}^{n \times q}$ for $i \in \mathbb{N}_{[1, k-1]}$ that are

obtained by solving Problem 2 and assign² $V_k := V_0$. Then, define the control law

$$\pi(x_t) := U_i \mu_i(x_t) \quad \text{if } t = kN + i, \quad N \in \mathbb{N}, \quad (13)$$

where, for all $i \in \mathbb{N}_{[0, k-1]}$, $\mu_i(x_t) \in \mathcal{M}_i(x_t)$, where

$$\mathcal{M}_0(x_t) := \{\mu \in \mathbb{R}_+^q : x_t = V_0 \mu, \quad \mathbf{1}_q^T \mu \leq 1\}, \quad (14)$$

$$\mathcal{M}_i(x_t) :=$$

$$\{\mu \in \mathbb{R}_+^q : V_i \mu = (AV_{i-1} + BU_{i-1})\mu_{i-1}(x_t), \quad \mathbf{1}_q^T \mu \leq 1\}, \quad (15)$$

for all $i \in \mathbb{N}_{[1, k-1]}$. The next result establishes that the control law (13)–(15) solves the constrained stabilization problem for system (7).

Proposition 3 Consider system (7) and the state and input constraint sets \mathbb{X} and \mathbb{U} as defined in (8) and (9), respectively. Consider also the controlled (k, λ) -contractive set \mathcal{S} as defined in (10) and the state-feedback control law defined in (13)–(15). Then, the closed-loop system

$$x_{t+1} = Ax_t + B\pi(x_t), \quad \forall t \in \mathbb{N} \quad (16)$$

is \mathcal{KL} -stable in \mathcal{S} , with respect to \mathbb{X} and \mathbb{U} .

Proof: For any $x_0 \in \mathcal{S}$ it holds that

$$x_{i+1} = (AV_i + BU_i)\mu_i(x_0), \quad i \in \mathbb{N}_{[0, k-1]}. \quad (17)$$

Since matrices V_i , $i \in \mathbb{N}_{[1, k-1]}$, U_i , $i \in \mathbb{N}_{[0, k-1]}$ are solutions of Problem 2, it follows from (12c) and (12d) that $[V_i]_{:j} \in \mathbb{X}$, for all $i \in \mathbb{N}_{[1, k]}$ and $j \in \mathbb{N}_{[1, q]}$, and that $[U_i]_{:j} \in \mathbb{U}$, for all $i \in \mathbb{N}_{[0, k-1]}$ and $j \in \mathbb{N}_{[1, k]}$. Thus, since for all $\mu_i(x_0) \in \mathcal{M}_i(x_0)$, $i \in \mathbb{N}_{[0, k-1]}$, it holds that $\mathbf{1}_q^T \mu_i(x_0) \leq 1$, it follows from (17) that $x_i \in \mathbb{X}$, $i \in \mathbb{N}_{[1, k]}$. Also, $\pi(x_0) \in \mathbb{U}$ by construction. Furthermore, from (12e) and (17) it holds that $x_k \in \lambda \mathcal{S}$, for any selection of $\mu_i(x_0) \in \mathcal{M}_i(x_0)$, $i \in \mathbb{N}_{[0, k-1]}$. Setting $g_i(x) := U_i \mu_i(x)$, $i \in \mathbb{N}_{[0, k-1]}$ and $\Phi(x, u) := Ax + Bu$, conditions (4) of Problem 1 are satisfied. Since $g_i(\cdot)$ satisfies Assumption 2 for all $i \in \mathbb{N}_{[0, k-1]}$, Proposition 2 holds and thus, the closed-loop system (17) is \mathcal{KL} -stable. ■

By Proposition 3, it can be observed that the solution of Problem 2 provides the information needed to construct stabilizing control strategies.

