



**QUEEN'S  
UNIVERSITY  
BELFAST**

## Injective dimension of sheaves of rational vector spaces

Sugrue, D. (2018). Injective dimension of sheaves of rational vector spaces. *Journal of Pure and Applied Algebra*. <https://doi.org/10.1016/j.jpaa.2018.10.011>

**Published in:**

Journal of Pure and Applied Algebra

**Document Version:**

Peer reviewed version

**Queen's University Belfast - Research Portal:**

[Link to publication record in Queen's University Belfast Research Portal](#)

**Publisher rights**

Copyright 2018 Elsevier.

This manuscript is distributed under a Creative Commons Attribution-NonCommercial-NoDerivs License

(<https://creativecommons.org/licenses/by-nc-nd/4.0/>), which permits distribution and reproduction for non-commercial purposes, provided the author and source are cited.

**General rights**

Copyright for the publications made accessible via the Queen's University Belfast Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

**Take down policy**

The Research Portal is Queen's institutional repository that provides access to Queen's research output. Every effort has been made to ensure that content in the Research Portal does not infringe any person's rights, or applicable UK laws. If you discover content in the Research Portal that you believe breaches copyright or violates any law, please contact [openaccess@qub.ac.uk](mailto:openaccess@qub.ac.uk).

# INJECTIVE DIMENSION OF SHEAVES OF RATIONAL VECTOR SPACES

DANNY SUGRUE

ABSTRACT. The Cantor-Bendixson rank of a topological space  $X$  is a measure of the complexity of the topology of  $X$ . We will be interested primarily in the case that the space is profinite: Hausdorff, compact and totally disconnected. In this paper, we prove that the injective dimension of the abelian category of sheaves of  $\mathbb{Q}$ -modules over a profinite space  $X$  is determined by the Cantor-Bendixson rank of  $X$ .

## 1. INTRODUCTION

The injective dimension of an object of an abelian category is the minimal number of non-zero terms of any injective resolution of the object. The injective dimension of an abelian category is the supremum of this value ranging over all objects of the category. This value gives us a bound  $n$  for which the groups of any injective resolution of the category at level  $k$  are trivial for  $k$  bigger than  $n$ . We are interested in the abelian category of sheaves of  $\mathbb{Q}$ -modules over a profinite space  $X$ . In this paper will show that the injective dimension of this category can be computed simply by looking at the Cantor-Bendixson rank of  $X$ .

If  $X$  is a topological space, we can transfinite inductively define the Cantor-Bendixson process on  $X$ . We set  $X^{(0)}$  to be  $X$  and given  $X^{(n)}$  we define  $X^{(n+1)}$  to be the complement in  $X^{(n)}$  of its isolated points. For a limit ordinal  $\lambda$ , we can define this stage of the process in terms of the successor ordinals  $\beta$  which converge to it. Namely by setting:

$$X^{(\lambda)} = \bigcap_{\beta < \lambda} X^{(\beta)}.$$

Furthermore, as in [GS10a, Lemma 2.7], if  $X$  is Hausdorff then there exists some ordinal for which this process stabilises. We call this ordinal the Cantor-Bendixson rank of  $X$ . If  $X$  has a larger Cantor-Bendixson rank then it has limit points which have a more complicated set of points which accumulate at it. We will show in this paper, that this measure of the complexity of the topology of  $X$  determines the injective dimension of sheaves of  $\mathbb{Q}$ -modules over  $X$ . The main result of the paper is given in the following theorem, see Theorems 4.4 and 4.5.

**Theorem.** *If  $X$  is a space which is scattered and of finite Cantor-Bendixson rank  $n$  then the injective dimension of sheaves of  $\mathbb{Q}$ -modules over  $X$  is  $n - 1$ . If  $X$  is any space with infinite Cantor-Bendixson rank then the injective dimension is also infinite.*

If  $X$  has finite Cantor-Bendixson rank and non-empty perfect hull then we conjecture that the injective dimension of sheaves over  $X$  is infinite, see Conjecture 4.6. The discussion after Theorems 4.4 and 4.5 will explain the difficulties which arise in this final case.

The results in this paper hold for any Hausdorff space  $X$ , however in the case where  $X$  is profinite the applications of these results are especially interesting. The details of these applications are in [Sug], where we work with the space of closed subgroups of a profinite group  $G$ , which is a profinite space. The main objective of the thesis [Sug] is to construct an algebraic model for rational  $G$ -spectra when  $G$  is profinite and calculate the injective dimension of this model. In [Sug], we are interested in the  $G$ -equivariant sheaves over  $SG$  which satisfy that the stalk at each  $K$  in  $SG$  is a  $N_G(K)/K$ -module. We call these Weyl- $G$ -sheaves and we define the category of Weyl- $G$ -sheaves over  $SG$  to be the full subcategory of  $G$ -equivariant sheaves over  $SG$  determined by these objects. In particular, the algebraic model is the category of chain complexes of Weyl- $G$ -sheaves over  $SG$ . The contents of this paper is contained in [Sug] and developed upon to include a  $G$ -equivariant analogue. This ultimately shows that the injective dimension of the algebraic model is determined by the Cantor-Bendixson rank of  $SG$ . A useful property of the space  $SG$ , is that in many cases the Cantor-Bendixson rank is determined by the algebraic properties of the group  $G$ . This is the focus of the papers by Gartside and Smith, [GS10a, GS10b].

The main theorems of this paper hold for sheaves of rational vector spaces. However these results apply to sheaves of  $R$ -modules for any semisimple ring  $R$ . This is because every  $R$ -module is injective (this is the key fact we use about  $\mathbb{Q}$ -modules). The first section of the paper will introduce the concept of the Cantor-Bendixson rank of a Hausdorff space and look at some useful applications of this concept. For the second section, we will set up the injective resolutions that we will use in the final calculations. In the final section we prove the main theorem and discuss the conjecture dealing with the remaining case.

## 2. CANTOR-BENDIXSON RANK

Given a profinite space  $X$ , the aim of this paper is to calculate the injective dimension of the category of sheaves of  $\mathbb{Q}$ -modules over  $X$  in terms of the Cantor-Bendixson rank of  $X$ , denoted  $\text{Rank}_{CB}(X)$ . We begin by defining and stating known properties of the Cantor-Bendixson

rank from [GS10b, GS10a]. Recall that an **isolated** point of a topological space  $X$  is a point  $x$  which satisfies that  $\{x\}$  is open in  $X$ .

**Definition 2.1.** For a topological space  $X$  we can define the **Cantor-Bendixson process** on  $X$ . Denote by  $X'$  the set of all isolated points of  $X$ . We define:

- (1) Let  $X^{(0)} = X$  and  $X^{(1)} = X \setminus X'$  have the subspace topology with respect to  $X$ .
- (2) For successor ordinals suppose we have  $X^{(\alpha)}$  for an ordinal  $\alpha$ , we define  $X^{(\alpha+1)} = X^{(\alpha)} \setminus X^{(\alpha)'}$ .
- (3) If  $\lambda$  is a limit ordinal we define  $X^{(\lambda)} = \operatorname{colim}_{\alpha < \lambda} X^{(\alpha)}$ .

Every Hausdorff topological space  $X$  has a minimal ordinal  $\alpha$  such that  $X^{(\alpha)} = X^{(\lambda)}$  for all  $\lambda \geq \alpha$ , see [GS10a, Lemma 2.7].

**Definition 2.2.** Let  $X$  be a Hausdorff topological space. Then we define the **Cantor-Bendixson rank** of  $X$  denoted  $\operatorname{Rank}_{CB}(X)$  to be the minimal ordinal  $\alpha$  such that  $X^{(\alpha)} = X^{(\lambda)}$  for all  $\lambda \geq \alpha$ .

A topological space  $X$  is called **perfect** if it has no isolated points.

**Definition 2.3.** If  $X$  is a Hausdorff space with Cantor-Bendixson rank  $\lambda$ , then we define the **perfect hull** of  $X$  to be the subspace  $X^{(\lambda)}$ .

