APPROXIMATIONS OF SPECTRA OF SCHRÖDINGER OPERATORS WITH COMPLEX POTENTIALS ON $\mathbb{R}^d$

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ABSTRACT. We study spectral approximations of Schrödinger operators $T = -\Delta + Q$ with complex potentials on $\Omega = \mathbb{R}^d$, or exterior domains $\Omega \subset \mathbb{R}^d$, by domain truncation. Our weak assumptions cover wide classes of potentials Q for which T has discrete spectrum, of approximating domains Ω_n, and of boundary conditions on $\partial\Omega_n$ such as mixed Dirichlet/Robin type. In particular, $\operatorname{Re} Q$ need not be bounded from below and Q may be singular. We prove generalized norm resolvent convergence and spectral exactness, *i.e.* approximation of *all* eigenvalues of T by those of the truncated operators T_n *without* spectral pollution. Moreover, we estimate the eigenvalue convergence rate and prove convergence of pseudospectra. Numerical computations for several examples, such as complex harmonic and cubic oscillators for $d = 1, 2, 3$, illustrate our results.

1. Introduction

Although domain truncation is one of the most commonly used techniques for approximating partial differential operators on unbounded domains, it is a major challenge to guarantee its reliability, even if the spectrum is purely discrete. Not only may the approximation produce *spurious limits* that are no true eigenvalues. It may also happen that some true eigenvalues are *not approximated*, in particular for non-selfadjoint operators. While very recent research and applications show that there is particular interest in Schrödinger operators on unbounded domains with complex potentials, *cf. e.g.* [23, 15, 25, 3, 34, 10, 4, 57], there are no general spectral convergence results for domain truncation for this basic class of operators.

In the present paper we fill this gap and prove *spectral exactness*, *i.e.* the absence of the two unwanted phenomena described above, for wide classes of Schrödinger operators $T = -\Delta + Q$ in $L^2(\Omega, \mathbb{C})$ where Ω is $\mathbb{R}^d$ or an exterior domain in $\mathbb{R}^d$. Our assumptions on the potential, the domains Ω_n approximating Ω, and the conditions on the artificial boundaries $\partial\Omega_n$ are very weak. For the complex-valued potential Q we only require $|Q(x)| \to \infty$ as $|x| \to \infty$ and some mild assumptions guaranteeing that T has discrete spectrum; in particular, $\operatorname{Re} Q$ need not be bounded from below and Q may be singular. For the approximating operators $T_n = -\Delta + Q$ in $L^2(\Omega_n, \mathbb{C})$ we require no regularity of the bounded domains Ω_n exhausting Ω as $n \to \infty$ for Dirichlet conditions on $\partial\Omega_n$, and only low regularity for mixed Dirichlet-Robin conditions. Moreover, we establish estimates for the convergence rate of the approximate eigenvalues and convergence of pseudospectra. Our abstract results are illustrated by numerical computations for several examples of different potentials, dimensions, domains, and boundary conditions.

The notion of *spectral exactness* was first introduced in [6] for regular approximations of singular selfadjoint Sturm-Liouville problems by interval truncation. It

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means that a sequence of approximating operators $\{T_n\}_n$ has the following two properties, *cf. e.g.* [15]:

i) *spectral inclusion*: for every eigenvalue $\lambda \in \sigma(T)$ there exist $\lambda_n \in \sigma(T_n)$, $n \in \mathbb{N}$, with $\lambda_n \to \lambda$ as $n \to \infty$;
ii) *no spectral pollution*: if there exists a sequence of eigenvalues $\lambda_n \in \sigma(T_n)$, $n \in \mathbb{N}$, with an accumulation point $\lambda \in \mathbb{C}$, then $\lambda \in \sigma(T)$.

For partial differential operators, results on spectral exactness in the literature are fragmented. Even in the case of Schrödinger operators, explicit proofs of spectral exactness are either confined to selfadjoint or elliptic problems, in both cases restricted to potentials with real part bounded from below, *cf.* [44, 14, 29] and references therein, or they cover only the one-dimensional case, *cf.* [21], or they concern Galerkin approximations, *cf.* [34]. Spectral exactness for domain truncation of *non-selfadjoint* differential operators was studied *e.g.* in [15, 16, 17], where tests for spectral exactness in terms of boundary conditions were developed. However, the verification of the assumptions therein proved to be difficult and sometimes impossible, *cf.* [17, Ex. 1]. Our new result yields spectral exactness also for this previously debated example, *cf.* Subsection 7.3.

In general, spectral exactness is a major challenge for non-selfadjoint problems. In the selfadjoint case, it is well-known that generalized strong resolvent convergence implies spectral inclusion, and if the resolvents converge even in norm, then spectral exactness prevails, *cf.* [60, Thm. 9.24 a), 9.26 b)] and also [61] for a survey on related results. Here "generalized" refers to the fact that the resolvents $(T_n - \lambda)^{-1}$ and $(T - \lambda)^{-1}$ do not act in the same space. In the non-selfadjoint case, norm resolvent convergence excludes spectral pollution, *cf.* [37, Sec. IV.3.1]; however, the approximation need not be spectrally inclusive, *cf.* [37, Ex. IV.3.8]. Moreover, in general, generalized strong resolvent convergence is not enough to guarantee spectral exactness even if all operators have compact resolvents, *cf.* the Galerkin approximation in [13, Ex. 5] where a spurious eigenvalue was proved to exist.

In this paper we establish spectral exactness by proving generalized norm resolvent convergence of T_n to $T = -\Delta + Q$ in $\mathbb{R}^d$, or in exterior domains in $\mathbb{R}^d$, for domain truncation. Striving for minimal assumptions on the potential Q, we exploit the interplay between the different parts of the potential Q if we decompose it as

$$Q = Q_0 - U + W$$

where Q_0 with $\operatorname{Re} Q_0 \geq 0$ is the "regular" part, $-U \leq 0$ is the "non-positive" part, and W is the "singular" part. More precisely, the required regularity of Q_0, and the way how we introduce the operators T and T_n, depend on the sectoriality angle θ of $Q_0 - U$:

I. If $\theta < \pi/2$, which requires $U \equiv 0$, we can allow for potentials with lower regularity and we use sectorial form techniques to introduce T and T_n, *cf.* Assumption I;
II. If $\theta \geq \pi/2$, where $\operatorname{Re} Q$ need not be bounded from below, we require more regularity and we use perturbation theory for m-accretive operator to introduce T and T_n, *cf.* Assumption II.

In both cases, the resulting operators T and T_n are quasi-sectorial in semigroup-sense, *cf.* [33, Sec. 2.8], and they coincide if both Assumptions I and II are satisfied. We emphasize that the formulation of our results is independent of the assumption that is satisfied.

The wide applicability of our abstract spectral convergence results, Theorem 5.1 and Theorem 6.1, may be seen from the following two one-dimensional examples illustrating the difference between the two different assumptions:

$$\theta < \pi/2, \text{ Assumption I}: \qquad Q(x) = (1+\mathrm{i})x^2 + \mathrm{i}\delta(x), \tag{1.1}$$

$$\theta \geq \pi/2, \text{ Assumption II}: \qquad Q(x) = \mathrm{i}x^3 - x^2 + \mathrm{i}|x|^{-\frac{1}{4}}, \tag{1.2}$$

as well as from the diversity of the examples for which we provide numerical computations that are backed up by our main results.

These examples include the one-dimensional Airy operator ($Q(x) = \mathrm{i}x$) and the imaginary cubic oscillator ($Q(x) = \mathrm{i}x^3$) which we truncate to finite intervals with Dirichlet, Neumann or Robin conditions. Both potentials satisfy Assumption II. While for the Airy operator it is known that the spectrum is empty, the imaginary cubic oscillator has non-empty spectrum which is real but not known in closed form. In both examples we observe, for increasing interval length, eigenvalues bifurcating from real to complex values that diverge eventually in complex conjugate pairs. This phenomenon is typical for problems that are selfadjoint in a Krein space, see *e.g.* [42, 43], and is also called PT-symmetry breaking/phase transition, see *e.g.* [9] and [19]. For the imaginary cubic oscillator this effect occurs only for some eigenvalue branches, while other branches remain real and converge to the true eigenvalues. For the Airy operator it occurs for all eigenvalue branches so that no finite limit points exist, thus leaving the spectrum of the Airy operator empty.

The three-dimensional harmonic oscillator ($Q(x) = |x|^2$) for which spectral exactness of eigenvalue approximations with Dirichlet conditions follow from classical results for selfadjoint operators, *cf.* [58, Thm. 4.5, 3.2], clearly satisfies Assumption I. Here our computations exemplify the effect of truncation to different subdomains. For cubes and balls in $\mathbb{R}^3$ one obtains different multiplicities of eigenvalue curves, while preserving the total multiplicity of the limiting eigenvalue. For the complex rotated oscillator on an exterior domain in $\mathbb{R}^2$ studied in [17], which satisfies the sectorial Assumption I, our theoretical results finally establish spectral exactness, which was not known until now.

The paper is organized as follows. In Section 2, we establish the two different sets of assumptions on the potential Q, introduce the operator $T = -\Delta + Q$ in $L^2\left(\mathbb{R}^d, \mathbb{C}\right)$ in two different ways, and provide the necessary results on the operator domain, graph norm, and resolvent estimates for T in both cases. In Section 3, we establish the assumptions on the truncated domains Ω_n and the boundary conditions on the artificial boundary $\partial\Omega_n$, introduce the corresponding approximating operators T_n, and study their properties. In particular, we show that the sequence $\{T_n\}_n$ is uniformly quasi-sectorial, *cf.* [33, Sec. 2.1], with semi-angle $< \pi/2$ in Case I and with $\geq \pi/2$ in Case II; moreover, in the latter case we derive uniform resolvent estimates in the complementary sector in the left half-plane. In Section 4, employing results on discretely or collectively compact approximations, *cf.* [53, 5, 47], we prove our main theorem on generalized norm resolvent convergence of T_n to T, *cf.* Theorem 4.1. In Section 5, we use this result to establish spectral exactness and estimates on the convergence rate of the approximate eigenvalues, *cf.* Theorems 5.1 and 5.2, as well as convergence of the pseudospectra of T_n to those of T in Attouch-Wets metric, which is a generalization of Hausdorff metric to unbounded subsets of $\mathbb{C}$, *cf.* Theorem 5.5. In Section 6, we show that all our theorems generalize to Schrödinger operators on exterior domains $\Omega \subset \mathbb{R}^d$ by sketching the necessary modifications in the assumptions and proofs. In the final Section 7, we illustrate the abstract results by numerical computations for several examples of different potentials Q, dimensions d, domains Ω, and boundary conditions on $\partial\Omega_n$, including complex cubic and harmonic oscillators.

Throughout this paper, we employ the following conventions. The Euclidean norm in $\mathbb{C}^d$ is denoted by $|\cdot|$, the corresponding scalar product by $\langle\cdot,\cdot\rangle_{\mathbb{C}^d}$, and the

Euclidean scalar product in $\mathbb{R}^d$ by a dot. A domain $\Omega \subset \mathbb{R}^d$ is an open connected subset; Ω is called exterior domain if $\mathbb{R}^d \setminus \Omega$ is compact. For a subset $\Omega \subset \mathbb{R}^d$, we tacitly view every function $f \in L^2(\Omega, \mathbb{C})$ as an element of $L^2(\mathbb{R}^d, \mathbb{C})$ by extending f by zero outside Ω; conversely, we view every $g \in L^2(\mathbb{R}^d, \mathbb{C})$ with $g \restriction \mathbb{R}^d \setminus \Omega = 0$ as an element of $L^2(\Omega, \mathbb{C})$. The norm and scalar product in $L^2\left(\mathbb{R}^d, \mathbb{C}\right)$ and $L^2\left(\Omega_n, \mathbb{C}\right)$ are denoted by $\|\cdot\|$, $\|\cdot\|_n$ and $\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_n$, respectively. All scalar products are linear in the first argument. Partial derivatives, always understood in the weak sense, are denoted by ∂_j and we systematically abbreviate $\langle \nabla f, \nabla g\rangle := \sum_{j=1}^d \langle \partial_j f, \partial_j g\rangle$, $\|\nabla f\| := \||\nabla f|\|$.

2. Schrödinger operators with complex potentials on $\mathbb{R}^d$

In this section, we establish mild criteria for Schrödinger operators $T = -\Delta + Q$ in $L^2\left(\mathbb{R}^d, \mathbb{C}\right)$ with complex-valued potential to have compact resolvent and to qualify for our main result on spectral exactness, *cf.* Assumption I or II. Our criteria allow for potentials Q of the form

$$Q = Q_0 - U + W, \quad \operatorname{Re} Q_0 \geq 0,\ U \geq 0,$$

with real part possibly unbounded from below ($U \not\equiv 0$) and with singular part ($W \not\equiv 0$). The assumptions and construction of the operator T are different for the case that $Q_0 - U$ is sectorial with semi-angle $\theta < \pi/2$ ($U \equiv 0$) or $\theta \geq \pi/2$ ($U \not\equiv 0$). The weaker sectoriality assumptions in the latter case necessitate more than the minimal regularity of Q_0 needed in the former case.

We remark that if Q satisfies both Assumptions I and II, then the operator T resulting in both cases is the same.

2.1. Semi-angle $\theta < \pi/2$.

We define the operator $T = -\Delta + Q$ through sectorial forms, *i.e.* via the first representation theorem, *cf.* [37, Thm. VI.2.1]. The potential Q is viewed as a form q that splits into two parts, $q = q_0 + w$.

The "regular" part q_0 is generated by $Q_0 \in L^1_{\mathrm{loc}}\left(\mathbb{R}^d, \mathbb{C}\right)$. The perturbation w is assumed to be bounded outside a ball $B_R(0)$ and $\|\nabla \cdot\|^2$-bounded in $L^2(B_R(0), \mathbb{C})$ as forms.

Since w need not be closable, also forms representing δ-like distributions comply with our assumptions.

Assumption I. The sesquilinear form q decomposes as $q = q_0 + w$ where q_0 and w have the following properties. The form q_0 is generated by $Q_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C})$, *i.e.*

$$q_0[\cdot] := \int_{\mathbb{R}^d} Q_0 |\cdot|^2 \,\mathrm{d}x, \quad \mathcal{D}(q_0) := \left\{ f \in L^2(\mathbb{R}^d, \mathbb{C}) \,:\, Q_0 |f|^2 \in L^1(\mathbb{R}^d, \mathbb{C}) \right\}, \tag{2.1}$$

such that

(I.i) *sectoriality of Q_0 with semi-angle $\theta < \pi/2$*: there exist $c_0 > 0$ and $\theta \in [0, \pi/2)$ with

$$\operatorname{Re} Q_0 \geq c_0, \quad |\operatorname{Im} Q_0| \leq \tan\theta \, \operatorname{Re} Q_0; \tag{2.2}$$

(I.ii) *unboundedness of Q_0 at infinity*:

$$|Q_0(x)| \to \infty \quad \text{as} \quad |x| \to \infty.$$

For the form w, there exist $R > r > 0$ and $\zeta \in C_0^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$ with

$$\operatorname{supp} \zeta \subset B_R(0), \quad 0 \leq \zeta \leq 1, \quad \zeta \restriction B_r(0) = 1,$$

and sesquilinear forms w_1, w_2 with $W_0^{1,2}(B_R(0), \mathbb{C}) \subset \mathcal{D}(w_1)$, $\mathcal{D}(w_2) = L^2(\mathbb{R}^d, \mathbb{C})$ with

$$\forall\, f \in \mathcal{D}(w): \quad \sqrt{\zeta} f \in W_0^{1,2}(B_R(0), \mathbb{C}), \quad w[f] = w_1[\sqrt{\zeta} f] + w_2[\sqrt{1-\zeta} f], \tag{2.3}$$

and such that

(I.iii) $\|\nabla\cdot\|^2$-boundedness of w_1 in $L^2(B_R(0),\mathbb{C})$: there exist $a_w\ge 0$, $b_w\in[0,1)$ so that, for every $f\in W_0^{1,2}(B_R(0),\mathbb{C})$,

$$|w_1[f]|\le a_w\|f\|^2+b_w\|\nabla f\|^2;$$

(I.iv) *boundedness of* w_2 *outside* $B_r(0)$: there exists $M_w\ge 0$ so that, for every $f\in L^2(\mathbb{R}^d,\mathbb{C})$,

$$|w_2[(1-\chi_r)f]|\le M_w\|f\|^2,$$

where χ_r is the characteristic function of $B_r(0)$.

The constants θ, c_0, a_w, b_w and M_w are the main characteristics of the potential Q. We highlight the dependence on w to keep track of terms caused by w in the proofs, thus allowing for straightforward simplifications if *e.g.* $w=0$.

Example 2.1. An example satisfying Assumption I in $d=1$ is given by $Q(x)=(1+\mathrm{i})x^2+\mathrm{i}\delta(x)$, *cf.* (1.1); here

$$\begin{aligned} q_0[f]&=(1+\mathrm{i})\|xf\|^2, &\mathcal{D}(q_0)&=\left\{f\in L^2(\mathbb{R},\mathbb{C})\,:\,x\mapsto xf(x)\in L^2(\mathbb{R},\mathbb{C})\right\},\\ w[f]&=\mathrm{i}|f(0)|^2, &\mathcal{D}(w)&=W^{1,2}(\mathbb{R},\mathbb{C}),\end{aligned}$$

and we can choose $w_2=0$ and b_w arbitrarily small since, by a well-known embedding inequality, there exists $C>0$ such that

$$\|f\|^2_{L^\infty(\mathbb{R})}\le C\|f\|_{W^{1,2}(\mathbb{R})}\|f\|,\quad f\in W^{1,2}(\mathbb{R},\mathbb{C}). \tag{2.4}$$

Remark 2.2. Assumption (I.i) can be weakened to

(I.i') *quasi-sectoriality of* Q_0 *with semi-angle* $\theta<\pi/2$ *and rotation angle* $\beta\in(-\pi/2,\pi/2)$: there exist $\beta\in(-\pi/2,\pi/2)$, $\mu\in\mathbb{C}$, and $\theta\in[0,\pi/2)$ with

$$|\arg(\mathrm{e}^{-\mathrm{i}\beta}(Q_0-\mu))|\le\theta.$$

Then all main results, *cf.* Theorems 4.1, 5.1, 5.2, and 5.5, continue to hold if

(I.iii') Assumption (I.iii) holds with $b_w\in[0,\cos\beta)$.

Note that (I.i), (I.iii) are the special case $\beta=0$, $\mu=0$ of (I.i'), (I.iii').

Proposition 2.3. *Let Assumption* I *be satisfied. Then*

i) *the form t given by*

$$t:=\|\nabla\cdot\|^2+q_0+w,\quad \mathcal{D}(t):=W^{1,2}\left(\mathbb{R}^d,\mathbb{C}\right)\cap\mathcal{D}(q_0),$$

is densely defined, closed, sectorial, and $C_0^\infty\left(\mathbb{R}^d,\mathbb{C}\right)$ *is a core of* t;

ii) *the m-sectorial operator T uniquely determined by t has compact resolvent.*

Proof. i) We write t in the form $t=t_0+w$ with

$$t_0:=\|\nabla\cdot\|^2+q_0,\quad \mathcal{D}(t_0):=\mathcal{D}(t).$$

By (2.2), for every $f\in\mathcal{D}(t)$,

$$|\operatorname{Im}t_0[f]|=|\operatorname{Im}q_0[f]|\le\tan\theta\,\operatorname{Re}t_0[f]. \tag{2.5}$$

Thus t_0 is sectorial and closed being the sum of two closed sectorial forms, *cf.* [37, Thm. VI.1.31]. The space $C_0^\infty\left(\mathbb{R}^d,\mathbb{C}\right)$ is a core of t_0 since it is a core of $\operatorname{Re}t_0$, *cf.* [22, Thm. 8.2.1] and [37, Thm. VI.1.21].