Remark 3 An alternative to the parameterized control law (13)–(15) is to directly compute a feasible control action (i.e., rather than a set of parameters) online, from the set of control sequences associated with the vertices of the initial set. This approach, which was already introduced in [17], can also be regarded as a utilization of the (k, λ) -contractive concept for linear systems. However, the concept itself, the equivalence with finite-time CLFs and the parametrization (13)–(15) of the control law was not identified therein. It should be mentioned also that there always exists a set of

²Note that the matrix V_k obtained by solving Problem 2 is discarded and V_k is set equal to V_0 .

parameters that recovers the control law obtained in [17] from the parametrization (13)–(15).

In what follows, two novel classes of state–feedback control laws, namely, periodic conewise linear control laws and periodic vertex–interpolation control laws, which allow for a tractable implementation, are presented.

A. Periodic conewise linear control laws

First, we consider a periodic conewise linear parametrization of the state–feedback control law. It is worth to note that the proposed parametrization recovers the parametrization proposed in [4] whenever \mathcal{S} is a controlled $(1, \lambda)$ –contractive set, i.e., a standard controlled λ –contractive set.

Let $\mathcal{S} \subseteq \mathbb{X}$ be a controlled (k, λ) –proper \mathcal{C} –polytopic set, with respect to system (7) and let $V_i, i \in \mathbb{N}_{[1, k-1]}$ and $U_i, i \in \mathbb{N}_{[0, k-1]}$ be a solution of Problem 2 and let sets $\mathcal{S}_i, i \in \mathbb{N}_{[0, k-1]}$, be defined by matrices $V_i, i \in \mathbb{N}_{[0, k-1]}$, i.e.,

$$\mathcal{S}_i := \text{convh}(\{[V_i]_{:j}\}_{j \in \mathbb{N}_{[1, q]}}). \quad (18)$$

For each set $\mathcal{S}_i, i \in \mathbb{N}_{[0, k-1]}$, we consider the induced simplicial decomposition $\{\mathcal{D}_i^s\}_{s \in \mathbb{N}_{[1, p_i]}}$, $p_i \in \mathbb{N}, i \in \mathbb{N}_{[0, k-1]}$, where

$$\mathcal{D}_i^s := \text{convh}(\{\mathbf{0}, \{[V_i]_{:j}\}_{j \in \mathcal{I}_i^s}\}), \quad \mathcal{I}_i^s \subseteq \mathbb{N}_{[1, q]},$$

such that the cardinality of each index set \mathcal{I}_i^s is equal to n and $\mathcal{S}_i \subseteq \bigcup_{s=1}^{p_i} \mathcal{D}_i^s$, for all $i \in \mathbb{N}_{[0, k-1]}$. Also, $\text{int}(\mathcal{D}_i^s) \cap \text{int}(\mathcal{D}_i^o) = \emptyset$, for all $s \in \mathbb{N}_{[1, p_i]}, o \in \mathbb{N}_{[1, p_i]}, s \neq o$, and for all $i \in \mathbb{N}_{[0, k-1]}$. We define the corresponding matrices $V_i^s \in \mathbb{R}^{n \times n}, U_i^s \in \mathbb{R}^{m \times n}, i \in \mathbb{N}_{[0, k-1]}, s \in \mathbb{N}_{[1, p_i]}$, by placing in the columns of V_i^s the generating vertices of the simplices \mathcal{D}_i^s (and in the columns of U_i^s the corresponding control actions), i.e.,

$$\begin{aligned} [V_i^s]_{:c} &:= [V_i]_{:j}, \quad \forall c \in \mathbb{N}_{[1, n]}, \forall j \in \mathcal{I}_i^s, \\ [U_i^s]_{:c} &:= [U_i]_{:j}, \quad \forall c \in \mathbb{N}_{[1, n]}, \forall j \in \mathcal{I}_i^s. \end{aligned}$$

