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Levene, R. H., Spronk, N., Todorov, I. G., & Turowska, L. (2017). Schur multipliers of Cartan pairs. *Proceedings of the Edinburgh Mathematical Society*, *60* (2), 413-440. https://doi.org/10.1017/S0013091516000067

Published in:
Proceedings of the Edinburgh Mathematical Society

Document Version:
Early version, also known as pre-print

SCHUR MULTIPLIERS OF CARTAN PAIRS

R. H. LEVENE, N. SPRONK, I. G. TODOROV AND L. TUROWSKA

ABSTRACT. We define the Schur multipliers of a separable von Neumann algebra $\mathcal{M}$ with Cartan masa $\mathcal{A}$, generalising the classical Schur multipliers of $\mathcal{B}(\ell^2)$. We characterise these as the normal $\mathcal{A}$-bimodule maps on $\mathcal{M}$. If $\mathcal{M}$ contains a direct summand isomorphic to the hyperfinite II_1 factor, then we show that the Schur multipliers arising from the extended Haagerup tensor product $\mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$ are strictly contained in the algebra of all Schur multipliers.

CONTENTS

1. INTRODUCTION

Let $\mathcal{B}(\ell^2)$ denote the space of bounded linear operators on ℓ^2. The Schur multipliers of $\mathcal{B}(\ell^2)$ have attracted considerable attention in the literature. These are the (necessarily bounded) maps of the form

$$M(\varphi)\colon \mathcal{B}(\ell^2) \to \mathcal{B}(\ell^2), \quad T \mapsto \varphi * T$$

where $\varphi = (\varphi(i,j))_{i,j\in\mathbb{N}}$ is a fixed matrix with the property that the Schur, or entry-wise, product $\varphi * T$ is in $\mathcal{B}(\ell^2)$ for every $T \in \mathcal{B}(\ell^2)$. Here we identify operators in $\mathcal{B}(\ell^2)$ with matrices indexed by $\mathbb{N} \times \mathbb{N}$ in a canonical way. It is well-known that if φ is itself the matrix of an element of $\mathcal{B}(\ell^2)$, then $M(\varphi)$ is a Schur multiplier, but that not every Schur multiplier of $\mathcal{B}(\ell^2)$ arises in this way.

In fact [13], Schur multipliers are precisely the normal (weak*-weak* continuous) $\mathcal{D}$-bimodule maps on $\mathcal{B}(\ell^2)$, where $\mathcal{D}$ is the maximal abelian self-adjoint algebra, or masa, consisting of the operators in $\mathcal{B}(\ell^2)$ whose matrix

Date: 30 July 2014.

is diagonal. By a result of R. R. Smith [20], each of these maps has completely bounded norm equal to its norm as linear map on $\mathcal{B}(\ell^2)$. Moreover, it follows from a classical result of A. Grothendieck [9] that the space of Schur multipliers of $\mathcal{B}(\ell^2)$ can be identified with $\mathcal{D} \otimes_{\mathrm{eh}} \mathcal{D}$, where $\otimes_{\mathrm{eh}}$ is the weak* (or extended) Haagerup tensor product introduced by D. P. Blecher and R. R. Smith in [3].

Recall [8, Definition 3.1] that a masa $\mathcal{A}$ in a von Neumann algebra $\mathcal{M}$ is a Cartan masa if there is a faithful normal conditional expectation of $\mathcal{M}$ onto $\mathcal{A}$, and the set of unitary normalizers of $\mathcal{A}$ in $\mathcal{M}$ generates $\mathcal{M}$.

Let $\mathcal{R}$ be the hyperfinite II_1-factor. For each Cartan masa $\mathcal{A} \subseteq \mathcal{R}$, F. Pop and R. R. Smith defined a Schur product $\star_{\mathcal{A}} \colon \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ using the Schur products of finite matrices and approximation techniques [15]. Using this product, they showed that every bounded $\mathcal{A}$-bimodule map $\mathcal{R} \to \mathcal{R}$ is completely bounded, with completely bounded norm equal to its norm. The separable von Neumann algebras $\mathcal{M}$ containing a Cartan masa $\mathcal{A}$ were coordinatised by J. Feldman and C. C. Moore [7, 8]. We use this coordinatisation to define the Schur multipliers of $(\mathcal{M}, \mathcal{A})$. Our definition generalises the classical notion of a Schur multiplier of $\mathcal{B}(\ell^2)$, and for $\mathcal{M} = \mathcal{R}$ and certain masas $\mathcal{A} \subseteq \mathcal{R}$, our definition of Schur multiplication extends the Schur product $\star_{\mathcal{A}}$ of [15].

In fact, the Schur multipliers of $\mathcal{M}$ turn out to be the adjoints of the multipliers of the Fourier algebra of the groupoid underlying the von Neumann algebra $\mathcal{M}$ (see [16, 17]). Our focus, however, is on algebraic properties such as idempotence, characterisation problems and connections with operator space tensor products, so we restrict our attention to Schur multipliers of von Neumann algebras with Cartan masas.