Let ζ be the function used in Assumption I. Note that $\|\zeta\|_\infty = 1$. By Assumption (I.iii), (I.iv), for every $f \in \mathcal{D}(t)$,

$$\begin{aligned} |w[f]| &\le |w_1[\sqrt{\zeta} f]| + |w_2[\sqrt{1-\zeta} f]| \\ &\le a_w \|\sqrt{\zeta} f\|^2 + b_w \|\nabla(\sqrt{\zeta} f)\|^2 + M_w \|f\|^2 \\ &\le b_w \Big(\|f \, \nabla\sqrt{\zeta}\| + \|\sqrt{\zeta}\, \nabla f\| \Big)^2 + (a_w + M_w)\|f\|^2 \\ &\le b_w(1+\varepsilon)\|\nabla f\|^2 + \left(a_w + M_w + b_w \left(1 + \frac{1}{\varepsilon}\right) \left\| \nabla \sqrt{\zeta} \right\|_\infty^2 \right) \|f\|^2 \\ &=: b_w(1+\varepsilon)\|\nabla f\|^2 + C_{w,\varepsilon}\|f\|^2 \end{aligned} \tag{2.6}$$

where $\varepsilon > 0$ may be chosen so small that $b_w(1+\varepsilon) < 1$. Note that $\|\nabla f\|^2 \le \operatorname{Re} t_0[f]$ by (2.2). Thus the form w is relatively bounded with respect to $\operatorname{Re} t_0$, and therefore also with respect to t_0 by (2.5), with relative bound smaller than 1. Hence the form t is closed and sectorial with $\mathcal{D}(t) = \mathcal{D}(t_0)$, $C_0^\infty(\mathbb{R}^d, \mathbb{C})$ is a core of t, and t uniquely determines an m-sectorial operator T, *cf.* [37, Thm. VI.3.4, VI.1.33, VI.2.1 i)].

ii) The embedding $\big(\mathcal{D}(t_0), (\operatorname{Re} t_0[\cdot] + \|\cdot\|^2)^{1/2}\big) \hookrightarrow L^2(\mathbb{R}^d, \mathbb{C})$ is compact by Rellich's criterion [48, Thm. XIII.65], Thus, by (2.6) and the choice of ε, so is the embedding $\big(\mathcal{D}(t), (\operatorname{Re} t[\cdot] + (C_{w,\varepsilon}+1)\|\cdot\|^2)^{1/2}\big) \hookrightarrow L^2(\mathbb{R}^d, \mathbb{C})$. Then, by [48, Thm. XIII.64, part (iv) $\Rightarrow$ (i)], the selfadjoint operator $\operatorname{Re} T$ has compact resolvent and hence so does T due to [37, Thm. VI.3.3]. $\square$

Remark 2.4 (quasi-sectorial case with semi-angle $\theta < \pi/2$)**.** For potentials Q_0 satisfying (I.i'), (I.iii') instead of (I.i), (I.iii), the form t uniquely determines a quasi-m-sectorial operator T with compact resolvent. Here quasi-m-sectorial means that the operator $\mathrm{e}^{-\mathrm{i}\beta}(T-\mu)$ is m-accretive and its numerical range $\mathrm{W}(\mathrm{e}^{-\mathrm{i}\beta}(T-\mu))$ satisfies

$$\mathrm{W}(\mathrm{e}^{-\mathrm{i}\beta}(T-\mu)) \subset \{z \in \mathbb{C} : |\arg(z)| \le \theta\},$$

cf. [27, Def. III.6.9]. In fact, one may show, analogously to Proposition 2.3, that the shifted and rotated form

$$\begin{aligned} \widetilde{t} &:= \mathrm{e}^{-\mathrm{i}\beta}\|\nabla \cdot\|^2 + \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\beta}(Q_0 - \mu)|\cdot|^2 \, \mathrm{d}x + \mathrm{e}^{-\mathrm{i}\beta} w, \\ \mathcal{D}(\widetilde{t}) &:= W^{1,2}(\mathbb{R}^d, \mathbb{C}) \cap \{f \in L^2(\mathbb{R}^d, \mathbb{C}) : \mathrm{e}^{-\mathrm{i}\beta}(Q_0 - \mu)|f|^2 \in L^1(\mathbb{R}^d, \mathbb{C})\}, \end{aligned}$$

uniquely determines an m-sectorial operator $\widetilde{T}$ with compact resolvent and $T := \mathrm{e}^{\mathrm{i}\beta}\widetilde{T} + \mu$. Note that $b_w < \cos\beta$ guarantees the relative boundedness of $\mathrm{e}^{-\mathrm{i}\beta} w$ with respect to $\operatorname{Re}(\mathrm{e}^{-\mathrm{i}\beta}\|\nabla \cdot\|^2)$ with relative bound smaller than 1.

2.2. **Semi-angle $\theta \ge \pi/2$.** As in the previous case, we split Q into a "regular" part Q_0 and perturbations. However, now the essential requirement is only $\operatorname{Re} Q_0 \ge 0$, which prevents us from using sectorial form techniques. Instead, we introduce an m-accretive operator $T_0 = -\Delta + Q_0$ using [27, Thm. VII.2.6, Cor. VII.2.7]. Then we add the qualitatively new, non-positive, part $-U$ (controlled by $\operatorname{Im} Q_0$) and the singular perturbation W (again bounded outside a ball $B_R(0)$, but inside now Δ-bounded in $L^2(B_R(0), \mathbb{C})$).

Assumption II. The function $Q \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C})$ decomposes as

$$Q = Q_0 - U + W$$

where $\operatorname{Re} Q_0 \ge 0$, $U \ge 0$, $U \operatorname{Re} Q_0 = 0$, $W \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C})$, and the following hold.

(II.i) *regularity of Q_0 and U*: $Q_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C})$, $U \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R})$, and there exist a_∇, b_∇, a_U, $b_U \ge 0$ such that

$$|\nabla Q_0|^2 \le a_\nabla + b_\nabla |Q_0|^2, \quad U^2 \le a_U + b_U |\operatorname{Im} Q_0|^2;$$

(II.ii) *unboundedness of* Q_0 *at infinity*:
$$|Q_0(x)| \to \infty \quad \text{as} \quad |x| \to \infty.$$

There exist $R > r > 0$ such that

(II.iii) Δ-*boundedness of* W *in* $L^2(B_R(0), \mathbb{C})$: there exist $a_W \geq 0$, $b_W \in [0,1)$ such that, for every $f \in W^{2,2}(B_R(0), \mathbb{C}) \cap W_0^{1,2}(B_R(0), \mathbb{C})$,
$$\|Wf\|^2 \leq a_W \|f\|^2 + b_W \|\Delta f\|^2;$$

(II.iv) *boundedness of* W *outside* $B_r(0)$: there exists $M_W \geq 0$ such that
$$\|(1 - \chi_r) W\|_\infty \leq M_W,$$
where χ_r is the characteristic function of $B_r(0)$.

Example 2.5. A simple example satisfying Assumption II in $d = 1$ is given by $Q(x) = \mathrm{i}x^3 - x^2 + \mathrm{i}|x|^{-\frac{1}{4}}$, *cf.* (1.2); here
$$Q_0(x) = \mathrm{i}x^3, \quad U(x) = x^2, \quad W(x) = \mathrm{i}|x|^{-\frac{1}{4}}, \quad x \in \mathbb{R},$$
and we can choose any $R > r > 0$ and b_W arbitrarily small. To see the latter, we note that by (2.4), for $f \in W^{2,2}(B_R(0), \mathbb{C}) \cap W_0^{1,2}(B_R(0), \mathbb{C})$,
$$\||x|^{-\frac{1}{4}} f\|^2_{L^2(B_R(0),\mathbb{C})} \leq C \||x|^{-\frac{1}{4}}\|^2_{L^2(B_R(0),\mathbb{C})} \|f\|_{W^{1,2}(B_R(0),\mathbb{C})} \|f\|_{L^2(B_R(0),\mathbb{C})}.$$
Integration by parts shows $\|f'\|^2_{L^2(B_R(0),\mathbb{C})} \leq \|f''\|_{L^2(B_R(0),\mathbb{C})} \|f\|_{L^2(B_R(0),\mathbb{C})}$ and so, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that, for $f \in W^{2,2}(B_R(0), \mathbb{C}) \cap W_0^{1,2}(B_R(0), \mathbb{C})$,
$$\||x|^{-\frac{1}{4}} f\|^2_{L^2(B_R(0),\mathbb{C})} \leq C_\varepsilon \|f\|^2_{L^2(B_R(0),\mathbb{C})} + \varepsilon \|f''\|^2_{L^2(B_R(0),\mathbb{C})}.$$

Proposition 2.6. *Let Assumption* II *be satisfied. Then*

i) *the minimal operator*
$$T_{\min} := -\Delta + Q, \quad \mathcal{D}(T_{\min}) := C_0^\infty\left(\mathbb{R}^d, \mathbb{C}\right), \tag{2.7}$$
is closable with closure
$$T = -\Delta + Q, \quad \mathcal{D}(T) = W^{2,2}\left(\mathbb{R}^d, \mathbb{C}\right) \cap \{f \in L^2\left(\mathbb{R}^d, \mathbb{C}\right) : Q_0 f \in L^2\left(\mathbb{R}^d, \mathbb{C}\right)\};$$

ii) *there exist* k, $K > 0$ *such that, for every* $f \in \mathcal{D}(T)$,
$$\begin{aligned} &k\left(\|\Delta f\|^2 + \|Q_0 f\|^2 + \|f\|^2\right) \\ &\leq \|Tf\|^2 + \|f\|^2 \\ &\leq K\left(\|\Delta f\|^2 + \|Q_0 f\|^2 + \|f\|^2\right); \end{aligned} \tag{2.8}$$

iii) *the embedding* $\left(\mathcal{D}(T), (\|T \cdot\|^2 + \|\cdot\|^2)^{1/2}\right) \hookrightarrow L^2\left(\mathbb{R}^d, \mathbb{C}\right)$ *is compact;*

iv) *if, in addition,* $b_U < 1$, *then the resolvent of* T *is compact. Moreover, for every* $b' \in (\max\{b_U, b_W\}, 1)$, *there exists* $a_{W-U}(b') \geq 0$ *such that the sector*
$$\mathcal{R}(b') := \left\{ \lambda \in \mathbb{C} : \operatorname{Re}\lambda < -\frac{a_{W-U}(b')}{1-\sqrt{b'}}, |\operatorname{Im}\lambda| < \frac{1-\sqrt{b'}}{\sqrt{b'}} |\operatorname{Re}\lambda| - \frac{a_{W-U}(b')}{\sqrt{b'}} \right\} \tag{2.9}$$
is a subset of $\varrho(T)$ *and, for all* $\lambda \in \mathcal{R}(b')$,
$$\|(T-\lambda)^{-1}\| \leq \frac{1}{(1-\sqrt{b'})|\operatorname{Re}\lambda| - \sqrt{b'}|\operatorname{Im}\lambda| - a_{W-U}(b')}. \tag{2.10}$$

Remark 2.7. Apart from the estimate of the spectrum that follows from iv), there are others which may further narrow down the spectral enclosure, at least for a certain range of $b' \in (0,1)$. For example, an estimate similar to the one in the proof of Lemma 2.9 shows that there exists $\widetilde{a}(b') \geq 0$ such that the hyperbolic region

$$\widetilde{\mathcal{R}}(b') := \left\{ \lambda \in \mathbb{C} : \operatorname{Re}\lambda < -\sqrt{\frac{2+b'}{1-b'}\,\widetilde{a}(b')}, |\operatorname{Im}\lambda|^2 < \frac{1-b'}{2+b'}|\operatorname{Re}\lambda|^2 - \widetilde{a}(b') \right\}$$

is a subset of $\varrho(T)$ and, with some $d(b') > 0$,

$$\|(T-\lambda)^{-1}\| \leq \frac{d(b')}{|\operatorname{Re}\lambda|}, \quad \lambda \in \widetilde{\mathcal{R}}(b').$$

In fact, the semi-angle $\widetilde{\vartheta} = \arctan\sqrt{\frac{1-b'}{2+b'}}$ of the asymptotes of $\widetilde{\mathcal{R}}(b')$ is larger than the semi-angle $\vartheta = \arctan\frac{1-\sqrt{b'}}{\sqrt{b'}}$ of the sector $\mathcal{R}(b')$ for $b' \in (b_0, 1)$ with some $b_0 \in (0,1)$, *i.e.* for these b' the set $\mathbb{C} \setminus \widetilde{\mathcal{R}}(b')$ gives a tighter spectral enclosure than $\mathbb{C} \setminus \mathcal{R}(b')$; here b_0 is a zero of a certain cubic polynomial, $b_0 \sim 0.46$.

Remark 2.8. Proposition 2.6 can be used to slightly extend the completeness result in [4]. Define $b := \max\{b_U, b_W\}$ and $\vartheta(b) := \arctan\left(\max\left\{\frac{1-\sqrt{b}}{\sqrt{b}}, \sqrt{\frac{1-b}{2+b}}\right\}\right)$. If the selfadjoint operator $(-\Delta + |Q_0| + 1)^{-1}$ in $L^2\left(\mathbb{R}^d, \mathbb{C}\right)$ belongs to the Schatten class $\mathcal{S}_p$ and

$$p < \frac{\pi}{2(\pi - \vartheta(b))}, \tag{2.11}$$

then the system of eigenfunctions and associated functions of T is complete.

This follows from [26, Cor. XI.9.31] combined with the bound (2.10) and the fact that the resolvent of T belongs to $\mathcal{S}_p$ if and only if so does $(-\Delta + |Q_0| + 1)^{-1}$; the latter is a consequence of (2.8) and the second resolvent identity.

An example which cannot be cast into the setting of [4] is the one-dimensional operator

$$T_{\alpha,\beta} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}|x|^\beta \operatorname{sgn} x - \alpha|x|^\beta, \quad \beta > 2, \ \alpha \in [0,1),$$

in $L^2(\mathbb{R}, \mathbb{C})$ for the case $\alpha \neq 0$. Nonetheless, our results now imply that its system of eigenfunctions and associated functions is complete if

$$\beta > 2\left(\frac{\pi}{\vartheta(\alpha^2)} - 1\right); \tag{2.12}$$

in fact, here $b_U = \alpha^2$, $b_W = 0$, and the eigenvalues $\{\mu_k\}_k$ of $-\mathrm{d}^2/\mathrm{d}x^2 + |x|^\beta$ satisfy $\mu_k k^{-\frac{2\beta}{\beta+2}} \to c > 0$ as $k \to \infty$, see *e.g.* [55], and hence (2.11) is equivalent to (2.12).

The proof of Proposition 2.6 uses three technical lemmas which are proved first.

Lemma 2.9. *Let Assumption* II *be satisfied and define*

$$T_{0,\min} := -\Delta + Q_0, \quad \mathcal{D}(T_{0,\min}) := \mathcal{D}(T_{\min}) = C_0^\infty\left(\mathbb{R}^d, \mathbb{C}\right). \tag{2.13}$$

Then, for every $\varepsilon_1 > 0$, there exists $C_1(\varepsilon_1) \geq 0$ such that, for every $f \in \mathcal{D}(T_{0,\min})$,

$$\|T_{0,\min}f\|^2 \geq (1-\varepsilon_1)\left(\|\Delta f\|^2 + \|Q_0 f\|^2\right) - C_1(\varepsilon_1)\|f\|^2.$$

Proof. Let $\varepsilon_1 > 0$. For $f \in \mathcal{D}(T_{0,\min})$,

$$\|T_{0,\min}f\|^2 = \|-\Delta f + Q_0 f\|^2 = \|\Delta f\|^2 + \|Q_0 f\|^2 + 2\operatorname{Re}\langle -\Delta f, Q_0 f\rangle. \tag{2.14}$$

Using $\operatorname{Re} Q_0 \geq 0$ and Assumption (II.i), we obtain

$$\begin{aligned} 2\operatorname{Re}\langle -\Delta f, Q_0 f\rangle &= 2\operatorname{Re}\langle \nabla f, f\nabla Q_0 + Q_0 \nabla f\rangle \geq 2\operatorname{Re}\langle \nabla f, f\nabla Q_0\rangle \\ &\geq -2\|\nabla f\|\|f\nabla Q_0\| \geq -\alpha\|f\nabla Q_0\|^2 - \frac{1}{\alpha}\|\nabla f\|^2 \\ &\geq -\alpha a_\nabla \|f\|^2 - \alpha b_\nabla \|Q_0 f\|^2 - \frac{1}{\alpha}\|\nabla f\|^2 \end{aligned}$$

where $\alpha > 0$ is arbitrary. Moreover, for every $\beta > 0$,

$$\|\nabla f\|^2 = \langle -\Delta f, f\rangle \leq \|\Delta f\|\,\|f\| \leq \frac{\beta}{2}\,\|\Delta f\|^2 + \frac{1}{2\beta}\,\|f\|^2 . \tag{2.15}$$

By inserting the above inequalities into (2.14), we obtain altogether

$$\|T_{0,\min} f\|^2 \geq \left(1 - \frac{\beta}{2\alpha}\right)\|\Delta f\|^2 + (1 - \alpha b_\nabla)\,\|Q_0 f\|^2 - \left(\alpha a_\nabla + \frac{1}{2\alpha\beta}\right)\|f\|^2.$$

Now the claim follows if we choose $\alpha = \varepsilon_1/b_\nabla$ and $\beta = 2\varepsilon_1^2/b_\nabla$. □

Lemma 2.10. *Let Assumption* II *be satisfied and let* $T_{0,\min}$ *be as in* (2.13)*. Then, for every* $\varepsilon_2 > 0$*, there exists* $C_2(\varepsilon_2) \geq 0$ *such that, for every* $f \in \mathcal{D}(T_{0,\min})$*,*

$$\|Wf\|^2 \leq (b_W + \varepsilon_2)\|\Delta f\|^2 + C_2(\varepsilon_2)\|f\|^2.$$

Proof. With the radii $R > r > 0$ used in Assumption II, we fix $\eta \in C_0^\infty(B_R(0), \mathbb{R})$ such that $0 \leq \eta \leq 1$ and $\eta \restriction B_r(0) = 1$. Since $\eta f \in C_0^\infty(B_R(0), \mathbb{C})$ for every $f \in C_0^\infty\left(\mathbb{R}^d, \mathbb{C}\right)$, it follows from Assumptions (II.iii), (II.iv) that

$$\begin{aligned} \|Wf\| &\leq \|W\eta f\| + \|W(1-\eta)f\| \leq \|W\eta f\| + M_W\|f\|, \\ \|W\eta f\|^2 &\leq a_W\|\eta f\|^2 + b_W\|\Delta(\eta f)\|^2. \end{aligned}$$

Moreover, we have $\Delta(\eta f) = (\Delta\eta)f + 2\nabla\eta.\nabla f + \eta\Delta f$, and the proof can be completed by straightforward estimates using (2.15). □

Lemma 2.11. *Let Assumption* II *be satisfied and let* $T_{\min}$ *be as in* (2.7)*. Then there exist* k*,* $K > 0$ *such that, for every* $f \in \mathcal{D}(T_{\min})$*,*

$$\begin{aligned} &k\left(\|\Delta f\|^2 + \|Q_0 f\|^2 + \|f\|^2\right) \\ &\leq \|T_{\min} f\|^2 + \|f\|^2 \\ &\leq K\left(\|\Delta f\|^2 + \|Q_0 f\|^2 + \|f\|^2\right). \end{aligned} \tag{2.16}$$

Proof. The upper bound in (2.16) is immediate from Assumption (II.i) and Lemma 2.10 as $T_{\min} = T_{0,\min} - U + W$. To show the lower bound, we start from

$$\begin{aligned} \|(T_{0,\min} - U + W)f\|^2 = \|T_{0,\min} f\|^2 + \|Uf\|^2 + \|Wf\|^2 - 2\operatorname{Re}\langle Uf, Wf\rangle \\ + 2\operatorname{Re}\langle \Delta f, (U - W)f\rangle + 2\operatorname{Re}\langle Q_0 f, Wf\rangle, \end{aligned} \tag{2.17}$$

where we used $2\operatorname{Re}\langle Uf, Q_0 f\rangle = 0$ since $U \operatorname{Re} Q_0 = 0$ by Assumption II. We set $\widetilde{\chi}_r := 1 - \chi_r$ where χ_r is the characteristic function of $B_r(0)$. Using Assumptions (II.i) and (II.iv), we obtain that, for arbitrary α, β, $\gamma > 0$,

$$\begin{aligned} 2|\langle Uf, Wf\rangle| &\leq 2|\langle Uf, \widetilde{\chi}_r Wf\rangle| + 2|\langle \chi_r Uf, Wf\rangle| \\ &\leq 2M_W\|Uf\|\|f\| + 2\|Wf\|\|\chi_r U\|_\infty\|f\| \\ &\leq \alpha\left(\|Uf\|^2 + \|Wf\|^2\right) + \frac{1}{\alpha}\left(M_W^2 + \|\chi_r U\|_\infty^2\right)\|f\|^2, \\ 2|\langle \Delta f, Uf\rangle| &\leq 2|\langle \widetilde{\chi}_r \Delta f, Uf\rangle| + 2|\langle \chi_r \Delta f, Uf\rangle| \\ &\leq \beta\|\widetilde{\chi}_r \Delta f\|^2 + \frac{1}{\beta}\|Uf\|^2 + \alpha\|\chi_r \Delta f\|^2 + \frac{1}{\alpha}\|\chi_r U\|_\infty^2\|f\|^2, \end{aligned} \tag{2.18}$$

and, analogously,

$$
\begin{aligned}
2|\langle \Delta f, Wf\rangle| &\le \gamma\|\chi_r \Delta f\|^2 + \frac{1}{\gamma}\|Wf\|^2 + \alpha\|\widetilde{\chi}_r \Delta f\|^2 + \frac{1}{\alpha} M_W^2 \|f\|^2,\\
2|\langle Q_0 f, Wf\rangle| &\le \alpha\left(\|Q_0 f\|^2 + \|Wf\|^2\right) + \frac{1}{\alpha}\left(M_W^2 + \|Q_0\chi_r\|_\infty^2\right)\|f\|^2.
\end{aligned}
$$

Inserting these estimates into (2.17) and applying Lemma 2.9 with arbitrary $\varepsilon_1 > 0$, we conclude that there exists $C_3(\varepsilon_1, \alpha) \ge 0$ such that

$$
\begin{aligned}
\|T_{\min} f\|^2 &= \|(T_{0,\min} - U + W)f\|^2\\
&\ge (1-\varepsilon_1)\left(\|\Delta f\|^2 + \|Q_0 f\|^2\right) + \|Uf\|^2 + \|Wf\|^2\\
&\quad - \beta\|\widetilde{\chi}_r \Delta f\|^2 - \gamma\|\chi_r \Delta f\|^2 - \frac{1}{\beta}\|Uf\|^2 - \frac{1}{\gamma}\|Wf\|^2\\
&\quad - \alpha\left(\|\Delta f\|^2 + 2\|Wf\|^2 + \|Uf\|^2 + \|Q_0 f\|^2\right) - C_3(\varepsilon_1,\alpha)\|f\|^2\\
&\ge (1-\varepsilon_1 - \max\{\beta,\gamma\} - \alpha)\|\Delta f\|^2 + (1-\varepsilon_1-\alpha)\|Q_0 f\|^2\\
&\quad - \Big(\frac{1}{\beta} + \alpha - 1\Big)\|Uf\|^2 - \Big(\frac{1}{\gamma} + 2\alpha - 1\Big)\|Wf\|^2 - C_3(\varepsilon_1,\alpha)\|f\|^2.
\end{aligned}
\tag{2.19}
$$

