Bifurcation of eigenvalues in nonlinear problems with antilinear symmetry


**Published in:**
Journal of Mathematical Physics

**Document Version:**
Publisher's PDF, also known as Version of record

**Queen's University Belfast - Research Portal:**
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Download date:31. Aug. 2021
Many physical systems can be described by eigenvalues of nonlinear equations and bifurcation problems with a linear part that is non-selfadjoint, e.g., due to the presence of loss and gain. The balance of these effects is reflected in an antilinear symmetry, e.g., the \( \mathcal{PT} \)-symmetry. Under the symmetry we show that the nonlinear eigenvalues bifurcating from real linear eigenvalues remain real and the corresponding nonlinear eigenfunctions remain symmetric. The abstract result is applied in a number of physical models of Bose-Einstein condensation, nonlinear optics, and superconductivity, and numerical examples are presented.
parameter $\epsilon$ the assumptions of the Banach fixed point theorem are satisfied for $f$ locally Lipschitz and we avoid the stronger condition $\|f(\psi)\| = O(\|\psi\|^2)$ as $\psi \to 0$ of Ref. 21. Moreover, we provide a more explicit expansion of $\mu$ and $\psi$.

Also bifurcation problems under symmetry constraints are a well studied problem but traditionally the presence of a continuous symmetry group is assumed.\textsuperscript{9,17} The standard tool in this setting is the equivariant branching lemma.\textsuperscript{22} However, this tool is not available for our discrete antilinear symmetry.

The interest in antilinear symmetries was initiated by a numerical observation in Ref. 2 that some Schrödinger operators with $\mathcal{PT}$-symmetric complex potentials have real eigenvalues. The question of real eigenvalues in non-selfadjoint nonlinear problems with symmetries has gained on physical relevance in the recent years due to the intensive research on nonlinear systems under the parity and time-reversal ($\mathcal{PT}$) symmetry mainly in Bose-Einstein condensates (BECs),\textsuperscript{25,19,22} nonlinear optics,\textsuperscript{30} see also Ref. 40 for an experimental breakthrough, or superconductivity.\textsuperscript{38} In these specific physical problems, the presence of real eigenvalues typically means the existence of stationary solutions of the form $e^{-i\mu t}\psi(x)$ with $\mu \in \mathbb{R}$ also if the system is subject to balanced gain and loss.

In the context of BECs, where the Gross-Pitaevskii equation models the dynamics of the condensate, a complex potential describes the injection and removal of particles and a balance of these two processes is reflected in the $\mathcal{PT}$-symmetry of the system. Numerical results on the bifurcation of eigenvalues, in particular in one dimensional models, can be found, e.g., in Refs. 6, 10, and 16. In optics under the paraxial approximation, the system can be modeled by the nonlinear Schrödinger equation (NLS) with a potential corresponding to the refractive index, which is complex if the amplification and damping of the light wave are present, a balance is again reflected in the $\mathcal{PT}$-symmetry. Numerical and formal mathematical results on the bifurcation from eigenvalues in the $\mathcal{PT}$-symmetric NLS include Refs. 4, 42, and 45 (one dimensional case) and Ref. 44 (two dimensions).

Superconducting wires driven with electric currents represent another example of a physical application of a nonlinear $\mathcal{PT}$-symmetric eigenvalue problem, cf., Refs. 38 and 39. The non-selfadjointness appears due to the dependence of the electric potential on the external current.

We demonstrate in Section IV that all of the above physical models are particular cases of (1.1) compliant with our assumptions. To our knowledge the only existing mathematically rigorous papers on bifurcation problems in specific examples under an antilinear symmetry are Refs. 24 and 39. A concrete example of the discrete NLS is considered in the former, the technique is based on the Lyapunov-Schmidt reduction and the implicit function theorem. In the latter, a one dimensional parabolic problem for superconducting wires is studied using the center manifold analysis.

Our approach to the bifurcation problem (1.1) is to deliberately use standard tools, the Lyapunov-Schmidt reduction and a fixed point iteration, and formulate the assumptions so that they can be checked (almost) straightforwardly for unbounded differential operators in applications. To facilitate the latter, we also provide an Appendix with a summary of known facts on non-selfadjoint differential operators, holomorphic families of operators, and basic nonlinearities.

II. NONLINEAR EIGENVALUE PROBLEM

The following basic assumption comprises a condition on a compatibility of the linear part $A$ with the nonlinearity $f$ and a spectral condition on $A$.

Assumption 1. Let $A$ be a densely defined, closed operator with a non-empty resolvent set in a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ with the induced norm $\| \cdot \|$, let $f$ be a mapping in $\mathcal{H}$ and let $(\mathcal{Y}, \| \cdot \|_Y)$ be a Banach space. Suppose that the following conditions are satisfied:

(a) $\mathcal{Y}$ is a subspace of $\mathcal{H}$, for some $n \in \mathbb{N}$ is $\text{Dom} (A^n) \subset \mathcal{Y} \subset \text{Dom} (A^{n-1})$, and there are $k_1, k_2 > 0$ such that, for all $\phi \in \text{Dom} (A^n)$,

$$\| \phi \|_{n-1} := \sum_{k=0}^{n-1} \| A^k \phi \| \leq k_1 \| \phi \|_Y \leq k_2 \sum_{k=0}^{n} \| A^k \phi \| =: k_2 \| \phi \|_n. \quad (2.1)$$

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(b) \( \mu_0 \in \mathbb{C} \) is an isolated simple (i.e., with the algebraic multiplicity one) eigenvalue of \( A \). Moreover, suppose that the normalizations of \( \psi_0 \in \text{Dom} (A), \psi_0' \in \text{Dom} (A^*) \),

\[
A \psi_0 = \mu_0 \psi_0, \quad A^* \psi_0' = \overline{\mu_0} \psi_0',
\]

are chosen such that

\[
\| \psi_0 \| = 1, \quad \langle \psi_0, \psi_0' \rangle = 1. \tag{2.2}
\]

(c) The mapping \( f : Y \to \text{Dom} (A^{n-1}) \) is Lipschitz in a neighborhood of the eigenvector \( \psi_0 \), more precisely: there exist \( r_L > 0 \) and \( L > 0 \) such that, for all \( \phi, \psi \in \{ \eta \in Y : \| \eta - \psi_0 \|_Y < r_L \} \),

\[
\| f(\phi) - f(\psi) \|_{n-1} \leq k_1 L \| \phi - \psi \|_Y. \tag{2.3}
\]

Let us give a few remarks on Assumption 1. The space \( Y \) is our working space in which we perform fixed point iterations. A natural choice for \( Y \) is (\( \text{Dom} (A), \| \cdot \|_1 \)), i.e., the domain of \( A \) equipped with its graph norm. Nonetheless, it may be convenient to work also with a different \( Y \), e.g., with the form-domain and the norm induced by the quadratic form of \( A \) since these can be much better accessible than \( (\text{Dom} (A), \| \cdot \|) \) itself, cf., Subsection 1 in the Appendix for examples. Obviously, if the Lipschitz continuity (2.3) is established with \( \| \cdot \|_1 \), it holds also with \( \| \cdot \|_n \) on the right-hand side. A motivation for considering \( n > 1 \) is given in Examples A.3 and A.7, see also Remark A.8.

Recall that if \( \mu_0 \) is a simple isolated eigenvalue of \( A \), then \( \overline{\mu_0} \) is a simple isolated eigenvalue of \( A^* \), cf., Ref. 23, Chap. III.6.5-6; moreover, it can be easily verified that the normalization (2.2) can be achieved.

The spectral (Riesz) projection \( P_0 \) on \( \text{Ker} (A - \mu_0) \), defined as a contour integral for sufficiently small \( \delta > 0 \), cf., Ref. 23, Chap. III.6, and the complementary projection \( Q_0 := I - P_0 \) can be written, using (2.2), as

\[
P_0 := -\frac{1}{2\pi i} \int_{\partial B_{\delta}(\mu_0)} (A - z)^{-1} \, dz = \langle \cdot, \psi_0' \rangle \psi_0, \quad Q_0 = I - \langle \cdot, \psi_0' \rangle \psi_0.
\]

Under the algebraic multiplicity of \( \mu_0 \) we understand the rank of \( P_0 \).

In the rest of this section we first prove the local existence and uniqueness of the solutions of the nonlinear eigenvalue problem (1.1) under Assumption 1. Next, we focus on homogeneous nonlinearities, for which (1.1) together with condition \( \| \psi \| = 1 \) can be solved.

A. Local existence and uniqueness

Proposition 2.1. Let \( A \) and \( f \) satisfy Assumption 1. Then every solution \( (\mu, \psi) \) of the nonlinear eigenvalue problem

\[
(A - \mu) \psi - \varepsilon f(\psi) = 0, \quad \langle \psi, \psi_0' \rangle = 1 \tag{2.4}
\]

can be written as

\[
\mu = \mu_0 + \varepsilon \nu + \varepsilon^2 \sigma, \quad \psi = \psi_0 + \varepsilon \phi + \chi, \tag{2.5}
\]

with \( \nu, \sigma \in \mathbb{C} \) and \( \phi, \chi \in Q_0 \text{Dom}(A) \) as follows:

\[
\nu = -\langle f(\psi_0), \psi_0' \rangle, \tag{2.6}
\]

\( \phi \) is the unique (in \( Q_0 \text{Dom}(A) \)) solution of

\[
(A - \mu_0) \phi = \nu \psi_0 + f(\psi_0), \tag{2.7}
\]

and \( (\sigma, \chi) \) solves the nonlinear system

\[
0 = \varepsilon \sigma + \langle f(\psi) - f(\psi_0), \psi_0' \rangle, \tag{2.8}
\]

\[
Q_0 (A - \mu_0) Q_0 \chi = \varepsilon \left[ (\nu + \varepsilon \sigma) (\chi + \varepsilon \phi) + Q_0 [f(\psi) - f(\psi_0)] \right]; \quad R(\chi). \tag{2.9}
\]

Moreover, there exists \( \varepsilon_0 > 0 \) such that, for all \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \), the nonlinear eigenvalue problem (2.8) and (2.9) has a unique solution in the neighborhood

\[
| \sigma | \leq r_1, \quad \| \chi \|_Y \leq r_2 \varepsilon^2,
\]

where \( r_1, r_2 = O(1) (\varepsilon \to 0) \) satisfy (2.13) and (2.16), respectively.
Proof: Without any loss of generality, for a given \((\mu_0, \psi_0)\) we can write the solution \((\mu, \psi)\) as in (2.5). Although at this point the representation is not unique, we already know that \(\varepsilon \phi + \chi = Q_0(\varepsilon\phi + \chi)\) due to the constraint \(\langle \psi, \psi_0^* \rangle = 1\), and the normalization \(\langle \psi_0, \psi_0^* \rangle = 1\).

