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Schmidt gap in random spin chains

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We numerically investigate the low-lying entanglement spectrum of the ground state of random one-dimensional spin chains obtained after partition of the chain into two equal halves. We consider two paradigmatic models: the spin-1/2 random transverse field Ising model, solved exactly, and the spin-1 random Heisenberg model, simulated using the density matrix renormalization group. In both cases we analyze the mean Schmidt gap, defined as the difference between the two largest eigenvalues of the reduced density matrix of one of the two partitions, averaged over many disorder realizations. We find that the Schmidt gap detects the critical point very well and scales with universal critical exponents.

I. INTRODUCTION

The entanglement spectrum (ES) [1], the set of eigenvalues of the reduced density matrix of a quantum many-body state, has now become a standard fingerprint that reveals much more information on a state compared to measures of bipartite entanglement, such as the von Neumann entropy and the negativity (see Refs. [2] and [3] for recent comprehensive reviews).

Originally introduced to study the transition to a topologically ordered state in the quantum Hall effect [1], ES has been used for the characterization of spin chains and other one-dimensional (1D) models in real and momentum space [4–13]. The distribution of the Schmidt eigenvalues in the middle of the spectrum has been studied by means of conformal field theory [14]. The study of the structure of the low-lying part of the ES in 1D models also reveals the Luttinger parameter [15, 16]. In 2D systems the situation is somewhat less clear and the universality of the ES has been challenged [17].

The Schmidt gap, the difference of the two largest Schmidt eigenvalues of the ES, originally introduced in [18] and [19] was shown to scale according to universal critical exponents in [20–24]. It was further employed in the characterization of 2D spin models in a region close to a topological spin liquid [25, 26]. The time evolution of the Schmidt gap was analyzed in [27–29] for the dynamics after a quantum quench in homogeneous systems and in [30] for a quench to a many-body localized Hamiltonian. Whether or not the Schmidt gap can be applied as an instrument to detect criticality in random models is still an open question.

The effect of randomness in spin models, whether introduced via disorder in coupling constants or through some random external field, has become an area of significant interest since the early studies on the random transverse field Ising model [31–33]. Randomness has been shown to modify the characteristics of phase transitions [34,35], as well as transition a spin system from one universality class to another [36,37] and is integral to the emergence of interesting phases such as the Griffiths and random singlet phases (RSPs) [38–40]. Recently a lot of attention has been devoted to the mechanism of many-body localization in 1D and 2D systems [41–44]. While these random models have usually been investigated using corresponding order parameters [36,45] and entanglement entropy [46–49], a few works have analyzed numerically the entanglement spectrum of the excited states of random spin chains [12,50–52].

In this paper, we study the Schmidt gap of the ground state of random spin-1/2 and spin-1 chains. We show for both models that the closing of the Schmidt gap, averaged over the disorder distribution, signals the occurrence of a quantum phase transition. Moreover, we are able to observe universal scaling of the Schmidt gap with critical exponents.

II. RANDOM TRANSVERSE-FIELD ISING MODEL

We consider L spin-12 arranged in a chain with open boundary conditions and Hamiltonian

\[ H = -\sum_i J_i \sigma_i^x \sigma_{i+1}^x - \sum_i h_i \sigma_i^z. \] (1)

The couplings \( J_i \) of the Ising interaction and the transverse magnetic fields \( h_i \) are independent random variables drawn from the distributions \( \pi(J)dJ \) and \( \rho(h)dh \), which can be gauged to be positive. In the following, we consider the distributions \( \pi(J) \) and \( \rho(h) \) to be uniform in the intervals \( J \in [0,1] \) and \( h \in [0,h_{\text{max}}] \), respectively, and 0 otherwise. This choice reduces the Hamiltonian parameters to only one variable, \( h_{\text{max}} \).

The physics underlying the ground state of this Hamiltonian is closely related to the finite-temperature behavior of a 2D classical Ising model with quenched randomness correlated along one direction [53, 54]. The quantum Hamiltonian in Eq. (1) is recovered by taking the

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continuum limit of the classical model, and it was first investigated with transfer matrix methods by Shankar and Murthy [55]. In particular, by a simple duality argument, it can be shown that a quantum phase transition takes place when the two distributions $\pi(J)$ and $\rho(h)$ are identical. By defining the magnetic-field parameter $\Delta_h = [\log h]_D$ (where $[\cdot]_D$ is the disorder average), the quantum critical point is found at $\Delta_c = [\log J]_D$, corresponding to $h_{\text{max}}^\text{max} = 1$ for our choice of distributions. The phase diagram features a paramagnetic phase ($\Delta_h > \Delta_c$) and a ferromagnetic phase ($\Delta_h < \Delta_c$) with non-zero spontaneous magnetization $m_h = [\sum_i \langle \sigma_i^x \rangle]_D \neq 0$, where $\langle \cdot \rangle$ denotes the ground-state average.