The proposed explicit state–feedback control law is uniquely defined by

$$\pi(x_t) := g_i(x_t) \quad \text{if } t = kN + i, \quad N \in \mathbb{N}, \quad (19)$$

where for all $i \in \mathbb{N}_{[0, k-1]}$

$$g_i(x_t) := U_i^s V_i^{s-1} x_{t+i} \quad \text{if } x_{t+i} \in \mathcal{D}_i^s. \quad (20)$$

Remark 4 The control law (19)–(20) can be written in the form (13)–(15). Indeed, (19) can be recovered by (13) by setting $[\mu_i(x_t)]_j := 0$, if $j \notin \mathcal{I}_i^s$, and $[\mu_i(x_t)]_j := [V_i^{s-1} x_t]_j$, if $j \in \mathcal{I}_i^s$, for all $i \in \mathbb{N}_{[0, k-1]}$. Thus, by Proposition 3, system (16) in closed–loop with the periodic conewise linear control law (19)–(20) is \mathcal{KL} –stable.

Remark 5 The control law gains $K_i^s := U_i^s V_i^{s-1}, i \in \mathbb{N}_{[0, k-1]}, s \in \mathbb{N}_{[1, p_i]}$ in (20) can be computed offline by solving a single linear program. Thus, evaluation of the resulting control law (19)–(20) can be done very efficiently in two steps. In the first step, given $k \in \mathbb{N}_{\geq 1}$, for any time instant t compute $N \in \mathbb{N}$ and $i \in \mathbb{N}_{[0, k-1]}$ such that

$t = kN + i$. In the second step, identify the simplex \mathcal{D}_i^s where state x_{t+i} lies by solving a point location problem. Observe that the simplicial partition $\{\mathcal{D}_i^s\}_{s \in \mathbb{N}_{[1, p_i]}}$ induces a conic partition, which allows for the point location problem to be solved efficiently.

B. Periodic vertex–interpolation control laws

The periodic conewise linear control law (19)–(20), belongs to the family of control laws defined in (13)–(15), and is computationally appealing because of its low complexity. Thus, it can be used in control applications where the computational resources are limited. However, it can be suboptimal, since, as indicated in Remark 4, the choice of $\mu_i(x_t) \in \mathcal{M}_i(x_t), t = kN + i$ does not result from an optimal selection strategy. Thus, it would be desirable to formulate a synthesis algorithm that allows an optimal selection of the control input, with respect to a suitable cost function.

To this aim, we consider an alternative class of stabilizing control laws, which is based on periodic vertex–interpolation and solving online a linear program. More specifically, we consider the same form of the control law (13)–(15), however, the choice of $\mu_i(x_t), i \in \mathbb{N}_{[0, k-1]}$ is uniquely defined from the solution of an optimization problem, solved online at every k time instants. First, consider the full column–rank matrices $H_{Z_i} \in \mathbb{R}^{c_i \times n}, i \in \mathbb{N}_{[1, k]}, c_i \in \mathbb{N}_{\geq n+1}$, which induce the proper \mathcal{C} –polytopic sets $\mathcal{Z}_i := \{x \in \mathbb{R}^n : H_{Z_i} x \leq \mathbf{1}_{c_i}\}, i \in \mathbb{N}_{[1, k]}$, and the positive scalars $\alpha_i, i \in \mathbb{N}_{[1, k]}$.

Problem 3 At time $t = kN, N \in \mathbb{N}$, given x_t , solve

$$\min_{\{\mu_i\}_{i \in \mathbb{N}_{[0, k]}}} \sum_{i=1}^k \alpha_i \text{gauge}(\mathcal{Z}_i, V_i \mu_i) \quad (21)$$

subject to

$$V_0 \mu_0 = x_t, \quad (22a)$$

$$V_{i+1} \mu_{i+1} = (AV_i + BU_i) \mu_i, \quad \forall i \in \mathbb{N}_{[0, k-1]}, \quad (22b)$$

$$\mathbf{1}_q^T \mu_i \leq \lambda^N, \quad \forall i \in \mathbb{N}_{[0, k-1]}, \quad (22c)$$

$$\mathbf{1}_q^T \mu_k \leq \lambda^{N+1}, \quad (22d)$$

$$\mu_i \geq \mathbf{0}_q, \quad \forall i \in \mathbb{N}_{[0, k]}. \quad (22e)$$

The resulting state–feedback control law is defined as

$$\pi(x_t) := U_i \mu_i^* \quad \text{if } t = kN + i, \quad N \in \mathbb{N}, \quad (23)$$

where $\mu_i^*, i \in \mathbb{N}_{[0, k]}$, denotes the optimal solution of Problem 3.