First, we apply the projection \(P_0\) to (2.4) and obtain
\[
v + \varepsilon\sigma = -(f(\psi), \psi_0^*) = -(f(\psi_0), \psi_0^*) - (f(\psi) - f(\psi_0), \psi_0^*).
\]
Defining \(v := -(f(\psi), \psi_0^*)\), we get Equation (2.8).

Second, we apply \(Q_0\) to (2.4), resulting in
\[
Q_0(A - \mu_0)Q_0(\varepsilon \phi + \chi) = (\varepsilon v + \varepsilon^2 \sigma)(\varepsilon \phi + \chi) + \varepsilon Q_0 f(\psi)
\]
\[
= \varepsilon \left[(v + \varepsilon\sigma)(\varepsilon \phi + \chi) + Q_0(f(\psi) - f(\psi_0)) + \nu \psi_0 + f(\psi_0)\right].
\]
Next, we define \(\phi\) to be the unique solution \(\phi \in Q_0(\text{Dom}(A))\) of (2.7). This solution exists because \(v \psi_0 + f(\psi_0) \in \text{Ker}(A^* - \overline{\mu_0})^k = \text{span} \{\psi_0^*\}^k\) and because \(\mu_0\) is an isolated simple eigenvalue, \(A - \mu_0\) is Fredholm, see Ref. 23, Theorem IV.5.28. The equation above thus becomes problem (2.9).

The rest of the proof deals with the existence of a unique solution \((\sigma, \chi)\), with \(\chi\) small and \(\sigma\) bounded, of (2.8) and (2.9). Note that \(Q_0(A - \mu_0)Q_0\) is boundedly invertible in \(Q_0\mathcal{H}\), hence Equation (2.9) can be rewritten as
\[
\chi = (Q_0(A - \mu_0)Q_0)^{-1}R(\chi) =: G(\chi).
\]
(2.10)

In the first step, for \(\varepsilon\) in a small neighborhood of 0, we use the fixed point argument to conclude the existence of a solution \(\chi\) of (2.10) with \(\|\chi\|_y = O(\varepsilon^2)\). We search for a fixed point
\[
\chi \in B_{r_2\varepsilon^2} := \{\eta \in Q_0\mathcal{Y} : \|\eta\|_y \leq r_2\varepsilon^2\},
\]
(2.11)
with some \(r_2 > 0\) independent of \(\varepsilon\). Its existence is guaranteed if we can show
(i) \(\chi \in B_{r_2\varepsilon^2} \Rightarrow G(\chi) \in B_{r_2\varepsilon^2}\).
(ii) there exists \(p \in (0, 1)\) such that \(\|G(\chi_1) - G(\chi_2)\|_y \leq p\|\chi_1 - \chi_2\|_y\) for all \(\chi_1, \chi_2 \in B_{r_2\varepsilon^2}\).

Note that \(\psi_0\), being an eigenfunction, satisfies \(\psi_0 \in \text{Dom}(A^m)\) for any \(m \in \mathbb{N}\). Thus the right hand side of (2.7) lies in \(\text{Dom}(A^{m-1})\), hence \(\phi \in Q_0(\text{Dom}(A^n)) \subset Q_0\mathcal{Y}\). Therefore if \(\chi \in Q_0\mathcal{Y}\), then \(Q_0f(\psi) \in Q_0(\text{Dom}(A^{m-1}))\) and \(R(\chi) \in Q_0(\text{Dom}(A^{m-1}))\). It is straightforward to check that then \(G(\chi) \in Q_0(\text{Dom}(A^n))\). The properties of the norm \(\| \cdot \|_y\), cf., Assumption 1(a), yield
\[
\|G(\chi)\|_y \leq \frac{k_2}{k_1}\|G(\chi)\|_n \leq \frac{k_2}{k_1}C_{\mu_0}\|R(\chi)\|_{n-1}.
\]
(2.12)

To show the second inequality in (2.12), note that because \(G(\chi) \in Q_0(\text{Dom}(A^n))\),
\[
AG(\chi) = (A - \mu_0)Q_0G(\chi) + \mu_0G(\chi)
\]
\[
= Q_0(A - \mu_0)Q_0G(\chi) + \mu_0G(\chi) = R(\chi) + \mu_0G(\chi).
\]
We thus have, for \(k \geq 1\),
\[
A^kG(\chi) = A^{k-1}(R(\chi) + \mu_0G(\chi)) = \cdots = \sum_{j=0}^{k-1}k^{k-1-j}A^jR(\chi) + \mu_0^kG(\chi).
\]
As a result, \(\|A^kG(\chi)\| \leq \sum_{j=0}^{k-1}|\mu_0|^{k-1-j}\|A^jR(\chi)\| + |\mu_0|^k\|Q_0(A - \mu_0)Q_0(\text{Dom}(A^{m-1}))\|\|R(\chi)\|\|\|Q_0(\text{Dom}(A^{m-1}))\|\|\|R(\chi)\|\|
\]
and the second inequality in (2.12) follows, where \(C_{\mu_0} > 0\) is a constant depending on \(|\mu_0|, n\) and \(\|Q_0(A - \mu_0)Q_0(\text{Dom}(A^{m-1}))\|\).

To ensure (i), we take \(\chi \in B_{r_2\varepsilon^2}\), estimate \(|R(\chi)|_{n-1}\), and select suitable \(r\) in the following. First note that \(P_0\) and \(Q_0\) are bounded on \(\text{Dom}(A^m)\), \(m \in \mathbb{N}_0\), moreover, since \(A^kP_0\psi = P_0A^k\psi\) and \(A^kQ_0\psi = Q_0A^k\psi\) for all \(\psi \in \text{Dom}(A^m)\) and \(k = 0, 1, \ldots, m\), we have
\[
\|P_0\|_m := \sup_{0 \neq \psi \in \text{Dom}(A^m)} \frac{\|P_0\psi\|_m}{\|\psi\|_m} = \sup_{0 \neq \psi \in \text{Dom}(A^m)} \frac{\sum_{k=0}^{m} \|P_0A^k\psi\|_m}{\sum_{k=0}^{m} \|A^k\psi\|_m} \leq \|P_0\|,
\]
and similarly \(\|Q_0\|_m \leq \|Q_0\|\). Next,
\[
|R(\chi)|_{n-1} \leq |\varepsilon|(\|v + \varepsilon\sigma\|_y + |\chi|_{n-1} + |\varepsilon||\phi|_{n-1}) + |Q_0(f(\psi) - f(\psi_0))|_{n-1}
\]
\[
\leq |\varepsilon|k_1(|v + \varepsilon\sigma|(r_2\varepsilon^2 + |\varepsilon||\phi|_y) + |Q_0|L|\varepsilon\phi + \chi|_y)
\]
\[
\leq \varepsilon^2k_1(|v + \varepsilon\sigma|(r_2\varepsilon^2 + |\varepsilon||\phi|_y) + |Q_0|L|\varepsilon\phi + \chi|_y).
\]
Hence, for all sufficiently small \( \epsilon \), we satisfy firstly \( \| \psi - \psi_0 \|_Y \leq \| \epsilon \| \| \phi \|_Y + r_2 \epsilon^2 < r_L \), hence the Lipschitz property of \( f \), cf., Assumption 1(c), can be indeed used. Second we satisfy condition (i).

It remains to prove (ii). Similarly as above, we obtain (with \( \psi_{1,2} := \psi_0 + \epsilon \phi + \chi_{1,2} \))

\[
\| R(\chi_1) - R(\chi_2) \|_{n-1} = |\epsilon| \| (1 + \epsilon \sigma)(\chi_1 - \chi_2) + Q_0(f(\psi_1) - f(\psi_2)) \|_{n-1} \\
\leq |\epsilon| \| k (1 + |\epsilon \sigma|) + \| Q_0 \|_L \| \chi_1 - \chi_2 \|_Y.
\]

Hence, for all sufficiently small \( \epsilon \), condition (ii) is also satisfied.

In summary, there exists \( \hat{\epsilon}_0 > 0 \), such that, for all \( \epsilon, |\epsilon| < \hat{\epsilon}_0 \), we have the function \( \chi \in B_{r_{2,\epsilon}} \) that solves (2.10); note that then \( \chi \in Q_0 \text{Dom}(A) \) as well. Note also that \( \chi \) and in particular \( \hat{\epsilon}_0 \) depend on \( \sigma \). However, we consider only \( |\sigma| \leq r_1 \), where \( r_1 \) satisfies (2.16), and an inspection of the estimates above shows that we can find \( \hat{\epsilon}_0 \) independent of \( \sigma \) (dependent only on \( r_1 \)).