Our main results are as follows. Let $\mathcal{M}$ be a separable von Neumann algebra with a Cartan masa $\mathcal{A}$. After defining the Schur multipliers of $(\mathcal{M}, \mathcal{A})$, we show in Theorem 4.11 that these are precisely the normal $\mathcal{A}$-bimodule maps $\mathcal{M} \to \mathcal{M}$, generalising the well-known result for $\mathcal{M} = \mathcal{B}(\ell^2)$, $\mathcal{A} = \mathcal{D}$. However, if $\mathcal{M} \neq \mathcal{B}(\ell^2)$, then the extended Haagerup tensor product $\mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$ need not exhaust the Schur multipliers; indeed we show in that if $\mathcal{M}$ contains a direct summand isomorphic to $\mathcal{R}$, then $\mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$ does not contain every Schur multiplier of $\mathcal{M}$. This is perhaps surprising, since in [15] Pop and Smith show that every (completely) bounded $\mathcal{A}$-bimodule map on $\mathcal{R}$ is the weak* pointwise limit of transformations corresponding to elements of $\mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$. Our result is a corollary to Theorem 6.12, in which we show that there are no non-trivial idempotent Schur multipliers of Toeplitz type on $\mathcal{R}$ that come from $\mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$.

Acknowledgements. The authors are grateful to Adam Fuller and David Pitts for providing Remark 4.12 and drawing our attention to [4]. We also wish to thank Jean Renault for illuminating discussions during the preparation of this paper.

2. Feldman-Moore relations and Cartan pairs

Here we recall some preliminary notions and results from the work of Feldman and Moore [7, 8]. Throughout, let X be a set and let $R \subseteq X \times X$ be an equivalence relation on X. We write $x \sim y$ to mean that $(x, y) \in R$. For $n \in \mathbb{N}$ with $n \geq 2$, we write

$$R^{(n)} = \{(x_0, x_1, \dots, x_n) \in X^{n+1} \colon x_0 \sim x_1 \sim \cdots \sim x_n\}.$$

The ith coordinate projection of R onto X will be written as $\pi_i \colon R \to X$, $(x_1, x_2) \mapsto x_i$.

Definition 2.1. A map $\sigma \colon R^{(2)} \to \mathbb{T}$ is a 2-*cocycle on* R if

$$\sigma(x, y, z)\sigma(x, z, w) = \sigma(x, y, w)\sigma(y, z, w)$$

for all $(x, y, z, w) \in R^{(3)}$. We say σ is *normalised* if $\sigma(x, y, z) = 1$ whenever two of x, y and z are equal. By [7, Proposition 7.8], any normalised 2-cocycle σ is *skew-symmetric*: for every permutation π on three elements,

$$\sigma(\pi(x, y, z)) = \begin{cases} \sigma(x, y, z) & \text{if } \pi \text{ is even,} \\ \sigma(x, y, z)^{-1} & \text{if } \pi \text{ is odd.} \end{cases}$$

Definition 2.2. An equivalence relation R on X is *countable* if for every $x \in X$, the equivalence class $[x]_R = \{y \in X \colon x \sim y\}$ is countable.

Now let (X, μ) be a standard Borel probability space and suppose that R is a countable equivalence relation which is also a Borel subset of $X \times X$, when $X \times X$ is equipped with the product Borel structure.

Definition 2.3. For $\alpha \subseteq X$, let $[\alpha]_R = \bigcup_{x \in \alpha} [x]_R$ be the R-saturation of α. We say that μ is *quasi-invariant under* R if

$$\mu(\alpha) = 0 \iff \mu([\alpha]_R) = 0$$

for any measurable set $\alpha \subseteq X$.

Definition 2.4. We say that (X, μ, R, σ) is a *Feldman-Moore relation* if (X, μ) is a standard Borel probability space, R is a countable Borel equivalence relation on X so that μ is quasi-invariant under R, and σ is a normalised 2-cocycle on R. When the context makes this unambiguous, for brevity we will simply refer to this Feldman-Moore relation as R.

Fix a Feldman-Moore relation (X, μ, R, σ).

Definition 2.5. Let $E \subseteq R$ and let $x, y \in X$. The horizontal slice of E at y is

$$E_y = \{z \in X \colon (z, y) \in E\} \times \{y\}$$

and the vertical slice of E at x is

$$E^x = \{x\} \times \{z \in X \colon (x, z) \in E\}.$$

We define

$$\mathbb{B}(E) = \sup_{x,y \in X} |E_x| + |E^y|,$$

and say that E is *band limited* if $\mathbb{B}(E) < \infty$. We call a bounded Borel function $a\colon R \to \mathbb{C}$ *left finite* if the support of a is band limited, and we write

$$\Sigma_0 = \Sigma_0(R)$$

for the set of all such left finite functions on R.

Definition 2.6. Equip R with the relative Borel structure from $X \times X$. The *right counting measure* for R is the measure ν on R defined by

$$\nu(E) = \int_X |E_y| \, d\mu(y)$$

for each measurable set $E \subseteq R$.

We shall also need a generalisation of the counting measure ν. For $n \geq 2$, let π_{n+1} be the projection of $R^{(n)}$ onto X defined by $\pi_{n+1}(x_0, x_1, \ldots, x_n) = x_n$, and let $\nu^{(n)}$ be the measure on $R^{(n)}$ given by

$$\nu^{(n)}(E) = \int_X |\pi_{n+1}^{-1}(y) \cap E| \, d\mu(y).$$

Now consider the Hilbert space $H = L^2(R, \nu)$, where ν is the right counting measure of R.