We choose $\varepsilon_2 > 0$ so small that $b_W' := b_W + \varepsilon_2 < 1$. Then, for β and γ such that $b_U/(b_U+1) < \beta < 1$ and $\max\{b_W', \beta\} < \gamma < 1$, we have

$$
b := 1 - b_U\left(\frac{1}{\beta} - 1\right) > 0, \quad c := 1 - \max\{\beta,\gamma\} - b_W'\left(\frac{1}{\gamma} - 1\right) > 0.
$$

In order to further estimate $\|Uf\|^2$, $\|Wf\|^2$ in (2.19), we note that, since β, $\gamma < 1$, their coefficients satisfy $\frac{1}{\beta} + \alpha - 1 > 0$ and $\frac{1}{\gamma} + 2\alpha - 1 > 0$. Assumption (II.i) and Lemma 2.10, applied with the chosen ε_2, imply that there exists $C_4(\varepsilon_1,\alpha) \ge 0$ with

$$
\begin{aligned}
\|T_{\min} f\|^2 &\ge \left(1 - \varepsilon_1 - \max\{\beta,\gamma\} - \alpha - b_W'\Big(\frac{1}{\gamma} + 2\alpha - 1\Big)\right)\|\Delta f\|^2\\
&\quad + \left(1 - \varepsilon_1 - \alpha - b_U\Big(\frac{1}{\beta} + \alpha - 1\Big)\right)\|Q_0 f\|^2 - C_4(\varepsilon_1,\alpha)\|f\|^2\\
&= \big(c - \varepsilon_1 - \alpha(1+2b_W')\big)\|\Delta f\|^2 + \big(b - \varepsilon_1 - \alpha(1+b_U)\big)\|Q_0 f\|^2 - C_4(\varepsilon_1,\alpha)\|f\|^2.
\end{aligned}
$$

Finally, choosing ε_1 and α sufficiently small, we find that there exists $C \ge 0$ with

$$
\|T_{\min} f\|^2 \ge C\left(\|\Delta f\|^2 + \|Q_0 f\|^2\right) - C_4(\varepsilon_1,\alpha)\|f\|^2,
$$

and hence

$$
\begin{aligned}
\big(C_4(\varepsilon_1,\alpha) + 1\big)\left(\|T_{\min} f\|^2 + \|f\|^2\right) &\ge \|T_{\min} f\|^2 + \big(C_4(\varepsilon_1,\alpha)+1\big)\|f\|^2\\
&\ge C\left(\|\Delta f\|^2 + \|Q_0 f\|^2\right) + \|f\|^2.
\end{aligned}
$$

Now the lower bound in (2.16) follows with $k := \min\{C, 1\}/(C_4(\varepsilon_1,\alpha)+1)$. □

Proof of Proposition 2.6. i) Since $\operatorname{Re} Q_0 \ge 0$, the operator $T_{0,\min}$ is closable and its closure T_0 has the domain

$$
\mathcal{D}(T_0) = \left\{ f \in W^{1,2}\left(\mathbb{R}^d, \mathbb{C}\right) : (-\Delta + Q_0) f \in L^2\left(\mathbb{R}^d, \mathbb{C}\right) \right\},
$$

cf. [27, Cor. VII.2.7]. Lemma 2.11 applied to $T_{\min}$ and $T_{0,\min}$ (which is $T_{\min}$ with $U = W = 0$) yields the existence of k, K, k_0, $K_0 > 0$ so that, for every $f \in \mathcal{D}(T_{\min})$,

$$
\frac{k}{K_0}\left(\|T_{0,\min} f\|^2 + \|f\|^2\right) \le \|T_{\min} f\|^2 + \|f\|^2 \le \frac{K}{k_0}\left(\|T_{0,\min} f\|^2 + \|f\|^2\right).
$$

Hence $T_{\min}$ is closable as well and its closure T satisfies $\mathcal{D}(T) = \mathcal{D}(T_0)$. The inclusion $W^{2,2}\left(\mathbb{R}^d, \mathbb{C}\right) \cap \{f \in L^2\left(\mathbb{R}^d, \mathbb{C}\right) : Q_0 f \in L^2\left(\mathbb{R}^d, \mathbb{C}\right)\} \subset \mathcal{D}(T)$ is obvious. It remains to prove the opposite inclusion. Since $\mathcal{D}(T_{\min}) = C_0^\infty\left(\mathbb{R}^d, \mathbb{C}\right)$ is a core of T, Lemma 2.11 and the equivalence of $(\|\Delta\cdot\|^2+\|\cdot\|^2)^{1/2}$ with $\|\cdot\|_{W^{2,2}(\mathbb{R}^d,\mathbb{C})}$ imply that

$$\begin{aligned}\mathcal{D}(T) &= \overline{C_0^\infty\left(\mathbb{R}^d,\mathbb{C}\right)}^{(\|T\cdot\|^2+\|\cdot\|^2)^{\frac12}} = \overline{C_0^\infty\left(\mathbb{R}^d,\mathbb{C}\right)}^{(\|\cdot\|^2_{W^{2,2}(\mathbb{R}^d,\mathbb{C})}+\|Q_0\cdot\|^2)^{\frac12}} \\ &\subset W^{2,2}\left(\mathbb{R}^d,\mathbb{C}\right)\cap\{f\in L^2\left(\mathbb{R}^d,\mathbb{C}\right): Q_0 f\in L^2\left(\mathbb{R}^d,\mathbb{C}\right)\}.\end{aligned}$$

ii) The claim follows from Lemma 2.11 and the fact that $\mathcal{D}(T_{\min})$ is a core of T.

iii) The embedding $\left(\mathcal{D}(T),(\|T\cdot\|^2+\|\cdot\|^2)^{1/2}\right)\hookrightarrow L^2\left(\mathbb{R}^d,\mathbb{C}\right)$ is compact due to (2.8) and Rellich's criterion [48, Thm. XIII.65].

iv) The compactness of the resolvent follows from claim iii) if we know that $\varrho(T)\neq\emptyset$. This will follow from the remaining claims in iv) since $\mathcal{R}(b')\neq\emptyset$.

To prove that $\mathcal{R}(b')\subset\varrho(T)$ for every $b'\in(\max\{b_U,b_W\},1)$, we first observe that, for every $f\in\mathcal{D}(T_{0,\min})=C_0^\infty\left(\mathbb{R}^d,\mathbb{C}\right)$, the first estimate in (2.18) yields

$$\|(W-U)f\|^2\le(1+\alpha)\left(\|Uf\|^2+\|Wf\|^2\right)+C(\alpha)\|f\|^2$$

where $\alpha>0$ is arbitrary and $C(\alpha)\ge0$. Assumption (II.i), Lemma 2.10 applied with $\varepsilon_2=\alpha b/(1+\alpha)$, and Lemma 2.9 applied with $\varepsilon_1=\alpha/(1+3\alpha)$, imply the existence of $\widetilde{C}_1(\alpha),\widetilde{C}_2(\alpha)\ge0$ such that, for all $f\in\mathcal{D}(T_{\min})$ and with $b:=\max\{b_U,b_W\}$,

$$\begin{aligned}\|(W-U)f\|^2&\le b(1+2\alpha)\left(\|\Delta f\|^2+\|Q_0f\|^2\right)+\widetilde{C}_1(\alpha)\|f\|^2\\ &\le b(1+3\alpha)\|T_0f\|^2+\widetilde{C}_2(\alpha)\|f\|^2.\end{aligned}$$

The latter remains valid for all $f\in\mathcal{D}(T_0)$ since $\mathcal{D}(T_{0,\min})$ is a core of T_0.

If $b'\in(\max\{b_U,b_W\},1)=(b,1)$ is arbitrary, we choose α such that $b'=b(1+3\alpha)$ and so there exists $a_{W-U}(b')\ge0$ such that, for every $f\in\mathcal{D}(T_0)$,

$$\|(W-U)f\|\le a_{W-U}(b')\|f\|+\sqrt{b'}\|T_0f\|. \tag{2.20}$$

Now let $\lambda\in\mathcal{R}(b')$. We verify the assumptions of [37, Thm. IV.3.17] with the unperturbed operator chosen as T_0, the perturbation as $W-U$, and $\zeta=\lambda$. Because T_0 is m-accretive and $\lambda\in\mathcal{R}(b')$ satisfies $\operatorname{Re}\lambda<0$, we have $\operatorname{Re}\lambda\in\varrho(T_0)$, $\|(T_0-\lambda)^{-1}\|\le|\operatorname{Re}\lambda|^{-1}$, and $\|T_0(T_0-\operatorname{Re}\lambda)^{-1}\|\le1$, *cf.* [37, Sec. V.10, Prob. V.3.31]. Notice that the first resolvent identity yields

$$\|T_0(T_0-\lambda)^{-1}\|=\|T_0(T_0-\operatorname{Re}\lambda)^{-1}(I+\mathrm{i}\operatorname{Im}\lambda(T_0-\lambda)^{-1})\|\le1+\frac{|\operatorname{Im}\lambda|}{|\operatorname{Re}\lambda|}.$$

Hence, for all $\lambda\in\mathcal{R}(b')$,

$$a_{W-U}(b')\|(T_0-\lambda)^{-1}\|+\sqrt{b'}\|T_0(T_0-\lambda)^{-1}\|\le\frac{a_{W-U}(b')+\sqrt{b'}|\operatorname{Im}\lambda|}{|\operatorname{Re}\lambda|}+\sqrt{b'}<1,$$

and so the inequality [37, IV.(3.12)] holds. Thus [37, Thm. IV.3.17] implies both $\lambda\in\varrho(T_0-U+W)$ and the estimate (2.8). □

3. Approximating operators in $\Omega_n\subset\mathbb{R}^d$

In this section, we define an approximating sequence $\{T_n\}_n$ of operators T_n in $L^2(\Omega_n,\mathbb{C})$ where $\Omega_n\subset\mathbb{R}^d$ are bounded domains, *i.e.* open and connected subsets, that exhaust $\mathbb{R}^d$ eventually. In order to work with operators with non-empty resolvent sets, we need to specify boundary conditions.

If the aim is to approximate T with simple operators T_n, then one can choose Ω_n for instance as expanding balls and impose Dirichlet boundary conditions. If the aim is to compare, or optimize, the convergence rate for the approximate eigenvalues, it may be necessary to consider other, more general, boundary conditions such as Robin conditions or mixed Dirichlet-Robin conditions.

Our approximation results cover both situations. For Dirichlet conditions only, we do not require any regularity of the boundary $\partial\Omega_n$. For mixed Dirichlet-Robin conditions on $\partial\Omega_n = \partial\Omega_n^{\mathrm{D}} \,\dot\cup\, \partial\Omega_n^{\mathrm{R}}$, formally given by

$$f \restriction \partial\Omega_n^{\mathrm{D}} = 0, \quad (\partial_\nu f + a_n f) \restriction \partial\Omega_n^{\mathrm{R}} = 0$$

where ∂_ν is the normal derivative on $\partial\Omega_n^{\mathrm{R}}$, we assume $\partial\Omega_n$ is Lipschitz and the functions $a_n : \partial\Omega_n^{\mathrm{R}} \to \mathbb{C}$ are suitably bounded, *cf.* Assumption III.

Assumption III. Let $\{\Omega_n\}_n \subset \mathbb{R}^d$ be a sequence of bounded domains satisfying

$$\partial\Omega_n = \partial\Omega_n^{\mathrm{D}} \,\dot\cup\, \partial\Omega_n^{\mathrm{R}}$$

where $\partial\Omega_n^{\mathrm{D}}$ is closed and the following hold.

(III.i) *exhausting property*: with the radius $R > 0$ used in Assumption I or II, there exists $\{r_n\}_n \subset \mathbb{R}$, $r_1 > R$, such that

$$\overline{B_{r_n+1}(0)} \subset \Omega_n, \quad r_{n+1} > r_n, \quad r_n \to \infty.$$

If $\partial\Omega_n^{\mathrm{R}} \neq \emptyset$ and $d \geq 2$, we additionally assume

(III.ii) *regularity of* $\partial\Omega_n$: Ω_n is Lipschitz.

If $a_n \neq 0$, $n \in \mathbb{N}$, we further assume

(III.iii) *control of Robin boundary terms*: $a_n \in L^\infty(\partial\Omega_n^{\mathrm{R}}, \mathbb{C})$, $n \in \mathbb{N}$, and

$$M_{\mathrm{Tr}} := \sup_n \|a_n\|_\infty K_n < \infty \tag{3.1}$$

where $K_n > 0$ are the constants in the trace embedding

$$\int_{\partial\Omega_n} |f|^p \,\mathrm{d}\sigma \leq K_n \left(\varepsilon^{1-\frac{1}{p}} \|\nabla f\|^p_{L^p(\Omega_n,\mathbb{C})} + \varepsilon^{-\frac{1}{p}} \|f\|^p_{L^p(\Omega_n,\mathbb{C})} \right) \tag{3.2}$$

valid for all $f \in W^{1,p}(\Omega_n, \mathbb{C})$, $\varepsilon \in (0,1)$, and $p \geq 1$, *cf.* [32, Thm. 1.5.1.10].

Remark 3.1. i) For balls or boxes, it can be shown that the constants K_n are uniformly bounded; then the condition (3.1) reduces to $\sup_n \|a_n\|_\infty < \infty$.

ii) Sometimes, *e.g.* in Propositions 3.4, 3.5 below, we indicate the dependence of the constants on the constant M_{Tr} in (3.1).

The operators T_n are introduced in several steps, analogously to the definition of T in the previous section. The main difference is in the first step, *cf.* Lemma 3.2, where we first introduce a Dirichlet-Robin Laplacian $S_{0,n} := -\Delta_n^{\mathrm{DR}}$ in $L^2(\Omega_n, \mathbb{C})$ via its quadratic form, see *e.g.* [22, Sec. 7] for more details on this approach.

We remark that if Q satisfies both Assumptions I and II, then also the approximating operators T_n introduced in the two different ways coincide.

Lemma 3.2. *Let Assumption* III *be satisfied. Then, for every $n \in \mathbb{N}$, the form*

$$s_{0,n} := \|\nabla \cdot\|_n^2 + \int_{\partial\Omega_n^{\mathrm{R}}} a_n |\cdot|^2 \,\mathrm{d}\sigma, \quad \mathcal{D}(s_{0,n}) := \overline{\mathcal{D}_n}^{\|\cdot\|_{W^{1,2}(\Omega_n,\mathbb{C})}}, \tag{3.3}$$

with

$$\mathcal{D}_n := \left\{ f \in C^\infty(\Omega_n, \mathbb{C}) : \exists f_0 \in C_0^\infty\left(\mathbb{R}^d, \mathbb{C}\right), f = f_0 \restriction \Omega_n, \operatorname{supp} f \cap \partial\Omega_n^{\mathrm{D}} = \emptyset \right\} \tag{3.4}$$

is densely defined, closed and sectorial and it uniquely determines an m-sectorial operator $S_{0,n} = -\Delta_n^{\mathrm{DR}}$ which has compact resolvent.

Proof. First observe that

$$W_0^{1,2}(\Omega_n, \mathbb{C}) \subset \overline{\mathcal{D}_n}^{\|\cdot\|_{W^{1,2}(\Omega_n,\mathbb{C})}} \subset W^{1,2}(\Omega_n, \mathbb{C}). \tag{3.5}$$

The symmetric form $\|\nabla \cdot\|_n^2$ defined on $\mathcal{D}(s_{0,n})$ is densely defined and closed since $(\mathcal{D}(s_{0,n}), \|\cdot\|_{W^{1,2}(\Omega_n,\mathbb{C})})$ is complete, *cf.* [37, Thm. VI.1.11]. The boundary trace embedding (3.2), applied with $p = 2$ and arbitrarily small $\varepsilon > 0$, together with (3.1) implies that the boundary term in (3.3) is a relatively bounded perturbation of $\|\nabla\cdot\|_n^2$ defined on $\mathcal{D}(s_{0,n})$ with relative bound 0. By [37, Thm. VI.1.33], the form $s_{0,n}$ is densely defined, closed, and sectorial, hence it uniquely determines an m-sectorial operator $S_{0,n}$, *cf.* [37, Thm. VI.2.1].

Moreover, for sufficiently large $c>0$, the norm $(\operatorname{Re} s_{0,n}[\cdot]+c\|\cdot\|_n^2)^{1/2}$ is equivalent to $\|\cdot\|_{W^{1,2}(\Omega_n,\mathbb{C})}$. Then, by the Rellich-Kondrachov theorem [1, Thm. 6.3], $(\mathcal{D}(s_{0,n}),(\operatorname{Re} s_{0,n}[\cdot]+c\|\cdot\|_n^2)^{1/2})$ is compactly embedded in $L^2(\Omega_n,\mathbb{C})$. Thus $\operatorname{Re} S_{0,n}$ has compact resolvent and hence so does $S_{0,n}$, *cf.* [37, Thm. VI.3.3]. $\square$

Remark 3.3. i) If Ω_n are sufficiently regular, *e.g.* $\partial\Omega_n$ is of class C^2, and either $\partial\Omega_n^{\mathrm{R}}=\emptyset$ or $\partial\Omega_n^{\mathrm{D}}=\emptyset$ where, in the latter case, $a_n\in W^{1,\infty}(\partial\Omega_n,\mathbb{C})$, then in Lemma 3.2 the usual domains of Dirichlet or Robin Laplacian are recovered,

$$\begin{aligned}\mathcal{D}(-\Delta_n^{\mathrm{D}})&=W^{2,2}(\Omega_n,\mathbb{C})\cap W_0^{1,2}(\Omega_n,\mathbb{C}),\\ \mathcal{D}(-\Delta_n^{\mathrm{R}})&=\{f\in W^{2,2}(\Omega_n,\mathbb{C}) : (\partial_\nu f+a_nf)\restriction\partial\Omega_n=0\},\end{aligned}$$

where ∂_ν denotes the normal derivative on $\partial\Omega_n=\partial\Omega_n^{\mathrm{R}}$.

ii) If the splitting $\partial\Omega_n=\partial\Omega_n^{\mathrm{D}}\,\dot\cup\,\partial\Omega_n^{\mathrm{R}}$ satisfies additional, very technical, regularity assumptions, *cf.* [41, Prop. 3.1], then

$$\overline{\mathcal{D}_n}^{\|\cdot\|_{W^{1,2}(\Omega_n,\mathbb{C})}}=\left\{f\in W^{1,2}(\Omega_n,\mathbb{C}) : f\restriction\partial\Omega_n^{\mathrm{D}}=0 \text{ a.e.}\right\}.$$

3.1. **Semi-angle $\theta<\pi/2$.** In this case, the operator T_n is introduced in one step by perturbation arguments using quadratic forms.

Proposition 3.4. *Let Assumptions* I, III *be satisfied and let $s_{0,n}$, q_0 be the forms defined in* (3.3), (2.1), *respectively. Then*

i) *for every $n\in\mathbb{N}$, the form*

$$t_n:=s_{0,n}+q_0+w,\quad \mathcal{D}(t_n):=\mathcal{D}(s_{0,n})\cap\mathcal{D}(q_0),$$

is densely defined, closed, and sectorial, and it uniquely determines an m-sectorial operator T_n which has compact resolvent.

ii) *the sequence $\{T_n\}_n$ is uniformly quasi-sectorial with semi-angle $<\pi/2$,* i.e. *there exist $\mu_0(M_{\mathrm{Tr}})\in\mathbb{C}$ and $\theta_0(M_{\mathrm{Tr}})\in[0,\pi/2)$ such that the numerical ranges and spectra of all T_n are contained in the uniform sector*

$$\sigma(T_n)\subset \mathrm{W}(T_n)\subset\mathcal{S}(M_{\mathrm{Tr}}):=\{z\in\mathbb{C} : |\arg(z-\mu_0(M_{\mathrm{Tr}}))|\le\theta_0(M_{\mathrm{Tr}})\}. \tag{3.6}$$

Proof. i) The form

$$t_{0,n}:=s_{0,n}+q_0,\quad \mathcal{D}(t_{0,n}):=\mathcal{D}(s_{0,n})\cap\mathcal{D}(q_0),$$

is the sum of two closed sectorial forms, hence it is closed and sectorial as well, *cf.* [37, Thm. VI.1.31]. So it uniquely determines an m-sectorial operator $T_{0,n}$. Notice that $\operatorname{Re} s_{0,n}[f]\le\operatorname{Re} t_{0,n}[f]$ for all $f\in\mathcal{D}(t_{0,n})$ and, with sufficiently large $c>0$, $(\operatorname{Re} s_{0,n}[\cdot]+c\|\cdot\|_n^2)^{1/2}$ is compactly embedded in $L^2(\Omega_n,\mathbb{C})$, *cf.* the proof of Lemma 3.2 for details. Hence $(\mathcal{D}(t_{0,n}),(\operatorname{Re} t_{0,n}[\cdot]+c\|\cdot\|_n^2)^{1/2})$ is compactly embedded in $L^2(\Omega_n,\mathbb{C})$ and consequently the resolvent of $\operatorname{Re} T_{0,n}$ is compact.