There are two ways that the Cantor-Bendixson process can stabilise. The first way is where the perfect hull is the empty set and the second is where it is a non-trivial subspace.

**Definition 2.4.** A compact Hausdorff space  $X$  of Cantor-Bendixson rank  $\alpha$  is called **scattered** if the space  $X^{(\alpha)}$  obtained by the definition above is equal to the empty set.

**Example 2.5.** If  $X$  is perfect or if  $X = \emptyset$  then  $\operatorname{Rank}_{CB}(X) = 0$ .

In the following example we will consider the space of closed subgroups of a profinite group  $G$ , which we denote by  $SG$ . We first give a description of this space. For a more detailed discussion of this construction, see [Dre71, Appendix pp. B4, B8].

**Definition 2.6.** Let  $G$  be a profinite space and  $SG$  be the set of closed subgroups of  $G$ . We define a topology on  $SG$  by considering the subbasis defined by the collection of subsets of the form:

$$O(N, NK) = \{A \in SG \mid NA = NK\},$$

where  $K$  ranges over the elements of  $SG$  and  $N$  over the open normal subgroups of  $G$ .

In [Dre71, Appendix pp. B4, B8] we see that with this topology  $SG$  is a profinite space. In particular it is the inverse limit over the finite discrete spaces of the form  $S(G/N)$ , where  $N$  is open and normal in  $G$ . We will now see what this construction looks like for the profinite group  $\mathbb{Z}_p$ .

**Proposition 2.7.** *The space  $S(\mathbb{Z}_p)$  is homeomorphic to:*

$$P = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \cup \{0\},$$

*which has the subspace topology with respect to  $\mathbb{R}$ .*

*Proof.* First notice that there is a bijection of sets given by  $p^k\mathbb{Z}_p \mapsto \frac{1}{k+1}$  and  $e \mapsto 0$ . We next observe that points of the form  $p^k\mathbb{Z}_p$  and  $\frac{1}{k+1}$  are isolated in their respective spaces. This is clear for  $P$  but for  $S(\mathbb{Z}_p)$  we can see the following:

$$\{p^k\mathbb{Z}_p\} = O(p^{k+1}\mathbb{Z}, p^k\mathbb{Z}_p).$$

To see that this is true observe that  $p^k\mathbb{Z}_p$  clearly belongs to this set. On the other hand, if  $A$  satisfies that  $Ap^{k+1}\mathbb{Z}_p = p^k\mathbb{Z}_p$  then  $A \leq p^k\mathbb{Z}_p$ . However  $A \leq p^{k+1}\mathbb{Z}_p$  has to be false since if it were true then we would have the following contradiction:

$$p^k\mathbb{Z}_p = Ap^{k+1}\mathbb{Z}_p = p^{k+1}\mathbb{Z}_p.$$

This shows that  $A$  must equal  $p^k\mathbb{Z}_p$ . A similar argument shows that there is a one to one correspondence between the neighbourhood basis of 0 and that of  $e$ . We can observe that:

$$O(p^k\mathbb{Z}_p, p^k\mathbb{Z}_p) = \{e\} \cup \{p^n\mathbb{Z}_p \mid n \geq k\},$$

which corresponds to the typical open neighbourhood of 0 in  $P$  of the form:

$$\{0\} \cup \left\{ \frac{1}{n} \mid n \geq k+1 \right\}.$$

□

This characterisation is stated in [Bar11, pp. 2115] and [GS10b, Example 3.2]

**Example 2.8.** Consider  $S(\mathbb{Z}_p)$  which by Proposition 2.7 is equivalent to  $P = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \cup \{0\}$  with the subspace topology of  $\mathbb{R}$ . Applying the first stage of the Cantor-Bendixson process to  $P$  results in removing the points of the form  $\frac{1}{n}$  leaving only the limit point 0. A second application of this process leaves us with the empty set since the singleton space consisting only of 0 is discrete. The process is stable from this point onwards so we therefore know that  $\text{Rank}_{CB}(P) = 2$ .

We will see more interesting examples after Proposition 2.12. The following proposition and theorem will explain how the perfect hull of a space  $X$  relates to  $X$  as a subspace. We shall allow  $X_H$  to denote the perfect hull of  $X$  and  $X_S$  to denote its complement, which we call the scattered part of  $X$ . Both  $X_S$  and  $X_H$  will be considered with the subspace topology with respect to  $X$ .

**Proposition 2.9.** *If  $X$  is a Hausdorff space, then  $X_H$  is always closed and  $X_S$  is always open.*

*Proof.* We shall prove that  $X_S$  is an open subset of  $X$ . If  $x$  is any point of  $X_S$  we will find an open subset of  $X$  containing  $x$  and contained in  $X_S$ . If  $x$  belongs to  $X_S$ , then by definition it is in the complement of  $X_H$  and hence there exist some ordinal  $\kappa$  such that  $x$  is isolated in  $X^{(\kappa)}$ . This in turn means that there exists some open subset  $U$  of  $X$  such that  $U \cap X^{(\kappa)} = \{x\}$ . This proves that each point belonging to  $U$  is eliminated in the Cantor-Bendixson process atleast before the stage of any ordinal strictly larger than  $\kappa$ . This is another way of saying that  $U$  is contained in  $X_S$  which proves the result.  $\square$

In general  $X_H$  may not be open. The next theorem shows that we can compute the cardinality of  $X_S$  in certain cases.

**Theorem 2.10** (Cantor-Bendixson Theorem). *Given a countably based Hausdorff topological space  $X$ , we can write  $X$  as a disjoint union of a countable scattered subset  $X_S$  with the perfect hull  $X_H$  of  $X$ .*

It is important to note that this theorem does not say that  $X$  can be written as the coproduct of  $X_H$  and  $X_S$  in the category of spaces. That is, this theorem does not claim that  $X_S$  and  $X_H$  provide a topological disconnection of  $X$ .

**Definition 2.11.** If  $X$  is a space and  $x \in X_S$ , we define the **height** of  $x$  denoted  $\text{ht}(X, x)$ , to be the ordinal  $\kappa$  such that  $x \in X^{(\kappa)}$  but  $x \notin X^{(\kappa+1)}$ . We sometimes denote this by  $\text{ht}(x)$  when the background space  $X$  is understood.

In the following proposition we will see how to calculate the Cantor-Bendixson rank of a product of two spaces. It is important to notice that this only works when both spaces have Cantor-Bendixson rank bigger than zero.

**Proposition 2.12.** *Let  $X$  be a space with  $\text{Rank}_{CB}(X) = n + 1$  and  $Y$  be a space with  $\text{Rank}_{CB}(Y) = m + 1$ , where  $m, n \in \mathbb{N}_0$ . Then  $\text{Rank}_{CB}(X \amalg Y) = m + n + 1 = \text{Rank}_{CB}(X) + \text{Rank}_{CB}(Y) - 1$ .*

*Proof.* We first prove this in the case where both  $X$  and  $Y$  are scattered. Let  $X_k$  denote the set of isolated points in  $X^{(k)}$ , and  $Y_k$  denote the set of isolated points in  $Y^{(k)}$ . First note that the isolated points of  $(X \amalg Y)$  are given by  $X_0 \amalg Y_0$ , and so:

$$(X \amalg Y)^{(1)} = (X \amalg Y) \setminus (X_0 \amalg Y_0).$$

To see this, first observe that  $X_0 \amalg Y_0$  consists of isolated points. On the other hand, take any point outside this set, say  $(x, y)$ , where either  $x$  or  $y$  has height bigger than or equal to 1. Assume without loss of generality that  $x$  is the point with non-trivial height. Then points of

the form  $(x', y)$ , where  $x'$  represents points which converge to  $x$ , belong to  $X \amalg Y$  and converge to  $(x, y)$ . Therefore  $(x, y)$  cannot be isolated in  $X \amalg Y$ .