Definition 2.7. We define a linear map

$$L_0\colon \Sigma_0 \to \mathcal{B}(H), \qquad L_0(a)\xi := a *_\sigma \xi$$

for $a \in \Sigma_0$ and $\xi \in H$, where

$$a *_\sigma \xi(x, z) = \sum_{y \sim x} a(x, y)\xi(y, z)\sigma(x, y, z), \quad \text{for } (x, z) \in R. \tag{1}$$

As shown in [8], this defines a bounded linear operator $L_0(a) \in \mathcal{B}(H)$ with $\|L_0(a)\| \leq \mathbb{B}(E)\|a\|_\infty$, where E is the support of a.

Definition 2.8. We define

$$\mathcal{M}_0(R, \sigma) = L_0(\Sigma_0)$$

to be the range of L_0.

Definition 2.9. The von Neumann algebra $\mathcal{M}(R, \sigma)$ of the Feldman-Moore relation (X, μ, R, σ) is the von Neumann subalgebra of $\mathcal{B}(H)$ generated by $\mathcal{M}_0(R, \sigma)$. We will abbreviate this as $\mathcal{M}(R)$ or simply $\mathcal{M}$ where the context allows.

Let $\Delta = \{(x, x)\colon x \in X\}$ be the diagonal of R, and let $\chi_\Delta\colon R \to \mathbb{C}$ be the characteristic function of Δ. Note that χ_Δ is a unit vector in H, since $\nu(\Delta) = \mu(X) = 1$.

Definition 2.10. The *symbol map* of R is the map

$$s\colon \mathcal{M} \to H, \quad T \mapsto T\chi_\Delta.$$

The *symbol set* for R is the range of s:

$$\Sigma(R,\sigma) = s(\mathcal{M}).$$

We often abbreviate this as $\Sigma(R)$ or Σ.

Since σ is normalised, equation (1) gives

$$s(L_0(a)) = a \quad \text{for } a \in \Sigma_0, \tag{2}$$

where equality holds almost everywhere. So we may view the Borel functions $a \in \Sigma_0$ as elements of $H = L^2(R,\nu)$. Moreover, for $T \in \mathcal{M}$ we have $\|s(T)\|_\infty \le \|T\|$ by [8, Proposition 2.6]. Hence

$$\Sigma_0 \subseteq \Sigma \subseteq H \cap L^\infty(R,\nu). \tag{3}$$

Definition 2.11. By [8], s is a bijection onto Σ, and its inverse

$$L\colon \Sigma \to \mathcal{M}$$

extends L_0. We call L the *inverse symbol map* of R. In fact, for any $a \in \Sigma$ we have $L(a)\xi = a *_\sigma \xi$ where $*_\sigma$ is the convolution product formally defined by equation (1).

If we equip Σ with the involution $a^*(x,y) = \overline{a(y,x)}$, the pointwise sum and the convolution product $*_\sigma$, then s is a $*$-isomorphism onto Σ: for all $a, b \in \Sigma$ and $\lambda, \mu \in \mathbb{C}$, we have

$$s(L(a)^*)(x,y) = \overline{a(y,x)},$$
$$s(L(\lambda a) + L(\mu b)) = \lambda a + \mu b \quad \text{and}$$
$$s(L(a)L(b)) = a *_\sigma b.$$

This is proven in [8]. By equation (2), $\Sigma_0(R)$ is a $*$-subalgebra of Σ, so $\mathcal{M}_0(R,\sigma)$ is a $*$-subalgebra of $\mathcal{M}(R,\sigma)$.

Definition 2.12. Given $\alpha \in L^\infty(X,\mu)$, let $d(\alpha)\colon R \to \mathbb{C}$ be given by

$$d(\alpha)(x,y) = \begin{cases} \alpha(x) & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $d(\alpha) \in \Sigma_0$. We write $D(\alpha) = L(d(\alpha)) \in \mathcal{M}$, and we define the *Cartan masa of* R to be

$$\mathcal{A} = \mathcal{A}(R) = \{D(\alpha)\colon \alpha \in L^\infty(X,\mu)\}.$$

By [8], $\mathcal{A}(R)$ is a Cartan masa in the von Neumann algebra $\mathcal{M}(R,\sigma)$.

Note that if $\xi \in H$ and $(x,y) \in R$, then

$$\begin{aligned} D(\alpha)\xi(x,y) &= \sum_{z\sim x} d(\alpha)(x,z)\xi(z,y)\sigma(x,z,y) = \alpha(x)\xi(x,y)\sigma(x,x,y) \\ &= \alpha(x)\xi(x,y). \end{aligned}$$

Since this does not depend on the normalised 2-cocycle σ, this shows that $\mathcal{A}(R)$ does not depend on σ.

Definition 2.13. If $\mathcal{A}$ is a Cartan masa in a von Neumann algebra $\mathcal{M}$, then we say that $(\mathcal{M}, \mathcal{A})$ is a *Cartan pair*. If $\mathcal{M} \subseteq \mathcal{B}(H)$ where H is a separable Hilbert space, then we say that $(\mathcal{M}, \mathcal{A})$ is a *separably acting* Cartan pair.