The magnetic properties, both at criticality and off criticality, can be derived following a renormalization group approach [31], where the short-wavelength modes are cut off from the system by targeting the strongest coupling $\Omega = \max\{J_i, h_i\}$. In practice, via perturbation theory, the excited states of the subspace for the local Hamiltonian describing the degrees of freedom connected to $\Omega$ are eliminated, leading to a new effective Hamiltonian with a lower energy scale $\Omega$ at each step. This iterative procedure allows the estimation of the correlation function and the various critical exponents [32]. In particular, the behavior of the typical correlation function $C(r) = \sum_i \langle \sigma_i^x \sigma_{i+r}^x \rangle$ is found to be very different from that of the average correlation function $[C(r)]_D = \langle \sum_i \langle \sigma_i^x \sigma_{i+r}^x \rangle \rangle_D$. At criticality, the typical correlation decays as $C(r) \sim \exp(-r^{\nu})$ while the average correlation follows a power-law decay $[C(r)]_D \sim 1/r^{2-\phi}$, with $\phi = (1 + \sqrt{5})/2$ is the golden mean. On the other hand, in the paramagnetic phase, both correlation functions decay exponentially with correlation length $\xi \sim (h_{\text{max}}^\text{max} - h_c^\text{max})^{-\nu}$. The critical exponent $\nu$ differs for the two cases, with $\nu = 1$ and $\nu = 2$ for the typical and average correlation function, respectively. The spontaneous magnetization in the ferromagnetic phase is $m_h(h_{\text{max}}^\text{max}) \sim (h_{\text{max}}^\text{max} - h_c^\text{max})^{2-\phi}$ with critical exponent $\beta = 2 - \phi = (3 - \sqrt{5})/2 \approx 0.381$.

The entanglement structure for the random transverse-field Ising chain can be calculated exactly by first mapping the spin degrees of freedom into a system of non-interacting fermions using the Jordan-Wigner transformation [56]. Within this representation, the reduced density matrix for a subsystem $S$ is simply given by $
 \rho_S = Z^{-1}\exp(-K)$, where $K$ is the entanglement Hamiltonian and $Z$ is the normalization constant [57]. Given the correlation matrices $C = \langle c_i^\dagger c_i \rangle$, $F = \langle c_i^\dagger c_j \rangle$, with $(c_i^\dagger, c)$ the fermionic creation and annihilation operators, the eigenvalues $\epsilon_k$ of $K$ can be calculated from the matrix $M = 2C - I - 2F$, where $I$ is the identity matrix, by singular value decomposition [58]. We can easily calculate the ES $\{\lambda_i\}$ directly from the full spectrum $\{\epsilon_k\}$ of the entanglement Hamiltonian following the approach explained in Ref. [58]. In all calculations we partition the chain into two equal halves.

In Fig. 1 we show the entanglement properties as a function of the upper bound $h_{\text{max}}^\text{max}$ of the magnetic-field distribution $\rho(h)$. In Fig. 1(a) we plot the six largest eigenvalues $\lambda_i$ in the ES plotted versus the upper bound $h_{\text{max}}^\text{max}$ of the magnetic-field distribution $\rho(h)$. (b) Schmidt gap $\Delta \lambda$ as a function of $h_{\text{max}}^\text{max}$ for different systems sizes. Each data point was computed as the average over $10^4$ realizations of disorder. The dashed line represents the expected critical value of $h_{\text{max}}^\text{max}$ at which the phase transition occurs.
We introduce disorder in the model by choosing:

\[ J_i = \zeta_i^\delta, \]  \(\text{(4)}\)

where \(\delta\) controls the strength of disorder and \(\zeta_i\) is a random variable distributed uniformly between 0 and 1. The probability distribution of \(J_i\) is:

\[ \pi_\delta(J) = \delta^{-1} J^{-1+1/\delta}. \]  \(\text{(5)}\)

The gapped Haldane is stable for \(J_{\min}/J_{\max} > 0.6\), where \(J_{\min}\) and \(J_{\max}\) are the smallest and largest couplings, respectively. For the power-law distribution, \(\text{(5)}\), and for \(\delta > 0\), we have \(J_{\min}/J_{\max} = 0\), and the gapped Haldane phase is immediately destroyed for an infinitesimal amount of this type of disorder. However, for small \(\delta\) the system enters the so-called gapless Haldane phase, a type of Griffith phase with closed Haldane gap, but exhibiting the hidden topological order characteristic of the gapped phase \([40, 60]\). For very strong disorder \(\delta \gg 1\), the ground state is in the random singlet phase, which is a gapless phase consisting of pairs of spins in singlets spanning arbitrarily long distances \([61-64]\).