The optimization Problem 3 yields a sequence of control laws that optimizes the weighted sum of Minkowski functions corresponding to the sets $\{\mathcal{Z}_i\}_{i \in \mathbb{N}_{[1, k]}}$, evaluated at $\{x_{t+i}\}_{i \in \mathbb{N}_{[1, k]}}$, respectively. The choice of sets $\mathcal{Z}_i, i \in \mathbb{N}_{[1, k]}$ and of the weights $\alpha_i, i \in \mathbb{N}_{[1, k]}$, is a design parameter related to the objectives of the control problem under study. It is worth noting that Problem 3 always admits a feasible solution, independent of the selection of the cost function.

Remark 6 The control law (23) can be written in the form (13)–(15). Indeed, for any $t = kN + i, N \in \mathbb{N}$, due to

the optimization constraints (22a)–(22e) in Problem 3, the proposed control law (23) is realized by a specific selection of $\mu_i^* \in \mathcal{M}_i(x_t)$, $i \in \mathbb{N}_{[0, k-1]}$. Thus, by Proposition 3, system (16) in closed-loop with the periodic vertex–interpolation control law (23) is \mathcal{KL} -stable.

Remark 7 Since the sets \mathcal{Z}_i , $i \in \mathbb{N}_{[1, k]}$, are chosen to be proper \mathcal{C} -polytopic sets, Problem 3 can be cast as a linear program, by introducing the additional slack variables δ_i , $i \in \mathbb{N}_{[i, k]}$. Thus, the optimization Problem 3 can be replaced by the equivalent linear programming problem

$$\min_{\delta_i, i \in \mathbb{N}_{[1, k]}, \{\mu_i\}_{i \in \mathbb{N}_{[0, k]}}} \sum_{i=1}^k \alpha_i \delta_i \quad (24)$$

subject to (22) and

$$H_{Z_i} V_i \mu_i \leq \delta_i \mathbf{1}_{c_i}, \quad \forall i \in \mathbb{N}_{[1, k]}. \quad (25)$$

Then, the implementation of the periodic vertex–interpolation control law is the following. First, given $k \in \mathbb{N}_{\geq 1}$, for any time instant t , compute $N \in \mathbb{N}$ and $i \in \mathbb{N}_{[0, k-1]}$ such that $t = kN + i$. If $i = 0$, solve the linear program (24), (22), (25) and implement control law (23). If $i \in \mathbb{N}_{[1, k-1]}$, implement control law (23) that was computed at step $t = kN$.

Remark 8 A proper cyclic rearrangement of the matrix sequences $\{V_i\}_{i \in \mathbb{N}_{[0, k-1]}}$, $\{U_i\}_{i \in \mathbb{N}_{[0, k-1]}}$ yields a variant of Problem 3 which can be solved at each time instant t . Then, according to the receding horizon principle, only the first element of the optimal sequence and the corresponding control law is applied. The remaining elements of the sequence are discarded and a corresponding optimization problem is solved again at time $t + 1$, for x_{t+1} . Under this receding horizon control scheme, asymptotic stability of the closed-loop system as well as recursive feasibility of the control law can be established. The full details of the receding horizon formulation are not reported here due to space limitations.