The set of isolated points of  $(X \amalg Y)^{(1)}$  are equal to the set

$$(X_0 \amalg Y_1) \amalg (X_1 \amalg Y_0).$$

To see that this is true, first observe that points in this set are isolated. On the other hand take a point  $(x, y)$  such that  $x$  has height greater than or equal to 2 and  $y$  is isolated. Then points of the form  $(x', y)$  would converge to  $(x, y)$ , where  $x'$  has height between 1 and the height of  $x$ . The points of the form  $(x', y)$  therefore belong to  $(X \amalg Y)^{(1)}$ . Similarly if we take  $(x, y)$  in  $X_1 \amalg Y_1$  we will have points in  $X_0 \amalg Y_1$  and  $X_1 \amalg Y_0$  accumulating at  $(x, y)$ , and these points belong to  $(X \amalg Y)^{(1)}$ . It follows that  $(x, y)$  cannot be isolated in  $(X \amalg Y)^{(1)}$ . We therefore have:

$$(X \amalg Y)^{(2)} = (X \amalg Y) \setminus [(X_0 \amalg Y_0) \amalg (X_0 \amalg Y_1) \amalg (X_1 \amalg Y_0)].$$

**Claim:** The isolated points in  $(X \amalg Y)^{(i)}$  are of the form

$$\amalg_{p+q=i} (X_p \amalg Y_q)$$

where  $0 \leq p \leq n$  and  $0 \leq q \leq m$ .

We have shown this holds for  $i = 0$  and  $i = 1$  so let the above claim be our inductive hypothesis, and suppose it holds for  $i$  and that  $\gamma$  is an isolated point of  $(X \amalg Y)^{(i+1)}$ .

Then if  $i$  is even and hence  $i + 1$  is odd, since all of the points accumulating at  $\gamma$  were eliminated in the previous stage of the Cantor-Bendixson process, and by our hypothesis each of these accumulation points which were isolated in  $(X \amalg Y)^{(i)}$  belong to some  $X_p \amalg Y_q$  where  $p + q = i$ , so  $\gamma$  must belong to  $X_{p+1} \amalg Y_q$  or  $X_p \amalg Y_{q+1}$ .

In the case where  $i$  is odd and  $i + 1$  is even we have the same possibilities plus the additional possibility where  $\gamma$  is in  $X_{\frac{i+1}{2}} \amalg Y_{\frac{i+1}{2}}$ . Therefore we have shown by induction that the isolated points are of the form

$$\amalg_{p+q=i+1} (X_p \amalg Y_q).$$

From this we can see that:

$$(X \amalg Y)^{(i)} = (X \amalg Y) \setminus \left[ \amalg_{0 \leq k \leq i-1} \left( \amalg_{p+q=k} X_p \amalg Y_q \right) \right].$$

We then have:

$$\begin{aligned} (X \amalg Y)^{(n+m-1)} &= (X_n \amalg Y_{m-1}) \amalg (X_{n-1} \amalg Y_m) \amalg (X_n \amalg Y_m) \\ (X \amalg Y)^{(n+m)} &= (X_n \amalg Y_m) \\ (X \amalg Y)^{(n+m+1)} &= \emptyset. \end{aligned}$$

This proves that

$$\text{Rank}_{CB}(X \amalg Y) = m + n + 1 = \text{Rank}_{CB}(X) + \text{Rank}_{CB}(Y) - 1.$$

The case where atleast one of  $X$  and  $Y$  are non-scattered is similar except we observe that we end up with

$$(X \amalg Y)^{(n+m+1)} = (X \amalg Y)_H$$

which may not be empty.  $\square$

The following example shows that the condition that both  $X$  and  $Y$  need to have non-zero Cantor-Bendixson rank in order for Proposition 2.12 to hold.

**Example 2.13.** If  $X = \emptyset$  and  $Y$  is any space with Cantor-Bendixson rank bigger than 1 then Proposition 2.12 fails. We know that  $X \amalg Y = \emptyset$  and therefore has Cantor-Bendixson rank 0. On the other hand:

$$\text{Rank}_{CB}(X) + \text{Rank}_{CB}(Y) - 1 = \text{Rank}_{CB}(Y) - 1 \neq 0.$$

Furthermore if  $X$  is perfect this fails. To see this take any point  $(x, y)$  in  $X \amalg Y$  where  $y$  is isolated. Since  $X$  is perfect we can find a net  $x_\gamma$  converging to  $x$  which is not constant. Therefore  $(x_\gamma, y)$  provides a non-constant net converging to  $(x, y)$  in  $X \amalg Y$ . Therefore in this case the Cantor-Bendixson rank of  $X \amalg Y$  is 0. We can see that Proposition 2.12 fails in this case similar to how it failed when  $X = \emptyset$ .

We now have the following two examples of Cantor-Bendixson rank calculations.

**Example 2.14.** From [GS10b, Proposition 2.5] we know that if  $q_1, q_2, \dots, q_n$  are a finite collection of distinct primes then there is an isomorphism:

$$S \left( \prod_{1 \leq i \leq n} \mathbb{Z}_{q_i} \right) \cong \prod_{1 \leq i \leq n} S(\mathbb{Z}_{q_i})$$

By Example 2.8 there is a homeomorphism of spaces:

$$S \left( \prod_{1 \leq i \leq n} \mathbb{Z}_{q_i} \right) \cong P^n$$



An application of Proposition 2.12 shows that:

$$\text{Rank}_{CB} \left( S \left( \prod_{1 \leq i \leq n} \mathbb{Z}_{q_i} \right) \right) = n + 1.$$

**Example 2.15.** The space  $\prod_{n \in \mathbb{N}} P^n$  gives an example of a space which has infinite Cantor-Bendixson rank. This is because we can set  $x_n$  to be the point in  $P^n$  with height equal to  $n + 1$  and we therefore have a sequence of points with unbounded height. Notice that this space is not profinite since it is not compact.

### 3. INJECTIVE RESOLUTIONS OF SHEAVES

In this section we construct an injective resolution of sheaves of  $\mathbb{Q}$ -modules over a space  $X$ . We do this by defining the Godement resolution of a sheaf and outlining why this is injective. In order to achieve this we record the following definition from [Ten75, Definition 2.5.4].

**Definition 3.1.** A **sheaf space** of  $\mathbb{Q}$ -modules over a space  $X$  is a pair  $(E, p)$  such that:

- $E$  is a topological space and  $p: E \rightarrow X$  is a continuous local homeomorphism.
- For every  $x \in X$ ,  $p^{-1}(x)$  is a  $\mathbb{Q}$ -module which is continuous with respect to  $E$ . More precisely, if  $U$  is any open subset of  $X$  then the following is a  $\mathbb{Q}$ -module:

$$E(U) = \{s: U \rightarrow E \mid p \circ s = \text{Id}, s \text{ cts}\}.$$

Every sheaf space determines a sheaf by considering the assignment  $U \mapsto E(U)$ . On the other hand a sheaf  $F$  determines a sheaf space  $(LF, \pi)$ . This space has underlying set  $\prod_{x \in X} F_x$  and is topologised as in [Ten75, Construction 2.3.8]. The map of spaces  $\pi$  assigns a germ  $s_x$  to  $x$ .

**Definition 3.2.** Let  $F$  be a sheaf of  $\mathbb{Q}$ -modules over a topological space  $X$ . Then we define the sheaf  $C^0(F)$  on the open sets  $U$  by taking  $C^0(F)(U)$  to be the collection of serrations, i.e., the set of not necessarily continuous functions  $\{f: U \rightarrow LF \mid \pi \circ f = \text{Id}\}$  which equate to  $\prod_{x \in U} F_x$ .

Note that every section is a serration so we have a natural inclusion  $\delta_0: F \rightarrow C^0(F)$  which is a monomorphism.

**Remark 3.3.** The map from a sheaf  $F$  into  $C^0(F)$  is given as follows:

$$\begin{aligned} F(U) &\rightarrow \prod_{y \in U} F_y \rightarrow \text{colim}_V \prod_{y \in V} F_y \\ s &\mapsto (s_y)_{y \in U} \mapsto ((s_y)_{y \in U})_x \end{aligned}$$

where the colimit ranges over all open neighbourhoods of  $x$ ,  $U$  is an open neighbourhood of a point  $x \in X$  and  $(-)_x$  is the germ at  $x$ . This induces a map  $\delta_{0x}$  on stalks as follows:

$$F_x \rightarrow \operatorname{colim}_V \prod_{y \in V} F_y$$

$$s_x \mapsto ((s_y)_{y \in U})_x$$

We call  $\delta_{0x}$  the serration map and denote it by  $S$  throughout to simplify notation.