The phase diagram for the spin-1 random antiferromagnetic Heisenberg chain when using a power-law disorder distribution is the following \([36, 65, 66]\): gapped Haldane at zero disorder \((\delta = 0)\), gapless Haldane (Griffiths) at \(0 < \delta < 1\), and, finally, RSP at \(\delta \geq 1\). This power-law distribution for the disorder is required in order to cross the phase transition between the Haldane and the RSPs \([36, 65]\). The critical point at which this phase transition takes place is known to be approximately \(\delta_c = 1\). A box-like disorder distribution is only able to reach a disorder distribution equivalent to that of \(\delta = 1\) and, thus, is unable to cross the quantum phase transition to the RSP \([67]\).

The results reported in this section of the paper have been obtained using finite-size density matrix renormalization-group (DMRG) calculations with open boundary conditions \([68, 69]\); between 2000 and 2500 random realizations were used. In an attempt to reduce issues in the calculations relating to degeneracy of the ground state, a staggered magnetic field of magnitude \(2.5 \times 10^{-3}\) was placed on the first two and last two spins. Due to the spin chain having zero spontaneous magnetization for all values of \(\delta\), we project over the total angular momentum \(M_z = \sum_i S_{zi} = 0\). In the DMRG calculations 100 states were kept during the renormalization process, resulting in a maximum discarded weight of \(10^{-6}\). We remark that an alternative method to deal with random spin chains is to employ a quantum parallel method in which disorder is simulated by means of auxiliary sites coupled to the physical sites \([70]\). However, within this method the calculation of the entanglement spectrum for each disorder realization would not be efficient.
\( \delta > \)

the central two spins are in a product state with cor-

Therefore, if no singlet crosses the center of the chain,

There is a nonzero probability of this product state occur-

In all plots, lines connect points and are a guide for the eye. The dashed vertical line represents the approximate critical value of \( \delta \) at which the disorder-induced phase transition takes place.

**A. String order and Schmidt gap**

We wish to investigate the disorder induced phase tran-

We then performed a finite-size scaling analysis [59] that, for critical disor-

It is known [36, 60, 61] that, for critical disorder, the correlation length diverges as \( \xi \sim (\delta - \delta_s)^{-\nu} \) with \( \nu = (1 + \sqrt{13})/2 \approx 2.3028 \) and that the string order parameter vanishes as \( O^\infty \sim (\delta - \delta_s)^{2\beta_s} \) with \( \beta_s = 2(3 - \sqrt{5})/(\sqrt{13} - 1) \approx 0.5864 \). Therefore, the string order decays with length as \( O^\infty(L) \sim L^{-\eta_s} \), with \( \eta_s = 2\beta_s/\nu_s \approx 0.5093 \). However, there is currently no conjecture for the theoretical decay rate of the Schmidt gap, thus we do not have a theoretical prediction for the value of \( \eta_{\Delta \Lambda} \). Due to the more conventional construction of the order parameter the Schmidt gap is expected to scale as \( \Delta \lambda \sim (\delta_c - \delta)^{\beta_{\Delta \Lambda}} \), therefore resulting in \( \eta_{\Delta \Lambda} = \beta_{\Delta \Lambda}/\nu_{\Delta \Lambda} \). The values we find for the critical exponent \( \eta \) obtained for the string order parameter and Schmidt gap are \( \eta_s = 0.20 \pm 0.04 \) and \( \eta_{\Delta \Lambda} = 0.37 \pm 0.07 \) respectively. Note that the value of \( \eta_{\Delta \Lambda} \) obtained is relatively far from the theoretical value. We expect this discrepancy to be due to the limited sizes of the chains we considered.

We then performed a finite-size scaling analysis [59]...
of our results for the string order parameter and the Schmidt gap in order to obtain another approximation of the critical decay exponent using Eq. (2). In this work we fix $\delta_c = 1$ and allow $\nu$ and $\beta_Q$ to vary until the best collapse of the finite-size results is obtained. The string order is known to scale, as above, with $\delta_c$, due to the construction of the order parameter.