We say that two Cartan pairs $(\mathcal{M}_1, \mathcal{A}_1)$ and $(\mathcal{M}_2, \mathcal{A}_2)$ are isomorphic, and write $(\mathcal{M}_1, \mathcal{A}_1) \cong (\mathcal{M}_2, \mathcal{A}_2)$, if there is a $*$-isomorphism of $\mathcal{M}_1$ onto $\mathcal{M}_2$ which carries $\mathcal{A}_1$ onto $\mathcal{A}_2$.

A *Feldman-Moore coordinatisation* of a Cartan pair $(\mathcal{M}, \mathcal{A})$ is a Feldman-Moore relation (X, μ, R, σ) so that

$$(\mathcal{M}, \mathcal{A}) \cong (\mathcal{M}(R, \sigma), \mathcal{A}(R)).$$

Definition 2.14. For $i = 1, 2$, let $R_i = (X_i, \mu_i, R_i, \sigma_i)$ be a Feldman-Moore relation with right counting measure ν_i. We say that these are isomorphic, and write $R_1 \cong R_2$, if there is a Borel isomorphism $\rho\colon X_1 \to X_2$ so that

(1) $\rho_*\mu_1$ is equivalent to μ_2, where $\rho_*\mu_1(E) = \mu_1(\rho^{-1}(E))$ for $E \subseteq X_2$;
(2) $\rho \times \rho(R_1) = R_2$, up to a ν_2-null set; and
(3) $\sigma_2(\rho(x), \rho(y), \rho(z)) = \sigma_1(x, y, z)$ for a.e. $(x, y, z) \in R_1^{(2)}$ with respect to $\nu_1^{(2)}$.

Our definition of the Schur multipliers of a von Neumann algebra $\mathcal{M}$ with a Cartan masa $\mathcal{A}$ will rest on:

Theorem 2.15 (The Feldman-Moore coordinatisation [8, Theorem 1])**.** *Every separably acting Cartan pair $(\mathcal{M}, \mathcal{A})$ has a Feldman-Moore coordinatisation. Moreover, if $R_i = (X_i, \mu_i, R_i, \sigma_i)$ is a Feldman-Moore coordinatisation of $(\mathcal{M}_i, \mathcal{A}_i)$ for $i = 1, 2$, then*

$$(\mathcal{M}_1, \mathcal{A}_1) \cong (\mathcal{M}_2, \mathcal{A}_2) \iff R_1 \cong R_2.$$

Remark 2.16. Suppose that we have isomorphic Feldman-Moore relations R_1 and R_2, with an isomorphism $\rho\colon X_1 \to X_2$ as in Definition 2.14. A calculation shows that if $h\colon X_2 \to \mathbb{R}$ is the Radon-Nikodym derivative of $\rho_*\mu_1$ with respect to μ_2, then the operator

$$U\colon L^2(R_2, \nu_2) \to L^2(R_1, \nu_1),$$

given for $(x, y) \in R_1$ and $f \in L^2(R_2, \nu_2)$ by

$$U(f)(x, y) = h(\rho(y))^{-1/2} f(\rho(x), \rho(y)),$$

is unitary. Moreover, writing L_i for the inverse symbol map of R_i, for $a \in \Sigma_0(R_1, \sigma_1)$ we have

$$U^* L_1(a) U = L_2(a \circ \rho^{-2}) \tag{4}$$

where

$$\rho^{-2}(u, v) = (\rho^{-1}(u), \rho^{-1}(v)), \quad (u, v) \in R_2.$$

It follows that

$$U^*\mathcal{M}(R_1, \sigma_1)U = \mathcal{M}(R_2, \sigma_2) \quad \text{and} \quad U^*\mathcal{A}(R_1)U = \mathcal{A}(R_2),$$

so conjugation by U implements an isomorphism

$$(\mathcal{M}(R_1,\sigma_1),\mathcal{A}(R_1)) \cong (\mathcal{M}(R_2,\sigma_2),\mathcal{A}(R_2))$$

whose existence is assured by Theorem 2.15.

3. Algebraic preliminaries

In this section, we collect some algebraic observations. Fix a Feldman-Moore relation $R = (X,\mu,R,\sigma)$ with right counting measure ν, let $H = L^2(R,\nu)$, let $\mathcal{M} = \mathcal{M}(R,\sigma)$ and let $\mathcal{A} = \mathcal{A}(R)$. Also let Σ_0 be the collection of left finite functions on R, and let s, L, Σ be the symbol map, inverse symbol map and the symbol set of R, respectively.

We can describe the bimodule action of $\mathcal{A}$ on $\mathcal{M}$ quite easily in terms of the pointwise product of symbols.

Definition 3.1. For $a, b \in L^\infty(R,\nu)$, let $a \star b$ be the pointwise product of a and b.