Notice that if a map  $f$  belongs to the set of serrations in Definition 3.2 then  $f$  does not have to be continuous. We can now define the Godement resolution using Definition 3.2 and [Bre97, pp. 36 - 37].

**Definition 3.4.** Let  $F$  be a sheaf of  $\mathbb{Q}$ -modules over a topological space  $X$ . Then as in Definition 3.2 we have  $C^0(F)$  and a monomorphism  $\delta_0: F \rightarrow C^0(F)$ .

Consider  $\operatorname{coker} \delta_0$ , if we replace  $F$  in the construction above with  $\operatorname{coker} \delta_0$  and set  $C^1(F) = C^0(\operatorname{coker} \delta_0)$  from Definition 3.2 we will get the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & F & \xrightarrow{\delta_0} & C^0(F) & \xrightarrow{\delta_1} & C^1(F) \\ & & & & \downarrow & \nearrow \delta_1' & \\ & & & & \operatorname{coker} \delta_0 & & \end{array}$$

where  $\delta_1'$  is the monomorphism from  $\operatorname{coker} \delta_0$  into  $C^1(F)$ . We can then continue to build the resolution inductively using this idea. This resolution which we have constructed is called the **Godement resolution**.

We consider the following example of a sheaf which will give us a more complete understanding of the Godement resolution.

**Example 3.5.** If  $x$  is any point of  $X$  and  $M$  any  $\mathbb{Q}$ -module, then we can define a sheaf  $\iota_x(M)$  over  $X$ . This takes value  $M$  at an open subset  $U$  of  $X$  if  $x$  belongs to  $U$  and 0 otherwise. We call this the skyscraper sheaf and the assignment  $M \mapsto \iota_x(M)$  defines a functor from the category of  $\mathbb{Q}$ -modules to the category of sheaves of  $\mathbb{Q}$ -modules. This functor is right adjoint to the functor from the category of sheaves to the category of  $\mathbb{Q}$ -modules which assigns  $F$  to the stalk  $F_x$ . We see this by considering the closed subset  $\{x\}$  of  $X$  and applying [Ten75, Theorem 3.7.13].

**Remark 3.6.** Each  $C^0(F)$  can be written as  $\prod_{y \in X} \iota_y(F_y)$ . To see this if  $U \subseteq X$  is open then  $C^0(F)(U) = \prod_{y \in U} F_y$ . On the other hand:

$$\left( \prod_{y \in X} \iota_y(F_y) \right) (U) = \prod_{y \in X} (\iota_y(F_y)(U)) = \prod_{y \in U} F_y$$

The following lemma relates the Cantor-Bendixson process to the Godement resolution. It shows that the  $k^{\text{th}}$  term of the Godement resolution is concentrated over  $X^{(k)}$ . This will ultimately provide an upper bound for the injective dimension of sheaves.

**Lemma 3.7.** *Let  $X$  be a topological space and  $F$  be a sheaf of  $\mathbb{Q}$ -modules over  $X$ . Then for every  $k \in \mathbb{N}$  we have that  $C^k(F)_x = 0$  for every  $x \in X \setminus X^{(k)}$ .*

*Proof.* We prove this using mathematical induction. For  $k = 0$  we will start by calculating  $C^0(F)_x$  when  $x$  is isolated. By definition we have:

$$C^0(F)_x = \operatorname{colim}_{U \ni x} \prod_{y \in U} F_y$$

where  $U$  ranges across all neighbourhoods of  $x$ . Since  $x$  is isolated it is clear that  $\{x\}$  is the minimal neighbourhood of  $x$ , so we have that  $C^0(F)_x = F_x$  as well as the fact that the monomorphism  $\delta_{0x}$  is an isomorphism. In particular this says that  $\operatorname{coker} \delta_{0x} = 0$ . It therefore follows that  $C^1(F)_x = 0$  since

$$C^1(F)_x = C^0(\operatorname{coker} \delta_0)_x = \operatorname{coker} \delta_{0x} = 0$$

Suppose  $\operatorname{coker} \delta_{k-1,x} = 0$  and hence  $C^k(F)_x = 0$  on  $X \setminus X^{(k)}$ , and take any  $x \in X \setminus X^{(k+1)}$  for some  $k \in \mathbb{N}$ . First observe that:

$$X \setminus X^{(k+1)} \supseteq X \setminus X^{(k)}.$$

If it happens that  $x \in X \setminus X^{(k)}$  and hence has height less than  $k$ , then by hypothesis  $\operatorname{coker} \delta_{k-1,x} = 0$  and hence  $C^k(F)_x = 0$ . Therefore since  $\operatorname{coker} \delta_{kx}$  is a quotient of  $C^k(F)_x$  which is zero it follows that  $\operatorname{coker} \delta_{kx} = 0$ . Since any point  $y$  which accumulates at  $x$  has scattered height less than that of  $x$ , and hence less than  $k$ , it follows that  $\operatorname{coker} \delta_{ky} = 0$ . We can then see immediately that  $C^{k+1}(F)_x = \operatorname{colim}_{V \ni x} \prod_{y \in V} \operatorname{coker} \delta_{ky} = 0$ .

The final case is the one where  $x$  is isolated in  $X^{(k)}$  and hence the scattered height of  $x$  is equal to  $k$ . This case yields the following diagram:

$$\begin{array}{ccccc} & & C^k(F) & & C^{k+1}(F) \\ & \nearrow^{\delta'_k} & & \searrow^{\delta'_{k+1}} & \\ \operatorname{coker} \delta'_{k-1} & & & & \operatorname{coker} \delta'_k \end{array}$$

Observe that all of the points  $y$  accumulating at  $x$  satisfy that the scattered height of  $y$  is less than that of  $x$  and hence less than  $k$ . It follows that  $\operatorname{coker} \delta'_{k-1,y} = 0$  by the inductive hypothesis for every such  $y$ . Similar to the  $k = 0$  step above we have:

$$C^k(F)_x = \operatorname{colim}_{U \ni x} \prod_{y \in U} \operatorname{coker} \delta_{k-1,y} = \operatorname{coker} \delta_{k-1,x}$$

and that  $\delta'_{kx}$  is an isomorphism so  $\text{coker } \delta'_{kx} = 0$ . All of the points which accumulate at  $x$  must belong to  $X \setminus X^{(k)}$  and we have already shown that these points  $y$  satisfy  $\text{coker } \delta'_{ky} = 0$ . This information combined proves that  $C^{k+1}(F)_x = \text{colim}_{U \ni x} \prod_{y \in U} \text{coker } \delta_{ky} = 0$ .  $\square$

Recall the following well-known proposition from category theory which will prove useful and is found in [Wei94, Proposition 2.3.10].

**Proposition 3.8.** *If  $(F, G)$  is an adjoint pair where*

$$F: \mathfrak{C} \rightarrow \mathfrak{D} \text{ and } G: \mathfrak{D} \rightarrow \mathfrak{C}$$

*are functors of Abelian categories, satisfying that  $F$  preserves monomorphisms then  $G$  preserves injective objects.*

**Proposition 3.9.** *If  $F$  is a sheaf of  $\mathbb{Q}$ -modules over  $X$  then  $C^k(F)$  is injective in the category of sheaves of  $\mathbb{Q}$ -modules.*

*Proof.* From the inductive way that  $C^k(F)$  is defined it is sufficient to prove that  $C^0(F)$  is injective. Remark 3.6 suggests that it is sufficient to prove that each  $\iota_x(F_x)$  is injective.

This follows from Proposition 3.8 applied to the adjoint pair of functors in Example 3.5,  $(Ev_x(-), \iota_x(-))$ , where  $Ev_x(F) = F_x$  for a sheaf  $F$ .