By the trace embedding (3.2) and Assumption (III.iii), the boundary term in (3.3) is relatively bounded with respect to $\|\nabla\cdot\|_n^2$ with relative bound 0.

For the form w we first note that, by assumption (2.3) in Assumption I, for every $f\in\mathcal{D}(t_n)\subset W^{1,2}(\Omega_n,\mathbb{C})$ we have $\sqrt{\zeta}f\in W_0^{1,2}(B_R(0),\mathbb{C})\subset\mathcal{D}(w_1)$ and thus $w[f]=w_1[\sqrt{\zeta}f]+w_2[\sqrt{1-\zeta}f]$ is well-defined. Using analogous arguments as in the proof of Proposition 2.3, one can verify that the form w is relatively bounded with respect to $\operatorname{Re} t_{0,n}+c$ with relative bound smaller than 1. Hence t_n

uniquely determines an m-sectorial operator T_n, *cf.* [37, Thm. VI.3.4]. The latter has compact resolvent since the resolvent of $\operatorname{Re} T_n$ is compact, *cf.* [37, Thm. VI.3.3].

ii) Using the trace embedding (3.2), Assumptions (III.iii), (I.i) and the estimate (2.6) on $|w|$, we obtain

$$\begin{aligned}|\operatorname{Im} t_n[f]| &\leq \left| \int_{\partial\Omega_n^{\mathrm{R}}} a_n |f|^2 \,\mathrm{d}\sigma \right| + |\operatorname{Im} q_0[f]| + |w[f]| \\ &\leq \|a_n\|_\infty \int_{\partial\Omega_n^{\mathrm{R}}} |f|^2 \,\mathrm{d}\sigma + \tan\theta \operatorname{Re} q_0[f] + b_w(1+\varepsilon)\|\nabla f\|_n^2 + C_{w,\varepsilon}\|f\|_n^2 \\ &\leq (\sqrt{\varepsilon} M_{\mathrm{Tr}} + b_w(1+\varepsilon))\|\nabla f\|_n^2 + \tan\theta \operatorname{Re} q_0[f] + \left(\frac{M_{\mathrm{Tr}}}{\sqrt{\varepsilon}} + C_{w,\varepsilon} \right) \|f\|_n^2 \\ &\leq \widetilde{C}_1(\varepsilon, M_{\mathrm{Tr}}) \left(\|\nabla f\|_n^2 + \operatorname{Re} q_0[f] \right) + \widetilde{C}_2(\varepsilon, M_{\mathrm{Tr}}) \|f\|_n^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\operatorname{Re} t_n[f] &\geq (1 - \sqrt{\varepsilon} M_{\mathrm{Tr}} - b_w(1+\varepsilon))\|\nabla f\|_n^2 + \operatorname{Re} q_0[f] - \left(\frac{M_{\mathrm{Tr}}}{\sqrt{\varepsilon}} + C_{w,\varepsilon} \right) \|f\|_n^2 \\ &\geq \widetilde{C}_3(\varepsilon, M_{\mathrm{Tr}}) \left(\|\nabla f\|_n^2 + \operatorname{Re} q_0[f] \right) - \widetilde{C}_4(\varepsilon, M_{\mathrm{Tr}}) \|f\|_n^2.\end{aligned} \tag{3.7}$$

Since $\widetilde{C}_i(\varepsilon, M_{\mathrm{Tr}})$, $i = 1, \dots, 4$, are independent of n and positive for $\varepsilon > 0$ sufficiently small because $b_w \in [0, 1)$, it follows that, for all $f \in \mathcal{D}(t_n)$,

$$|\operatorname{Im} t_n[f]| \leq \frac{\widetilde{C}_1(\varepsilon, M_{\mathrm{Tr}})}{\widetilde{C}_3(\varepsilon, M_{\mathrm{Tr}})} \left(\operatorname{Re} t_n[f] + \widetilde{C}_4(\varepsilon, M_{\mathrm{Tr}}) \|f\|_n^2 \right) + \widetilde{C}_2(\varepsilon, M_{\mathrm{Tr}}) \|f\|_n^2.$$

Now the inclusion (3.6) for the numerical range follows easily. The inclusion for the spectrum follows *e.g.* since T_n has compact resolvent and so $\varrho(T_n) \cap (\mathbb{C} \setminus \mathcal{S}(M_{\mathrm{Tr}})) \neq \emptyset$, *cf.* [37, Thm. V.3.2]. □

3.2. **Semi-angle $\theta \geq \pi/2$.** Since Q_0 is assumed to be locally bounded, *cf.* Assumption (II.i), $Q_0 f$ is well-defined for all functions $f \in L^2(\Omega_n, \mathbb{C})$. We define $T_{0,n}$ as the operator sum

$$T_{0,n} := S_{0,n} + Q_0, \quad \mathcal{D}(T_{0,n}) := \mathcal{D}(S_{0,n}), \quad n \in \mathbb{N}. \tag{3.8}$$

The operator T_n is introduced in the following proposition by further adding the locally bounded non-positive part $-U$ and the "singular" part W which turns out to be Δ_n^{DR}-bounded with relative bound smaller than 1, *cf.* Lemma 3.7.

Proposition 3.5. *Let Assumptions* II, III *be satisfied and let $T_{0,n}$ be as in* (3.8). *Then, for every $n \in \mathbb{N}$,*

i) *the operator*

$$T_n := T_{0,n} - U + W, \quad \mathcal{D}(T_n) := \mathcal{D}(T_{0,n}),$$

is closed and has compact resolvent;

ii) *there exist $\widetilde{k}(M_{\mathrm{Tr}})$, $\widetilde{K}(M_{\mathrm{Tr}}) > 0$, independent of n, such that, for every $f \in \mathcal{D}(T_n)$,*

$$\begin{aligned}&\widetilde{k}(M_{\mathrm{Tr}}) \left(\|\Delta_n^{\mathrm{DR}} f\|_n^2 + \|Q_0 f\|_n^2 + \|f\|_n^2 \right) \\ &\leq \|T_n f\|_n^2 + \|f\|_n^2 \\ &\leq \widetilde{K}(M_{\mathrm{Tr}}) \left(\|\Delta_n^{\mathrm{DR}} f\|_n^2 + \|Q_0 f\|_n^2 + \|f\|_n^2 \right);\end{aligned} \tag{3.9}$$

iii) *if* $b_U < 1$, *then the sequence* $\{T_n\}_n$ *is uniformly quasi-sectorial, more precisely, for every* $b' \in (\max\{b_U, b_W\}, 1)$, *there is* $a_{W-U}(b', M_{\mathrm{Tr}}) \geq 0$, *independent of* n, *such that the sector*

$$\begin{aligned}\mathcal{R}(b', M_{\mathrm{Tr}}) := \Big\{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda < -M_{\mathrm{Tr}}^2 - \frac{a_{W-U}(b', M_{\mathrm{Tr}})}{1-\sqrt{b'}}, \\ |\operatorname{Im} \lambda| < \frac{1-\sqrt{b'}}{\sqrt{b'}} |\operatorname{Re} \lambda + M_{\mathrm{Tr}}^2| - \frac{a_{W-U}(b', M_{\mathrm{Tr}})}{\sqrt{b'}} \Big\}\end{aligned} \tag{3.10}$$

is a subset of $\varrho(T_n)$ *for all* $n \in \mathbb{N}$ *and, for all* $\lambda \in \mathcal{R}(b', M_{\mathrm{Tr}})$,

$$\|(T_n - \lambda)^{-1}\| \leq \frac{1}{(1-\sqrt{b'})|\operatorname{Re} \lambda + M_{\mathrm{Tr}}^2| - \sqrt{b'}|\operatorname{Im} \lambda| - a_{W-U}(b', M_{\mathrm{Tr}})}. \tag{3.11}$$

We mention that, formally, for $M_{\mathrm{Tr}} = 0$ the set $\mathcal{R}(b', M_{\mathrm{Tr}})$ and the resolvent estimate in Proposition 3.5 iii) coincide with the set $\mathcal{R}(b')$ and the resolvent estimate in Proposition 2.6 iv).

Before we prove Proposition 3.5, we establish analogues of Lemmas 2.9, 2.10 for the approximating operators T_n where we need to account for the boundary terms; here we omit the dependence of the constants on M_{Tr}.

Lemma 3.6. *Let Assumptions* II, III *be satisfied and let* $T_{0,n}$ *be as in* (3.8). *Then, for every* $\varepsilon_3 > 0$, *there exists* $C_3(\varepsilon_3) \geq 0$, *independent of* n, *such that, for every* $f \in \mathcal{D}(T_{0,n})$,

$$\|T_{0,n} f\|_n^2 \geq (1 - \varepsilon_3) \left(\|\Delta_n^{\mathrm{DR}} f\|_n^2 + \|Q_0 f\|_n^2 \right) - C_3(\varepsilon_3) \|f\|_n^2.$$

Proof. We adapt the proof of Lemma 2.9. Let $\varepsilon_3 > 0$. For $f \in \mathcal{D}(T_{0,n})$,

$$\|T_{0,n} f\|_n^2 = \|\Delta_n^{\mathrm{DR}} f\|_n^2 + \|Q_0 f\|_n^2 + 2 \operatorname{Re}\langle -\Delta_n^{\mathrm{DR}} f, Q_0 f\rangle_n. \tag{3.12}$$

Before we estimate the individual terms, we prove two estimates that are used later on. First, for arbitrary $\alpha, \beta > 0$, by the trace embedding (3.2) with $\sqrt{\varepsilon} = \frac{\beta}{M_{\mathrm{Tr}}}$ and Assumption (III.iii), we obtain

$$\begin{aligned}\|\nabla f\|_n^2 &= \langle -\Delta_n^{\mathrm{DR}} f, f\rangle_n - \int_{\partial\Omega_n^{\mathrm{R}}} a_n |f|^2 \,\mathrm{d}\sigma \\ &\leq \alpha \|\Delta_n^{\mathrm{DR}} f\|_n^2 + \frac{1}{4\alpha}\|f\|_n^2 + \beta \|\nabla f\|_n^2 + \frac{M_{\mathrm{Tr}}^2}{\beta}\|f\|_n^2;\end{aligned}$$

hence, for $\beta := 1/2$,

$$\|\nabla f\|_n^2 \leq 2\alpha \|\Delta_n^{\mathrm{DR}} f\|_n^2 + \left(\frac{1}{2\alpha} + 4M_{\mathrm{Tr}}^2\right)\|f\|_n^2. \tag{3.13}$$

Secondly, Assumption (II.i) implies

$$\|f \nabla Q_0\|_n^2 \leq a_\nabla \|f\|_n^2 + b_\nabla \|Q_0 f\|_n^2. \tag{3.14}$$

Now we estimate the last term on the right-hand side of (3.12).

First we verify that $Q_0 f \in \mathcal{D}(s_{0,n})$. Since $Q_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C})$, *cf.* Assumption (II.i), and $f \in \mathcal{D}(T_{0,n}) \subset \mathcal{D}(s_{0,n}) \subset W^{1,2}(\Omega_n, \mathbb{C})$, *cf.* (3.8) and (3.5), we have $Q_0 f \in W^{1,2}(\Omega_n, \mathbb{C})$. Moreover, using the definition of $\mathcal{D}(s_{0,n})$, *cf.* (3.3), we find $\{f_k\}_k \subset C_0^\infty(\mathbb{R}^d, \mathbb{C})$ such that

$$\operatorname{dist}(\operatorname{supp} f_k, \partial\Omega^{\mathrm{D}}) := \inf_{\substack{x_1 \in \operatorname{supp} f_k \\ x_2 \in \partial\Omega^{\mathrm{D}}}} \|x_1 - x_2\| > 0, \quad k \in \mathbb{N}, \tag{3.15}$$

$$\|f_k \restriction \Omega_n - f\|_{W^{1,2}(\Omega_n, \mathbb{C})} \to 0, \quad k \to \infty. \tag{3.16}$$

Since $Q_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C})$, we have $\{Q_0 f_k\}_k \subset W^{1,2}\left(\mathbb{R}^d,\mathbb{C}\right)$. Let J_ε, $\varepsilon>0$, be the standard mollifier, *cf.* [1, Par. 2.28], and let $k\in\mathbb{N}$. Due to (3.15) and properties of mollifiers, *cf.* [1, Par. 2.28, Lem. 3.16], there exists $\varepsilon_k>0$ such that

$$\begin{aligned}&(J_\varepsilon * Q_0 f_k)\restriction\Omega_n \in \mathcal{D}_n, \quad 0<\varepsilon<\varepsilon_k,\\ &\|(J_\varepsilon * Q_0 f_k)\restriction\Omega_n - Q_0 f_k\restriction\Omega_n\|_{W^{1,2}(\Omega_n,\mathbb{C})}\to 0, \quad \varepsilon\searrow 0.\end{aligned}$$

Thus $\{Q_0 f_k\restriction\Omega_n\}_k \subset \mathcal{D}(s_{0,n})$, hence $Q_0 f\in\mathcal{D}(s_{0,n})$ by (3.16) and $Q_0\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C})$.

Now we continue the estimates. For every $f\in\mathcal{D}(T_{0,n})$,

$$\begin{aligned}2\,\mathrm{Re}\langle -\Delta_n^{\mathrm{DR}} f, Q_0 f\rangle_n &= 2\,\mathrm{Re}\langle \nabla f, f\nabla Q_0 + Q_0\nabla f\rangle_n + 2\,\mathrm{Re}\int_{\partial\Omega_n^{\mathrm{R}}} a_n\overline{Q_0}|f|^2\,\mathrm{d}\sigma\\ &\geq 2\,\mathrm{Re}\langle\nabla f, f\nabla Q_0\rangle_n + 2\,\mathrm{Re}\int_{\partial\Omega_n^{\mathrm{R}}} a_n\overline{Q_0}|f|^2\,\mathrm{d}\sigma\\ &\geq -\gamma\|f\nabla Q_0\|_n^2 - \frac{1}{\gamma}\|\nabla f\|_n^2 - \left|2\int_{\partial\Omega_n^{\mathrm{R}}} a_n\overline{Q_0}|f|^2\,\mathrm{d}\sigma\right|\end{aligned} \tag{3.17}$$

for arbitrary $\gamma>0$. We have $Q_0 f^2\in W^{1,1}(\Omega_n,\mathbb{C})$, hence the trace embedding (3.2) with $p=1$, $\varepsilon=1$, and Assumption (III.iii) yield

$$\begin{aligned}\left|2\int_{\partial\Omega_n^{\mathrm{R}}} a_n\overline{Q_0}|f|^2\,\mathrm{d}\sigma\right| &\leq 2\|a_n\|_\infty\int_{\partial\Omega_n^{\mathrm{R}}}\left|Q_0 f^2\right|\mathrm{d}\sigma\\ &\leq 2\|a_n\|_\infty K_n\int_{\Omega_n}\Big(\left|\nabla(Q_0 f^2)\right| + \left|Q_0 f^2\right|\Big)\,\mathrm{d}x\\ &\leq 2M_{\mathrm{Tr}}\Big(\|f\nabla Q_0\|_n\|f\|_n + \|Q_0 f\|_n(2\|\nabla f\|_n + \|f\|_n)\Big)\\ &\leq \delta\|f\nabla Q_0\|_n^2 + 2\delta\|Q_0 f\|_n^2 + \frac{2M_{\mathrm{Tr}}^2}{\delta}(2\|\nabla f\|_n^2 + \|f\|_n^2)\end{aligned} \tag{3.18}$$

where $\delta>0$ is arbitrary. Using (3.18) and (3.13), (3.14) to estimate (3.17), and by choosing γ, δ and then α sufficiently small, we arrive at

$$2\,\mathrm{Re}\langle -\Delta_n^{\mathrm{DR}} f, Q_0 f\rangle_n \geq \varepsilon_3\left(\|\Delta_n^{\mathrm{DR}} f\|_n^2 + \|Q_0 f\|_n^2\right) - C_3(\varepsilon_3)\|f\|_n^2$$

for some $C_3(\varepsilon_3)\geq 0$, so the claim follows from (3.12). □

Lemma 3.7. *Let Assumptions* II, III *be satisfied and let* $T_{0,n}$ *be as in* (3.8). *Then, for every* $\varepsilon_4>0$, *there exists* $C_4(\varepsilon_4)\geq 0$, *independent of* n, *such that, for every* $f\in\mathcal{D}(T_{0,n})$,

$$\|Wf\|_n^2 \leq (b_W+\varepsilon_4)\|\Delta_n^{\mathrm{DR}} f\|_n^2 + C_4(\varepsilon_4)\|f\|_n^2.$$

Proof. Let $\eta\in C_0^\infty(B_R(0),\mathbb{R})$ be as in the proof of Lemma 2.10. For $f\in\mathcal{D}(T_{0,n})\subset\mathcal{D}(s_{0,n})\subset W^{1,2}(\Omega_n,\mathbb{C})$, we have $\eta f\in W_0^{1,2}(B_R(0),\mathbb{C})\subset\mathcal{D}(s_{0,n})$ since $B_R(0)\subset\Omega_n$, *cf.* (3.5). Let $\psi\in\mathcal{D}_n$, *cf.* (3.4). Then $\eta\psi\in C_0^\infty(B_R(0),\mathbb{C})$ and, integrating by parts, we can verify that

$$s_{0,n}(\eta f,\psi) = \langle -2\nabla\eta.\nabla f - f\Delta\eta - \eta\Delta_n^{\mathrm{DR}} f, \psi\rangle_n.$$

Since $\mathcal{D}_n$ is a core of $s_{0,n}$, the first representation theorem [37, Thm. VI.2.1] implies that $\eta f\in\mathcal{D}(-\Delta_n^{\mathrm{DR}})$ and $\Delta_n^{\mathrm{DR}}(\eta f) = f\Delta\eta + 2\nabla\eta.\nabla f + \eta\Delta_n^{\mathrm{DR}} f$. With the help of (3.13) instead of (2.15), the proof can be finished in the same way as the proof of Lemma 2.10. □

Proof of Proposition 3.5. i) Since Q_0, $U\in L^\infty_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C})$ by Assumption (II.i), and W is $S_{0,n}$-bounded with relative bound smaller than 1 by Lemma 3.7, the operator T_n is closed, *cf.* [37, Thm. IV.1.1]. Moreover, by [37, Thm. IV.1.16], it has compact resolvent since $S_{0,n}$ is m-sectorial with compact resolvent, *cf.* Lemma 3.2.

ii) The equivalence of the norms can be proved by a straightforward adaptation of the arguments in the proof of Lemma 2.11; note that Lemma 3.7 is used instead of Lemma 2.10.

iii) By the trace embedding (3.2) and Assumption (III.iii), we have

$$\operatorname{Re}\langle T_{0,n} f, f\rangle_n \geq (1 - M_{\mathrm{Tr}}\sqrt{\varepsilon})\|\nabla f\|_n^2 - \frac{M_{\mathrm{Tr}}}{\sqrt{\varepsilon}}\|f\|_n^2 = -M_{\mathrm{Tr}}^2\|f\|_n^2,$$

where we have chosen $\sqrt{\varepsilon} = 1/M_{\mathrm{Tr}}$ in the last step. Hence the numerical range of $T_{0,n}$ lies in the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -M_{\mathrm{Tr}}^2\}$ and so does the spectrum since $T_{0,n}$ has compact resolvent, *cf.* [37, Thm. V.3.2]; moreover, $\|(T_{0,n} - \lambda)^{-1}\| \leq |\operatorname{Re}\lambda + M_{\mathrm{Tr}}^2|^{-1}$ if $\operatorname{Re}\lambda < -M_{\mathrm{Tr}}^2$.

Since $b_U < 1$, the claims in (3.10)–(3.11) are now obtained by an argument based on [37, Thm. IV.3.17], similarly as in the proof of Proposition 2.6; here Lemmas 3.6, 3.7 are used, and $a_{W-U}(b', M_{\mathrm{Tr}})$ is the constant in the relative boundedness inequality of $W - U$ with respect to $T_{0,n}$, in analogy to (2.20). □

4. Convergence of T_n to T

In this section, we prove that the operators T_n converge to T in generalized norm resolvent sense, *cf.* Theorem 4.1. The proof relies on two ingredients.

First, in Lemma 4.3, we show generalized *strong* resolvent convergence of T_n to T. Here, for semi-angle $\theta \geq \pi/2$, we employ the so-called common core property of approximations, *cf.* [59, Thm. 1]. For semi-angle $\theta < \pi/2$, where it is not even guaranteed that $C_0^\infty(\mathbb{R}^d, \mathbb{C}) \subset \mathcal{D}(T)$, we use form techniques inspired by the approach in [40, Prop. 5.4] for a selfadjoint Laplacian in twisted tubes.