V. CONSTRAINED STABILIZATION OF THE BUCK DC–DC CONVERTER

The circuit schematics of the Buck DC–DC converter is shown in Figure 1. The output voltage v_C can be less or equal than the power supply v_s and its value depends on the on/off ratio of a switch which is controlled by a fixed frequency pulse-width modulated (PWM) signal. The state variables $z \in \mathbb{R}^2$ for the buck converter are the voltage v_C across the output capacitor and the current i_L through the filter inductor, i.e. $z := (v_C \quad i_L)^T$. The input variable is the state of the switching node q (open or closed) which is realized by the two transistors shown in Figure 1. The averaged discrete-time dynamics of the converter is computed in two steps [22].

First, a continuous-time averaged model of the Buck converter of the form

$$\dot{z}_t = A_c z_t + B_c q_t$$

is obtained, with the corresponding matrices

$$A_c = \begin{pmatrix} -\frac{1}{R_H C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_L}{L} \end{pmatrix}, B_c = \begin{pmatrix} 0 \\ \frac{v_s}{L} \end{pmatrix},$$

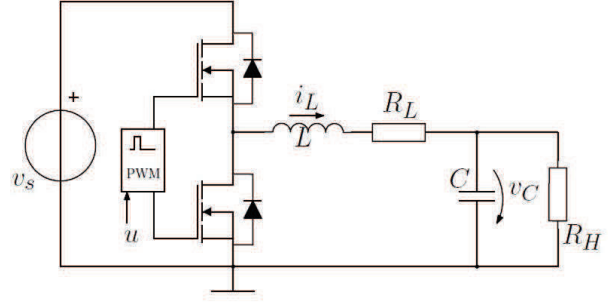


Fig. 1. The Buck DC–DC power converter.

where v_s is the supply voltage and q is the state of the switching node, i.e., $q = 1$ when the high-side transistor is conducting, while $q = 0$ when the high-side transistor is not conducting. Next, the continuous-time averaged model is discretized with a sampling rate of $T_s = 10 \mu s$, resulting to the description $z_{t+1} = A z_t + B d_t$. The input $d \in \mathbb{R}_{[0, 1]}$ of the discrete-time model corresponds to the duty-cycle ratio of the control signal applied to the switching node. The numerical values for the circuit components are $R_L = 0.2 \Omega$, $C = 22 \mu F$, $L = 220 \mu H$ and the sampling time is $T_s = 10 \mu s$. The system matrices for the discrete-time system are

$$A = \begin{pmatrix} 0.9456 & 0.4388 \\ -0.0439 & 0.9719 \end{pmatrix}, B = \begin{pmatrix} 0.2019 \\ 0.8978 \end{pmatrix}. \quad (26)$$

The values of the eigenvalues of matrix A are $0.9587 \pm i0.1382$, thus the system is open-loop stable. However, the challenges in the control problem stem from the presence of inherent hard polytopic state and input constraints $z \in \mathbb{Z}$ and $d \in \mathbb{D}$ respectively, which must be satisfied at all times for safety issues. In detail,

$$\mathbb{Z} := \{z \in \mathbb{R}^2 : 0 \leq z_1 \leq 22.5, 0 \leq z_2 \leq 3\}, \quad (27)$$

$$\mathbb{D} := \{d \in \mathbb{R} : 0 \leq d \leq 1\}. \quad (28)$$

Usually, a specific set-point $(z_s, d_s) \in \mathbb{D} \times \mathbb{Q}$ is imposed for the converter. For the case under study, it holds that $z_s := (10 \quad 1)^T$, while the corresponding duty-cycle ratio is $d_s := 0.5203$. Applying the coordinate transformation $x = z - z_s$, $u = d - d_s$, the description of the resulting system is

$$x_{t+1} = A x_t + B u_t, \quad (29)$$

where A, B are defined in (26). Moreover, the translated state and input constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$ are of the form (8) and (9) respectively, with

$$H_x = \begin{pmatrix} 12.5^{-1} & 0 \\ 0 & 2^{-2} \\ -10^{-1} & 0 \\ 0 & -1 \end{pmatrix}, H_u = \begin{pmatrix} 0.5203^{-1} \\ -0.4797^{-1} \end{pmatrix}. \quad (30)$$

The control problem is formulated as follows. Given a (k, λ) -controlled contractive set \mathcal{S} , compute an admissible state–feedback control law such that the closed-loop system is \mathcal{KL} -stable with respect to the set–point.