In order to solve the first equation in (2.8), we prove first that the solution \( \chi \) is continuous in \( \sigma \). More precisely,

\[
\| \chi(\sigma_1) - \chi(\sigma_2) \|_Y \leq \frac{k_2}{k_1} C_{\mu_0} |\epsilon| \| (\chi(\sigma_1) - \chi(\sigma_2)) + \epsilon (\sigma_1 \chi(\sigma_1) - \sigma_2 \chi(\sigma_2)) + \epsilon^2 (\sigma_1 - \sigma_2) \phi \\
+ Q_0(f(\psi_0 + \epsilon \phi + \chi_{1,2})) \|_{n-1} \\
\leq k_2 C_{\mu_0} |\epsilon| \| (\chi(\sigma_1) - \chi(\sigma_2)) + \epsilon (\sigma_1 - \sigma_2) \phi \|_Y \\
+ \| Q_0 \|_L \| \chi(\sigma_1) - \chi(\sigma_2) \|_Y \\
\leq k_2 C_{\mu_0} |\epsilon| \left( \frac{1}{2} \| \chi(\sigma_1) + \chi(\sigma_2) \|_Y + |\epsilon| \| \phi \|_Y \right) \| \sigma_1 - \sigma_2 \|_Y.
\]

Hence, for \( |\sigma_1, \sigma_2| \leq r_1 \),

\[
\| \chi(\sigma_1) - \chi(\sigma_2) \|_Y \left( 1 - k_2 C_{\mu_0} |\epsilon| \| (\chi(\sigma_1) - \sigma_1) + \| Q_0 \|_L \right) \\
\leq k_2 C_{\mu_0} |\epsilon| \left( r_2 |\epsilon| + \| \phi \|_Y \right) \| \sigma_1 - \sigma_2 \|_Y.
\]

(2.14)

As the final step, we use the fixed point argument on

\[
\sigma = -\frac{1}{\epsilon} (f(\psi) - f(\psi_0), \psi_0^*) =: S(\sigma),
\]

(2.15)

where we search for a fixed point in \( \{ \sigma \in \mathbb{C} : |\sigma| \leq r_1 \} \) with a suitable \( r_1 \) selected below. Since

\[
|S(\sigma)| \leq k_1 L \| P_0 \| \| \phi \|_Y + r_2 |\epsilon|,
\]

we choose \( r_1 \) such that

\[
r_1 > k_1 L \| P_0 \| \| \phi \|_Y,
\]

(2.16)

hence, for sufficiently small \( |\epsilon| \), \( |\sigma| \leq r_1 \) implies \( |S(\sigma)| \leq r_1 \). Moreover, using the continuity of \( \chi \) in \( \sigma \), cf., (2.14), we obtain

\[
|S(\sigma_1) - S(\sigma_2)| \leq \frac{1}{|\epsilon|} \| f(\psi(\sigma_1)) - f(\psi(\sigma_2)) \| \\
\leq \frac{k_1 L}{|\epsilon|} \| \chi(\sigma_1) - \chi(\sigma_2) \|_Y \leq C \epsilon^2 |\sigma_1 - \sigma_2|,
\]

hence, for small \( |\epsilon| \), the fixed point argument yields the sought solution of (2.15). \( \square \)
If the nonlinearity is homogeneous, e.g., \( f(\psi) = |\psi|^q \psi \), solutions with norm one can be generated from the nonlinear eigenfunctions of Proposition 2.1 by a scaling.

**Corollary 2.2 (Nonlinear eigenfunction with norm one).** Let \( A \) and \( f \) satisfy Assumption 1 and suppose that \( f \) is a homogeneous nonlinearity, i.e., for all \( a > 0 \), it satisfies the scaling property \( f(a\psi) = af(\psi) \) with some \( q \in \mathbb{R} \). Given the nonlinear eigenpair \((\mu(\epsilon), \psi(\epsilon))\) for \( |\epsilon| < \epsilon_0 \), from Proposition 2.1, the pair

\[
(\tilde{\mu}(\epsilon), \tilde{\psi}(\epsilon)) = \left( \mu(\epsilon), \|\psi(\epsilon)\|^{-1}\psi(\epsilon) \right) \quad \text{with} \quad \tilde{\epsilon} = \epsilon \|\psi(\epsilon)\|^{-1}
\]

(2.17)

solves \((A - \tilde{\mu})\psi - \tilde{\epsilon} f(\tilde{\psi}) = 0\) and satisfies \(\|\tilde{\psi}(\epsilon)\| = 1\). The mapping \( \epsilon \mapsto \|\psi(\epsilon)\|^{-1} \) is injective for \( |\epsilon| < \epsilon_1 \), where \( \epsilon_1 \leq \epsilon_0 \) is small enough.

**Proof.** From the scaling property we immediately get

\[
(A - \mu(\epsilon))\|\psi(\epsilon)\|^{-1}\psi(\epsilon) - \epsilon f(\|\psi(\epsilon)\|^{-1}\psi(\epsilon)) = 0.
\]

The injectivity follows from the asymptotic equivalence \( \epsilon \|\psi(\epsilon)\|^{-1} = \epsilon \|\psi_0 + \epsilon \phi + \chi\|^{-1} \sim \epsilon \|\psi_0\|^{-1} = \epsilon \) for \( \epsilon \to 0 \).

\( \square \)

### III. ANTILINEAR SYMMETRIES

First we recall the notion of antilinear symmetry.

**Assumption 2 (Antilinear symmetries of \( A \) and \( f \)).** In a Hilbert space \( \mathcal{H} \) let \( C \) be an antilinear, isometric, and involutive operator, i.e., for all \( \phi, \psi \in \mathcal{H} \) and \( \lambda \in \mathbb{C} \), \( C(\lambda \phi + \psi) = \overline{\lambda} C\phi + C\psi \), \( \langle C\phi, C\psi \rangle = \langle \psi, \phi \rangle \), and \( C^2 = I \). Let \( A \) be a densely defined and closed operator in \( \mathcal{H} \) such that

(a) for all \( \psi \in \text{Dom}(A) \),

\[
C\psi \in \text{Dom}(A) \quad \text{and} \quad AC\psi = CA\psi,
\]

(3.1)

(b) for all \( \psi \in \mathcal{Y} \), where \( \mathcal{Y} \) is the space from Assumption 1(c),

\[
Cf(\psi) = f(C\psi).
\]

(3.2)

The operator \( C \) is referred to as the *antilinear symmetry* of \( A \) and \( f \).

**Example 3.1 (\( \mathcal{P}\mathcal{T} \) symmetry).** Let \( \mathcal{H} = L^2(\Omega) \), where \( \Omega = \mathbb{R}^d \) or \( \Omega = (-r, r)^d \) with \( r \in (0, +\infty) \). Define

\[
\mathcal{P}\psi(x) := \psi(-x), \quad \mathcal{T}\psi := \overline{\psi}.
\]

In quantum mechanics, \( \mathcal{P} \) corresponds to the space reflection (parity) and \( \mathcal{T} \) is the time-reversal. The antilinear \( \mathcal{P}\mathcal{T} \) symmetry is the composition, i.e., \( C := \mathcal{P}\mathcal{T} \). In more dimensional domains, the so-called partial \( \mathcal{P}\mathcal{T} \) symmetries, where

\[
(\mathcal{P}\phi)(x_1, \ldots, x_i, \ldots, x_d) := \psi(x_1, \ldots, -x_i, \ldots, x_d),
\]

are sometimes considered, cf., Ref. 4 or Ref. 44.

Schrödinger operators \( -\Delta + V \) with complex potentials \( V \), cf., Section IV and Examples A.1 and A.3, are \( \mathcal{P}\mathcal{T} \)-symmetric if \( V \) is \( \mathcal{P}\mathcal{T} \)-symmetric, i.e., \( [\mathcal{P}\mathcal{T}, V] = 0 \), or, in other words, the real and imaginary parts of \( V \) satisfy \( \text{Re} V(-x) = \text{Re} V(x) \) and \( \text{Im} V(-x) = -\text{Im} V(x) \). The Schrödinger operators with singular potentials from Example A.2 possess this symmetry if, for every \( \phi, \psi \in \text{Dom}(a) \), the sesquilinear form \( v_2 \) satisfies \( v_2(\phi, \mathcal{P}\mathcal{T}\psi) = v_2(\mathcal{P}\mathcal{T}\phi, \psi) \), see Section IV for examples of such \( v_2 \).

Regarding nonlinearities, the most common \( f \) in BECs is \( f(\psi) = |\psi|^q \psi \), which is \( \mathcal{P}\mathcal{T} \)-symmetric. General polynomial nonlinearities with \( x \)-dependent coefficients are \( \mathcal{P}\mathcal{T} \)-symmetric if and only if the coefficients \( a_{pq} \) are so, i.e., \( a_{pq}(-x) = a_{pq}(x) \), see Example A.7. Also the non-local \( f_2(\psi)(x) := \psi \int_0^\infty \text{Im} \left( \left( \psi(s)\overline{\psi}_s(s) \right) ds \right. \) from the superconductor model in Ref. 39 is \( \mathcal{P}\mathcal{T} \)-symmetric.
Theorem 3.2. Let A and f satisfy Assumptions 1 and 2. Suppose in addition that \( \mu_0 \in \mathbb{R} \) and choose the corresponding eigenvector \( \psi_0 \) as \( C \)-symmetric, i.e., \( C\psi_0 = \psi_0 \). Then, for all \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \), the nonlinear eigenpair \((\mu, \psi)\) from Proposition 2.1 satisfies \( \mu \in \mathbb{R} \) and \( C\psi = \psi \).

Proof. Recall that \( \mu = \mu_0 + \varepsilon \nu + \varepsilon^2 \sigma \), \( \psi = \psi_0 + \varepsilon \phi + \chi \), and \( P_0 \) is the spectral projection of \( A \) corresponding to the eigenvalue \( \mu_0 \). Moreover, the eigenvector \( \psi_0 \) can be indeed selected as \( C \)-symmetric, see Lemma A.5.

In the first step, we show that \( \nu \) is real and \( C\phi = \phi \). The spectral projection \( P_0 \) can be written as \( P_0 = \langle \cdot, \psi_0^* \rangle \psi_0 \), where \( \psi_0^* \) is as in Proposition 2.1, therefore \( -\nu \psi_0 = P_0(f(\psi_0)) \), cf., (2.6). Using symmetries, namely (3.2), \( C\psi_0 = \psi_0 \) and \( P_0 C = CP_0 \), see Lemma A.5 for details, we obtain

\[
C(-\nu \psi_0) = CP_0(f(\psi_0)) = P_0(f(C\psi_0)) = P_0(f(\psi_0)) = -\nu \psi_0, \tag{3.3}
\]

thus \( -\nu \psi_0 = -\nu \psi_0 \), hence \( \nu \in \mathbb{R} \). Applying \( C \) to Equation (2.7), using the symmetries of \( A \), \( f \), and \( \psi_0 \), i.e., (3.1), (3.2), and \( C\psi_0 = \psi_0 \), we get \( (A - \mu_0)C\phi = \nu \psi_0 + f(\psi_0) \). As \( P_0 C = CP_0 \) and since the solution of (2.7) with \( P_0 \phi = 0 \) is unique, we get \( C\phi = \phi \).