Secondly, in Lemma 4.4, we establish discrete compactness, *cf.* [53, Def. 3.1.(k)], of the sequence of embeddings

$$\Big(\mathcal{D}(T_n), \big(\|T_n \cdot\|_n^2 + \|\cdot\|_n^2\big)^{\frac{1}{2}}\Big) \hookrightarrow L^2(\Omega_n, \mathbb{C}), \quad n \in \mathbb{N}. \tag{4.1}$$

Theorem 4.1. *Let Assumption* III *be satisfied and assume that*

I. *in the case of semi-angle $\theta < \pi/2$, Assumption* I *holds and T, T_n, $n \in \mathbb{N}$, are the operators defined in Propositions* 2.3, 3.4*, respectively;*
II. *in the case of semi-angle $\theta \geq \pi/2$, Assumption* II *holds with $b_U < 1$ and T, T_n, $n \in \mathbb{N}$, are the operators defined in Propositions* 2.6, 3.5*, respectively.*

Then, for every $\lambda \in \varrho(T)$, there exists $n_\lambda \in \mathbb{N}$ such that, for all $n \geq n_\lambda$, $\lambda \in \varrho(T_n)$ and

$$\big\|(T_n - \lambda)^{-1}\chi_{\Omega_n} - (T - \lambda)^{-1}\big\| \to 0, \quad n \to \infty. \tag{4.2}$$

Remark 4.2. The generalized norm resolvent convergence in (4.2) is even locally uniform, *i.e.* for all $\lambda \in \varrho(T)$, there exist $r_\lambda > 0$ and $n_\lambda \in \mathbb{N}$ such that $B_{r_\lambda}(\lambda) \subset \bigcap_{n \geq n_\lambda} \varrho(T_n) \cap \varrho(T)$ and the convergence is uniform in $B_{r_\lambda}(\lambda)$.

To see the latter, let $n_\lambda \in \mathbb{N}$ be so large that $\|(T_n - \lambda)^{-1}\chi_{\Omega_n} - (T - \lambda)^{-1}\| \leq 2\|(T - \lambda)^{-1}\|$ for all $n \geq n_\lambda$. If we choose $r_\lambda := \|(T - \lambda)^{-1}\|^{-1}/4$, then a Neumann series argument yields that, for every $\mu \in B_{r_\lambda}(\lambda)$, the resolvents $(T - \mu)^{-1}$, $(T_n - \mu)^{-1}$, $n \geq n_\lambda$, exist and are uniformly bounded (in n and μ). Then the uniform convergence of the resolvents follows from (4.12) below with λ_0, λ replaced by λ, μ.

To prove Theorem 4.1, we first show the two lemmas described above.

Lemma 4.3. *Let the assumptions of Theorem* 4.1 *be satisfied. Then there exists $\gamma > 0$ such that $(-\infty, -\gamma) \subset \bigcap_n \varrho(T_n) \cap \varrho(T) \neq \emptyset$ and for all $\lambda_0 \in (-\infty, -\gamma)$ and for every $f \in L^2(\mathbb{R}^d, \mathbb{C})$,*

$$\big\|\big((T_n - \lambda_0)^{-1}\chi_{\Omega_n} - (T - \lambda_0)^{-1}\big) f\big\| \to 0, \quad n \to \infty.$$

Proof. I. *Semi-angle* $\theta < \pi/2$: Since T, T_n are m-sectorial and the numerical ranges of T_n satisfy (3.6), the set $\mathcal{S}(M_{\mathrm{Tr}}) \cup \overline{\mathrm{W}(T)}$ contains all spectra and is itself contained in some right half-plane. In particular, $(-\infty, \delta_1) \subset \bigcap_n \varrho(T_n) \cap \varrho(T)$ for some $\delta_1 \in \mathbb{R}$.

Now let $\lambda_0 \notin \mathcal{S}(M_{\mathrm{Tr}}) \cup \overline{\mathrm{W}(T)}$ be arbitrary; without loss of generality, we assume that $\lambda_0 = 0$, *i.e.* $0 \notin \mathcal{S}(M_{\mathrm{Tr}}) \cup \overline{\mathrm{W}(T)}$; otherwise we replace T, T_n by $T - \lambda_0$, $T_n - \lambda_0$, respectively. Then there exists $d_0 > 0$ such that $\operatorname{dist}(0, \overline{\mathrm{W}(T_n)}) \geq d_0$ and $\operatorname{dist}(0, \overline{\mathrm{W}(T)}) \geq d_0$.

We prove the claim by contradiction. Suppose that $T_n^{-1}\chi_{\Omega_n} f \to T^{-1} f$ in $L^2(\mathbb{R}^d, \mathbb{C})$ does not hold. Then there exist $\delta > 0$ and an infinite subset $I \subset \mathbb{N}$ such that $\|T_n^{-1}\chi_{\Omega_n} f - T^{-1} f\| \geq \delta$ for all $n \in I$. We will show that $\{T_n^{-1}\chi_{\Omega_n} f\}_{n\in I}$ contains a subsequence converging to $T^{-1}f$, a contradiction.

To simplify the notation, we set $f_n := \chi_{\Omega_n} f$, $\phi_n := T_n^{-1} f_n$. Note that

$$\|\phi_n\| = \|T_n^{-1} f_n\| \leq \|T_n^{-1}\| \|f\| \leq \frac{\|f\|}{d_0}$$

and rewrite $T_n \phi_n = f_n$ in terms of forms,

$$\forall\, \phi \in \mathcal{D}(t_n): \quad t_n(\phi_n, \varphi) = \langle f_n, \phi \rangle_n. \tag{4.3}$$

If we insert $\phi = \phi_n$ and take real parts in the equation in (4.3), we obtain

$$\|\nabla \phi_n\|_n^2 + \operatorname{Re} \int_{\partial \Omega_n^{\mathrm{R}}} a_n |\phi_n|^2 \,\mathrm{d}\sigma + \operatorname{Re} q_0[\phi_n] + \operatorname{Re} w[\phi_n] = \operatorname{Re}\langle f_n, \phi_n \rangle_n.$$

Taking absolute values on both sides and using the relative $\|\nabla \cdot\|^2$-bounds of the boundary term and of w, *cf.* the trace embedding (3.2), Assumption (III.iii), and (2.6), we arrive at

$$\begin{aligned}
&(1 - \sqrt{\varepsilon} M_{\mathrm{Tr}} - b_w(1+\varepsilon)) \|\nabla \phi_n\|_n^2 + \operatorname{Re} q_0[\phi_n] \\
&\leq \|f\| \|\phi_n\| + \left(\frac{M_{\mathrm{Tr}}}{\sqrt{\varepsilon}} + C_{w,\varepsilon} \right) \|\phi_n\|^2 \\
&\leq \left(\frac{1}{d_0} + \left(\frac{M_{\mathrm{Tr}}}{\sqrt{\varepsilon}} + C_{w,\varepsilon} \right) \frac{1}{d_0^2} \right) \|f\|^2 =: K_1(\varepsilon),
\end{aligned}$$

where $K_1(\varepsilon) > 0$ is independent of n. If we choose $\varepsilon > 0$ sufficiently small, we find that there exists $K_2 \geq 0$, independent of n, such that

$$\|\nabla \phi_n\|_n \leq K_2, \qquad \operatorname{Re} q_0[\phi_n] \leq K_2. \tag{4.4}$$

Let $\zeta_n \in C_0^\infty(B_{r_n+1}(0), \mathbb{R}) \subset C_0^\infty(\Omega_n, \mathbb{R})$ be such that

$$0 \leq \zeta_n \leq 1, \quad \zeta_n \restriction B_{r_n}(0) = 1, \quad \|\zeta_n\|_\infty + \|\nabla \zeta_n\|_\infty \leq M_1, \tag{4.5}$$

where r_n are the radii used in Assumption III and $M_1 > 0$ is independent of n. We define $\psi_n := \zeta_n \phi_n$. Note that ψ_n coincides with ϕ_n on $B_{r_n}(0)$ and its support is contained in Ω_n. It is easy to see that $\psi_n \in W^{1,2}\left(\mathbb{R}^d, \mathbb{C}\right)$ and that, by (4.4) and (4.5), there exists $K \geq 0$ such that

$$\|\psi_n\| \leq \|\phi_n\| \leq \frac{\|f\|}{d_0}, \quad \|\nabla \psi_n\| \leq K, \quad \operatorname{Re} q_0[\psi_n] \leq K.$$

Hence $\{\psi_n\}_{n\in I}$ is a bounded sequence in $\mathcal{H}_1 := W^{1,2}\left(\mathbb{R}^d, \mathbb{C}\right) \cap \mathcal{D}(q_0)$ equipped with the norm $(\|\nabla \cdot\|^2 + \operatorname{Re} q_0[\cdot] + \|\cdot\|^2)^{1/2}$. Therefore there exists a subsequence $\{n_k\}_k \subset I$ such that $\{\psi_{n_k}\}_k$ converges weakly in $\mathcal{H}_1$ to some $\psi \in \mathcal{H}_1$. Since $\mathcal{H}_1$ is compactly embedded in $L^2\left(\mathbb{R}^d, \mathbb{C}\right)$, *cf.* Rellich's criterion [48, Thm. XIII.65], the sequence $\{\psi_{n_k}\}_k$ converges to ψ in $L^2\left(\mathbb{R}^d, \mathbb{C}\right)$.

Now we prove that $\{\phi_{n_k}\}_k$ converges to ψ. The properties of ζ_n, *cf.* (4.5), imply

$$\|\phi_n - \psi_n\|^2 \leq \int_{|x| \geq r_n} |\phi_n|^2 \,\mathrm{d}x. \tag{4.6}$$

Using $\operatorname{Re} Q_0 > 0$ and $\operatorname{Re} Q_0(x) \to \infty$ as $|x| \to \infty$, we obtain

$$K_2 \geq \operatorname{Re} q_0[\phi_n] \geq \int_{|x|\geq r_n} \operatorname{Re} Q_0 |\phi_n|^2 \,\mathrm{d}x \geq \left(\operatorname*{ess\,inf}_{|x|\geq r_n} \operatorname{Re} Q_0 \right) \int_{|x|\geq r_n} |\phi_n|^2 \,\mathrm{d}x; \quad (4.7)$$

hence, since $r_n \to \infty$,

$$\|\phi_n - \psi_n\| \to 0, \quad n \to \infty, \quad (4.8)$$

and thus $\{\phi_{n_k}\}_k$ converges to ψ.

Finally, to obtain the contradiction, we prove that $\psi = T^{-1} f$, *i.e.*

$$\psi \in \mathcal{D}(T), \quad f = T\psi. \quad (4.9)$$

To this end, we show that

$$\forall\, \varphi \in C_0^\infty\left(\mathbb{R}^d, \mathbb{C}\right): \quad t_{n_k}(\phi_{n_k}, \varphi) \to t(\psi, \varphi), \quad k \to \infty;$$

then $\langle f, \varphi\rangle = \lim_{k\to\infty} \langle f_{n_k}, \varphi\rangle_{n_k} = t(\psi, \varphi)$ by (4.3), and hence (4.9) follows from the representation theorem [37, Thm. VI.2.1] and the fact that $C_0^\infty\left(\mathbb{R}^d, \mathbb{C}\right)$ is a core of t, *cf.* Proposition 2.3.

Let $\varphi \in C_0^\infty\left(\mathbb{R}^d, \mathbb{C}\right)$. There exists $n(\varphi) \in \mathbb{N}$ such that, for all $n \geq n(\varphi)$, we have $\operatorname{supp} \varphi \subset B_{r_n}(0) \subset \Omega_n$ and therefore $\zeta_n \varphi = \varphi$, $\zeta_n \nabla\varphi = \nabla\varphi$ by (4.5) and

$$\langle \nabla\phi_n, \nabla\varphi\rangle_n = \langle \nabla\psi_n, \nabla\varphi\rangle, \quad \int_{\Omega_n} Q_0\, \phi_n \overline{\varphi} \,\mathrm{d}x = \int_{\mathbb{R}^d} Q_0\, \psi_n \overline{\varphi} \,\mathrm{d}x,$$

$$\int_{\partial\Omega_n^{\mathrm{R}}} a_n\, \phi_n \overline{\varphi} \,\mathrm{d}\sigma = 0, \qquad \langle f_n, \varphi\rangle_n = \langle f, \varphi\rangle.$$

Since w is relatively bounded with respect to $\|\nabla \cdot\|^2$, it is a bounded form on $\mathcal{H}_1$. Therefore, $w(\psi_{n_k}, g) \to w(\psi, g)$ for any $g \in \mathcal{H}_1$. Recall that the function ζ in Assumption I satisfies $\operatorname{supp} \zeta \subset B_R(0)$. Hence, since ψ_{n_k} and ϕ_{n_k} coincide on $B_{r_n}(0) \supset B_R(0)$, we have $(\phi_{n_k} - \psi_{n_k})\sqrt{\zeta} = 0$. Thus the splitting property (2.3) of w in Assumption I and the polarization identity [37, Eq. VI.(1.1)] imply

$$w(\phi_{n_k}, \varphi) - w(\psi_{n_k}, \varphi) = w_2(\sqrt{1-\zeta}\,(\phi_{n_k} - \psi_{n_k}), \sqrt{1-\zeta}\,\varphi).$$

Because the form w_2 is bounded, *cf.* Assumption (I.iv), by (4.8), we have $w(\phi_{n_k}, \varphi) - w(\psi_{n_k}, \varphi) \to 0$. Altogether, we obtain

$$\lim_{k\to\infty} t_{n_k}(\phi_{n_k}, \varphi) = \lim_{k\to\infty} t(\psi_{n_k}, \varphi) + (w(\phi_{n_k}, \varphi) - w(\psi_{n_k}, \varphi)) = \lim_{k\to\infty} t(\psi_{n_k}, \varphi),$$

and the latter equals $t(\psi, \varphi)$ since ψ is the weak limit of $\{\psi_{n_k}\}_k$ in $\mathcal{H}_1$.

II. *Semi-angle* $\theta \geq \pi/2$: The intersection of the resolvent sets is non-empty since there exists $\delta_2 \in \mathbb{R}$ such that $(-\infty, \delta_2) \subset \mathcal{R}(b') \cap \mathcal{R}(b', M_{\mathrm{Tr}}) \neq \emptyset$, *cf.* (2.9), (3.10).

Let $\lambda_0 \in \mathcal{R}(b') \cap \mathcal{R}(b', M_{\mathrm{Tr}}) \supset (-\infty, \delta_2)$ be arbitrary. We can follow [59, Thm. 1] which can be straightforwardly generalized to the non-selfadjoint case if a uniform bound on $\|(T_n - \lambda_0)^{-1}\|$ is available; such a bound is given by (3.11). In order to check the assumptions of [59, Thm. 1], recall that $\mathcal{D}(T_{\min}) = C_0^\infty\left(\mathbb{R}^d, \mathbb{C}\right)$ is a core of T, *cf.* Proposition 2.6. For every $f \in \mathcal{D}(T_{\min})$ there exists $n_0(f) \in \mathbb{N}$ such that $\operatorname{supp} f \subset \Omega_n$ for all $n \geq n_0(f)$. Then, for $n \geq n_0(f)$, we have $f_n := \chi_{\Omega_n} f \in \mathcal{D}(T_n)$ and $Tf = T_n f_n$. Notice that, since $\mathcal{D}(T_n)$ is not described explicitly, we use the first representation theorem [37, Thm. VI.2.1] to verify the latter. $\square$

Lemma 4.4. *Let the assumptions of Theorem* 4.1 *be satisfied and let* $I \subset \mathbb{N}$ *be an arbitrary infinite subset. Then every sequence of elements* $\phi_n \in \mathcal{D}(T_n)$, $n \in I$, *such that* $\left\{\|T_n\phi_n\|_n^2 + \|\phi_n\|_n^2\right\}_n$ *is bounded has a convergent subsequence in* $L^2\left(\mathbb{R}^d, \mathbb{C}\right)$.

Proof. Let $\phi_n \in \mathcal{D}(T_n)$, $n \in I$, and $M \geq 0$ be such that

$$\|T_n\phi_n\|_n^2 + \|\phi_n\|_n^2 \leq M. \tag{4.10}$$

I. *Semi-angle* $\theta < \pi/2$: The bound (4.10) implies that $|t_n[\phi_n]| \leq M/2$. Define $f_n := T_n\phi_n$ and note that $\|f_n\|_n^2 \leq M$. Now we proceed analogously as in (4.3)–(4.8) to find a convergent subsequence of $\{\phi_n\}_n$.

II. *Semi-angle* $\theta \geq \pi/2$: Let $\{\zeta_n\}_n$ be the family of functions defined in (4.5). We set $\psi_n := \zeta_n\phi_n$. Using the inequality (3.13) and the equivalence of norms in (3.9), we obtain the existence of $\widetilde{M} \geq 0$ such that $\|\nabla\psi_n\| + \|Q_0\psi_n\| \leq \widetilde{M}$. Hence, it follows from Rellich's criterion [48, Thm. XIII.65] that $\{\psi_n\}_n$ is contained in a compact subset of $L^2(\mathbb{R}^d, \mathbb{C})$, thus it has a convergent subsequence $\{\psi_{n_k}\}_k$. Finally, using an analogous argument as (4.6)–(4.7) with (4.7) replaced by

$$\widetilde{M}^2 \geq \int_{|x|\geq r_n} |Q_0|^2|\phi_n|^2 \,\mathrm{d}x \geq \left(\operatorname*{ess\,inf}_{|x|\geq r_n} |Q_0(x)|^2\right) \int_{|x|\geq r_n} |\phi_n|^2 \,\mathrm{d}x, \tag{4.11}$$

one may show that $\{\phi_{n_k}\}_k$ has the same limit as $\{\psi_{n_k}\}_k$. □

Proof of Theorem 4.1. By Lemma 4.3, there exists $\gamma > 0$ such that we have generalized strong resolvent convergence at all $\lambda_0 \in (-\infty, -\gamma)$. One may verify that Assumptions I, II, III remain valid under complex conjugation of q, Q, a_n, W, and with w replaced by the adjoint form w^*. Hence Propositions 2.3, 2.6, 3.4, 3.5 define closed operators $\widehat{T}, \widehat{T}_n$, and the latter coincide with the adjoints T^*, T_n^*, *cf.* [37, Thm. VI.2.5] (for semi-angle $\theta < \pi/2$) and [27, Thm. VII.2.5, 2.6, Cor. 2.7], [36, Cor. 1] (for semi-angle $\theta \geq \pi/2$). Moreover, Lemma 4.3 implies that $(T_n^* - \lambda_0)^{-1}\chi_{\Omega_n}$ converges strongly to $(T^* - \lambda_0)^{-1}$. Then [5, Thm. 3.4] yields that the resolvents converge even in norm provided we verify that $\{(T_n - \lambda_0)^{-1}\chi_{\Omega_n} : n \in \mathbb{N}\}$ and $\{(T_n^* - \lambda_0)^{-1}\chi_{\Omega_n} : n \in \mathbb{N}\}$ are collectively compact sets. The claim for the former set follows from [5, Prop. 2.1] since every $(T_n - \lambda_0)^{-1}$ is compact and the sequence of embeddings (4.1) is discretely compact, *cf.* Lemma 4.4; the reasoning for the set of adjoint operators is analogous.

Now let $\lambda \in \varrho(T)$ with $\lambda \neq \lambda_0$ be arbitrary. By the spectral mapping theorem we have $\mu := (\lambda - \lambda_0)^{-1} \in \varrho\big((T_n - \lambda_0)^{-1}\big)$. By [37, Thm. IV.2.25], together with (4.2) for $\lambda = \lambda_0$, there exists $n_\lambda \in \mathbb{N}$ such that, for all $n \geq n_\lambda$, we have $\mu \in \varrho\big((T_n - \lambda_0)^{-1}\big)$ and so $\lambda \in \varrho(T_n)$. A straightforward application of the first resolvent identity yields

$$\begin{aligned} &\big((T_n - \lambda)^{-1}\chi_{\Omega_n} - (T - \lambda)^{-1}\big)\, S_n \\ &= \big(I + (\lambda - \lambda_0)(T - \lambda)^{-1}\big)\,\big((T_n - \lambda_0)^{-1}\chi_{\Omega_n} - (T - \lambda_0)^{-1}\big) \end{aligned} \tag{4.12}$$

with $S_n = I - (\lambda - \lambda_0)(T_n - \lambda_0)^{-1}\chi_{\Omega_n}$. Since $S := \lim_{n\to\infty} S_n$ has a bounded inverse, the operator S_n is boundedly invertible for all sufficiently large n and $\|S_n^{-1}\|$ is uniformly bounded, *cf.* [37, Thm. IV.1.16]. Now the convergence (4.2) follows from the convergence at λ_0 and (4.12). □

5. Convergence of spectra and pseudospectra

In the following theorem, we prove that $\{T_n\}_n$ is a spectrally exact approximation of T, i.e. *all* eigenvalues of T are approximated and *no* spectral pollution occurs. In addition, we prove norm convergence of the spectral projections.