Definition 3.2. For $\alpha \in L^\infty(X,\mu)$ we write

$$c(\alpha)\colon R \to \mathbb{C}, \quad (x,y) \mapsto \alpha(x) \quad \text{and} \quad r(\alpha)\colon R \to \mathbb{C}, \quad (x,y) \mapsto \alpha(y).$$

Lemma 3.3. *For $a \in \Sigma$ and $\beta,\gamma \in L^\infty(X,\mu)$, we have*

$$D(\beta)L(a)D(\gamma) = L(c(\beta)\star a \star r(\gamma)).$$

Proof. The statement follows from the identity $s\big(D(\beta)L(a)D(\gamma)\big) = c(\beta)\star a\star r(\gamma)$; its verification is straightforward, but we include it for completeness:

$$\begin{aligned} s\big(D(\beta)L(a)D(\gamma)\big)(x,y) &= \big(D(\beta)L(a)D(\gamma)\chi_\Delta\big)(x,y)\\ &= \beta(x)\big(L(a)D(\gamma)\chi_\Delta\big)(x,y)\\ &= \beta(x)\sum_{z\sim x} a(x,z)\big(D(\gamma)\chi_\Delta\big)(z,y)\sigma(x,z,y)\\ &= \beta(x)\sum_{z\sim y} a(x,z)\gamma(z)\chi_\Delta(z,y)\sigma(x,z,y)\\ &= \beta(x)a(x,y)\gamma(y)\sigma(x,y,y)\\ &= \beta(x)a(x,y)\gamma(y)\\ &= \left(c(\beta)\star a\star r(\gamma)\right)(x,y). \end{aligned}$$

□

Recall the standard way to associate an inverse semigroup to R. Suppose that $f\colon \delta \to \rho$ is a Borel isomorphism between two Borel subsets $\delta,\rho \subseteq X$. Such a map will be called a *partial Borel isomorphism of X*. If $g\colon \delta' \to \rho'$ is another partial Borel isomorphism of X, then we can (partially) compose them as follows:

$$g\circ f\colon f^{-1}(\rho\cap\delta') \to g(\rho\cap\delta'), \quad x \mapsto g(f(x)).$$

Let us write $\operatorname{Gr} f = \{(x,f(x))\colon x \text{ is in the domain of } f\}$ for the graph of f. Under (partial) composition, the set

$$\mathcal{I}(R) = \{f\colon f \text{ is a partial Borel isomorphism of } X \text{ with } \operatorname{Gr} f \subseteq R\}$$

is an inverse semigroup, where the inverse of $f\colon \delta \to \rho$ in $\mathcal{I}(R)$ is the inverse function $f^{-1}\colon \rho \to \delta$.

If $f \in \mathcal{I}(R)$, then $\mathbb{B}(\operatorname{Gr} f) \leq 2$, so $\chi_{\operatorname{Gr} f} \in \Sigma_0$. We define an operator $V(f) \in \mathcal{M}$ by

$$V(f) = L(\chi_{\operatorname{Gr} f}).$$

If δ is a Borel subset of X, we will write $P(\delta) = V(\mathrm{id}_\delta)$ where id_δ is the identity map on the Borel set $\delta \subseteq X$. Note that $P(\delta) = D(\chi_\delta)$.

Lemma 3.4.

(1) *If $f \in \mathcal{I}(R)$, then $V(f)^* = V(f^{-1})$.*

(2) *If $f \in \mathcal{I}(R)$ and δ, ρ are Borel subsets of X, then*

$$P(\delta)V(f)P(\rho) = V(\mathrm{id}_\rho \circ f \circ \mathrm{id}_\delta).$$

(3) *If δ is a Borel subset of X, then $P(\delta)$ is a projection in $\mathcal{A}$, and every projection in $\mathcal{A}$ is of this form.*

(4) *If ρ is a Borel subset of X, then $V(f)P(\rho) = P(f^{-1}(\rho))V(f)$.*

(5) *If $f : \delta \to \rho$ is in $\mathcal{I}(R)$, then $V(f)$ is a partial isometry with initial projection $P(\rho)$ and final projection $P(\delta)$.*

Proof. (1) It is straightforward that $\chi_{\operatorname{Gr}(f^{-1})} = (\chi_{\operatorname{Gr} f})^*$ (where the * on the right hand side is the involution on Σ discussed in §2 above). Since L is a $*$-isomorphism, $V(f^{-1}) = V(f)^*$.

(2) Note that

$$(\delta \times X) \cap \operatorname{Gr} f \cap (X \times \rho) = \operatorname{Gr}(\mathrm{id}_\rho \circ f \circ \mathrm{id}_\delta),$$

so

$$c(\chi_\delta) \star \chi_{\operatorname{Gr} f} \star r(\chi_\rho) = \chi_{\operatorname{Gr}(\mathrm{id}_\rho \circ f \circ \mathrm{id}_\delta)}.$$

By Lemma 3.3,

$$P(\delta)V(f)P(\rho) = L(c(\chi_\delta) \star \chi_{\operatorname{Gr} f} \star r(\chi_\rho)) = V(\mathrm{id}_\rho \circ f \circ \mathrm{id}_\delta).$$

(3) Taking $f = \mathrm{id}_\delta$ in (1) shows that $P(\delta) = V(\chi_{\mathrm{id}_\delta})$ is self-adjoint; and taking $f = \mathrm{id}_\Delta$ and $\delta = \rho$ in (2) shows that $P(\delta)$ is idempotent. So $P(\delta)$ is a projection. Since $P(\delta) = D(\chi_\delta)$, we have $P(\delta) \in \mathcal{A}$. Conversely, since L is a $*$-isomorphism, any projection P in $\mathcal{A}$ is equal to $D(\alpha)$ for some projection $\alpha \in L^\infty(X, \mu)$. So $\alpha = \chi_\delta$ for some Borel set $\delta \subseteq X$, and hence $P = P(\delta)$ for some Borel set $\delta \subseteq X$.