Since \( \sigma \) is the solution of the fixed point problem \( \sigma = S(\sigma) \) with \( S(\sigma) = -\frac{1}{\varepsilon}(f(\psi(\sigma)) - f(\psi_0), \psi_0^* \rangle \) and \( \psi(\sigma) = \psi_0 + \varepsilon \phi + \chi(\sigma) \) and \( \chi \) solves the fixed point equation \( \chi = G(\chi; \sigma) \), it remains to show that the coupled fixed point problem preserves the realness of \( \sigma \) and the \( C \)-symmetry of \( \chi \).

Given \( \sigma \in \mathbb{R} \) (with \( |\sigma| \leq \tau_1 \)), we prove that

\[
C\chi = \chi \implies CG(\chi; \sigma) = G(\chi; \sigma). \tag{3.4}
\]

As \( G(\chi; \sigma) = (Q_0(A - \mu_0)Q_0)^{-1} R(\chi; \sigma) \), we first show the analogous property for \( R \) and then the commutation of \( (Q_0(A - \mu_0)Q_0)^{-1} \) with \( C \). Since \( Q_0 = I - P_0 \), we get from (A5) that \( Q_0 C = C Q_0 \) as well. Also note that for \( C\chi = \chi \) the full solution \( \psi = \psi_0 + \phi + \chi \) is \( C \)-symmetric. Hence, for \( \varepsilon, \sigma \in \mathbb{R} \) and \( C\chi = \chi \),

\[
C R(\chi; \sigma) = \varepsilon \left\{ \langle \nu + \varepsilon \sigma \rangle (\chi + \varepsilon \phi) + Q_0 C(f(\psi) - f(\psi_0)) \right\}
= \varepsilon \left\{ \langle \nu + \varepsilon \sigma \rangle (\chi + \varepsilon \phi) + Q_0 f(C\psi) - f(C\psi_0) \right\} = R(\chi; \sigma).
\]

To prove (3.4), it remains to show that \( C(Q_0(A - \mu_0)Q_0)^{-1} = (Q_0(A - \mu_0)Q_0)^{-1} C \). To this end, take any \( \varepsilon \in Q_0 \mathcal{H} \), then

\[
C(Q_0(A - \mu_0)Q_0)^{-1} \varepsilon = (Q_0(A - \mu_0)Q_0)^{-1} Q_0(A - \mu_0)Q_0 C(Q_0(A - \mu_0)Q_0)^{-1} \varepsilon = (Q_0(A - \mu_0)Q_0)^{-1} C \varepsilon.
\]

Property (3.4) implies that the fixed point of \( \chi = G(\chi) \) in \( B_{r_{2 \varepsilon}^2} \), cf., (2.11), lies in \( B_{r_{2 \varepsilon}^2} \cap \{ \eta \in \mathcal{H} : C\eta = \eta \} \).

Finally, we need to show that \( C\chi = \chi \) implies \( S(\sigma) \in \mathbb{R} \). Once again, because \( C\chi = \chi \) implies \( C\psi = \psi \), we get by a manipulation analogous to (3.3),

\[
C(-\varepsilon S(\sigma)\psi_0) = CP_0(f(\psi) - f(\psi_0)) = P_0(f(C\psi) - f(C\psi_0)) = -\varepsilon S(\sigma)\psi_0,
\]

hence \( S(\sigma) \in \mathbb{R} \) by the same arguments as (3.3) below. \( \square \)

Remark 3.3. The operator \( A \) and the nonlinearity \( f \) may possess also a linear symmetry, i.e., \( S : \mathcal{H} \to \mathcal{H} \) such that for all \( \phi, \psi \in \mathcal{H} \) and \( \lambda \in \mathbb{C} \), \( S(\lambda \phi + \psi) = \lambda S\phi + S\psi \), \( \langle S\phi, \psi \rangle = \langle \phi, S\psi \rangle \) and \( S^2 = I \). Then, for a simple \( \mu_0 \in \mathbb{R} \), the symmetry or antisymmetry of the nonlinear eigenfunctions \( \psi(\varepsilon) \) is preserved, however, as already linear examples show, it cannot be concluded from the linear symmetry that \( \mu(\varepsilon) \) remains real.

IV. APPLICATIONS

We demonstrate how our results can be applied in a number of physics problems from the literature. We indicate what working spaces \( \mathcal{Y} \) can be used. Not surprisingly for Schrödinger operators, we work in

\[
(\mathcal{Y}, \| \cdot \|_\mathcal{Y}) = (\mathcal{H}^2_Q(\Omega), \| \cdot \|_{\mathcal{H}^2_Q(\Omega)}) := ((\mathcal{H}^2(\Omega) \cap \text{Dom}(Q)), \| \cdot \|_{\mathcal{H}^2(\Omega)}) \tag{4.1}
\]
where, for simplicity, $\Omega$ is $\mathbb{R}^d$ or an interval, $s > 0$, $Q \in L^2_{\omega} = L^2_{\omega}((\Omega))$ and $\operatorname{Dom}(Q) := \{\psi \in L^2(\Omega) : Q\psi \in L^2(\Omega)\}$. In Subsection 2 in the Appendix we collect some results on non-selfadjoint differential operators and typical nonlinearities. In the examples below we indicate how checking our assumptions can be reduced to those facts.

Notice that all our examples can be viewed as a holomorphic family $A(\gamma)$ with $A(0) = A(0)^*$ and thus $\sigma(A(0)) \subset \mathbb{R}$. The parameter $\gamma$ controls the strength of the non-symmetric part of the operator. The spectral condition in Assumption 1(b) and in Theorem 3.2 can be frequently verified using the reasoning in Subsection 2 in the Appendix.

A. Toy model

Let $\mathcal{H} = L^2((-r, r))$ with $r = \pi/2$, $\gamma \in \mathbb{R}$, and let $A_\gamma$ be the m-sectorial operator associated with the form

$$a_\gamma[\psi] := \|\psi\|^2 + i\gamma (|\psi(r)|^2 - |\psi(-r)|^2), \quad \operatorname{Dom}(a_\gamma) := H^1((-r, r)).$$

By standard arguments, cf., Ref. 23, Example VI.2.16 $A_\gamma = -\partial^2_x$ with

$$\operatorname{Dom}(A_\gamma) = \{\psi \in H^1((-r, r)) : \psi'(\pm r) + i\gamma \psi(\pm r) = 0\},$$

moreover, $(A_\gamma)^* = A_{-\gamma}$ and $A_\gamma$ is $\mathcal{P}\mathcal{T}$-symmetric, cf., Ref. 28. In the notation of Example A.2, $V_1 = 0$ so we take $\mathcal{J} = \mathcal{H}_0^1((-r, r), r) = H^1((-r, r))$. The resolvent of $A_\gamma$ is compact, for $\gamma \notin \mathbb{Z}$ all eigenvalues are explicit: $\sigma(A_\gamma) = \{\gamma^2\} \cup \{n^2\}_{n=1}^\infty$, see, e.g., Refs. 28 and 30. If $\gamma \in \mathbb{Z}$, then all eigenvalues are simple, so Assumption 1(b) is satisfied for any $\mu_0 \in \sigma(A_\gamma)$. The eigenfunctions $\{\xi_n\}_{n=0}^\infty$, $\{\xi_n^*\}_{n=0}^\infty$ of $A_\gamma$ and $A_{\gamma}^*$, respectively, with normalization satisfying (2.2), read

$$\xi_0(x) = 1, \quad \xi_n(x) = \sqrt{\frac{1}{r}} \frac{n}{\sqrt{n^2 + \gamma^2}} \left(\cos(n(x + r)) - \frac{i\gamma}{n} \sin(n(x + r))\right),$$

$$\xi_0^*(x) = \frac{\gamma \pi}{\sin(\gamma \pi)} \xi_0(x), \quad \xi_n^*(x) = \frac{n^2 + \gamma^2}{n^2 - \gamma^2} \xi_n(x), \quad n \in \mathbb{N}.$$

We consider the cubic nonlinearity $f(\psi) = |\psi|^2 \psi$, see Examples A.7 and 3.1 for details on checking Assumptions 1 and 2.

Theorem 3.2 yields a nonlinear eigenpair $(\mu(\epsilon), \phi(\epsilon))$ with $\|\phi(\epsilon)\| = 1$ and $\mu(\epsilon) \in \mathbb{R}$ for any $\mu_0 \in \sigma(A_\gamma)$ and small $\epsilon$. For $\mu_0 = n^2, n \geq 1$, straightforward calculations lead to $\nu_n = -\langle \xi_n^*, \xi_n^2 \rangle = -3\pi/2$ and also $\phi_n$ can be calculated explicitly by solving (2.7). For the special eigenvalue $\mu_0 = \alpha^2$ and the corresponding eigenfunction $\psi_0 = \xi_0$, the normalized nonlinear eigenpair can be found even explicitly, namely $(\mu(\epsilon), \phi(\epsilon)) = (\gamma^2 - \alpha^2, \psi(0))$ which is in agreement with the expansion of $\mu$ and $\phi$ from Proposition 2.1.

B. BECs and nonlinear optics

The nonlinear Schrödinger equation with cubic nonlinearity

$$(\Delta - V)\psi + \epsilon |\psi|^2 \psi = \mu \psi \quad (4.2)$$

is a frequently used model for time harmonic states $u(x,t) = e^{-i\omega t} \phi(x)$ in both Bose-Einstein condensates and nonlinear optics. The potential (refractive index in optics) is assumed to be complex, the coupling constant for the imaginary part is the parameter $\gamma$. It models either the injection and removal of particles in condensates, cf., for instance Refs. 6, 10, and 16, or the gain and loss of the medium in optics, cf., for instance Refs. 34 and 44. The balance of the injection and removal or of the gains and losses is reflected in the $\mathcal{P}\mathcal{T}$ symmetry of $V$.