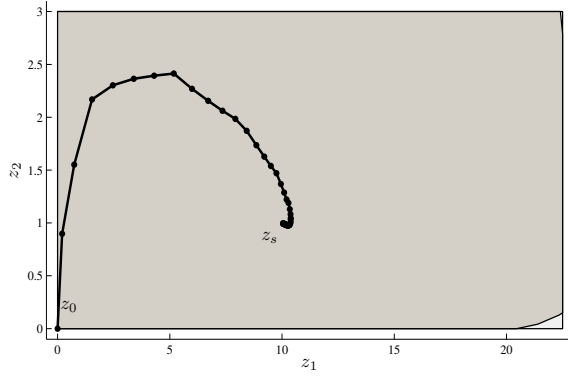


Fig. 2. The maximal controlled invariant set \mathcal{S} (grey), the state constraint set \mathbb{Z} (light grey), the trajectory of the closed-loop system under the periodic vertex-interpolation control law (23), starting from the zero initial condition, and the set-point $z_s = (10 \ 1)^T$.

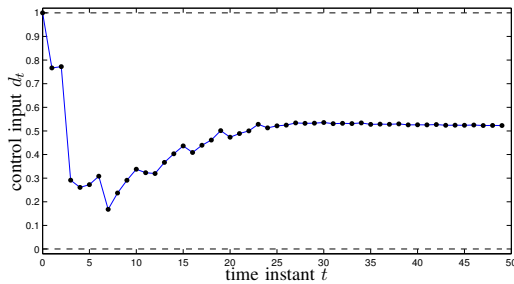


Fig. 3. The control input $d_t = \pi(z_t - z_s) + d_s$, resulting from the vertex-interpolation control law (23), starting from the zero initial condition.

First, the periodic vertex-interpolation based control law (23) was employed.

By utilizing the existing algorithms in the literature, e.g., [2], [3], [10]–[14], the maximal controlled invariant set \mathcal{S} was computed, which was found to be a proper \mathcal{C} -polytopic set. For the case under study, \mathcal{S} includes the relevant state which corresponds to the case when the power converter is turned off, i.e., $z = 0$.

Remark 9 It is worth noting that the zero state does not belong in any constraint admissible controlled λ -contractive set. However, in a recent work [22], controlled λ -contractive sets that contained the zero state were utilized for the constrained stabilization of the converter, by relaxing the state constraint set, allowing for negative values for the voltage v_C of the capacitor. Although the relaxation was made within some prespecified limits, negative voltage implies an increase in ohmic losses which may in turn lead to the overheating of the converter.

The maximal controlled invariant \mathcal{S} for the setting under study is shown in Figure 2 in grey color. It can be seen that it captures almost all the domain of operation \mathbb{Z} , which is shown in Fig. 2 in light grey.

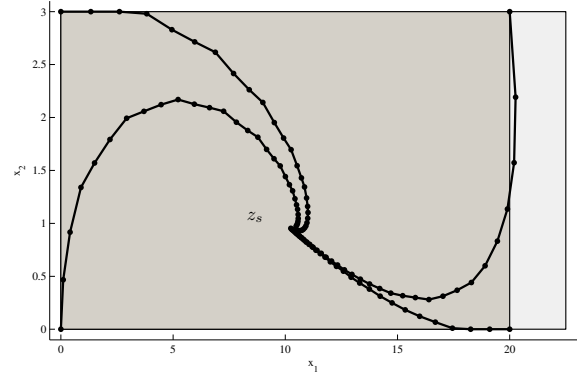


Fig. 4. The controlled $(3, 0.99)$ -contractive set \mathcal{S}_0 (grey), the state constraints set \mathbb{Z} (light grey), the trajectories of the closed-loop system under the periodic conewise linear control law (19),(20), starting from the vertices of the set \mathcal{S}_0 , and the set-point $z_s = (10 \ 1)^T$.