Note that the left adjoint preserves monomorphisms since a monomorphism of sheaves is a morphism of sheaves such that the map at each stalk is a monomorphism of  $\mathbb{Q}$ -modules. Using that each  $F_x$  is a  $\mathbb{Q}$ -module, we apply the fact that every object in the category of  $\mathbb{Q}$ -modules is injective to deduce that  $\iota_x(F_x)$  is an injective sheaf.  $\square$

We next look at a lemma which helps us with our injective dimension calculations since it will ultimately enable us to calculate Ext groups.

Recall from Definition 3.4 that if  $x \in X$  and  $k < \text{ht}(X, x)$  then:

$$\text{coker } \delta_{kx} = \left[ \text{colim}_{U \ni x} \prod_{y \in U} \text{coker } \delta_{k-1y} \right] / S$$

where  $S$  is the serration map from Remark 3.3. Explicitly if  $a \in \text{coker } \delta_{k-1x}$  we define  $(a, \underline{0})_x$  to be the element in  $\text{coker } \delta_{kx}$  which is the germ at  $x$  of the family which is  $a$  in place  $x$  and zero elsewhere.

**Lemma 3.10.** *Suppose  $X$  is a space with  $\text{Rank}_{CB}(X) = n$  for  $n \in \mathbb{N}$  such that  $X^{(n)} = \emptyset$ . Then for  $j \leq n - 1$ ,  $x \in X^{(j)}$  and  $F$  a sheaf over  $X$ , we have an isomorphism  $\text{hom}(\iota_x(\mathbb{Q}), C^k(F)) \cong \text{coker } \delta_{k-1x}$  for  $k < j$ , and the map:*

$$\delta_{k+1*}: \text{hom}(\iota_x(\mathbb{Q}), C^k(F)) \rightarrow \text{hom}(\iota_x(\mathbb{Q}), C^{k+1}(F))$$

is given by the map:

$$\begin{aligned} \alpha_{k+1}: \operatorname{coker} \delta_{k-1,x} &\rightarrow \operatorname{coker} \delta_{k,x} \\ a &\mapsto (a, \underline{0})_x. \end{aligned}$$

*Proof.* Firstly notice that  $C^k(F)$  is defined to be  $C^0(\operatorname{coker} \delta_{k-1})$ , so we begin by proving that  $\operatorname{hom}(\iota_x(\mathbb{Q}), C^0(F)) \cong F_x$ . Observe that  $C^0(F) = \prod_{y \in X} \iota_y(F_y)$  so we can write:

$$\begin{aligned} \operatorname{hom}(\iota_x(\mathbb{Q}), C^0(F)) &= \operatorname{hom}(\iota_x(\mathbb{Q}), \prod_{y \in X} \iota_y(F_y)) = \prod_{y \in X} \operatorname{hom}(\iota_x(\mathbb{Q}), \iota_y(F_y)) \\ &= \prod_{y \in X} \operatorname{hom}(\iota_x(\mathbb{Q})_y, F_y) = F_x. \end{aligned}$$

In particular if  $f \in \operatorname{hom}(\iota_x(\mathbb{Q}), C^k(F))$ , then this is determined by a map in  $\operatorname{hom}(\iota_x(\mathbb{Q}), \iota_x(\operatorname{coker} \delta_{k-1,x}))$ . This corresponds to a point  $f_x \in \operatorname{coker} \delta_{k-1,x}$ , so  $f$  is given by the element:

$$[f_x, \underline{0}]_x \in \operatorname{colim}_{V \ni x} \left( \prod_{y \in V} \operatorname{coker} \delta_{k-1,y} \right) = C^0(\operatorname{coker} \delta_{k-1})_x,$$

with this germ at  $x$  of the family taking value  $f_x$  in position  $x$  and 0 elsewhere. It follows that  $\delta_{k+1,*}(f)$  corresponds to  $\delta_{k+1}([f_x, \underline{0}]_x)$ .

But we therefore have:

$$\delta_{k+1}([f_x, \underline{0}]) = \left( [(f_x, \underline{0})_x]^S, \left( [(f_x, \underline{0})_y]^S \right)_{y \in U} \right)_x$$

in  $\operatorname{colim}_{V \ni x} \left( \prod_{y \in V} \operatorname{coker} \delta_{k,y} \right) = C^0(\operatorname{coker} \delta_k)_x$ , where  $[-]^S$  represents the class in the cokernel of the map  $S$ . This follows from the definition of the maps  $\delta$  in Definition 3.4.

However if  $x \neq y$  then since  $X$  is Hausdorff there is a neighbourhood of  $y$  not containing  $x$  so  $[(f_x, \underline{0})_y]^S = [(\underline{0})_y]^S$ . Therefore  $\delta_{k+1}((f_x, \underline{0})_x)$  can be written as  $\left( [(f_x, \underline{0})_x]^S, \left( [(\underline{0})_y]^S \right)_{y \in U} \right)$ .

We therefore have that  $\delta_{k+1,*}$  is defined as follows:

$$\begin{aligned} \delta_{k+1,*}: \operatorname{hom}(\iota_x(\mathbb{Q}), C^k(F)) &\rightarrow \operatorname{hom}(\iota_x(\mathbb{Q}), C^{k+1}(F)) \\ [f_x, \underline{0}]_x &\mapsto \left( [(f_x, \underline{0})_x]^S, \left( [(\underline{0})_y]^S \right)_{y \in U} \right) \end{aligned}$$

Since we are interested in what the maps correspond to as maps between the  $x$  components of the products of  $C^0(\operatorname{coker} \delta_{k-1})$  and  $C^0(\operatorname{coker} \delta_k)$ , we observe that it sends  $f_x$  to  $[(f_x, \underline{0})_x]^S$ .  $\square$

Now we consider the preceding lemma for points in  $X$  which either have infinite height or belong to the hull of  $X$ .

**Lemma 3.11.** *Suppose  $X$  is a space with infinite Cantor-Bendixson rank. Then for  $x \in X^{(n)}$  for any  $n \in \mathbb{N}$ , and  $F$  a sheaf over  $X$ , we have an isomorphism  $\mathrm{hom}(\iota_x(\mathbb{Q}), C^k(F)) \cong \mathrm{coker} \delta_{k-1_x}$  for  $k < n$ , and the map:*

$$\delta_{k+1_*} : \mathrm{hom}(\iota_x(\mathbb{Q}), C^k(F)) \rightarrow \mathrm{hom}(\iota_x(\mathbb{Q}), C^{k+1}(F))$$

is given by the map:

$$\begin{aligned} \alpha_{k+1} : \mathrm{coker} \delta_{k-1_x} &\rightarrow \mathrm{coker} \delta_{k_x} \\ a &\mapsto (a, \underline{0})_x. \end{aligned}$$

This also holds for points of infinite height and points in the hull.

*Proof.* The proof of this result is almost the same as the proof of Lemma 3.10. The only differences arise from the fact that in Lemma 3.10  $X$  has finite Cantor-Bendixson rank, say  $n$ , and so  $\mathrm{coker} \delta_k$  is zero if  $k$  is bigger than  $n$ . Therefore the argument in Lemma 3.10 is only interesting provided we are applying it to  $\mathrm{coker} \delta_k$  for  $k$  small enough. If  $X$  has infinite Cantor-Bendixson rank then we know that for each  $n \in \mathbb{N}$  there exists a point  $x_n$  with height  $n$ , as well as points of infinite height. Therefore the argument on  $\mathrm{coker} \delta_k$  doesn't become trivial eventually. The same observation is true if  $X$  has any point in the perfect hull of  $X$ .  $\square$

The previous two lemmas indicate how the calculations in this paper differ from sheaf cohomology. In this setting we apply the functor  $\mathrm{hom}(\iota_x(\mathbb{Q}), -)$  to an injective resolution and this is different from sheaf cohomology where we apply a functor  $\mathrm{hom}(A, -)$  for some constant sheaf  $A$ .

#### 4. INJECTIVE DIMENSION CALCULATION

We now formally give the definition of the injective dimension of sheaves of  $\mathbb{Q}$ -modules over a space  $X$  as seen in [Wei94, Definition 4.1.1, Definition 10.5.10].