(4) Since $\mathrm{id}_\rho \circ f = f \circ \mathrm{id}_{f^{-1}(\rho)}$ and $P(X) = I$, this follows by taking $\delta = X$ in (2).

(5) Using the fact that σ is normalised, a simple calculation yields

$$\chi_{\operatorname{Gr} f} *_\sigma \chi_{\operatorname{Gr} f^{-1}} = \chi_{\operatorname{Gr}(\mathrm{id}_\delta)}.$$

Applying the $*$-isomorphism L and using (1) gives $V(f)V(f)^* = P(\delta)$ and replacing f with f^{-1} gives $V(f)^*V(f) = P(\rho)$. $\square$

Proposition 3.5. *Let $\Phi\colon \mathcal{M} \to \mathcal{M}$ be a linear $\mathcal{A}$-bimodule map.*

(1) *If $f \in \mathcal{I}(R)$ and $V = V(f)$, then $s(\Phi(V)) = \chi_{\operatorname{Gr} f} \star s(\Phi(V))$.*

(2) *For* $i = 1, 2$, *let* $f_i\colon \delta_i \to \rho_i$ *be in* $\mathcal{I}(R)$ *and let* $V_i = V(f_i)$. *If* $G = \mathrm{Gr}(f_1) \cap \mathrm{Gr}(f_2)$, *then*

$$\chi_G \star s(\Phi(V_1)) = \chi_G \star s(\Phi(V_2)).$$

Proof. (1) Let $f \in \mathcal{I}(R)$ and let $\rho \subseteq X$ be a Borel set. Since Φ is an $\mathcal{A}$-bimodule map, Lemma 3.4 implies that

$$\begin{aligned} V^*\Phi(V)P(\rho) &= V^*\Phi(VP(\rho)) = V^*\Phi(P(f^{-1}(\rho))V) \\ &= V^*P(f^{-1}(\rho))\Phi(V) \\ &= (P(f^{-1}(\rho))V)^*\Phi(V) \\ &= (VP(\rho))^*\Phi(V) = P(\rho)V^*\Phi(V). \end{aligned}$$

Hence $V^*\Phi(V)$ commutes with all projections in $\mathcal{A}$, and since $\mathcal{A}$ is a masa, $V^*\Phi(V) \in \mathcal{A}$. If δ is the domain of f, then by Lemma 3.4(5), $P(\delta)$ is the final projection of V, and therefore

$$\Phi(V) = \Phi(P(\delta)V) = P(\delta)\Phi(V) = VV^*\Phi(V) \in V\mathcal{A}.$$

So $\Phi(V) = VD(\gamma)$ for some $\gamma \in L^\infty(X, \mu)$. By Lemma 3.3,

$$s(\Phi(V)) = s(VD(\gamma)) = s(L(\chi_{\mathrm{Gr}\, f})D(\gamma)) = \chi_{\mathrm{Gr}\, f} \star d(\gamma),$$

so $s(\Phi(V)) = \chi_{\mathrm{Gr}\, f} \star s(\Phi(V))$.

(2) Let $\delta = \pi_1(G)$ where $\pi_1(x, y) = x$ for $(x, y) \in R$. It is easy to see that $\chi_G = c(\chi_\delta) \star \chi_{\mathrm{Gr}\, f_i}$ for $i = 1, 2$. By part (1), $s(\Phi(V_i)) = \chi_{\mathrm{Gr}\, f_i} \star s(\Phi(V_i))$. Hence by Lemmas 3.3 and 3.4,

$$\begin{aligned} \chi_G \star s(\Phi(V_i)) &= c(\chi_\delta) \star \chi_{\mathrm{Gr}\, f_i} \star s(\Phi(V_i)) \\ &= c(\chi_\delta) \star s(\Phi(V_i)) = s(P(\delta)\Phi(V_i)) \\ &= s(\Phi(P(\delta)V_i)) = s(\Phi(V(f_i \circ \mathrm{id}_\delta))). \end{aligned}$$

The definition of δ ensures that $f_1 \circ \mathrm{id}_\delta = f_2 \circ \mathrm{id}_\delta$, so $\chi_G \star s(\Phi(V_1)) = \chi_G \star s(\Phi(V_2))$. □

4. Schur multipliers: definition and characterisation

Let (X, μ, R, σ) be a Feldman-Moore coordinatisation of a separably acting Cartan pair $(\mathcal{M}, \mathcal{A})$, and let Σ_0, Σ be as in Section 2. In this section we define the class $\mathfrak{S}(R, \sigma)$ of Schur multipliers of the von Neumann algebra $\mathcal{M}$ with respect to the Feldman-Moore relation R. The main result in this section, Theorem 4.11, characterises these multipliers as normal bimodule maps. From this it follows that $\mathfrak{S}(R, \sigma)$ depends only on the Cartan pair $(\mathcal{M}, \mathcal{A})$. We also show that isomorphic Feldman-Moore relations yield isomorphic classes of Schur multipliers.