Theorem 5.1. *Let Assumption* III *be satisfied and assume that*

I. *in the case of semi-angle $\theta < \pi/2$, Assumption* I *holds and T, T_n, $n \in \mathbb{N}$, are the operators defined in Propositions* 2.3, 3.4, *respectively;*
II. *in the case of semi-angle $\theta \geq \pi/2$, Assumption* II *holds with $b_U < 1$ and T, T_n, $n \in \mathbb{N}$, are the operators defined in Propositions* 2.6, 3.5, *respectively.*

Then the following hold:

i) Spectral inclusion with preservation of algebraic multiplicity: *If* $\lambda \in \mathbb{C}$ *is an eigenvalue of* T *of algebraic multiplicity* m*, then, for* n *large enough,* T_n *has exactly* m *eigenvalues* (*repeated according to their algebraic multiplicities*) *in a neighbourhood of* λ *which converge to* λ *as* $n \to \infty$ *and the corresponding spectral projections converge in norm.*

ii) No spectral pollution: *If* $\{\lambda_n\}_n \subset \mathbb{C}$ *is a sequence of eigenvalues* $\lambda_n \in \sigma(T_n)$*,* $n \in \mathbb{N}$*, such that there exists* $\lambda \in \mathbb{C}$ *and a subsequence* $\{\lambda_{n_k}\}_k \subset \{\lambda_n\}_n$ *with* $\lambda_{n_k} \to \lambda$ *as* $k \to \infty$*, then* λ *is an eigenvalue of* T*.*

Proof. i) Since T has compact resolvent, every eigenvalue $\lambda \in \sigma(T)$ is isolated, *i.e.* there exists $\varepsilon > 0$ such that $\overline{B_\varepsilon(\lambda)} \backslash \{\lambda\} \subset \varrho(T)$. By Theorem 4.1, we have $\|(T-z)^{-1} - (T_n - z)^{-1}\chi_{\Omega_n}\| \to 0$ for every $z \in \partial B_\varepsilon(\lambda)$ and the convergence is uniform in $\partial B_\varepsilon(\lambda)$ since $\partial B_\varepsilon(\lambda)$ is compact, *cf.* Remark 4.2. Hence the spectral projections

$$E := -\frac{1}{2\pi \mathrm{i}} \int_{\partial B_\varepsilon(\lambda)} (T-z)^{-1}\,\mathrm{d}z, \qquad E_n := -\frac{1}{2\pi \mathrm{i}} \int_{\partial B_\varepsilon(\lambda)} (T_n-z)^{-1}\,\mathrm{d}z$$

satisfy $\|E - E_n\chi_{\Omega_n}\| \to 0$ and therefore there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $\operatorname{rank} E_n = \operatorname{rank}(E_n\chi_{\Omega_n}) = \operatorname{rank} E = m$.

ii) Spectral pollution cannot occur since it would contradict the locally uniform convergence of the resolvents, *cf.* Remark 4.2. □

Based on [47, Thm. 2], we prove an estimate on the convergence rate of the arithmetic mean of the eigenvalues in terms of the decay rate of the functions in the corresponding algebraic eigenspace. An analogous result can be obtained, using [47, Thm. 6], for the individual eigenvalues instead of their arithmetic mean; if, however, λ is not semi-simple, *i.e.* λ has ascent greater than one, then the convergence of the individual eigenvalues is slower than the one of their arithmetic mean.

Theorem 5.2. *Let Assumption* III *be satisfied and assume that*

I. *in the case of semi-angle* $\theta < \pi/2$*, Assumption* I *holds and for every* $\varphi \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$ *and* $f, g \in \mathcal{D}(w)$*,*

$$w(\varphi f, g) = w(f, \overline{\varphi} g), \tag{5.1}$$

and T*,* T_n*,* $n \in \mathbb{N}$*, are the operators defined in Propositions* 2.3, 3.4*, respectively;*

II. *in the case of semi-angle* $\theta \geq \pi/2$*, Assumption* II *holds with* $b_U < 1$ *and* T*,* T_n*,* $n \in \mathbb{N}$*, are the operators defined in Propositions* 2.6, 3.5*, respectively.*

Let $\lambda \in \sigma(T)$ *be an eigenvalue of algebraic multiplicity* m*, let* $\mathcal{L}_\lambda(T)$ *be the corresponding algebraic eigenspace and let* $\{\lambda_{1;n}, \dots, \lambda_{m;n}\} \subset \sigma(T_n)$ *be the eigenvalues of* T_n *converging to* λ *as* $n \to \infty$*,* cf. *Theorem* 5.1*. Then there exists* $C \geq 0$*, independent of* n*, such that*

$$\left|\lambda - \frac{1}{m}\sum_{j=1}^{m} \lambda_{j;n}\right| \leq C \max_{\substack{\phi \in \mathcal{L}_\lambda(T) \\ \|\phi\|=1}} \left\|\phi \restriction \mathbb{R}^d \backslash B_{r_n}(0)\right\| \tag{5.2}$$

where r_n *are the radii used in Assumption* (III.i)*.*

Remark 5.3. The decay rate of $\phi \in \mathcal{L}_\lambda(T)$ can be further estimated as

$$\max_{\substack{\phi \in \mathcal{L}_\lambda(T) \\ \|\phi\|=1}} \left\|\phi \restriction \mathbb{R}^d \backslash B_{r_n}(0)\right\| \leq \frac{D}{\operatorname*{ess\,inf}_{|x| \geq r_n} |Q_0(x)|^\iota},$$

where $D \geq 0$ is independent of n and $\iota = 1/2$ if $\theta < \pi/2$ and $\iota = 1$ if $\theta \geq \pi/2$, respectively, *cf.* (4.7) and (4.11). However, the decay rate of $\phi \in \mathcal{L}_\lambda(T)$ is typically much faster than the growth of $|Q_0|$, in fact exponential, *cf.* [2, 52, 20] or [50] for complex polynomial potentials or [38] for new general results in the case of semi-angle $\theta \geq \pi/2$.

Proof of Theorem 5.2. Let $\mu \in \varrho(T)$. Theorem 4.1 implies that $\mu \in \varrho(T_n)$ for all sufficiently large n and $\|(T_n - \mu)^{-1}\chi_{\Omega_n} - (T-\mu)^{-1}\| \to 0$. The spectral mapping theorem yields $\nu := (\lambda-\mu)^{-1} \in \sigma((T-\mu)^{-1})$ and the eigenvalues $\nu_{j;n} := (\lambda_{j;n}-\mu)^{-1} \in \sigma((T_n-\mu)^{-1}) \subset \sigma((T_n-\mu)^{-1}\chi_{\Omega_n})$ satisfy $\nu_{j;n} \to \nu$ as $n \to \infty$. Now the identity $|\lambda - \lambda_{j;n}| = |\nu\nu_{j;n}|^{-1}|\nu - \nu_{j;n}|$ implies that it suffices to study the convergence rate for $\nu_{j;n}$.

By [47, Thm. 2], there exists $C_1 \geq 0$, independent of n, such that,

$$\left|\nu - \frac{1}{m}\sum_{j=1}^{m} \nu_{j;n}\right| \leq C_1 \left\| \left((T-\mu)^{-1} - (T_n-\mu)^{-1}\chi_{\Omega_n}\right) \restriction \mathcal{L}_\lambda(T) \right\|.$$

Below we show that there exists $\widetilde{C} > 0$, independent of n, such that, for every $\phi \in \mathcal{L}_\lambda(T)$,

$$\begin{aligned} \left\| \left((T-\mu)^{-1} - (T_n-\mu)^{-1}\chi_{\Omega_n}\right)\phi \right\| \leq \widetilde{C}\big(&\left\|\phi \restriction \mathbb{R}^d \backslash B_{r_n}(0)\right\| \\ &+ \left\|((T-\mu)^{-1}\phi) \restriction \mathbb{R}^d \backslash B_{r_n}(0)\right\|\big). \end{aligned} \tag{5.3}$$

Since $\mathcal{L}_\lambda(T)$ is an invariant subspace of T, we have

$$\begin{aligned} \max_{\substack{\phi \in \mathcal{L}_\lambda(T) \\ \|\phi\|=1}} \left\|((T-\mu)^{-1}\phi) \restriction \mathbb{R}^d \backslash B_{r_n}(0)\right\| &\leq \|(T-\mu)^{-1}\| \max_{\substack{\phi \in \mathcal{L}_\lambda(T) \\ \|\phi\|=1}} \frac{\left\|((T-\mu)^{-1}\phi) \restriction \mathbb{R}^d \backslash B_{r_n}(0)\right\|}{\left\|(T-\mu)^{-1}\phi\right\|} \\ &\leq \|(T-\mu)^{-1}\| \max_{\substack{\psi \in \mathcal{L}_\lambda(T) \\ \|\psi\|=1}} \|\psi \restriction \mathbb{R}^d \backslash B_{r_n}(0)\|, \end{aligned}$$

hence the estimate (5.2) in the claim follows.

To prove (5.3), let $\{\zeta_n\}_n \subset C_0^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$ be such that, with $\widetilde{\zeta}_n := 1 - \zeta_n$,

$$\begin{aligned} &0 \leq \zeta_n \leq 1, \quad \zeta_n \restriction B_{r_n}(0) = 1, \quad \operatorname{supp} \zeta_n \subset B_{r_n+1}(0), \\ &\|\nabla\zeta_n\|_\infty + \|\Delta\zeta_n\|_\infty = \|\nabla\widetilde{\zeta}_n\|_\infty + \|\Delta\widetilde{\zeta}_n\|_\infty \leq C_2 \end{aligned}$$

where $C_2 > 0$ is independent of n. Let $\phi \in \mathcal{L}_\lambda(T)$ and set $\psi := (T-\mu)^{-1}\phi$. First we adapt the approach of [28] or [39, Prop. 5.3] based on

$$\|g\| = \sup_{f \neq 0} \frac{|\langle g, f\rangle|}{\|f\|}.$$

Let $f \in L^2(\mathbb{R}^d, \mathbb{C})$, $f \neq 0$. Then, with $\widetilde{\chi}_{\Omega_n} := 1 - \chi_{\Omega_n}$, we write

$$\begin{aligned} &\langle \left((T-\mu)^{-1} - (T_n-\mu)^{-1}\chi_{\Omega_n}\right)\phi, f\rangle \\ &= \langle (T-\mu)^{-1}\phi, \chi_{\Omega_n} f\rangle + \langle (T-\mu)^{-1}\phi, \widetilde{\chi}_{\Omega_n} f\rangle - \langle (T_n-\mu)^{-1}\chi_{\Omega_n}\phi, \chi_{\Omega_n} f\rangle, \end{aligned} \tag{5.4}$$

and the second term satisfies

$$\left|\langle (T-\mu)^{-1}\phi, \widetilde{\chi}_{\Omega_n} f\rangle\right| = |\langle \widetilde{\chi}_{\Omega_n}\psi, f\rangle| \leq \|f\|\|\psi \restriction \mathbb{R}^d \backslash B_{r_n}(0)\|. \tag{5.5}$$

Since $\mu \in \varrho(T_n)$, we have $\overline{\mu} \in \varrho(T_n^*)$. Define $g_n := (T_n^* - \overline{\mu})^{-1}\chi_{\Omega_n} f \in \mathcal{D}(T_n^*)$. Note that the functions g_n are uniformly bounded, $\|g_n\| \leq \sup_n \|(T_n-\mu)^{-1}\| \, \|f\|$. Now the remaining terms on the right-hand side of (5.4) can be written as

$$\begin{aligned} &\langle (T-\mu)^{-1}\phi, \chi_{\Omega_n} f\rangle - \langle (T_n-\mu)^{-1}\chi_{\Omega_n}\phi, \chi_{\Omega_n} f\rangle \\ &= \langle \psi, (T_n^* - \overline{\mu})g_n\rangle - \langle \chi_{\Omega_n}(T-\mu)\psi, g_n\rangle \\ &= \langle \zeta_n\psi, (T_n^* - \overline{\mu})g_n\rangle + \langle \widetilde{\zeta}_n\psi, (T_n^* - \overline{\mu})g_n\rangle \\ &\quad - \langle \chi_{\Omega_n}(T-\mu)\psi, \zeta_n g_n\rangle - \langle \chi_{\Omega_n}(T-\mu)\psi, \widetilde{\zeta}_n g_n\rangle \\ &= \langle \zeta_n\psi, T_n^* g_n\rangle - \langle T\psi, \zeta_n g_n\rangle + \langle \widetilde{\zeta}_n\psi, \chi_{\Omega_n} f\rangle - \langle \chi_{\Omega_n}\phi, \widetilde{\zeta}_n g_n\rangle. \end{aligned} \tag{5.6}$$

The last two terms can be estimated easily,

$$\begin{aligned}&\left|\langle\widetilde{\zeta}_n\psi,\chi_{\Omega_n}f\rangle-\langle\chi_{\Omega_n}\phi,\widetilde{\zeta}_n g_n\rangle\right|\le\|\widetilde{\zeta}_n\psi\|\|f\|+\|\widetilde{\zeta}_n\phi\|\|g_n\|\\&\le\|f\|\big(\big\|\psi\restriction\mathbb{R}^d\backslash B_{r_n}(0)\big\|+\sup_n\big\|(T_n-\mu)^{-1}\big\|\big\|\phi\restriction\mathbb{R}^d\backslash B_{r_n}(0)\big\|\big).\end{aligned}$$

Below we show, separately for the cases $\theta<\pi/2$ and $\theta\ge\pi/2$, that the first two terms of (5.6) satisfy

$$\langle\zeta_n\psi,T_n^*g_n\rangle-\langle T\psi,\zeta_n g_n\rangle=\langle\psi\Delta\zeta_n,g_n\rangle+2\langle\psi\nabla\zeta_n,\nabla g_n\rangle \tag{5.7}$$

and there exists $C_3\ge 0$ such that

$$\|\nabla g_n\|_n^2\le C_3\|f\|^2. \tag{5.8}$$

Then, since $\nabla\zeta_n\restriction B_{r_n}(0)=0$, $\Delta\zeta_n\restriction B_{r_n}(0)=0$, it follows that there is $C_4\ge 0$ with

$$\begin{aligned}&|\langle\zeta_n\psi,T_n^*g_n\rangle-\langle T\psi,\zeta_n g_n\rangle|\\&\le(\|g_n\|_n\|\Delta\zeta_n\|_\infty+2\|\nabla g_n\|_n\|\nabla\zeta_n\|_\infty)\big\|\psi\restriction\mathbb{R}^d\backslash B_{r_n}(0)\big\|\\&\le C_4\|f\|\big\|\psi\restriction\mathbb{R}^d\backslash B_{r_n}(0)\big\|.\end{aligned} \tag{5.9}$$

Thus summarizing (5.5)–(5.9) we obtain (5.3).

It remains to prove (5.7) and (5.8). First we study the case $\theta<\pi/2$. Since $\psi\in\mathcal{D}(T)\subset W^{1,2}\left(\mathbb{R}^d,\mathbb{C}\right)\cap\mathcal{D}(q_0)$ and $\zeta_n\in C_0^\infty\left(\mathbb{R}^d,\mathbb{R}\right)$ with $\operatorname{supp}\zeta_n\subset B_{r_n+1}(0)\subset\Omega_n$, we conclude that $\zeta_n\psi\in\mathcal{D}(t_n)=\mathcal{D}(t_n^*)$ and

$$\langle\zeta_n\psi,T_n^*g_n\rangle=\overline{\langle T_n^*g_n,\zeta_n\psi\rangle}=\overline{t_n^*(g_n,\zeta_n\psi)}=t_n(\zeta_n\psi,g_n).$$

Moreover, it follows from $g_n\in\mathcal{D}(T_n^*)\subset W^{1,2}\left(\Omega_n,\mathbb{C}\right)\cap\mathcal{D}(q_0)$ and the properties of ζ_n that $\zeta_n g_n\in\mathcal{D}(t)$. Hence, using $\operatorname{supp}\zeta_n\subset\Omega_n$, assumption (5.1), and integration by parts, we obtain

$$\begin{aligned}\langle\zeta_n\psi,T_n^*g_n\rangle-\langle T\psi,\zeta_n g_n\rangle&=t_n(\zeta_n\psi,g_n)-t(\psi,\zeta_n g_n)\\&=\langle\nabla(\zeta_n\psi),\nabla g_n\rangle+\int_{\partial\Omega_n^{\mathrm{R}}}a_n\zeta_n\psi\,\overline{g_n}\,\mathrm{d}\sigma+q_0(\zeta_n\psi,g_n)+w(\zeta_n\psi,g_n)\\&\quad-\langle\nabla\psi,\nabla(\zeta_n g_n)\rangle-q_0(\psi,\zeta_n g_n)-w(\psi,\zeta_n g_n)\\&=\langle\nabla(\zeta_n\psi),\nabla g_n\rangle-\langle\nabla\psi,\nabla(\zeta_n g_n)\rangle\\&=\langle\psi\nabla\zeta_n,\nabla g_n\rangle-\langle\nabla\psi,g_n\nabla\zeta_n\rangle\\&=\langle\psi\Delta\zeta_n,g_n\rangle+2\langle\psi\nabla\zeta_n,\nabla g_n\rangle,\end{aligned}$$

which proves (5.7). The estimate (3.7) implies that there exist $C_5,C_6\ge 0$ with

$$|t_n^*[g_n]|=|t_n[g_n]|\ge\operatorname{Re}t_n[g_n]\ge C_5\left(\|\nabla g_n\|_n^2+\operatorname{Re}q_0[g_n]\right)-C_6\|g_n\|_n^2.$$

Since $\operatorname{Re}q_0[g_n]\ge 0$ and $T_n^*g_n=\overline{\mu}g_n+\chi_{\Omega_n}f$, there exists $C_3\ge 0$ such that

$$\|\nabla g_n\|_n^2\le\frac{1}{C_5}\left(|t_n^*[g_n]|+C_6\|g_n\|_n^2\right)=\frac{1}{C_5}\left(|\langle T_n^*g_n,g_n\rangle_n|+C_6\|g_n\|_n^2\right)\le C_3\|f\|^2.$$

For $\theta\ge\pi/2$, we first note that $T_n^*=-(\Delta_n^{\mathrm{DR}})^*+(Q_0-U+W)^*$ where the adjoint of the potential is simply its complex conjugate, *cf.* the proof of Theorem 4.1 for details. Hence, with $\zeta_n\psi\in\mathcal{D}(s_{0,n})=\mathcal{D}(s_{0,n}^*)$ and integrating by parts in the last step, we get

$$\begin{aligned}\langle\zeta_n\psi,T_n^*g_n\rangle-\langle T\psi,\zeta_n g_n\rangle&=\langle\zeta_n\psi,-(\Delta_n^{\mathrm{DR}})^*g_n\rangle+\langle(Q_0-U+W)\zeta_n\psi,g_n\rangle\\&\quad-\langle-\Delta\psi,\zeta_n g_n\rangle-\langle(Q_0-U+W)\zeta_n\psi,g_n\rangle\\&=\langle\psi\Delta\zeta_n,g_n\rangle+2\langle\psi\nabla\zeta_n,\nabla g_n\rangle.\end{aligned}$$

Finally, adapting (3.13) and (3.9) for T_n^*, we see that there are $C_7,C_3\ge 0$ with

$$\|\nabla g_n\|_n^2\le C_7(\|T_n^*g_n\|^2+\|g_n\|^2)\le C_3\|f\|^2.$$

□

To conclude this section, we study the convergence properties of the pseudospectra of the operators T_n; here we use the following definition, *cf.* [56] for an overview.

Definition 5.4. Let $\varepsilon > 0$. The *ε-pseudospectrum* $\sigma_\varepsilon(A)$ of a closed operator A is

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{ \lambda \in \varrho(A) : \|(A-\lambda)^{-1}\| > \frac{1}{\varepsilon} \right\}.$$

To study convergence of the sequence of $\sigma_\varepsilon(T_n)$, $n \in \mathbb{N}$, we need a suitable metric for closed unbounded subsets of $\mathbb{C}$. We use convergence in *Attouch-Wets metric* d_{AW} which is a generalization of convergence in Hausdorff metric for unbounded sets, *cf.* [8, Chap. 3] for details and further discussions. We refrain from giving the definition of d_{AW} here since we only use the following, equivalent, characterization, *cf.* [8, Cor. 3.1.8].

Let $\Lambda, \Lambda_n \subset \mathbb{C}$ be closed non-empty subsets. Then the sequence $\{\Lambda_n\}_n$ converges to Λ in Attouch-Wets metric, $d_{\mathrm{AW}}(\Lambda_n, \Lambda) \to 0$, $n \to \infty$, if and only if for all closed balls $\overline{B_\varrho(0)}$, $\varrho > 0$,

$$\max \left\{ \sup_{w \in \Lambda_n \cap \overline{B_\varrho(0)}} \operatorname{dist}(w, \Lambda), \sup_{z \in \Lambda \cap \overline{B_\varrho(0)}} \operatorname{dist}(z, \Lambda_n) \right\} \to 0, \quad n \to \infty. \tag{5.10}$$

Theorem 5.5. *Let Assumption* III *be satisfied and assume that*

I. *in the case of semi-angle $\theta < \pi/2$, Assumption* I *holds and T, T_n, $n \in \mathbb{N}$, are the operators defined in Propositions* 2.3, 3.4*, respectively;*
II. *in the case of semi-angle $\theta \geq \pi/2$, Assumption* II *holds with $b_U < 1$ and T, T_n, $n \in \mathbb{N}$, are the operators defined in Propositions* 2.6, 3.5*, respectively.*

Then, for any $\varepsilon > 0$,

$$d_{\mathrm{AW}}\big(\overline{\sigma_\varepsilon(T_n)}, \overline{\sigma_\varepsilon(T)}\big) \to 0, \quad n \to \infty.$$

Proof. Since T has compact resolvent, *cf.* Proposition 2.3 (for semi-angle $\theta < \pi/2$) and Proposition 2.6 (for semi-angle $\theta \geq \pi/2$), its resolvent norm is not constant on any open subset of $\varrho(T)$, *cf.* [24, Thm. 2.2]. Then the claim follows from the generalized norm resolvent convergence of T_n to T, similarly as in [11, Thm. 2.1]. In fact, without assuming that condition (ii) in [11, Thm. 2.1] holds, the claim of [11, Lem. 4.3] can be modified as follows. For every $\delta > 0$ and $\mathcal{K} \subset \mathbb{C}$ compact, there exists $n_0 \in \mathbb{N}$ such that

$$\overline{\sigma_\varepsilon(T_n)} \cap \mathcal{K} \subset B_\delta(\overline{\sigma_\varepsilon(T)}), \quad \overline{\sigma_\varepsilon(T)} \cap \mathcal{K} \subset B_\delta(\overline{\sigma_\varepsilon(T_n)}), \quad n \geq n_0, \tag{5.11}$$

where $B_\delta(\Lambda)$ denotes the open δ-neighbourhood of the set Λ (called $\omega_\delta(\Lambda)$ in [11]). Now (5.11) yields the convergence (5.10) which, in turn, implies convergence in Attouch-Wets metric. □

6. Exterior domains

In this section, we extend our results to the situation of a Schrödinger operator T_Ω acting in $L^2(\Omega, \mathbb{C})$ where $\Omega \subset \mathbb{R}^d$ is an exterior domain, *i.e.* $\mathbb{R}^d \setminus \Omega$ is compact. We focus on dimension $d \geq 2$ since in $d = 1$ an exterior domain is not connected, although we can also treat the case when $\Omega \subset \mathbb{R}$ is a half-line. The generalization is almost straightforward and the proofs are analogous. Therefore we only mention major differences and additional ingredients.