**Definition 4.1.** The **injective dimension** of a sheaf  $F$  over  $X$  denoted by  $\mathrm{ID}(F)$  is the minimum positive integer  $n$  (if it exists) such that there is an injective resolution of the form

$$0 \longrightarrow X \xrightarrow{\epsilon} I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} I_n \longrightarrow 0.$$

where  $I_j \neq 0$  for  $j \leq n$ . It is infinite if such a value doesn't exist.

From [Wei94, Theorem 4.1.2] we define the injective dimension of the category of sheaves of  $\mathbb{Q}$ -modules to be:

$$\sup \{ \mathrm{ID}(F) \mid F \in \mathrm{Sheaf}_{\mathbb{Q}}(X) \},$$

where  $\text{Sheaf}_{\mathbb{Q}}(X)$  denotes the category of sheaves of  $\mathbb{Q}$ -modules over  $X$ . We can now verify that the injective dimension of sheaves of  $\mathbb{Q}$ -modules is bounded above for a particular class of space.

**Proposition 4.2.** *If  $X$  is a scattered space with  $\text{Rank}_{CB}(X) = n$  for  $n \in \mathbb{N}$  then the injective dimension of sheaves of  $\mathbb{Q}$ -modules over  $X$  is bounded above by  $n - 1$ .*

*Proof.* By Proposition 3.9 the Godement resolution is an injective resolution. An application of Lemma 3.7 shows that the terms of the Godement resolution are zero after term  $n - 1$ . Therefore the injective dimension of each sheaf is less than or equal to  $n - 1$ .  $\square$

In order to get equality it is sufficient to find a particular sheaf  $F$  for which  $\text{ID}(F) \geq \text{Rank}_{CB}(X) - 1$ . To achieve this we look at [Wei94, Lemma 4.1.8, Exercise 10.7.2] which says  $\text{ID}(F) \leq \text{Rank}_{CB}(X) - 2$  if and only if  $\text{Ext}^{\text{Rank}_{CB}(X)-1}(A, F) = 0$  for every sheaf  $A$ . In particular if we can find sheaves  $A$  and  $F$  such that  $\text{Ext}^{\text{Rank}_{CB}(X)-1}(A, F) \neq 0$  then we must have that  $\text{ID}(F) > \text{Rank}_{CB}(X) - 2$ . This then forces  $\text{ID}(F) = \text{Rank}_{CB}(X) - 1$ .

We now work towards verifying the lower bound. We will look at the following lemma which will illustrate that the Godement resolution is non-zero at term  $k$  provided  $k$  is less than  $\text{Rank}_{CB}(X)$ .

**Lemma 4.3.** *Let  $X$  be a non-empty scattered space and  $k \in \mathbb{N}$  be less than or equal to  $\text{Rank}_{CB}(X)$ . For each  $x \in X^{(k)}$ ,  $\text{coker } \delta_{k-1,x} \neq 0$  in the Godement resolution of  $c\mathbb{Q}$  the constant sheaf at  $\mathbb{Q}$ .*

*Furthermore if  $X$  is any space with a non-empty perfect hull and  $k \in \mathbb{N}$ , then for each  $x \in X^{(k)}$  we have  $\text{coker } \delta_{k-1,x} \neq 0$  in the Godement resolution of  $c\mathbb{Q}$ .*

*Proof.* We begin with the scattered case and we will prove this using an induction argument. Since

$$\text{colim}_{U \ni x} c\mathbb{Q}(U) = c\mathbb{Q}_x = \text{colim}_{U \ni x} Pc\mathbb{Q}(U) = \mathbb{Q}$$

where  $Pc\mathbb{Q}$  represents the presheaf, any  $q_x \in \mathbb{Q}_x$  is represented by some  $q \in \mathbb{Q}$ . We therefore have the following diagram by [Bre97, pp. 36-37]:

$$\begin{aligned} \mathbb{Q} &\rightarrow \prod_{y \in U} \mathbb{Q} \rightarrow \text{colim}_{V \ni x} \prod_{y \in V} \mathbb{Q} \\ q &\mapsto (q_y)_{y \in U} \mapsto ((q_y)_{y \in U})_x \end{aligned}$$

which induces a map:

$$\begin{aligned} \mathbb{Q} &\rightarrow \text{colim}_{V \ni x} \prod_{y \in V} \mathbb{Q} \\ q_x &\mapsto ((q_y)_{y \in U})_x \end{aligned}$$

We call this map the serration map and denote it by  $S$ . This is not surjective since we have a point  $(0_x, \underline{1})_x$  not in the image of  $S$ . This

point is non-zero since if  $S(a) = [(0_x, \underline{1})_x]^S$  then  $a_x = 0$  and so there is an open neighbourhood  $U$  of  $x$  such that  $a_y = 0$  for  $y \in U$ . However the definition of the serration map shows that  $a_y = 1$  also for  $y \neq x$  which is a contradiction. Therefore  $\text{coker } \delta_{0_x} \neq 0$  for  $x \in X^{(1)}$ .

Suppose this holds up to some  $n \in \mathbb{N}$  and for any  $x \in X^{(n+1)}$ . By assumption we have that

$$0 \neq \text{coker } \delta_{n_x} = \text{colim}_{U \ni x} \left( \prod_{y \in U} \text{coker } \delta_{n-1_y} \right) / S$$

Using the fact that sheafification preserves stalks of presheaves we have a map using [Bre97, pp. 36-37] as follows:

$$\begin{aligned} \left( \prod_{y \in U} \text{coker } \delta_{n-1_y} \right) / S &\rightarrow \prod_{z \in U} \text{coker } \delta_{n_z} \rightarrow \text{colim}_{V \ni x} \prod_{z \in V} \text{coker } \delta_{n_z} \\ [(a_y)_{y \in U}]^S &\mapsto \left( \left[ \left( (a_y)_{y \in U} \right)^S \right]_z \right)_{z \in U} \mapsto \left( \left( \left[ \left( (a_y)_{y \in U} \right)^S \right]_z \right)_{z \in U} \right)_x \end{aligned}$$

which induces a map:

$$\begin{aligned} \text{coker } \delta_{n_x} &\rightarrow \text{colim}_{V \ni x} \prod_{y \in V} \text{coker } \delta_{n_y} \\ \left( \left[ \left( (a_y)_{y \in U} \right)^S \right]_x \right) &\mapsto \left( \left( \left[ \left( (a_y)_{y \in U} \right)^S \right]_z \right)_{z \in U} \right)_x \end{aligned}$$

Let  $U$  be any open neighbourhood of  $x$ . Then for each  $y \in U$  such that  $y \in X^{(n)}, X^{(n+1)}$  or  $X^{(n+2)}$  we can choose  $0 \neq a_y \in \text{coker } \delta_{n-1_y}$  by the inductive hypothesis. Set  $s^y = (0_y, a_z)_{z \in U \setminus \{y\}}$ , then  $\left[ (s^y)_y \right]^S \neq 0$  in  $\text{coker } \delta_{n_y}$  for  $y \in X^{(n+1)}, X^{(n+2)}$  and we denote this by  $b_y$ . This is shown to be non-zero by following a similar argument to that seen earlier in this proof. We can therefore consider for any  $x \in X^{(n+2)}$ :

$$\left( \left[ \left( (0_x, b_y)_{y \in U \setminus \{x\}} \right)^S \right]_x \right)$$

which is not in the image of the serration map so  $\text{coker } \delta_{n+1_x} \neq 0$ . This is also seen by referring to the previous argument seen earlier in this proof.

Note if  $\text{Rank}_{CB}(X)$  is infinite then for each  $k \in \mathbb{N}$  we have that each  $X^{(k)}$  has isolated points to remove, so this is true for every  $k$ . If  $\text{Rank}_{CB}(X) = n$  and  $X^{(n)} = \emptyset$  then  $X^{(n-1)}$  is discrete and therefore satisfies that  $C^n(F) = 0$  by Lemma 3.7. It follows that the argument therefore only results in non-zero stalks for  $k \leq n - 1$ . If  $x$  belongs to the perfect hull of  $X$  then this argument also holds for each  $k \in \mathbb{N}$  since the hull is contained in each  $X^{(k)}$ .  $\square$

We now use the above calculations to verify the injective dimension of sheaves using the Cantor-Bendixson dimension.