Definition 4.1. Let $R = (X, \mu, R, \sigma)$ be a Feldman-Moore coordinatisation of a Cartan pair $(\mathcal{M}, \mathcal{A})$. We say that $\varphi \in L^\infty(R, \nu)$ is a *Schur multiplier of* $(\mathcal{M}, \mathcal{A})$ *with respect to* R, or simply a *Schur multiplier of* $\mathcal{M}$, if

$$a \in \Sigma(R, \sigma) \implies \varphi \star a \in \Sigma(R, \sigma)$$

where $\star$ is the pointwise product on $L^\infty(R,\nu)$. We then write

$$m(\varphi)\colon \Sigma(R,\sigma) \to \Sigma(R,\sigma), \quad a \mapsto \varphi \star a$$

and

$$M(\varphi)\colon \mathcal{M} \to \mathcal{M}, \quad T \mapsto L(\varphi \star s(T)).$$

Set

$$\mathfrak{S} = \mathfrak{S}(R,\sigma) = \{\varphi \in L^\infty(R,\nu)\colon \varphi \text{ is a Schur multiplier of } \mathcal{M}\}.$$

It is clear from Definition 4.1 that $\mathfrak{S}(R,\sigma)$ is an algebra with respect to pointwise addition and multiplication of functions.

Example 4.2. For a suitable choice of Feldman-Moore coordinatisation, $\mathfrak{S}(R,\sigma)$ is precisely the set of classical Schur multipliers of $\mathcal{B}(\ell^2)$. Indeed, let $X = \mathbb{N}$, equipped with the (atomic) probability measure μ given by $\mu(\{i\}) = p_i$, $i \in \mathbb{N}$, and set $R = X \times X$. If $p_i > 0$ for every $i \in \mathbb{N}$, then μ is quasi-invariant under R. Let σ be the trivial 2-cocycle $\sigma \equiv 1$. The right counting measure for the Feldman-Moore relation (X,μ,R,σ) is $\nu = \kappa \times \mu$ where κ is counting measure on $\mathbb{N}$. Indeed, for $E \subseteq R$,

$$\nu(E) = \sum_{y\in\mathbb{N}} |E_y|\mu(\{y\}) = \sum_{y\in\mathbb{N}} \kappa \times \mu(E_y) = \kappa \times \mu(E).$$

Hence $L^2(R,\nu)$ is canonically isometric to the Hilbert space tensor product $\ell^2 \otimes \ell^2(\mathbb{N},\mu)$. Let $T \in \mathcal{M}(R,\sigma)$. For an elementary tensor $\xi \otimes \eta \in L^2(R,\nu)$, we have

$$T(\xi\otimes\eta)(i,j) = L_{s(T)}(\xi\otimes\eta)(i,j) = \sum_{k=1}^{\infty} s(T)(i,k)\xi(k)\eta(j) = (A_{s(T)}\xi\otimes\eta)(i,j)$$

where $A_a \in \mathcal{B}(\ell^2)$ is the operator with matrix $a\colon \mathbb{N}\times\mathbb{N} \to \mathbb{C}$. It follows that the map $T \mapsto A_{s(T)} \otimes I$ is an isomorphism between $\mathcal{M}(R,\sigma)$ and $\mathcal{B}(\ell^2) \otimes I$, so

$$\Sigma(R,\sigma) = \{a\colon \mathbb{N}\times\mathbb{N} \to \mathbb{C} \mid a \text{ is the matrix of } A \text{ for some } A \in \mathcal{B}(\ell^2)\}.$$

In particular, a function $\varphi : \mathbb{N}\times\mathbb{N} \to \mathbb{C}$ is in $\mathfrak{S}(R,\sigma)$ if and only if φ is a (classical) Schur multiplier of $\mathcal{B}(\ell^2)$.

Example 4.3. If (X,μ,R,σ) is a Feldman-Moore relation and Δ is the diagonal of R, then $\chi_\Delta \in \mathfrak{S}(R,\sigma)$ since for any $a \in L^\infty(R,\nu)$, the function

$$\chi_\Delta \star a = d(x \mapsto a(x,x))$$

belongs to Σ_0 and hence to Σ.

More generally:

Proposition 4.4. *For any Feldman-Moore relation (X,μ,R,σ), we have $\Sigma_0(R,\sigma) \subseteq \mathfrak{S}(R,\sigma)$.*

Proof. Let $\varphi \in \Sigma_0(R, \sigma)$ and let $a \in \Sigma(R, \sigma)$. Recall that $a \in L^\infty(R, \nu)$, so we can choose a bounded Borel function $\alpha \colon R \to \mathbb{C}$ with $\alpha = a$ almost everywhere with respect to ν. The function $\varphi \star \alpha$ is then bounded, and its support is a subset of the support of φ, which is band limited. Hence $\varphi \star \alpha \in \Sigma_0$, and $\varphi \star \alpha = \varphi \star a$ almost everywhere. By equation (3), we have $\varphi \star a \in \Sigma(R, \sigma)$, so $\varphi \in \mathfrak{S}(R, \sigma)$. □

We now embark on the proof of our main result.

Lemma 4.5. *Let $\mathfrak{X}$ be a Banach space, let V be a complex normed vector space, and let α, β and h be linear maps so that the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\;h\;} & V \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ \mathfrak{X} & \xrightarrow{\;h\;} & V \end{array}$$

If h and β are continuous and h is injective, then α is continuous.