Various potentials like perturbations of the harmonic oscillator or of delta interactions can be found in the literature on condensates. In optics, a typical $V$ is bounded and, for a heterogeneous material, it can be discontinuous, which is however not an obstacle in view of Example A.1. The latter together with Example A.7 explains why Assumptions 1(a) and 1(c) are satisfied for classes of
complex potentials for \( \mathcal{Y} \) being appropriately chosen as \( \mathcal{H}^2_\Omega(\Omega) \), see (4.1). We consider two \( \mathcal{PT} \) or \( \mathcal{PT}_T \)-symmetric examples from the literature, both fitting with Example A.1,

\[
V(x) = x^2 + v_0 e^{-\sigma x^2} + i\gamma e^{-\rho x^2}, \quad \gamma, v_0 \in \mathbb{R}, \sigma, \rho \geq 0,
\]

(4.3)

\[
V(x, y) = -(v_0 + i\gamma y) e^{-x^2 - y^2}, \quad \gamma, v_0 \in \mathbb{R},
\]

(4.4)

see Refs. 10 and 44. While the spectrum for (4.3) is discrete, there is the essential spectrum \([0, \infty)\) for (4.4), cf., Subsection 2 c of the Appendix.

For (4.4) all eigenvalues are simple and real if \( v_0 \) and \( |\gamma| \) are small, cf., Remark A.6. Moreover, it follows, e.g., from Ref. 33 that the number of non-real eigenvalues is finite for any \( v_0, \gamma \in \mathbb{R} \); a numerical analysis of eigenvalues for (4.3) is in Ref. 10. For (4.4) there are discrete negative eigenvalues \(\mu \) such that \(\mu \approx -v_0 \) (and \(\text{Re} V \gamma \approx 0\)), which are simple for sufficiently small \(v_0\), cf., Subsection 2 c of the Appendix.

In summary, Theorem 3.2 proves the effects observed in the physics literature, i.e., for \( \varepsilon \neq 0 \), nonlinear eigenvalues are shifted with respect to linear ones and those \(\mu\) that start from real simple linear eigenvalues \(\mu_0\) are real and the antilinear symmetry of the nonlinear solution \(\psi\) is preserved.

A numerical analysis of (4.2) with \(d = 2\) and two potentials, one of which is qualitatively similar to (4.3), is performed in Section V.

The applicability of our results can be explained similarly also for the spin-orbit-coupled Bose-Einstein condensate, cf., Ref. 22, as well as for other nonlinearities like monopolar or dipolar interactions.\(^{10}\) Also other potentials are allowed. For instance, for periodic \(V\) our results apply to nonlinear eigenvalues of the Bloch eigenvalue problem on the periodicity cell. Note, however, that the bifurcation of spatial solitons in \(\Omega = \mathbb{R}^n\) from the spectrum in the case of a periodic \(V\), as in Ref. 34, is not covered due to the absence of eigenvalues. Yet another example, where Theorem 3.2 applies, is the discrete \(\mathcal{PT}\)-symmetric NLS used in modeling optical lattices.\(^{24}\)

C. Coupled mode equations in optics

In Kerr-nonlinear optical fibers with a Bragg grating and a localized defect the propagation of asymptotically broad wavepackets can be described by the system of “coupled mode equations,”

\[
i(\partial_t E_1 + \partial_x E_1) + \kappa(x) E_2 + V(x) E_2 + (|E_1|^2 + 2|E_2|^2) E_1 = 0,
\]

\[
i(\partial_t E_2 - \partial_x E_2) + \kappa(x) E_1 + V(x) E_1 + (|E_2|^2 + 2|E_1|^2) E_2 = 0,
\]

with \(\kappa(x) \to \kappa_\infty > 0\) and \(V(x) \to 0\) as \(|x| \to \infty\), see Ref. 18. The potentials \(\kappa(x) - \kappa_\infty\) and \(V(x)\) describe the defect of the material and are determined by the refractive index. Once again, we consider the time harmonic ansatz \(E(x, t) = e^{-i\omega t}\psi(x)\) and after the rescaling \(\psi := e^{-i\omega t/2}\phi\) (with \(\varepsilon > 0\)), we obtain eigenvalue problem (1.1) with \(A\) being a Dirac type operator as in Example A.4, and

\[
f(\psi) = \left(\frac{(|\psi_1|^2 + 2|\psi_2|^2) \psi_1}{(|\psi_2|^2 + 2|\psi_1|^2) \psi_2}\right).
\]

As mentioned in Subsection 2 d of the Appendix, real smooth and bounded potentials \(\kappa\) and \(V\) exist such that \(A\) has a simple isolated eigenvalue. Example A.4 and analogous procedure to Example A.7 show that Assumption 1 is satisfied with \(\mathcal{H} = L^2(\mathbb{R}) \times L^2(\mathbb{R})\) and \(\mathcal{Y} = H^2(\mathbb{R}) \times H^1(\mathbb{R})\) provided \(V, \kappa \in L^\infty\) and \(\kappa(x) \to \kappa_\infty > 0\) and \(V(x) \to 0\) as \(|x| \to \infty\).

For materials with loss/gain, the potentials \(V\) and \(\kappa\) become complex and choosing them \(\mathcal{PT}\)-symmetric, we satisfy also Assumption 2. The existence of a real simple isolated eigenvalue \(\mu_0\) of \(A\) is guaranteed at least for small imaginary parts of \(\kappa\) and \(V\) by the analytic dependence as in Remark A.6. Hence (in the language of Ref. 18) our results show that conservative nonlinear defect modes bifurcate from linear ones in the \(\mathcal{PT}\)-symmetric case.

D. Superconductivity

A model of a finite superconducting wire is discussed in Refs. 38 and 39 and the bifurcation of nonlinear states for a nonlinear parabolic Equation \((d = 1)\) on \(\Omega = (-1, 1)\) is studied. In detail, the
problem

\[ w_t = w_{xx} + ixIw + \Gamma w + N[w], \quad x \in (-1, 1), \quad w(-1) = w(1) = 0, \]  

\[ N[w] = -|w|^2 w + iw \int_0^x \text{Im} \left( w(s,t) \overline{w}_x(s,t) \right) ds, \quad (4.5) \]

where \( I \) and \( \Gamma \) are real parameters, is considered. In Ref. 39, Sec. 6 the authors study the bifurcation of nonlinear (generally \( t \)-dependent) solutions from the zero solution at the smallest eigenvalue \( \lambda_1 \in \mathbb{R} \) of

\[ A := -\partial_x^2 - ixI, \quad \text{Dom}(A) := H^2((-1,1)) \cap H^1_0((-1,1)). \]

The potential \(-ixI\) is \( PT \)-symmetric, so the spectrum of \( A \) remains real if the parameter \( I \) is chosen small enough, cf., Remark A.6, and the number of non-real eigenvalues remains finite for any \( I \in \mathbb{R} \). For the bifurcation problem the authors set \( \Gamma = \Re \lambda_1 + \varepsilon, 0 < \varepsilon \ll 1 \) and use the center manifold reduction, where the center manifold is one dimensional and corresponds to the zero eigenvalue of \( A - \Re \lambda_1 \). On the manifold they study \( t \)-dependent, but also stationary nonlinear solutions. The asymptotics of the latter are given by

\[ w(x) \sim e^{1/2\alpha u_1(x)}, \]

where \( u_1 \) is the linear eigenfunction corresponding to \( \lambda_1 \) and \( \alpha \in \mathbb{R} \) is the projection coefficient on the center subspace and solves an algebraic equation. For \( I \) small enough \( \lambda_1 \in \mathbb{R} \), such that a real nonlinear eigenvalue \( \Gamma \) bifurcates. The eigenfunction \( w \) is \( PT \)-symmetric due to the \( PT \)-invariance of the center manifold.

In the formal part of Ref. 39 the more detailed expansion

\[ w(x) \sim e^{1/2\alpha u_1(x)} + e^{3/2}w_1(x) \]

is given, where the correction \( w_1 \) solves

\[ (A - \lambda_1)w_1 = \alpha u_1 + N[\alpha u_1]. \]

(4.6)

\( \alpha \in \mathbb{R} \) can then be selected via the solvability condition of the above equation and agrees to leading order with the \( \alpha \) from the center manifold approach.

To relate this work to our results, we rescale the \( t \)-independent solution \( w(x) = e^{1/2\psi(x)} \) and recover from (4.5) a problem of type (1.1), namely

\[ (A - \Gamma)\psi - \varepsilon N[\psi] = 0, \]

cf., Examples A.7 and A.9, Equation (4.6) thus corresponds to our (2.7). Observe that \( A \) fits the setting of Example A.2 with \( V_1(x) = ixI, v_2 = 0 \), and \( \mathcal{Y} = H^1_{0,\text{Dir}}((-1,1)) = H^1_0((-1,1)). \) The nonlinearities are discussed in Examples A.7, A.9 and 3.1 and it is shown that \( H^1 \) is a suitable space for the Lipschitz condition (2.3). Hence, Theorem 3.2 is applicable and provides real nonlinear eigenvalues \( \mu \) with \( PT \)-symmetric nonlinear solutions \( \psi \) (2.7). Observe that \( A \) fits into the setting of Example A.2 with \( V_1(x) = ixI, v_2 = 0 \), and \( \mathcal{Y} = H^1_{0,\text{Dir}}((-1,1)) = H^1_0((-1,1)). \) The nonlinearities are discussed in Examples A.7, A.9 and 3.1 and it is shown that \( H^1 \) is a suitable space for the Lipschitz condition (2.3). Hence, Theorem 3.2 is applicable and provides real nonlinear eigenvalues \( \mu \) with \( PT \)-symmetric nonlinear solutions \( \psi \).