**Theorem 4.4.** *Suppose  $X$  is a space with  $\text{Rank}_{CB}(X) = n$  such that  $X^{(n)} = \emptyset$ . Then the category of sheaves over  $X$  has injective dimension equal to  $n - 1$ .*

*Proof.* To see this we need to find an object in the category of sheaves over  $X$  so that  $\text{ID}(X) = n - 1$ , we will show that  $c\mathbb{Q}$  satisfies  $\text{Id}(c\mathbb{Q}) = n - 1$ . Firstly by Lemma 3.7 we know that  $\text{ID}(c\mathbb{Q}) \leq n - 1$  since the Godement resolution gives an injective resolution of the form:

$$0 \longrightarrow c\mathbb{Q} \xrightarrow{\delta_0} I^0 \xrightarrow{\delta_1} \dots \xrightarrow{\delta_{n-2}} I^{n-2} \xrightarrow{\delta_{n-1}} I^{n-1} \longrightarrow 0$$

From Lemma 4.3 we know that each  $I_j \neq 0$ . We will show that the  $\text{Ext}^{n-1}(\iota_x(\mathbb{Q}), c\mathbb{Q})$  group calculated by the above injective resolution is non-zero. Let  $x$  be an element of  $X$  with  $\text{ht}(X, x) = n - 1$  (any point of  $X$  with maximal height).

We apply the functor  $\text{Hom}(\iota_x(\mathbb{Q}), -)$  and forget the  $c\mathbb{Q}$  term to get:

$$\begin{array}{ccccccc} \text{Hom}(\iota_x(\mathbb{Q}), I^0) & \xrightarrow{\delta_{1*}} & \text{Hom}(\iota_x(\mathbb{Q}), I^1) & \xrightarrow{\delta_{2*}} & \dots & \xrightarrow{\delta_{n-2*}} & \text{Hom}(\iota_x(\mathbb{Q}), I^{n-2}) \\ & & & & & & \downarrow \delta_{n-2*} \\ & & & & & & 0 \longleftarrow \text{Hom}(\iota_x(\mathbb{Q}), I^{n-1}) \end{array}$$

which we can no longer assume to be exact. This is equal to the following sequence:

$$\begin{array}{ccccccc} \mathbb{Q} & \xrightarrow{\alpha_1} & \text{coker } \delta_{0x} & \xrightarrow{\alpha_2} & \dots & \xrightarrow{\alpha_{n-3}} & \text{coker } \delta_{n-4x} \\ & & & & & & \downarrow \alpha_{n-2} \\ & & & & & & 0 \longleftarrow \text{coker } \delta_{n-2x} \xleftarrow{\alpha_{n-1}} \text{coker } \delta_{n-3x} \end{array}$$

We want to show that  $\text{Ext}^{(n-1)}(\iota_x(\mathbb{Q}), c\mathbb{Q}) = \ker \alpha_n / \text{Im } \alpha_{n-1} \neq 0$  and  $\text{Ext}^n(\iota_x(\mathbb{Q}), c\mathbb{Q}) = \ker \alpha_{n+1} / \text{Im } \alpha_n = 0$ . It is clear that  $\text{Ext}^n(\iota_x(\mathbb{Q}), c\mathbb{Q})$  is 0. For the other we need to prove that the map:

$$\begin{aligned} \alpha_{n-1}: \text{coker } \delta_{n-3x} &\rightarrow \text{coker } \delta_{n-2x} \\ a &\mapsto (a, \underline{0})_x \end{aligned}$$

is not surjective. This is done in a similar fashion to Proposition 4.3.

For any open neighbourhood  $U$  of  $x$  there are infinitely many points  $z$  of  $U$  such that  $z \in X^{(n-2)}$  and  $\text{coker } \delta_{n-3z} \neq 0$  so we can choose such a point  $a_z$  for each  $z$ . Consider:

$$a = [((0_x, a_z)_{z \in U \setminus \{x\}})_x]^S \in \text{coker } \delta_{n-2x}.$$

If  $t_x \in \text{coker } \delta_{n-3x}$  is in the preimage of  $a$  with respect to  $\alpha_{n-1}$  we would have:

$$[(t_x, 0)_x]^S = \alpha_{n-1}(t_x) = [((0_x, a_z)_{z \in U \setminus \{x\}})_x]^S$$

so  $t_x = 0$  which implies that  $\left[ \left( (0_x, a_z)_{z \in U \setminus \{x\}} \right)_x \right]^S = 0$ . But this cannot be the case since there are infinitely many  $z$  satisfying that  $a_z \neq \underline{0}$  by construction, so we have a contradiction and  $\alpha_{n-1}$  cannot be surjective.  $\square$

We now deal with the case where the Cantor-Bendixson dimension is infinite.

**Theorem 4.5.** *If  $X$  is a space with infinite Cantor-Bendixson rank, then the injective dimension of sheaves of  $\mathbb{Q}$ -modules over  $X$  is infinite.*

*Proof.* Since the Cantor-Bendixson rank of  $X$  is infinite there exists a sequence of points  $x_n$  each having height  $n$ . As a consequence of Theorem 4.4 for each  $x_n$  we know that  $\text{Ext}^n(\iota_{x_n}(\mathbb{Q}), c\mathbb{Q}) \neq 0$  and this happens for each  $n$  since we do not have a maximal height. This proves the result.  $\square$

We are left to deal with the case where  $X$  is not scattered but has finite Cantor-Bendixson rank. In the above cases when we resolve with respect to  $\iota_x(\mathbb{Q})$  the resolved sequence becomes zero after  $\text{ht}(x) - 1$  so the kernel of the map  $\alpha_{\text{ht}(x)+1}$  is  $\text{coker } \delta_{\text{ht}(x)-1, x}$  which is advantageous since we can choose any point of  $\text{coker } \delta_{\text{ht}(x)-1}$  not in the image of  $\alpha_{\text{ht}(x)}$ . This changes when we are working in the case where  $X$  has a perfect hull.

We know that the following Godement resolution is infinite:

$$0 \longrightarrow c\mathbb{Q} \xrightarrow{\delta_0} I^0 \xrightarrow{\delta_1} \dots \xrightarrow{\delta_{n-1}} I^{n-1} \xrightarrow{\delta_n} I^n \xrightarrow{\delta_n} \dots$$

Therefore after resolving like above for a point  $x$  in the perfect hull we obtain the following infinite sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Q} & \xrightarrow{\alpha_1} & \text{coker } \delta_{0x} & \xrightarrow{\alpha_2} & \dots & \xrightarrow{\alpha_{n-2}} & \text{coker } \delta_{n-3x} \\ & & & & & & & & \downarrow \alpha_{n-1} \\ & & & & & & \dots & \xleftarrow{\alpha_{n+1}} & \text{coker } \delta_{n-1x} & \xleftarrow{\alpha_n} & \text{coker } \delta_{n-2x} \end{array}$$

The important thing to notice is that since this is non-zero at infinitely many places, when calculating the group  $\text{Ext}^n$  we can't just choose any representative of  $\text{coker } \delta_{n-1x}$  since the kernel is not everything.

In order to choose something in the kernel we need to adjust our argument above, namely instead of choosing a representative  $(0_x, s^y)_{y \in U \setminus \{x\}}$  with  $0 \neq s^y \in \text{coker } \delta_{n-2y}$  arbitrary, we need  $(s^y)_{y \in U \setminus \{x\}}$  to be determined by a section  $s$  over  $\text{coker } \delta_{n-2}$ . That is we want each  $s^y$  to be of the form  $s_y$  for that section  $s$ , and such that each open neighbourhood  $U$  of  $x$  contains infinitely many  $y$  such that  $s^y \neq 0$ .