For an exterior domain Ω we define the corresponding operator T_Ω in an analogous way as in Section 2. Since $\partial\Omega$ is non-empty, we now have to impose boundary conditions ensuring that T_Ω has non-empty resolvent set. While for the m-sectorial case $\theta < \pi/2$ we can allow a combination of Dirichlet and Robin conditions, determined by a function $a_{\mathrm{in}} : \partial\Omega^{\mathrm{R}} \to \mathbb{C}$, for the case of semi-angle $\theta \geq \pi/2$ only

Dirichlet conditions are allowed; this restriction is due to Kato's Theorem [27, Thm. VII.2.5] which we use to define T_Ω as a perturbation of an m-accretive operator.

Assumption IV. Let $d \geq 2$ and let $\Omega \subset \mathbb{R}^d$ be an exterior domain, *i.e.* $\mathbb{R}^d \backslash \Omega \neq \emptyset$ is compact,

$$\partial\Omega = \partial\Omega^{\mathrm{D}} \,\dot\cup\, \partial\Omega^{\mathrm{R}}$$

with $\partial\Omega^{\mathrm{D}}$ closed and with $\partial\Omega^{\mathrm{R}} = \emptyset$ if $\theta \geq \pi/2$. If $\theta < \pi/2$ and $\partial\Omega^{\mathrm{R}} \neq \emptyset$, we assume

(IV.i) *regularity of* $\partial\Omega$: Ω is Lipschitz;

(IV.ii) *control of Robin boundary term*: $a_{\mathrm{in}} \in L^\infty(\partial\Omega^{\mathrm{R}}, \mathbb{C})$.

The main results in Sections 4 and 5 generalize in a straightforward way to the situation of an exterior domain if analogues of the claims in Sections 2 and 3 are available. In Subsections 6.1, 6.2 below, we provide proof ideas of the latter and indicate additional modifications in order to prove the following theorem. Here, by $\|\cdot\|_\Omega$ we denote the norm of $L^2(\Omega, \mathbb{C})$.

Assumption I.Ω. The sesquilinear form q decomposes as $q = q_0 + w$ where q_0 and w have the following properties. The form q_0 is generated by $Q_0 \in L^1_{\mathrm{loc}}(\overline{\Omega}, \mathbb{C})$, *i.e.*

$$q_0[\cdot] := \int_\Omega Q_0 |\cdot|^2 \, \mathrm{d}x, \quad \mathcal{D}(q_0) := \left\{ f \in L^2(\Omega, \mathbb{C}) \,:\, Q_0|f|^2 \in L^1(\Omega, \mathbb{C}) \right\},$$

such that

(I.i.Ω) *sectoriality of Q_0 with semi-angle $\theta < \pi/2$*: there exist $c_0 > 0$ and $\theta \in [0, \pi/2)$ with

$$\operatorname{Re} Q_0 \geq c_0, \quad |\operatorname{Im} Q_0| \leq \tan\theta \, \operatorname{Re} Q_0;$$

(I.ii.Ω) *unboundedness of Q_0 at infinity*:

$$|Q_0(x)| \to \infty \quad \text{as} \quad |x| \to \infty.$$

For the form w, there exist $R > r > 0$ and $\zeta \in C_0^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$ with

$$\mathbb{R}^d \setminus \Omega \subset B_r(0), \quad \operatorname{supp}\zeta \subset B_R(0), \quad 0 \leq \zeta \leq 1, \quad \zeta \restriction B_r(0) = 1,$$

sesquilinear forms w_1, w_2 with $\overline{\mathcal{D}_{\Omega,R}}^{\|\cdot\|_{W^{1,2}(\Omega,\mathbb{C})}} \subset \mathcal{D}(w_1)$, $\mathcal{D}(w_2) = L^2(\Omega,\mathbb{C})$ where $\mathcal{D}_{\Omega,R} := \{f \in C^\infty(B_R(0) \cap \Omega, \mathbb{C}) : \exists f_0 \in C_0^\infty(B_R(0), \mathbb{C}), f = f_0 \restriction \Omega, \operatorname{supp} f \cap \partial\Omega^{\mathrm{D}} = \emptyset\}$ with

$$\forall\, f \in \mathcal{D}(w): \quad \sqrt{\zeta} f \in \overline{\mathcal{D}_{\Omega,R}}^{\|\cdot\|_{W^{1,2}(\Omega,\mathbb{C})}}, \quad w[f] = w_1[\sqrt{\zeta} f] + w_2[\sqrt{1-\zeta} f],$$

such that

(I.iii.Ω) *$\|\nabla\cdot\|^2_\Omega$-boundedness of w_1 in $L^2(B_R(0) \cap \Omega, \mathbb{C})$*: there are $a_w \geq 0$, $b_w \in [0,1)$ so that, for every $f \in \overline{\mathcal{D}_{\Omega,R}}^{\|\cdot\|_{W^{1,2}(\Omega,\mathbb{C})}}$,

$$|w_1[f]| \leq a_w \|f\|^2_\Omega + b_w \|\nabla f\|^2_\Omega;$$

(I.iv.Ω) *boundedness of w_2 outside $B_r(0)$*: there exists $M_w \geq 0$ so that, for every $f \in L^2(\Omega, \mathbb{C})$,

$$|w_2[(1 - \chi_{r,\Omega}) f]| \leq M_w \|f\|^2_\Omega,$$

where $\chi_{r,\Omega}$ is the characteristic function of $B_r(0) \cap \Omega$.

Assumption II.Ω. The function $Q \in L^2_{\mathrm{loc}}(\Omega, \mathbb{C})$ decomposes as

$$Q = Q_0 - U + W$$

where $\operatorname{Re} Q_0 \geq 0$, $U \geq 0$, $U \operatorname{Re} Q_0 = 0$, $W \in L^2_{\mathrm{loc}}(\overline{\Omega}, \mathbb{C})$, and the following hold.

(II.i.Ω) *regularity of Q_0 and U:* $Q_0 \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega}, \mathbb{C})$, $U \in L^\infty_{\mathrm{loc}}(\overline{\Omega}, \mathbb{R})$, and there exist a_∇, b_∇, a_U, $b_U \geq 0$ such that

$$|\nabla Q_0|^2 \leq a_\nabla + b_\nabla |Q_0|^2, \quad U^2 \leq a_U + b_U |\operatorname{Im} Q_0|^2;$$

(II.ii.Ω) *unboundedness of Q_0 at infinity:*

$$|Q_0(x)| \to \infty \quad \text{as} \quad |x| \to \infty.$$

There exist $R > r > 0$ such that $\mathbb{R}^d \setminus \Omega \subset B_r(0)$ and

(II.iii.Ω) *Δ-boundedness of W in $L^2(B_R(0)\cap\Omega, \mathbb{C})$:* there exist $a_W \geq 0$, $b_W \in [0, 1)$ such that, for every $f \in \{f \in W^{1,2}_0(B_R(0)\cap\Omega, \mathbb{C}) : \Delta f \in L^2(B_R(0)\cap\Omega, \mathbb{C})\}$,

$$\|Wf\|^2_\Omega \leq a_W \|f\|^2_\Omega + b_W \|\Delta f\|^2_\Omega;$$

(II.iv.Ω) *boundedness of W outside $B_r(0)$:* there exists $M_W \geq 0$ such that

$$\|(1 - \chi_{r,\Omega})W\|_\infty \leq M_W,$$

where $\chi_{r,\Omega}$ is the characteristic function of $B_r(0) \cap \Omega$.

Theorem 6.1. *Let Assumptions* IV *and* III *hold with $\mathbb{R}^d$ replaced by Ω and, if applicable, the regularity assumption* (III.ii) *with $\partial\Omega_n$ replaced by the smaller set $\partial\Omega_n \setminus \partial\Omega$. Further assume that*

I. *for semi-angle $\theta < \pi/2$, Assumption* I.Ω *holds;*
II. *for semi-angle $\theta \geq \pi/2$, Assumption* II.Ω *holds with $b_U < 1$.*

Then the statements of Theorems 4.1, 5.1, 5.2, *and* 5.5 *continue to hold with T, T_n replaced by T_Ω, $(T_\Omega)_n$, respectively .*

Proof of Theorem 6.1*:* Sketch of modifications of the proofs.

6.1. **Semi-angle $\theta < \pi/2$.** The analogue of Proposition 2.3 holds for the m-sectorial operator T_Ω which is uniquely determined by the closed sectorial form

$$\begin{aligned} t_\Omega[f] &:= \|\nabla f\|^2_\Omega + \int_{\partial\Omega^{\mathrm{R}}} a_{\mathrm{in}} |f|^2 \,\mathrm{d}\sigma + q_0[f] + w[f], \\ \mathcal{D}(t_\Omega) &:= \overline{\mathcal{D}_\Omega}^{(\|\cdot\|^2_{W^{1,2}(\Omega,\mathbb{C})} + \operatorname{Re} q_0[\cdot])^{\frac{1}{2}}}, \end{aligned} \tag{6.1}$$

where

$$\mathcal{D}_\Omega := \left\{ f \in C^\infty(\Omega, \mathbb{C}) : \exists f_0 \in C^\infty_0\left(\mathbb{R}^d, \mathbb{C}\right), f = f_0 \restriction \Omega, \operatorname{supp} f \cap \partial\Omega^{\mathrm{D}} = \emptyset \right\}. \tag{6.2}$$

To show that T_Ω has compact resolvent, first observe that

$$\mathcal{D}(t_\Omega) \subset W^{1,2}(\Omega, \mathbb{C}) \cap \mathcal{D}(q_0)$$

and that, by Assumptions (I.iii.Ω), (I.iv.Ω), and a trace embedding analogous to (3.2), there is a constant $c > 0$ so that $(\operatorname{Re} t_\Omega[\cdot] + c\|\cdot\|^2_\Omega)^{1/2}$ is equivalent to $(\|\cdot\|^2_{W^{1,2}(\Omega,\mathbb{C})} + \operatorname{Re} q_0[\cdot])^{1/2}$. Next, similarly as in (4.7), there is $C > 0$ such that, for all sufficiently large $n \in \mathbb{N}$ and all $f \in \mathcal{D}(t_\Omega)$,

$$\int_{x\in\Omega, |x|\geq n} |f|^2 \,\mathrm{d}x \leq C \frac{\operatorname{Re} t_\Omega[f] + c\|f\|^2_\Omega}{\operatorname*{ess\,inf}_{x\in\Omega, |x|\geq n} \operatorname{Re} Q_0}. \tag{6.3}$$

Therefore [1, Thm. 2.33] and Assumption (I.ii.Ω) imply that the embedding $(\mathcal{D}(t_\Omega), (\operatorname{Re} t_\Omega[\cdot] + c\|\cdot\|^2_\Omega)^{1/2}) \hookrightarrow L^2(\Omega, \mathbb{C})$ is compact.

The approximating operators $(T_\Omega)_n$ are introduced analogously as in Section 3. In fact, under Assumption III with $\mathbb{R}^d$ replaced by Ω, Lemma 3.2 and Proposition 3.4 are generalized in a straightforward way; there appears an additional boundary term as in (6.1) which is a harmless relatively bounded perturbation with relative bound 0.

In order to prove an analogue of the generalized strong resolvent convergence in Lemma 4.3, we use that $\mathcal{D}_\Omega$ in (6.2) is a core of t_Ω and rely on the estimate (6.3); the latter is also used to prove the analogue of Lemma 4.4, *i.e.* the discrete compactness of the embeddings.

Norm resolvent convergence and convergence of spectra and pseudospectra then follow in a straightforward way from these analogues of Lemmas 4.3, 4.4.

6.2. **Semi-angle $\theta \geq \pi/2$.** In order to prove an analogue of Proposition 2.6, take $R_\Omega > 0$ sufficiently large and $\xi \in C_0^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$ such that

$$\mathbb{R}^d \setminus \Omega \subset B_{R_\Omega - 2}(0), \quad \xi \restriction B_{R_\Omega}(0) = 1, \quad \operatorname{supp} \xi \subset B_{R_\Omega+1}(0).$$

Then the closure of

$$\begin{aligned}(T_\Omega)_{\min} &:= -\Delta + Q,\\ \mathcal{D}((T_\Omega)_{\min}) &:= \{f \in W_0^{1,2}(\Omega, \mathbb{C}) \,:\, (-\Delta + Q_0)f \in L^2(\Omega, \mathbb{C}),\, (1-\xi)f \in C_0^\infty(\Omega, \mathbb{C})\},\end{aligned}$$

is given by

$$T_\Omega = -\Delta + Q, \quad \mathcal{D}(T_\Omega) = \{f \in W_0^{1,2}(\Omega, \mathbb{C}) \,:\, (-\Delta + Q_0)f \in L^2(\Omega, \mathbb{C})\}. \tag{6.4}$$

In fact, we may proceed similarly as in the proof of Proposition 2.6, starting with

$$(T_\Omega)_{0,\min} := -\Delta + Q_0, \quad \mathcal{D}((T_\Omega)_{0,\min}) := \mathcal{D}((T_\Omega)_{\min}).$$

Then Lemmas 2.9–2.11 may be generalized for $(T_\Omega)_{0,\min}$ and $(T_\Omega)_{\min}$, with similar arguments as for the generalizations for the operators $T_{0,n}$, *cf.* Lemmas 3.6 and 3.7. To this end, we use that every function $f \in \mathcal{D}((T_\Omega)_{0,\min})$ has compact support and belongs to the domain of the Laplacian defined in $L^2(\Omega, \mathbb{C})$ (because ξf and $(1-\xi)f$ both belong to the latter domain); note that the quadratic form has no boundary term because $\partial\Omega = \partial\Omega^{\mathrm{D}}$ by the assumptions. We thus arrive at the analogue of the estimate (2.15) and at a norm equivalence similar to (2.16), *i.e.* there exist $\beta_\Omega, k_\Omega, K_\Omega > 0$ such that, for all $f \in \mathcal{D}((T_\Omega)_{0,\min}) = \mathcal{D}((T_\Omega)_{\min})$,

$$\|\nabla f\|_\Omega^2 \leq \frac{\beta_\Omega}{2} \|\Delta f\|_\Omega^2 + \frac{1}{2\beta_\Omega} \|f\|_\Omega^2 \tag{6.5}$$

and

$$\begin{aligned}&k_\Omega \left(\|\Delta f\|_\Omega^2 + \|Q_0 f\|_\Omega^2 + \|f\|_\Omega^2\right)\\ &\leq \|(T_\Omega)_{\min} f\|_\Omega^2 + \|f\|_\Omega^2\\ &\leq K_\Omega \left(\|\Delta f\|_\Omega^2 + \|Q_0 f\|_\Omega^2 + \|f\|_\Omega^2\right).\end{aligned} \tag{6.6}$$

The latter continues to hold for the closure of $(T_\Omega)_{\min}$. Below we prove that the closure of $(T_\Omega)_{0,\min}$ is $(T_\Omega)_0$ where

$$(T_\Omega)_0 := -\Delta + Q_0, \quad \mathcal{D}((T_\Omega)_0) := \{f \in W_0^{1,2}(\Omega, \mathbb{C}) \,:\, (-\Delta + Q_0)f \in L^2(\Omega, \mathbb{C})\};$$

then (6.4) follows from (6.6).

To justify that $(T_\Omega)_0$ is the closure of $(T_\Omega)_{0,\min}$, we employ two cut-off functions $\zeta_i \in C^\infty(\Omega, \mathbb{R})$, $i = 0, 1$, that satisfy

$$\begin{aligned}&0 \leq \zeta_0 \leq 1, \quad \zeta_0 \restriction \Omega \setminus B_{R_\Omega - 1}(0) = 1, \quad \zeta_0 \restriction B_{R_\Omega - 2}(0) \cap \Omega = 0,\\ &0 \leq \zeta_1 \leq 1, \quad \zeta_1 \restriction \Omega \setminus B_{R_\Omega}(0) = 1, \quad \zeta_1 \restriction B_{R_\Omega - 1}(0) \cap \Omega = 0.\end{aligned}$$

Note that these properties yield

$$\zeta_0 \zeta_1 = \zeta_1, \quad \zeta_i(1-\xi) = 1 - \xi, \quad i = 0, 1. \tag{6.7}$$

The potential $\widetilde{Q}_0 := \zeta_0 Q_0$ satisfies Assumption II. Thus Proposition 2.6 and its proof imply that

$$T_0 := -\Delta + \widetilde{Q}_0, \quad \mathcal{D}(T_0) := \left\{f \in W^{1,2}\left(\mathbb{R}^d, \mathbb{C}\right) : (-\Delta + \widetilde{Q}_0)f \in L^2\left(\mathbb{R}^d, \mathbb{C}\right)\right\},$$

has the separation property, *i.e.* $\mathcal{D}(T_0) = W^{2,2}(\mathbb{R}^d,\mathbb{C}) \cap \{f \in L^2(\mathbb{R}^d,\mathbb{C}) : \widetilde{Q}_0 f \in L^2(\mathbb{R}^d,\mathbb{C})\}$, and $C_0^\infty(\mathbb{R}^d,\mathbb{C})$ is a core of T_0. Let $-\Delta^{\mathrm{D}}_{B_{R_\Omega+1}(0)\cap\Omega}$ be the Dirichlet Laplacian in $L^2(B_{R_\Omega+1}(0)\cap\Omega,\mathbb{C})$ defined via its quadratic form. Observe that if $f \in \mathcal{D}((T_\Omega)_0)$, then $\xi f \in \mathcal{D}(-\Delta^{\mathrm{D}}_{B_{R_\Omega+1}(0)\cap\Omega})$ and $(1-\xi)f \in \mathcal{D}(T_0)$. Since $C_0^\infty(\mathbb{R}^d,\mathbb{C})$ is a core of T_0, there exists a sequence $\{f_n\}_n \subset C_0^\infty(\mathbb{R}^d,\mathbb{C})$ that converges to the function $(1-\xi)f \in \mathcal{D}(T_0)$ in the graph norm of T_0. Using (6.7) and (6.5), (6.6), one may verify that the same holds for the sequence $\{\zeta_1 f_n\}_n \subset C_0^\infty(\Omega,\mathbb{C})$. Then

$$(T_\Omega)_0(1-\xi)f = T_0(1-\xi)f, \quad (T_\Omega)_0\zeta_1 f_n = T_0\zeta_1 f_n$$

implies that $\{\xi f + \zeta_1 f_n\}_n \subset \mathcal{D}((T_\Omega)_{0,\min})$ approximates f in the graph norm of $(T_\Omega)_0$, and so the claim follows.

The operator $(T_\Omega)_0$ is m-accretive, *cf.* Kato's theorem [27, Thm. VII.2.5]. Using Assumption (II.ii.Ω), we obtain that the resolvent of $(T_\Omega)_0$ is compact. The same holds for T_Ω by a perturbation argument as in the proof of Proposition 2.6. In the same way, we prove a resolvent estimate similar to (2.10).

The approximating operators are introduced similarly as in Section 3.2.

Generalized strong resolvent convergence, *cf.* Lemma 4.3, and discrete compactness of the embeddings, *cf.* Lemma 4.4, can be verified by straightforward generalizations of the given proofs; here we make use of the analogue of (4.11) which follows from (6.6).

As in the case $\theta < \pi/2$, the claims in Theorem 6.1 then follow from these analogues of Lemmas 4.3, 4.4.

7. Examples

In this section, we present numerical examples for dimensions $d = 1, 2, 3$ which are backed up by our spectral convergence results Theorem 5.1 and Theorem 6.1. We study the Airy operator and the imaginary cubic oscillator in $\mathbb{R}$ with various boundary conditions, the harmonic oscillator in $\mathbb{R}^3$ with different truncated domains, and the complex rotated oscillator on an exterior domain in $\mathbb{R}^2$ considered in [17] for which spectral exactness has not been known up to now.

In the figures below the real and imaginary parts of the numerical computations of a selection of eigenvalues $\lambda_n^{(k)}$, $k \in \mathbb{N}$, of T_n are displayed when the truncated domains Ω_n are increased. All numerical computations arising in this section were performed on a standard dual-core Linux machine with the use of the software Wolfram Mathematica 9. The differential equations on the finite domains Ω_n were solved numerically by implementing a shooting method in Mathematica.