V. NUMERICAL EXAMPLES

We analyze numerically problem (4.2) in \( L^2(\mathbb{R}^2) \) with two potentials \( V \). Our choice of \( d = 2 \) rather than the numerically simpler \( d = 1 \) allows for the investigation of partial \( PT \)-symmetries as well as the interplay between antilinear and linear symmetries in a single problem. The numerics are performed using the package \texttt{pde2path}\textsuperscript{12,13} which employs linear finite elements for the discretization, Newton’s iteration for the computation of nonlinear solutions, and an arclength continuation of solution branches. The free complex phase of the solution was fixed by forcing \( \text{Im}(\psi(x_0)) = 0 \) at a selected point \( x_0 \) within the computational domain. For the plots we take \( x_0 = (0,0) \) in the \( PT \)-symmetric case. The numerical grid is selected symmetric with respect to both coordinate axes as well as with respect to the reflection \( x \rightarrow -x \). This is crucial for recovering symmetries of eigenfunctions and realness of eigenvalues.
A. $\mathcal{PT}$ and $\mathcal{PT}_1$-symmetric $V$

Here we consider (4.2) in $L^2(\mathbb{R}^2)$ with

$$V(x_1,x_2) = \frac{1}{2}(x_1^2 + x_2^2) + i\gamma x_1 \frac{2}{x_1^2 + x_2^2 + 2},$$

(5.1)

and the norm condition $\|\psi\|_{L^2} = 1$.

Clearly, the problem is $\mathcal{PT}$ and $\mathcal{PT}_1$-symmetric (it has also the linear $\mathcal{P}_2$ symmetry). We choose the computational domain $x \in [-8,8]^2$ with homogeneous Dirichlet boundary conditions and the mesh of $2 \times 80^2 = 12800$ isosceles right triangles of equal size. The first four eigenfunctions are well localized within the selected domain.

For $\gamma = 0$, the eigenvalues of $A$ are known explicitly: $\lambda_k = \sqrt{2}k$, $k \in \mathbb{N}$, and the multiplicity of $\lambda_k$ is $k$. Enumerating the eigenvalues including their multiplicity, we obtain our eigenvalues $\mu_n$ for $\varepsilon = \gamma = 0$. For $\gamma = 2$, the numerically obtained first four eigenvalues (for $\varepsilon = 0$) are $\mu_1 \approx 2.096$, $\mu_2 \approx 2.583$, $\mu_3 \approx 3.155$, and $\mu_4 \approx 4.256$, and they are all simple.

In Fig. 1(a) the bifurcation diagram in $\varepsilon$ for $\mu_1, \mu_2, \mu_3$, and $\mu_4$ is plotted for $\gamma = 2$. The numerics suggest that all the four simple eigenvalues (as well as the eigenvalue family 3b bifurcating from the branch 3) remain real for at least $\varepsilon \leq 10$. Note that the secondary bifurcation branches in this and following diagrams are purely numerical observations and are not described by our analytical results.

In Figs. 1(b) and 1(c) we perform continuation in the parameter $\gamma$ for $\varepsilon = 2$. The results are qualitatively similar to those in 1D from Ref. 10. When two real eigenvalues collide, they leave the real axis and become a complex conjugate pair. In addition, however, a secondary bifurcation can occur, like, e.g., from $\mu_3$ at $\varepsilon \approx 1.5$ (see the inset in Fig. 1(b)).

The nonlinear eigenfunctions for $\gamma = 2$ at the labeled points in Fig. 1(a) are plotted in Fig. 2. Eigenfunctions 1 and 2 satisfy all the three symmetries, i.e., $\mathcal{PT}$, $\mathcal{PT}_1$, as well as the linear $\mathcal{P}_2$.
symmetry $\psi(x_1, -x_2) = \pm \psi(x_1, x_2)$. Eigenfunctions 3 and 4 satisfy the linear (anti)symmetry and can be chosen either $\mathcal{P}\mathcal{T}$- or $\mathcal{P}_1\mathcal{T}$-symmetric after a suitable multiplication by $i$. The eigenfunction 3b on the dotted branch (bifurcating from the primary branch) is only $\mathcal{P}_1\mathcal{T}$-symmetric.

For the nonlinear eigenfunctions at $\varepsilon = 2$ in the $\gamma$-bifurcation diagram in Fig. 1(b) the following symmetries hold. The real branches 1-4 have the expected symmetries ($\mathcal{P}\mathcal{T}$, $\mathcal{P}_1\mathcal{T}$, $\mathcal{P}_2$). The complex branches 5 and 6 are only linearly ($\mathcal{P}_2$) symmetric. The secondary bifurcation branch (with a real eigenvalue) has only the $\mathcal{P}_1\mathcal{T}$ symmetry.

**B. $\mathcal{P}_1\mathcal{T}$-symmetric $V$**

In this second example we take (4.2) in $L^2(\mathbb{R}^2)$ with the $\mathcal{P}_1\mathcal{T}$-symmetric (but not $\mathcal{P}\mathcal{T}$-symmetric) potential,

$$V(x_1, x_2) = -(e^{-(x_1-a)^2} + e^{-(x_1+a)^2}) \left[ (3 + 2i\gamma)e^{-(x_2-a)^2} + (2 + i\gamma)e^{-(x_2+a)^2} \right],$$  \hspace{1cm} (5.2)

with $\gamma, a \in \mathbb{R}$, studied also in Ref. 44. We choose $a = 3/2$ and $\gamma = 0.1$. At least the first four linear eigenvalues appear to be simple numerically. The discretization is like in Sec. V A except the domain is $x \in [-13, 13]^2$ with 14,400 triangles.

We present in Fig. 3 only the $\varepsilon$-bifurcation diagram (the $\gamma$-diagram is qualitatively similar to that in Sec. V A). Again, for $\varepsilon$ small the nonlinear eigenvalues stay real but later secondary bifurcations result in complex conjugate pairs of eigenvalues. In Fig. 4, we plot the six eigenfunctions labeled in Fig. 3(a). Clearly, all the four eigenfunctions on the primary branches (labels 1–4) with real eigenvalues are $\mathcal{P}_1\mathcal{T}$-symmetric. The bifurcating solutions (labels 1b and 3b) are asymmetric.
For complex conjugate pairs we plot only one eigenfunction as the two can be chosen to be related by \( \psi_2(x) = (P_T \psi_1)(x) \).

**ACKNOWLEDGMENTS**

P.S. thanks G. Wunner and H. Cartarius for drawing his attention to this topic. The research of P.S. is supported by the Swiss National Foundation, SNF Ambizione Gigrant No. PZ00P2_154786. The research of T.D. is partly supported by the German Research Foundation, DFG Grant No. DO1467/3-1.

**APPENDIX: SCHröDINGER AND DIRAC OPERATORS AND SOME NONLINEARITIES SATISFYING ASSUMPTION 1**

We collect some known facts, both classical and recent ones, on Schrödinger and Dirac operators as well as on some non-linearities and explain that they satisfy Assumption 1. The selection is inspired by various physical models from the literature, cf., Section IV. The space \( \mathcal{H}_0^1(\Omega) \) is as in (4.1).

1. **Schrödinger operators and the space \( \mathcal{Y} \) in Assumption 1(a)**

**Example A.1** (Schrödinger operators with complex potentials, \( n = 1 \)). Let \( \mathcal{H} = L^2(\mathbb{R}^d) \), \( V_1 \in W^{1, \infty}_{\text{loc}}(\mathbb{R}^d) \), and \( V_2 \in L^2_{\text{loc}}(\mathbb{R}^d) \). Let \( V_1 \) and \( V_2 \) satisfy further:

(i) Re \( V_1 \geq 0 \) and \( |\nabla V_1| \leq C_1|V_1| + C_2 \) with some \( C_1, C_2 > 0 \),

(ii) there exist \( \alpha \in [0, 1) \), \( \beta \geq 0 \) such that, for all \( \psi \in \mathcal{H}^2_{V_1}(\mathbb{R}^d) \),

\[
\|V_2\psi\| \leq \alpha (\|\Delta \psi\| + \|V_1\psi\|) + \beta \|\psi\|.
\]

Then the operator \( A := -\Delta + V_1 + V_2 \), Dom \( (A) := \mathcal{H}^2_{V_1}(\mathbb{R}^d) \) and the space \( \mathcal{Y} = \mathcal{H}^2_{V_1}(\mathbb{R}^d) \) satisfy Assumption 1(a) with \( n = 1 \). The main step needed to justify (2.1) is the fact that, for all \( \psi \in \text{Dom}(A) \),

\[
c_1\|\psi\|^2_{\mathcal{H}^2_{V_1}} \leq \|A\psi\|^2 + \|\psi\|^2 \leq c_2\|\psi\|^2_{\mathcal{H}^2_{V_1}},
\]

where \( c_1, c_2 > 0 \) are independent of \( \psi \), cf., for instance Ref. 3 for details; in fact, (A1) holds also if \( \mathbb{R}^d \) is replaced by a general open domain \( \Omega \subset \mathbb{R}^d \), Dirichlet boundary conditions are imposed at \( \partial \Omega \), the second condition in (i) is relaxed to \( |\nabla V_1| \leq \varepsilon |V_1|^2 + C_\varepsilon \) for every \( \varepsilon > 0 \) and a magnetic field is added, see Ref. 29.

**Example A.2** (Schrödinger operators with singular potentials, \( n = 1 \)). Let \( \mathcal{H} = L^2((-r, r)) \) with \( r \in (0, +\infty] \), \( V_1 \in L^1_{\text{loc}}((-r, r)) \) and \( v_2 \) be a sesquilinear form.