Figure 4 (a) shows the collapse for the string order parameter. The best finite-length collapse was obtained for $\beta_{st} = 0.24 \pm 0.05$ and $\nu_{st} = 2.3 \pm 0.4$, corresponding to a value of $\eta_{st} = 0.21 \pm 0.04$. This, again, is relatively far from the theoretical value of $\eta$ but is in close agreement with the value found previously using the finite-size extrapolation. Figure 4 (b) shows the results when the same finite-size scaling is applied to the Schmidt gap data, with the best collapse obtained at $\beta_{\Delta \lambda} = 0.9 \pm 0.1$ and $\nu_{\Delta \lambda} = 2.3 \pm 0.4$, corresponding to $\eta_{\Delta \lambda} = 0.38 \pm 0.08$. This is significantly closer to the theoretical value of $\eta$ while also being in good agreement with the value found from the extrapolation. In both cases, the critical exponent $\nu$ is found to be very close to the theoretical value, thus validating the numerical simulations.

Finally, we investigate the disorder-averaged ES for the first 12 eigenvalues of the reduced density matrix. The Haldane phase has a known [18, 21] degeneracy sequence of $[2, 4, 2, 4, \ldots]$ in the eigenvalues of the reduced density matrix. In the RSP, the ES is dependent on the number of singlets cut at the center of the chain, with eigenvalues $\lambda_1$ to $\lambda_{2N}$ having a value proportional to $1/3^N$ (with $N$ being the number of spin-1 singlets cut). This leads to an expected eigenvalue degeneracy distribution of $[1, 2, 6, 18, \ldots]$ which can also be written as $[3^0, 3^1, 3^2, \ldots]$.

Figure 5 shows the disorder averaged ES for the first 12 eigenvalues of a chain of 72 spins. For this fixed length, the eigenvalues separate significantly for $\delta > 0.4$. We expect that in the thermodynamic limit this separation would occur close to $\delta_c$. We can observe quite clearly the transition between the Haldane phase and the RSP in the structure of the ES, shown in Fig. 5. In particular, we expect that for larger values of $\delta$, where all contributions but those of singlets are almost eradicated, $\lambda_4$ will group closer with the eigenvalues $\lambda_5$ and $\lambda_6$.

Particularly interesting is also the probability distribution $P$ of the entanglement entropy, as it is directly related to the distribution of the eigenvalues. We calculate the von Neumann entropy:

$$E = -\text{Tr} \rho_{\ell} \log_2 \rho_{\ell}$$

of the reduced density matrix

$$\rho_{\ell} = \text{Tr}_{L-\ell} |\psi_G\rangle \langle \psi_G|,$$

where $|\psi_G\rangle$ is the ground state of Hamiltonian (3) and $\ell = L/2$.

Figure 6 shows the probability distribution of the von Neumann entropy for two values of disorder, one corresponding to the Griffiths phase and one to the RSP. The
distribution is plotted such that the horizontal axis represents the ratio $E/E_s$, where $E_s = \log_2(3) \approx 1.585$ is the entanglement of a spin-1 singlet. Therefore, peaks at integer values represent an integer number of singlets crossing the center of the chain. A significant change in the distribution of entanglement is seen as we move from the Griffiths phase to the RSP. Specifically, the distribution becomes much broader in the RSP but at the same time we see a dominance of one entanglement value (and thus ES), which is unseen in the Griffiths phase. Our ability to only investigate smaller values of $\delta$ explains the relatively rare occurrence of zero ($\sim 0.5\%$) spin-1 singlets being cut. As such, we fully expect these peaks to become more prominent for larger disorders and for larger lengths.

It is well understood [37, 73] that, further in the RSP for large disorder, the smearing between contributions to the von Neumann entropy from singlets decreases, and the same would be seen in the distribution of the eigenvalues. We assume that, for strong enough disorder, the ES will depend exclusively on the number of singlets being cut, and thus the disorder-averaged spectrum will depend on the probability of cutting a number of singlets $N$, with this probability varying as the disorder increases.

IV. CONCLUSIONS

In summary, we have numerically investigated the entanglement spectrum of the ground state of random spin-1/2 and spin-1 chains. The structure and degeneracy of the low-lying levels of the entanglement spectrum reveal the emergence of a quantum phase transition even in these disordered models. Remarkably, even for the two inequivalent random models we studied, the Schmidt gap detects the corresponding critical points and scales with universal critical exponents. These results reinforce the role of the Schmidt gap as a useful probe in quantum critical phenomena and open the way to possible extensions to dynamics in the presence of disorder and noise.

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