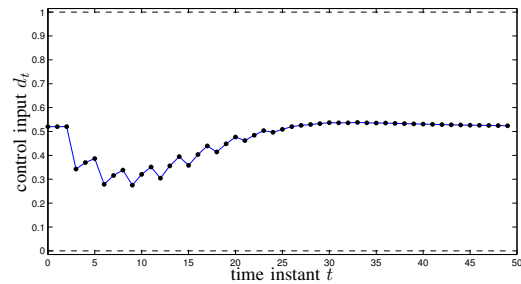


Fig. 5. The control input $d_t = \pi(z_t - z_s) + d_s$, resulting from the periodic conewise linear control law (19),(20), starting from the zero initial condition.

The first step towards constructing a stabilizing periodic vertex-interpolation control law concerns the characterization of the set \mathcal{S} as controlled (k, λ) -contractive. To this end, a linear program, similar to the feasibility Problem 2, was solved in an iterative fashion. In specific, the optimization problem

$$\min_{\lambda} \quad \lambda$$

$$\{U_i\}_{i \in \mathbb{N}_{[0, k-1]}}, \{V_i\}_{i \in \mathbb{N}_{[1, k]}}$$

subject to constraints (12), was solved iteratively for increasing values of $k \in \mathbb{N}$, starting from $k = 1$, until the optimal solution λ^* was less than one. The set \mathcal{S} was characterized with the (k, λ) -contractivity property with $k = 4$ and $\lambda = 0.99 \in \mathbb{R}_{[0, 1]}$. Moreover, the optimal values of matrices $\{U_i\}_{i \in \mathbb{N}_{[0, 3]}}$, $\{V_i\}_{i \in \mathbb{N}_{[1, 3]}}$ were used to formulate the optimization Problem 3 which produces the optimal control strategy (23), where V_0 and V_4 contain the vertices of set \mathcal{S} . For the case under study, the optimization cost was set to $\text{gauge}(\mathcal{S}, V_4 \mu_4)$, i.e., the Minkowski function of the maximal controlled invariant set evaluated at x_{t+4} . The linear program was solved in a modern desktop computer, using the command `linprog` from MATLAB. The worst case computational time needed for the solution of the optimization Problem 3 was found to be equal to 0.08sec. The resulting trajectory

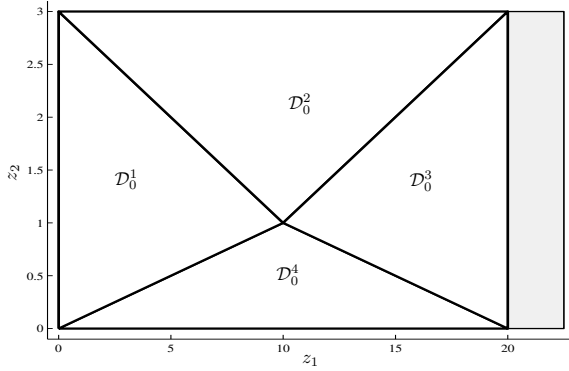


Fig. 6. The state constraint set \mathbb{Z} (grey), simplices \mathcal{D}_0^j , $j \in \mathbb{N}_{[1,4]}$ (light grey) and the set $\mathcal{S}_0 = \cup_{j=1}^4 \mathcal{D}_0^j$.