Let \( V_1 \) and \( v_2 \) further satisfy:

(i) \( \text{Re} V_1 > 0 \) and \( |\text{Im} V_1| < \tan \theta \text{Re} V_1 \) with \( \theta \in [0, \pi/2) \),

(ii) \( \mathcal{H}^1_{\sqrt{\text{Re} V_1}}((-r, r)) \subset \text{Dom}(v_2) \) and there exist \( \alpha \in [0, 1) \), \( \beta \geq 0 \) such that, for all \( \psi \in \mathcal{H}^1_{\sqrt{\text{Re} V_1}}((-r, r)) \),

\[
|v_2(\psi)| \leq \alpha (\|\psi\|^2 + \|\sqrt{\text{Re} V_1} \psi\|^2) + \beta \|\psi\|^2.
\]

To check that condition (A2) is satisfied, one can follow the same procedure, e.g., in Example 4.1, use the following inequality valid for every \( \varepsilon > 0 \):

\[
\|\psi\|^2_2 \leq C\|\psi\|^2_{\mathcal{H}^1} \leq \varepsilon \|\psi\|^2_{\mathcal{H}^1} + C(\varepsilon)\|\psi\|^2.
\]

The \( m \)-sectorial operator \( A \) associated (via the first representation theorem [Ref. 23, Theorem VI.2.1]) with the closed sectorial form

\[
a[\psi] := \|\psi\|^2 + \int_{-r}^r V_1(x)|\psi(x)|^2 \, dx + v_2(\psi), \quad \text{Dom}(a) := \mathcal{H}^1_{\sqrt{\text{Re} V_1}}((-r, r)),
\]

and the space \( \mathcal{Y} = \mathcal{H}^1_{\sqrt{\text{Re} V_1}}((-r, r)) \) satisfy Assumption 1(a) with \( n = 1 \).
The domain of \( a \) and the space \( \mathcal{Y} \) can be selected also, for instance, as
\[
\mathcal{H}^{1,\text{D}}_{\sqrt{\text{Re} V_1}} := H^1_0((-r,r)) \cap \text{Dom}(\sqrt{\text{Re} V_1}),
\]
i.e., Dirichlet boundary conditions are imposed at \( \pm r \), and the analogues of all claims above remain true.

**Example A.3 (Schrödinger operators with bounded and regular potentials, \( n > 1 \)).** We present an example of Schrödinger operators satisfying Assumption 1(a) for \( n > 1 \). The motivation for \( n > 1 \) comes from the condition \( s > d/2 \) for \( H^s(\mathbb{R}^d) \) needed for polynomial nonlinearities in Example A.7 below, guaranteeing that the polynomial nonlinearities satisfy Assumption 1(c).

Let \( \mathcal{H} = L^2(\mathbb{R}^d) \) and let \( V \in W^{2(m-1),\infty}(\mathbb{R}^d) \) for some \( m \in \mathbb{N} \) be a possibly complex potential. Then the operator
\[
A := -\Delta + V, \quad \text{Dom}(A) := H^2(\mathbb{R}^d),
\]
and the space and \( \mathcal{Y} = \mathcal{H}^{2n}(\mathbb{R}^d) = H^{2n}(\mathbb{R}^d) \) satisfy Assumption 1(a) with any \( n \leq m \). The claim holds since \( \text{Dom}(A^n) = H^{2n}(\mathbb{R}^d) \) and, for all \( \psi \in H^{2n}(\mathbb{R}^d) \),
\[
c_1\|\psi\|_{H^{2n}}^2 \leq \|A^n\psi\|^2 + \|\psi\|^2 \leq c_2\|\psi\|_{H^{2n}}^2,
\]
where \( c_1, c_2 > 0 \). The second inequality above follows from \( V \in W^{2(m-1),\infty}(\mathbb{R}^d) \) since \( A^n\psi \) consists of terms \( \pm \Delta^i V/j!\Delta^j V/\partial x^j \cdots \Delta^n V/\partial x^n \psi \) with \( i, j, k \in \{0,1\} \) for all \( k = 1, \ldots, n \) and such that the highest derivative acting on \( V \) is of order \( 2(n-1) \). Each term can be thus estimated by \( c\|\psi\|_{H^{2n}} \). Remaining estimates follow from the equivalence of norms on \( H^{2n}(\mathbb{R}^d) \), see, e.g., Ref. 11, Lemma 3.7.2.

**Example A.4 (First order Dirac type operator).** An example of a physically interesting operator other than a Schrödinger one is
\[
A := \begin{pmatrix} -i\partial_x - V(x) & -\kappa(x) \\ -\kappa(x) & i\partial_x - V(x) \end{pmatrix},
\]
(A4)
with \( V, \kappa \in L^\infty(\mathbb{R}) \). This operator occurs in a model for optical waves in fibers with a Bragg grating and a localized defect, see also Section IV C. Assumption 1(a) is satisfied under the natural choice \( \mathcal{H} = L^2(\mathbb{R}) \times L^2(\mathbb{R}), \mathcal{Y} = \text{Dom}(A) = H^1(\mathbb{R}) \times H^1(\mathbb{R}) \), where \( \|\psi\|_{\mathcal{Y}} := \|\psi_1\|_{H^1} + \|\psi_2\|_{H^1} \).

2. **The spectral condition (Assumption 1(b))**

The presence of simple isolated eigenvalues can be often justified by perturbation results for holomorphic families. We also briefly summarize basic spectral results on Schrödinger and Dirac operators.

**a. Holomorphic families of operators**

Standard results on the spectrum of a holomorphic family of operators \( A(\gamma) \), \( \gamma \in \mathbb{C} \), cf., Ref. 23, Chap. VII, yield that if the spectrum \( \sigma(0) \) is separated into two parts, then this remains true also for \( \sigma(\gamma) \) with \( |\gamma| \) sufficiently small, cf., Ref. 23, Theorem VII.1.7. Moreover, isolated eigenvalues with finite multiplicities depend analytically on \( \gamma \) and their algebraic multiplicities are preserved, cf., Ref. 23, Theorem VII.1.8. Criteria for the holomorphicity of an operator family can be found in Ref. 23, Chap. VII, Theorems VII.2.6 and VII.4.8. A sufficient condition for the analyticity of operators or quadratic forms of the type
\[
A(\gamma) = A_0 + \gamma B, \quad a(\gamma) = a_0 + \gamma b, \quad \gamma \in \mathbb{C},
\]
where \( A_0 \) is a densely defined closable operator and \( a_0 \) is a densely defined closable sectorial form, is the relative boundedness of \( B \) or \( b \) with respect to \( A_0 \) or \( \text{Re} a_0 \), respectively, i.e., with some \( \alpha, \beta \geq 0 \),
\[
\begin{align*}
A(\gamma) : \text{Dom}(A_0) & \subset \text{Dom}(B), \quad \|B\psi\| \leq \alpha\|A_0\psi\| + \beta\|\psi\|, \quad \psi \in \text{Dom}(A_0), \\
a(\gamma) : \text{Dom}(a_0) & \subset \text{Dom}(b), \quad |b(\psi)|^2 \leq \alpha\text{Re} a_0|\psi| + \beta\|\psi\|^2, \quad \psi \in \text{Dom}(a_0).
\end{align*}
\]

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b. Basic spectral properties of C-symmetric operators and stability of real simple eigenvalues

First we recall a simple lemma on C-symmetric operators; the proof is straightforward, see, e.g., Refs. 5 and 31 for some details.

**Lemma A.5.** Let A be a closed operator possessing a C-symmetry, cf., Assumption 2, and let \( \mu_0 \) be an eigenvalue of A. Then

(i) \( \overline{\mu_0} \) is an eigenvalue of A and if there is a C-symmetric eigenvector \( \psi_0 \) corresponding to \( \mu_0 \), i.e.,

\[
(A - \mu_0)\psi_0 = 0, \quad C\psi_0 = \psi_0,
\]

then \( \mu_0 \in \mathbb{R} \). Moreover, if \( \mu_0 \) is real and simple, then the corresponding eigenvector can be chosen C-symmetric.

(ii) if \( \mu_0 \) is isolated simple and real, then, in addition, the spectral (Riesz) projection \( P_0 \) of A corresponding to \( \mu_0 \) commutes with C, i.e.,

\[
P_0C = CP_0.
\]

**Remark A.6 (Stability of real simple eigenvalues).** For a C-symmetric family \( A(\gamma) \), cf., Subsection 2 a of the Appendix, isolated simple real eigenvalues of \( A(0) \) remain isolated simple and real for small \( |\gamma| \) since eigenvalues are analytic in \( \gamma \) and always form complex conjugated pairs, cf., Lemma A.5(i). This basic fact from perturbation theory for eigenvalues was used for linear \( PT \)-symmetric problems, e.g., in Refs. 5 and 31.

c. Spectra of Schrödinger operators

We consider Schrödinger operators in the setting of Examples A.1–A.3; many of the following spectral properties are valid in a much greater generality, cf., Ref. 15 for instance.

Let \( A \) be the operator from Example A.1 and let \( \lim_{|x| \to \infty} |V_1(x)| = \infty \). Then the resolvent of \( A \) is compact, hence \( \sigma(A) = \sigma_{\text{ess}}(A) \). Moreover, in the selfadjoint setting, the ground state is simple, cf., Ref. 37, Theorem XIII.47 and XIII.48. For \( d = 1 \), a Wronskian argument can be used to conclude the simplicity of all eigenvalues for (real) single-well potentials like \( A_0 = -\partial_x^2 + |x|^\beta \), \( \beta > 0 \). For the singular Schrödinger operators from Example A.2, the resolvent is compact if \( r < \infty \) (also if Dirichlet boundary conditions are considered) or if \( r = \infty \) and \( \lim_{|x| \to \infty} |V_1(x)| = \infty \).

Let \( A \) be the Schrödinger operator from Example A.1 with \( V = V_1 + V_2 \in L^\infty(\mathbb{R}^d) \) and \( \lim_{|x| \to \infty} (V_1 + V_2)(x) = 0 \), then \( \sigma_{\text{ess}}(A) = [0, +\infty) \), cf., Ref. 15, Corollary X.4.2, Example X.4.3 and, e.g., Ref. 37, Example XIII.4.6. (Note that there are several different definitions of the essential spectrum for non-selfadjoint operators, cf., Ref. 15, Chap. IX, nevertheless, all coincide for this special situation.) Discrete eigenvalues may appear outside the essential spectrum. Particularly in \( d = 1, 2 \), the selfadjoint operator \( A_0 := -\Delta + \varepsilon Q \) with \( \int Q \ dx < 0 \) and \( Q \) decaying sufficiently fast, cf., Refs. 41 and 26 for precise assumptions on \( Q \), possesses a unique negative simple eigenvalue for all sufficiently small \( \varepsilon > 0 \); some non-selfadjoint extensions can be found in Ref. 35.

d. Spectrum of the Dirac type operator (A4)

If \( \kappa(x) \to \kappa_\infty > 0 \) and \( V(x) \to 0 \) as \( |x| \to \infty \), then the essential spectrum of \( A \) in (A4) is \( (-\infty, -\kappa_\infty) \cup \{ \kappa_\infty, \infty \} \). To see this, notice the perturbation result Ref. 8, Prop.6.6 and the unitary equivalence of \( A \) in (A4) and \( H \) in Ref. 8 for \( \kappa \to \infty \). For real \( \kappa \) and \( V \) the operator is self-adjoint. Bounds on eigenvalues of non-selfadjoint perturbations outside of the essential spectrum are proved in Ref. 8. Special choices of real \( \kappa \) and \( V \) with simple eigenvalues in \( (-\kappa_\infty, \kappa_\infty) \) have been found in Ref. 18, Sec. 4.B. Using a connection to the spectral problem of the inverse scattering theory for the nonlinear Schrödinger equation. For example, for \( \kappa(x) = (\mu_0^2 + k^2 \tanh^2(kx))^{1/2} \) and \( V(x) = k^2 \mu_0^2 k^{-2}(x) \text{sech}^2(kx) \) with \( \mu_0 \in (-\kappa_\infty, \kappa_\infty) \) and \( k \in \mathbb{R} \setminus \{0\} \), the operator \( A \) has a simple eigenvalue \( \mu_0 \).
3. Nonlinearities (Assumption 1(c))

We present several nonlinearities \( f \) satisfying Assumption 1(c). The considered spaces \( \mathcal{Y} \) are those arising for Schrödinger operators in Subsection 1 of the Appendix.