Recalling a fact from sheaf theory that a section  $s$  over an open neighbourhood  $U$  of  $x$  has germ  $s_x = 0$  if and only if  $s$  restricts to some smaller neighbourhood to give the zero section. Also recall that we can

build a section in  $\text{coker } \delta_{n-2}(U)$  by considering  $\prod_{y \in U} \text{coker } \delta_{n-3}/S$ , where  $S$  represents the map defined in Remark 3.3. This means that if we can construct the family  $[(a^y)_{y \in U \setminus \{x\}}]_S$  to be an alternating family where infinitely many  $a^y \neq 0$  in  $\text{coker } \delta_{n-3y}$  and infinitely many do equal zero, then we may have a suitable section  $s$  to proceed with the proof. This approach needs the following condition to proceed:

If  $a^y = 0$ , then any neighbourhood  $U$  of  $y$  contains infinitely many points  $z$  such that  $a^z \neq 0$ .

If we can construct given any net converging to  $x$ , two term-wise disjoint subnets then we can do the above construction to show that the injective dimension of sheaves in this case is infinite, provided the sequence is set up to satisfy the condition. With this in mind we have the following conjecture.

**Conjecture 4.6.** *If  $X$  has finite Cantor-Bendixson rank and non-empty perfect hull then the injective dimension of sheaves of  $\mathbb{Q}$ -modules over  $X$  is infinite.*

We now look at examples of spaces and the application of the result relating injective dimension of sheaves of  $\mathbb{Q}$ -modules over  $X$  to the Cantor-Bendixson rank of  $X$ . Our primary interest is in the space of closed subgroups of a profinite group  $G$ , as defined in Definition 2.6.

**Example 4.7.** If  $G$  is a discrete group then  $SG$  is a finite discrete space and hence has Cantor-Bendixson dimension 1. Therefore Theorem 4.4 implies that the injective dimension of sheaves of  $\mathbb{Q}$ -modules over  $SG$  is 0.

The above example works equally for any discrete space. Another way of seeing that the injective dimension of sheaves of  $\mathbb{Q}$ -modules over a discrete space is zero is by observing that such a sheaf is equivalent to a product of  $\mathbb{Q}$ -modules. We can see this by noticing that since each point  $x$  in  $X$  is isolated, the stalk of a sheaf  $F$  at  $x$  is determined by evaluating the sheaf at the open subset  $\{x\}$ . This becomes even clearer by considering a sheaf  $F$  from the point of view of Definition 3.1. Let  $(LF, \pi)$  be the sheaf space for  $F$  over a discrete space  $X$ . Then since every map of sets from  $X$  to  $LF$  is continuous, it follows that  $F(U) = \prod_{x \in U} F_x$  for any open subset  $U$ . This is an alternative way of saying that  $F$  is equivalent to  $C^0(F)$  from Definition 3.2, which we know to be injective by Proposition 3.9.

**Example 4.8.** If  $G = \mathbb{Z}_p$  for any prime number  $p$ , then  $S(\mathbb{Z}_p)$  is homeomorphic to the space  $P$  from Proposition 2.7. This space has Cantor-Bendixson rank 2 as seen in Example 2.8. Therefore the category of sheaves of  $\mathbb{Q}$ -modules over  $S(\mathbb{Z}_p)$  has injective dimension 1 by Theorem 4.4.

**Example 4.9.** Consider distinct primes  $p_1, p_2, \dots, p_n$ . We have a profinite group  $\prod_{1 \leq i \leq n} \mathbb{Z}_{p_i}$  with corresponding profinite space  $S\left(\prod_{1 \leq i \leq n} \mathbb{Z}_{p_i}\right)$ . This space is homeomorphic to  $P^n$  by [GS10a, Proposition 2.5]. Then by Proposition 2.12 we have that  $\text{Rank}_{CB}(P^n) = n + 1$  and  $(P^n)^{(n+1)} = \emptyset$ . We can now apply Theorem 4.4 to deduce that the injective dimension of sheaves over  $S\left(\prod_{1 \leq i \leq n} \mathbb{Z}_{p_i}\right)$  is exactly  $n$ .

The following example demonstrates a particular case where the category of sheaves has infinite injective dimension.

**Example 4.10.** In Example 2.15 we observed that the space  $\prod_{n \in \mathbb{N}} P^n$  has infinite Cantor-Bendixson rank. Therefore an application of Theorem 4.5 shows that the injective dimension of sheaves of  $\mathbb{Q}$ -modules over this space is also infinite.

Furthermore if we consider Conjecture 4.6 we can see the possible implications.

**Example 4.11.** The profinite completion of  $\mathbb{Z}$  is defined to be:

$$\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$$

where the product runs over the collection of prime numbers  $p$ . This is a profinite group under the product topology and we can see that  $S(\hat{\mathbb{Z}})$  is perfect. If proven to be correct, Conjecture 4.6 would imply that the injective dimension of sheaves over this space is infinite.

Another important example of a space is defined in [GS10a, Definition 2.8], and this construction is similar to the Cantor space.

**Definition 4.12.** Let  $F_0 = P_0 = [0, 1]$ , the closed unit interval. We set  $F_1 = F_0 \setminus (\frac{1}{3}, \frac{2}{3})$  and  $B_1 = F_1 \cup \{\frac{1}{2}\}$ . That is, to form  $F_1$  we remove the middle third of the interval of  $F_0$  and to form  $B_1$  we reinsert the midpoint of the deleted interval to  $F_1$ .

Given  $F_{i-1}$  we define  $F_i$  by deleting the middle third intervals of the remaining segments of  $F_{i-1}$  and we define  $B_i$  by reinserting midpoints of the deleted intervals to  $F_i$ . We set  $F = \bigcap_{n \in \mathbb{N}} F_n$  and  $B = \bigcap_{n \in \mathbb{N}} B_n$ .

The main focus of [GS10a] is on proving that the algebraic structure of a profinite group  $G$  can tell us about  $SG$ . Specifically, throughout [GS10a] there are many assumptions on the algebraic structure of  $G$  which lead to the conclusion that  $SG$  is homeomorphic to  $B$  from Definition 4.12.

**Example 4.13.** Consider the spaces  $B$  and  $F$  defined in Definition 4.12. From [GS10a, Definition 2.8] we know that the space  $B$  has perfect hull given by the Cantor space  $F$  and that  $\text{Rank}_{CB}(B) = 1$ . If

Conjecture 4.6 were true then it would follow that the injective dimension of sheaves over  $B$  is infinite.

I would like to thank my PhD supervisor Dr David Barnes for his excellent guidance and support throughout many interesting discussions about this work. I would also like to thank Dr Martin Mathieu for his very helpful advice about this paper.

#### REFERENCES

- [Bar11] David Barnes. Rational  $\mathbb{Z}_p$ -equivariant spectra. *Algebr. Geom. Topol.*, 11(4):2107–2135, 2011.
- [Bre97] Glen E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.
- [Dre71] Andreas W. M. Dress. *Notes on the theory of representations of finite groups. Part I: The Burnside ring of a finite group and some AGN-applications*. Universität Bielefeld, Fakultät für Mathematik, Bielefeld, 1971. With the aid of lecture notes, taken by Manfred Küchler.
- [GS10a] Paul Gartside and Michael Smith. Classifying spaces of subgroups of profinite groups. *J. Group Theory*, 13(3):315–336, 2010.
- [GS10b] Paul Gartside and Michael Smith. Counting the closed subgroups of profinite groups. *J. Group Theory*, 13(1):41–61, 2010.
- [Sug] Danny Sugrue. Rational  $G$ -equivariant cohomology theories for  $G$  profinite. Thesis in progress 2018.
- [Ten75] Barry Roy Tennison. *Sheaf theory*. Cambridge University Press, Cambridge, England-New York-Melbourne, 1975. London Mathematical Society Lecture Note Series, No. 20.
- [Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

MATHEMATICAL SCIENCES RESEARCH CENTRE, SCHOOL OF MATHEMATICS AND PHYSICS, QUEEN'S UNIVERSITY BELFAST, UNIVERSITY ROAD, BELFAST, BT7 1NN, NORTHERN IRELAND, UK

*Email address:* dsugrue01@qub.ac.uk