7.1. **Potentials $Q(x) = \mathrm{i}x$ and $Q(x) = \mathrm{i}x^3$ in $\mathbb{R}$.** In both cases the sets Ω_n are chosen as intervals $\Omega_n = (-s_n, s_n)$ with $s_n \nearrow \infty$ as $n \to \infty$ and we impose various boundary conditions at the endpoints $\pm s_n$.

Potential $Q(x) = \mathrm{i}x$. The resolvent of $T = -\Delta + \mathrm{i}x$ is compact and the spectrum of T is empty, *cf. e.g.* [50, 3], whereas the spectrum of T_n in $L^2((-s_n, s_n),\mathbb{C})$ is *not*,

$$\sigma(T) = \emptyset, \quad \sigma(T_n) = \{\lambda_n^{(k)} : k \in \mathbb{N}\} \neq \emptyset,\ n \in \mathbb{N}, \tag{7.1}$$

since T_n is a bounded perturbation of $-\mathrm{d}^2/\mathrm{d}x^2$ with separated boundary conditions; moreover, the system of eigenfunctions and associated functions of the operator T_n forms a Riesz basis, *cf.* [46]. The pseudospectra of T are also well-studied, *cf.* [12]. For T_n with Dirichlet conditions at $\pm s_n$, a detailed analysis of the bottom of the spectrum showed that, *cf.* [7, Thm. 3.1],

$$\lim_{n\to\infty}(\inf \operatorname{Re}\sigma(T_n)) = \frac{|\mu_1|}{2} \approx 1.169 \tag{7.2}$$

where $0 > \mu_1 \approx -2.338$ is the first zero of the Airy function.

Figure 1 illustrates the behaviour of the eigenvalue branches of T_n with Dirichlet conditions for increasing $s_n \in [0, 10]$; for Robin conditions, the plots look similar.

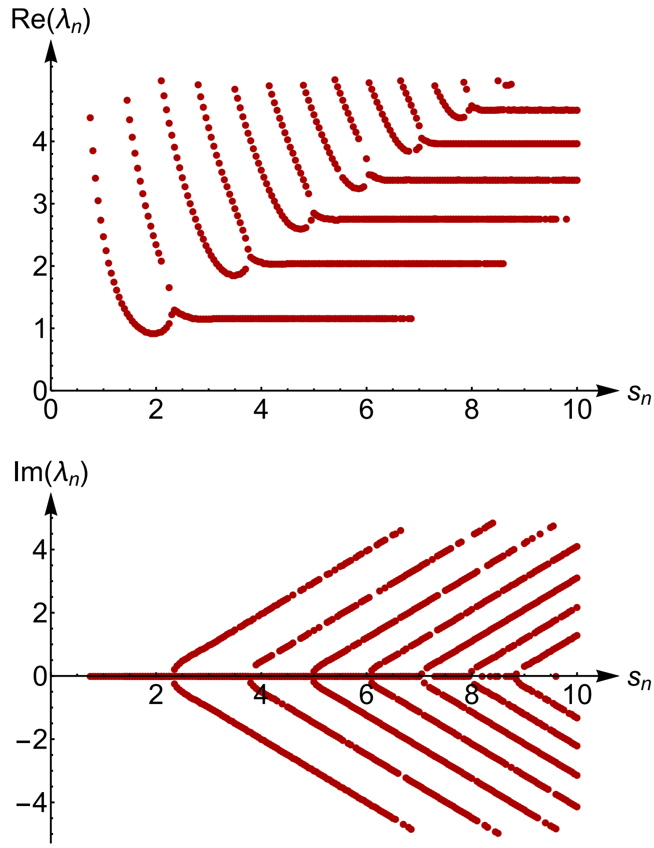

Figure 1. $Q(x)=\mathrm{i}x$: Real (top) and imaginary (bottom) parts of approximate eigenvalues of $-\Delta+\mathrm{i}x$ on $(-s_n,s_n)$ with Dirichlet conditions for $s_n=0.05n\in[0,10]$, $n=0,1,\dots,200$.

For $s_n \leq 2$, the eigenvalues are real and decrease monotonically when s_n increases, until at some point the gaps between two consecutive eigenvalues start to shrink and, as s_n increases further, these pairs meet and form a complex conjugate pair. The real parts of each pair seem to converge, whereas their imaginary parts diverge to $\pm\infty$ in almost straight lines.

Hence in the limit $s_n \nearrow \infty$ there are no eigenvalues left, which is in agreement with (7.1) and illustrates our result on spectral exactness, *cf.* Theorem 5.1. The behaviour of the eigenvalue with smallest real part agrees with the result (7.2) in [7].

Potential $Q(x) = \mathrm{i}x^3$. The imaginary cubic oscillator $T = -\Delta + \mathrm{i}x^3$ and related operators have been studied extensively, *cf.* [18, 9, 49, 51, 31, 35, 30]. It is known that its spectrum is non-empty and consists of eigenvalues $\lambda^{(k)}$, $k \in \mathbb{N}$, which are all real, simple and behave asymptotically as $k^{6/5}$, but the eigenvalues are not known in closed form. The system of eigenfunctions of T is complete in $L^2(\mathbb{R}, \mathbb{C})$, but it does not form a basis, *cf.* [51, 35], while, as in the previous example, the system of eigenfunctions of T_n does form a Riesz basis.

Figure 2 shows the behaviour of the eigenvalues of T_n for increasing $s_n \in [0, 4.5]$ for Dirichlet (asterisks/red) and Neumann (squares/blue) conditions at $\pm s_n$ simultaneously; for Robin conditions the eigenvalue behaviour is similar.

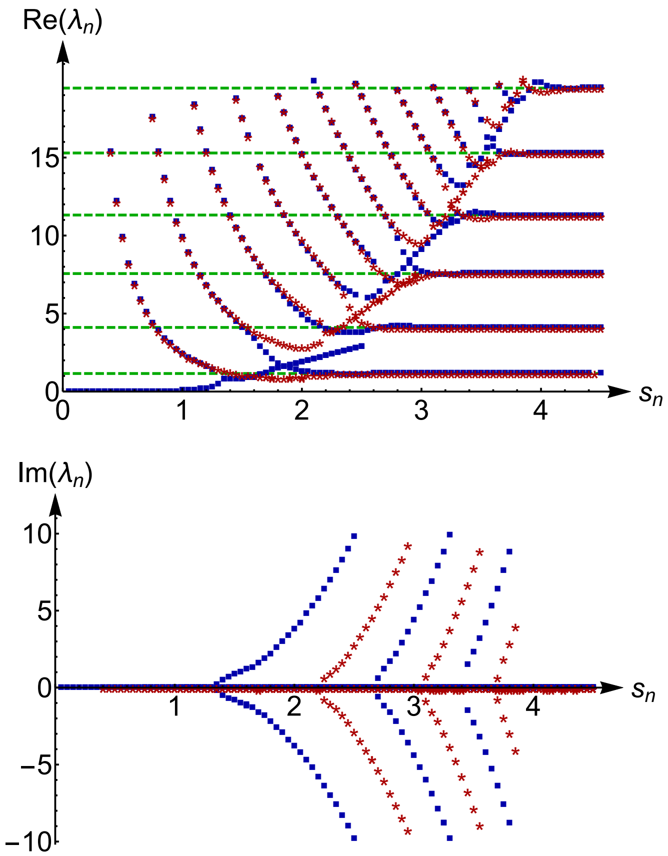

FIGURE 2. $Q(x) = \mathrm{i}x^3$: Real (top) and imaginary (bottom) parts of approximate eigenvalues of $-\Delta + \mathrm{i}x^3$ on $(-s_n, s_n)$ with Dirichlet (asterisks/red) and Neumann (squares/blue) conditions for $s_n = 0.01n \in [0, 4.5]$, $n = 0, 1, \dots, 450$, and smallest 6 (real) eigenvalues of of $-\Delta + \mathrm{i}x^3$ on $\mathbb{R}$ (dashed/green horizontal lines).

For small s_n, the eigenvalues are again all real. As s_n is increased, two groups of eigenvalue branches develop: Some eigenvalues form complex conjugate pairs and their imaginary parts diverge to $\pm\infty$, so they have *no* limit in $\mathbb{C}$, while other eigenvalues *do* converge to a finite limit which must be an eigenvalue $\lambda^{(k)}$ of T due to the spectral exactness Theorem 5.1. Our result also guarantees that all eigenvalues of T are approximated in this way and it confirms that the numerically computed eigenvalues in [9, Tab. 1] or the following ones computed by M. Tater, *cf.* [54],

$$\lambda^{(1)}_{\mathrm{M.T.}} = 1.1562671, \quad \lambda^{(2)}_{\mathrm{M.T.}} = 4.1092288, \quad \lambda^{(3)}_{\mathrm{M.T.}} = 7.5622739,$$
$$\lambda^{(4)}_{\mathrm{M.T.}} = 11.314422, \quad \lambda^{(5)}_{\mathrm{M.T.}} = 15.291554, \quad \lambda^{(6)}_{\mathrm{M.T.}} = 19.451529,$$

do indeed approximate true eigenvalues.

Moreover, Figure 2 shows that, for $s_n \geq 4$, the differences between our numerical approximations and their limits, *i.e.* the 6 smallest true eigenvalues of T marked by dashed/green horizontal lines, are already very small. Figure 3 illustrates the convergence rate of $|\lambda^{(1)} - \lambda^{(1)}_n|$ for Dirichlet conditions at the endpoints $\pm s_n$, where

$$\lambda^{(1)} := \min \sigma(T) \approx \lambda^{(1)}_{\mathrm{M.T.}}, \quad \lambda^{(1)}_n := \min \sigma(T_n).$$

The top plot in Figure 3, which is a zoom of Figure 2 near the smallest eigenvalue $\lambda^{(1)}$ of T, reveals that $\lambda_n^{(1)}$ converges to $\lambda^{(1)}$ in an oscillatory manner; the bottom plot, showing the values of $\log|\lambda^{(1)} - \lambda_n^{(1)}|$ for better visibility of the oscillations, suggests an exponential convergence rate of the eigenvalues as $s_n \nearrow \infty$.

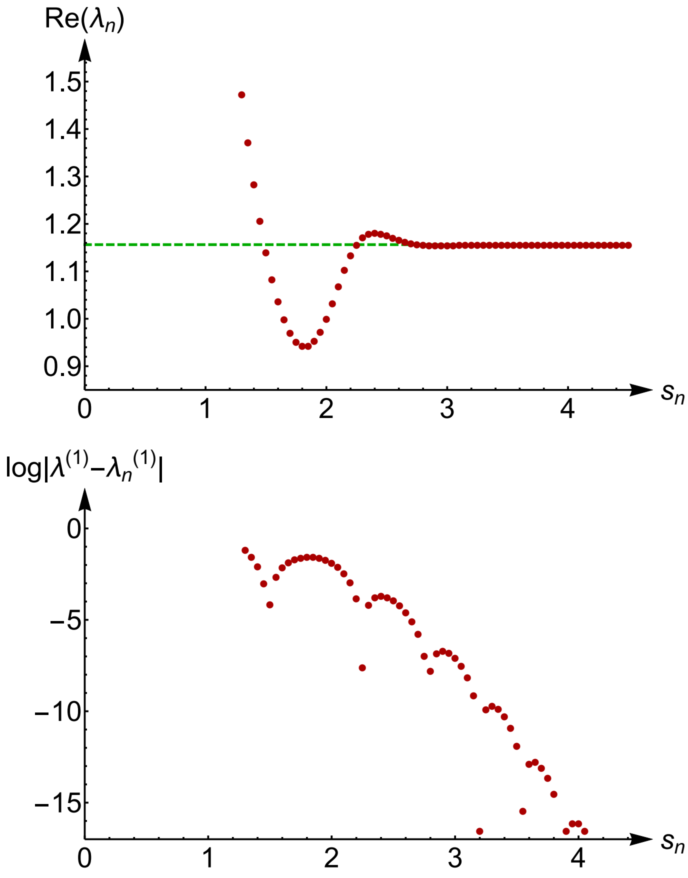

FIGURE 3. $Q(x) = \mathrm{i}x^3$: Approximation of smallest eigenvalue $\lambda^{(1)}$ of $-\Delta + \mathrm{i}x^3$ on $\mathbb{R}$ (dashed/green horizontal line on the top) by smallest eigenvalues $\lambda_n^{(1)}$ of $-\Delta + \mathrm{i}x^3$ on $(-s_n, s_n)$ with Dirichlet conditions for $s_n = 0.05n \in [0, 4.5]$, $n = 0, 1, \dots, 90$.

7.2. **Harmonic oscillator in $\mathbb{R}^3$.** Since the eigenvalues $\lambda^{(k)} = 2k+1$, $k \in \mathbb{N}$, of $T = -\Delta + |x|^2$ on $\mathbb{R}^3$ are well-known, this is a nice benchmark example. In several dimensions, the simplest choice of Ω_n are cubes and balls. While cubes are natural for potentials allowing for separation in Cartesian coordinates, balls are suited for radial potentials, *i.e.* $Q(x) = Q(|x|)$. The harmonic oscillator, *i.e.* $Q(x) := |x|^2$, allows for both separations, so we can compare the approximations for cubes and balls.

For cubes $\Omega_n = (-s_n, s_n)^3$ with $s_n \nearrow \infty$, the spectral problem for $T_n = -\Delta + |x|^2$ is reduced to the one-dimensional problem

$$-f''(x) + x^2 f(x) = \mu f(x), \quad x \in (-s_n, s_n), \tag{7.3}$$

subject to Robin conditions at the artificial endpoints $\pm s_n$. Every eigenvalue $\lambda_n^{(k)}$, $k \in \mathbb{N}$, of T_n can be expressed as $\lambda_n^{(k)} = \mu_n^{(k_1)} + \mu_n^{(k_2)} + \mu_n^{(k_3)}$ where $\{\mu_n^{(k)}\}_k$ are the eigenvalues of the corresponding one-dimensional problem (7.3).

For balls $\Omega_n = B_{s_n}(0)$ the operator T_n can be written in spherical coordinates and the eigenfunctions of T_n can be factorized as $f(r, \theta, \varphi) = g_l(r) Y_l^m(\theta, \varphi)$; here

$Y_l^m(\theta,\varphi)$, $m=-l,\dots,l,\ l\in\mathbb{N}_0$, are the spherical harmonics which satisfy $\Delta Y_l^m = \frac{l(l+1)}{r^2}Y_l^m$, and g_l is an eigenfunction of the (m-independent) radial problem

$$-g''(r)-\frac{2}{r}g'(r)+\left(\frac{l(l+1)}{r^2}+r^2\right)g(r)=\lambda g(r),\quad r\in(0,s_n), \tag{7.4}$$

with some Robin condition at s_n.

In Figure 4 we compare the eigenvalues of T_n for cubes and balls with Dirichlet conditions on $\partial\Omega_n$ for increasing $s_n\in[0,5]$; for Robin conditions the plots are similar. The behaviour of the eigenvalue approximations does not differ much for cubes and balls, both converge to true eigenvalues $\lambda^{(k)}=2k+1$, $k\in\mathbb{N}$, and all true eigenvalues are approximated in this way, *cf.* Theorem 5.1.

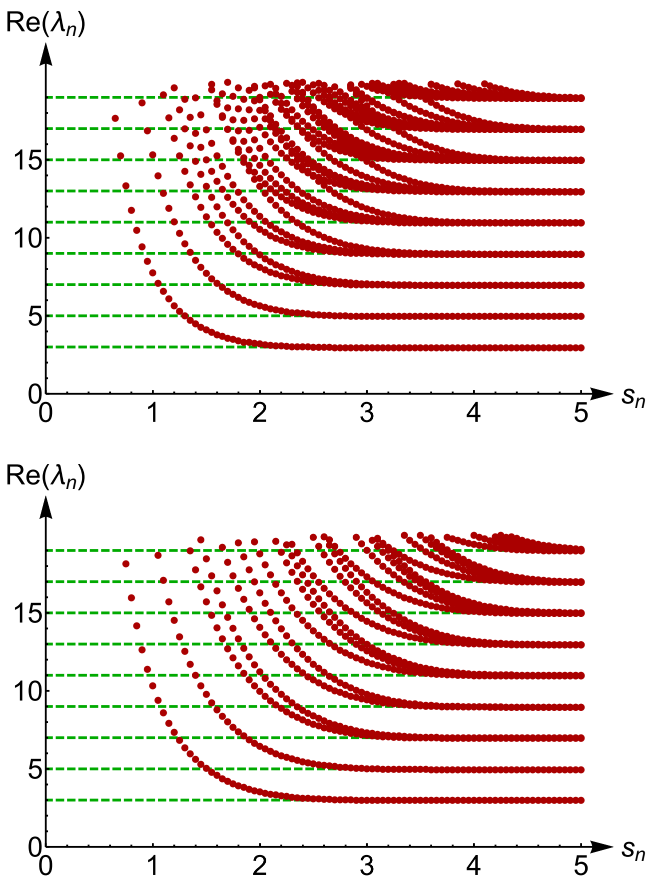

FIGURE 4. Eigenvalues of harmonic oscillator $-\Delta+|x|^2$ in $\mathbb{R}^3$ approximated on cubes $\Omega_n=(-s_n,s_n)^3$ (top) and balls $\Omega_n=B_{s_n}(0)$ (bottom) for Dirichlet conditions on $\partial\Omega_n$ for $s_n=0.05n\in[0,5]$, $n=0,1,\dots,100$.

The main difference lies in how the degeneracy $\frac{k(k+1)}{2}$ of the k-th eigenvalue $\lambda^{(k)}=2k+1$ of T is reflected in the approximations. In fact, in Figure 4, the eigenvalue branches in general represent eigenvalues of higher multiplicities which differ for cubes and balls. Nevertheless, in both cases the sum of the multiplicities of all branches converging to $\lambda^{(k)}$ equals its degeneracy $\frac{k(k+1)}{2}$. The different multiplicities are due to the fact that on cubes eigenfunctions with permuted coordinate axes correspond to the same eigenvalue, while for balls each eigenvalue curve corresponds to one l (the subscript of the spherical harmonics) since the eigenfunction g_l in (7.4) is independent of $m=-l,\dots,l$.

7.3. **Complex rotated oscillator on exterior domain in $\mathbb{R}^2$.** In [17, Ex. 1], the problem

$$\begin{aligned} -\Delta f(x,y) + (1+3\mathrm{i})(x^2+y^2) f(x,y) &= \lambda f(x,y), \qquad & x^2+y^2 &\geq 1, \\ f(x,y) &= 0, & x^2+y^2 &= 1, \end{aligned}$$

was studied, but it could not be decided whether domain truncation is spectrally exact. Our new result for exterior domains, *cf.* Theorem 6.1, yields a definite and positive answer.

Indeed, Theorem 6.1 proves that if we truncate the exterior domain $\mathbb{R}^2 \backslash B_1(0)$ to $\Omega_n := B_{s_n}(0) \backslash \overline{B_1(0)}$ with $s_n \nearrow \infty$ and impose Dirichlet conditions also on the outer boundary of Ω_n, i.e. $\Omega_n^{\mathrm{D}} = \Omega_n$, $\Omega_n^{\mathrm{R}} = \emptyset$ in Assumption III, we do obtain a spectrally exact approximation. In polar coordinates, the truncated problem decouples into an infinite system of problems that depend on $l \in \mathbb{N}_0$ (representing the angular part),

$$\begin{aligned} -f''(r) - \frac{1}{r} f'(r) + \left((1+3\mathrm{i}) r^2 + \frac{l^2}{r^2} \right) f(r) &= \lambda f(r), \qquad & r &\in (1, s_n), \\ f(r) &= 0, & r &\in \{1, s_n\}. \end{aligned} \tag{7.5}$$

We performed numerical computations to find and approximate the eigenvalues $\lambda_n^{(k,l)}$, $k \in \mathbb{N}$, in the box $[0,20] + [0,15]\,\mathrm{i}$ for different $l \in \mathbb{N}_0$ and increasing s_n. For $l \geq 7$, no eigenvalue was found in this box. For $l = 0, 1, \dots, 6$, the eigenvalues in the box change very little (less than 10^{-7}) for $s_n \in [5, 10]$. So the numerical approximations for $s_n = 10$ shown in Table 1 already lie near true eigenvalues.

Value of l	Approximate eigenvalues $\lambda = \lambda_n^{(k,l)}$ up to 7 digits
$l = 0$	$\lambda \approx 8.1962583 + 9.8951098\,\mathrm{i}$
$l = 1$	$\lambda \approx 8.5747825 + 9.9950630\,\mathrm{i}$
$l = 2$	$\lambda \approx 9.6945118 + 10.3061585\,\mathrm{i}$
$l = 3$	$\lambda \approx 11.5061205 + 10.8625746\,\mathrm{i}$
$l = 4$	$\lambda \approx 13.9201983 + 11.7211938\,\mathrm{i}$
$l = 5$	$\lambda \approx 16.7923324 + 12.9529682\,\mathrm{i}$
$l = 6$	$\lambda \approx 19.9029928 + 14.6018978\,\mathrm{i}$

TABLE 1. Approximate eigenvalues $\lambda_n^{(k,l)} \in [0,20] + [0,15]\,\mathrm{i}$ of (7.5) for $s_n = 10$.

Remark 7.1. We mention that the numerical values in [17] in fact correspond to a different potential; a recomputation by the authors agrees with the values in Table 1, *cf.* the personal communication [45].

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