Example A.7 (Polynomial nonlinearity). Let \( \mathcal{H} = L^2(\Omega) \) and \( \mathcal{Y} \) be as in (4.1) with \( s > d/2, s \in \mathbb{N}, \) and \( \Omega = \mathbb{R}^d \). Let \( N \in \mathbb{N} \) and

\[
f_{\text{pol}}(\psi) := \sum_{p,q=0}^{N} a_{pq} \psi^{p} \overline{\psi}^{q} \quad \text{with} \quad a_{pq} \in C_b^0(\mathbb{R}^d, \mathbb{C}) \quad \text{for all} \quad p, q,
\]

where \( C_b^0 \) is the space of functions with continuous and bounded derivatives up to order \( s \). A classical example is the cubic nonlinearity \( |\psi|^2 \psi \), i.e., \( a_{21} = 1, a_{pq} = 0 \) otherwise. We indicate below how one can prove that \( f_{\text{pol}} \) in (A6) satisfies Assumption 1(c) with any \( \eta_0 \in \mathcal{Y} = \mathcal{H}_Q^s(\mathbb{R}^d) \) and \( r_L > 0 \).

First note that, for \( s > d/2 \), the norm \( \| \cdot \|_{\mathcal{H}^s} \) satisfies the so-called algebra property: there exists \( C_a > 0 \) such that, for all \( \phi, \psi \in H^s(\mathbb{R}^d) \),

\[
\|\phi \psi\|_{\mathcal{H}^s} \leq C_a \|\phi\|_{\mathcal{H}^s} \|\psi\|_{\mathcal{H}^s},
\]

cf., Ref. 1, Theorem 4.39 or Ref. 14, Lemma 4.2. Moreover, the Sobolev embedding of \( H^s(\Omega) \) in \( L^\infty(\Omega) \) holds, cf., Ref. 1, Theorem 4.12, i.e., there exists \( C_s > 0 \) such that, for all \( \phi \in H^s(\mathbb{R}^d) \), we have \( \|\phi\|_{L^\infty} \leq C_s \|\phi\|_{H^s} \). Thus, for all \( \phi, \psi \in \mathcal{H}_Q^s(\mathbb{R}^d) \),

\[
\|\phi \psi\|_{\mathcal{H}_Q^s} \leq C_s \|\phi\|_{\mathcal{H}^s} \|\psi\|_{\mathcal{H}^s} + \sqrt{\|\phi\|_{L^\infty} \|\psi\|_{L^\infty}} \|Q\phi\| \quad \|Q\psi\|
\]

\[
\leq \max \left\{ C_a, \frac{C_s}{\sqrt{2}} \right\} \|\phi\|_{\mathcal{H}_Q^s} \|\psi\|_{\mathcal{H}_Q^s},
\]

hence the norm \( \| \cdot \|_{\mathcal{H}_Q^s} \) satisfies the algebra property as well.

Clearly, it suffices to check (2.3) for a single term \( \psi^p \overline{\psi}^q \). We give details only for \( p = 2, q = 0 \); the other cases are similar. Using (A7), we obtain

\[
\|\phi^2 - \psi^2\|_{\mathcal{H}_Q^s} \leq C_1 \left( \|\phi - \psi\|_{\mathcal{H}_Q^s} \|\phi + \psi\|_{\mathcal{H}_Q^s} \right) \leq C_2 \left( \|\phi\|_{\mathcal{H}_Q^s} + \|\psi\|_{\mathcal{H}_Q^s} \right) \|\phi - \psi\|_{\mathcal{H}_Q^s},
\]

for all \( \phi, \psi \in \{ \eta \in \mathcal{H}_Q^s(\mathbb{R}^d) : \|\eta - \eta_0\|_{\mathcal{H}^s} < r_L \} \) with any \( \eta_0 \in \mathcal{H}_Q^s(\mathbb{R}^d) \) and \( r_L > 0 \).

Finally, the validity of (2.3) in Assumption 1(c) can be checked by noticing that for \( a \in C_b^0(\mathbb{R}^d) \) and \( \psi \in \mathcal{H}_Q^s(\mathbb{R}^d) \),

\[
\|a \psi\|_{\mathcal{H}_Q^s} \leq \|a\|_{C^s} \|\psi\|_{\mathcal{H}^s} + \|a\|_{C^0} \|Q\psi\| \leq \|a\|_{C^s} \|\psi\|_{\mathcal{H}^s},
\]

Remark A.8 (Motivation for \( n > 1 \) in Assumption 1(a)). Examples A.3 and A.7 constitute the primary motivation for \( n > 1 \) in the choice of a space \( \mathcal{Y} \) with \( \text{Dom}(A^n) \subset \mathcal{Y} \subset \text{Dom}(A^{n-1}) \). The condition \( \mathbb{N} \ni s > d/2 \) in Example A.7 implies that for \( d \geq 4 \) we need \( s \geq 3 \). The natural \( H^2 \)-space of Example A.1 is, therefore, not suitable as our working space. On the other hand, the space \( H^{2n} \) of Example A.3 with \( n > d/4 \) is sufficiently small.

In summary, if we choose

\[
d \leq \frac{4}{3}, \quad \frac{d}{2} \leq n \leq m \quad \text{with} \quad s, n, m \in \mathbb{N},
\]

then \( A \) in (A3) with \( V \in W^{2(m-1),\infty}(\mathbb{R}^d), f = f_{\text{pol}}, \) and \( \mathcal{Y} = H^{2n}(\mathbb{R}^d) \) satisfies Assumptions 1(a) and 1(c). For \( d = 1,2 \) inequality (A8) holds also with \( s = 2, n = 1, \) such that \( \mathcal{Y} = \mathcal{H}_Q^1(\mathbb{R}^d) \) of Example A.1 can be used with \( f = f_{\text{pol}} \). For \( d = 1 \) we can use even \( s = 1, n = 1 \) and, hence, the spaces \( \mathcal{H}^{1,\text{D}}_{\text{Re} V_1}((-r,r)), \mathcal{H}^{1,\text{D}}_{\text{Re} V_1}((-r,r)) \) of Example A.2 are admissible.

Example A.9 (Nonlocal nonlinearity from Ref. 39). Let \( \mathcal{H} = L^2((-1,1)), \mathcal{Y} = \mathcal{H}_Q^{1,\text{D}}((-1,1)) = H_0^1((-1,1)) \) and \( (f_\psi(\psi))(x) := \psi \int_0^1 \text{Im} \left( \psi(s) \overline{\psi}_s(s) \right) \, ds \). The validity of (2.3) is easily checked since
f_N satisfies
\[
\left\| f_N(\phi) - f_N(\psi) \right\| = \frac{1}{2} \left\| (\phi - \psi) \int_0^x \text{Im} \left( \phi \bar{\psi}_x + \psi \bar{\phi}_x \right) ds + (\phi + \psi) \int_0^x \text{Im} \left( \phi \bar{\phi}_x - \psi \bar{\psi}_x \right) ds \right\|
\]
\[
\leq \frac{1}{2} \left( \| \phi - \psi \|_{L^\infty} \left\| \int_0^x \text{Im} \left( \phi \bar{\phi}_x + \psi \bar{\psi}_x \right) ds \right\| + \| \phi + \psi \|_{L^\infty} \left\| \int_0^x \text{Im} \left( \phi \bar{\phi}_x - \psi \bar{\psi}_x \right) ds \right\| \right)
\]
For any \( \eta_0 \in H_0^1((-1, 1)), r_L > 0, \) and \( \phi, \psi \in \{ \eta \in H_0^1((-1, 1)) : \| \eta - \eta_0 \|_{H^1} < r_L \}, \) we have
\[
\left\| \int_0^x \text{Im} \left( \phi \bar{\phi}_x + \psi \bar{\psi}_x \right) ds \right\| \leq \left\| \| \phi \| \| \phi_x \| + \| \psi \| \| \psi_x \| \right\| \leq \frac{1}{\sqrt{2}} \left( \| \phi \|_{H^1}^2 + \| \psi \|_{H^1}^2 \right)
\]
\[
\left\| \int_0^x \text{Im} \left( \phi \bar{\phi}_x - \psi \bar{\psi}_x \right) ds \right\| \leq \frac{1}{2} \left( \left\| \phi - \psi \right\| \left\| \phi_x + \psi_x \right\| + \left\| \phi + \psi \right\| \left\| \phi_x - \psi_x \right\| \right)
\]
\[
\leq \sqrt{2} \left( \| \phi \|_{H^1} \| \phi - \psi \|_{H^1} \right)
\]
\[
\leq 2 \sqrt{2} \left( r_L + \| \eta_0 \|_{H^1} \right) \left( \| \phi - \psi \|_{H^1} \right)
\]
Thus, by the embedding of \( H^1((-1, 1)) \) in \( L^\infty((-1, 1)) \) as in Example A.7, we obtain (2.3). In summary, Assumption I(c) holds with \( n = 1. \)