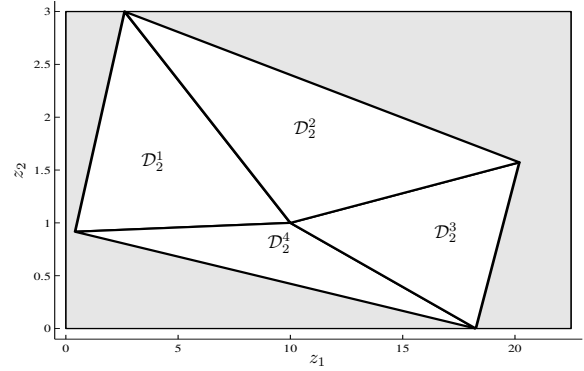


Fig. 8. The state constraint set \mathbb{Z} (grey) and simplices \mathcal{D}_2^j , $j \in \mathbb{N}_{[1,4]}$ (light grey).

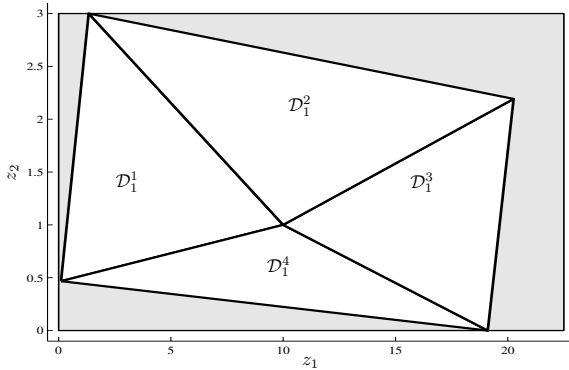


Fig. 7. The state constraint set \mathbb{Z} (grey) and simplices \mathcal{D}_1^j , $j \in \mathbb{N}_{[1,4]}$ (light grey).

of the closed-loop system with the initial condition equal to the zero state $z_0 = (0 \ 0)^T$ is shown in Figure 2. Moreover, the corresponding control effort $d_t := \pi(z_t - z_s) + d_s$, where $\pi(\cdot)$ denotes the control law (23), is shown in Figure 3.

Next, a periodic conewise linear law (19),(20) was applied to the constrained stabilization problem. To enhance computability, the low complexity (k, λ) -contractive set $\mathcal{S}_0 := \{z \in \mathbb{R}^2 : 0 \leq z_1 \leq 20, 0 \leq z_2 \leq 3\}$, shown in Figure 4, which captures most of the relevant states, was selected as the set of initial conditions for which the stabilizing control law is computed. The set \mathcal{S}_0 was characterized with the (k, λ) -contractivity property, by applying the same procedure as for the maximal controlled invariant set \mathcal{S} for the periodic vertex-interpolation control law. The corresponding values were found to be $k = 3$ and $\lambda = 0.99$. The simplicial decompositions $\{\mathcal{D}_i^s\}_{s \in \mathbb{N}_{[1,4]}}$, $i \in \mathbb{N}_{[0,2]}$, of the sets \mathcal{S}_i , $i \in \mathbb{N}_{[0,2]}$ (18) are shown in Figures 6,7 and 8 respectively. The trajectories of the closed-loop system with initial conditions the vertices of \mathcal{S}_0 , including the zero state $z_0 = (0 \ 0)^T$ can be seen in Figure 4. Moreover, the control effort $d_t := \pi(z_t - z_s) + d_s$ for the zero initial state, where $\pi(\cdot)$ corresponds to the control law (19),(20), is shown in Figure 5. The worst

case computational time needed to solve the point location problem for the selection of the control law (20) was found to be equal to $60 \mu\text{sec}$, which is significantly lower than the time needed to compute the periodic vertex-interpolation control law. It is worth noticing that the computational time is expected to be much lower in an FPGA implementation.

VI. CONCLUSIONS

The concepts of controlled (k, λ) -contractive sets and finite-time control Lyapunov functions were introduced in this paper. Two novel synthesis methods for constrained stabilization of linear systems that exploit these concepts were proposed, namely, periodic conewise linear control laws and periodic vertex-interpolation control laws. The benefits of these synthesis methods were demonstrated in the constrained stabilization problem of the DC-DC buck converter.

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