Proof. If $x_n \in \mathfrak{X}$ with $x_n \to 0$ and $\alpha(x_n) \to y$ as $n \to \infty$ for some $y \in \mathfrak{X}$, then

$$\begin{aligned} h(y) = h(\lim_{n\to\infty} \alpha(x_n)) &= \lim_{n\to\infty} h(\alpha(x_n)) \\ &= \lim_{n\to\infty} \beta(h(x_n)) = \beta(h(\lim_{n\to\infty} x_n)) = \beta(h(0)) = 0. \end{aligned}$$

Since h is injective, $y = 0$ and α is continuous by the closed graph theorem. □

If φ is a Schur multiplier of $\mathcal{M}$, then we have the following commutative diagram of linear maps:

$$\begin{array}{ccc} \mathcal{M} & \underset{s}{\overset{L}{\leftrightarrows}} & \Sigma(R, \sigma) \\ {\scriptstyle M(\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle m(\varphi)} \\ \mathcal{M} & \underset{s}{\overset{L}{\leftrightarrows}} & \Sigma(R, \sigma) \end{array}$$

We now record some continuity properties of this diagram.

Proposition 4.6. *Let (X, μ, R, σ) be a Feldman-Moore relation, let $(\mathcal{M}, \mathcal{A}) = (\mathcal{M}(R, \sigma), \mathcal{A}(R))$, let $\mathcal{H} = L^2(R, \nu)$ where ν is the right counting measure of R, and write $\Sigma = \Sigma(R, \sigma)$. Let $\varphi \in \mathfrak{S}(R, \sigma)$.*

1. *$m(\varphi)$ is continuous as a map on $(\Sigma, \|\cdot\|_\infty)$.*
2. *s is a contraction from $(\mathcal{M}, \|\cdot\|_{\mathcal{B}(\mathcal{H})})$ to $(\Sigma, \|\cdot\|_\infty)$.*
3. *$M(\varphi)$ is norm-continuous.*

(4) *$m(\varphi)$ is continuous as a map on* $(\Sigma, \|\cdot\|_2)$.
(5) *s is a contraction from* $(\mathcal{M}, \|\cdot\|_{\mathcal{B}(\mathcal{H})})$ *to* $(\Sigma, \|\cdot\|_2)$.
(6) *s is continuous from* $(\mathcal{M}, \mathrm{SOT})$ *to* $(\Sigma, \|\cdot\|_2)$, *where SOT is the strong operator topology on $\mathcal{M}$.*

Proof. (1) and (4) follow from the fact that φ is essentially bounded.
(2) See [8, Proposition 2.6].
(3) This follows from (2) and Lemma 4.5.
(5) follows from the fact that χ_Δ is a unit vector in $\mathcal{H}$.
(6) Let $\{T_\lambda\}$ be a net in $\mathcal{M}$ which converges in the SOT to $T \in \mathcal{M}$. Then $s(T_\lambda) = T_\lambda(\chi_\Delta) \to T(\chi_\Delta) = s(T)$ in $\|\cdot\|_2$. □

If R is a Feldman-Moore relation with right counting measure ν, let ν^{-1} be the measure on R given by

$$\nu^{-1}(E) = \nu(\{(y, x) \colon (x, y) \in E\}).$$

We will need the following facts, which are established in [8]:

Proposition 4.7.
(1) *ν and ν^{-1} are mutually absolutely continuous;*
(2) *if $d = \frac{d\nu^{-1}}{d\nu}$, then the set $d^{1/2}\Sigma_0 = \{d^{1/2}a \colon a \in \Sigma_0\}$ of* right finite *functions on R has the property that for $b \in d^{1/2}\Sigma_0$, the formula*

$$R_0(b)\xi = \xi *_\sigma b, \quad \xi \in H$$

defines a bounded linear operator $R_0(b) \in \mathcal{B}(H)$; and
(3) *for $b \in d^{1/2}\Sigma_0$, we have $R_0(b) \in \mathcal{M}'$ and $R_0(b)(\chi_\Delta) = b$.*

We will now see that the SOT-convergence of a *bounded* net in $\mathcal{M}$ is equivalent to the $\|\cdot\|_2$ convergence of its image under s.

Proposition 4.8. *Let $\{T_\lambda\} \subseteq \mathcal{M}(R)$ be a norm bounded net.*
(1) *$\{T_\lambda\}$ converges in the* SOT *if and only if $\{s(T_\lambda)\}$ converges with respect to $\|\cdot\|_2$.*
(2) *For $T \in \mathcal{M}$, we have*

$$T_\lambda \to_{\mathrm{SOT}} T \iff s(T_\lambda) \to_{\|\cdot\|_2} s(T).$$

Proof. (1) The “only if” is addressed by Proposition 4.6(6).

Conversely, suppose that $s(T_\lambda) = T_\lambda(\chi_\Delta)$ converges with respect to $\|\cdot\|_2$ on H. For a right finite function $b \in d^{1/2}\Sigma_0$, we have

$$R_0(b)T_\lambda(\chi_\Delta) = T_\lambda R_0(b)(\chi_\Delta) = T_\lambda(b)$$

which converges in H. By [8, Proposition 2.3], the set of right finite functions is dense in H. Since $\{T_\lambda\}$ is bounded, we conclude that $T_\lambda(\xi)$ converges for every $\xi \in H$. So we may define a linear operator $T \colon H \to H$ by $T(\xi) = \lim_\lambda T_\lambda(\xi)$; then $\|T(\xi)\| \leq \sup_\lambda \|T_\lambda\|\|\xi\|$, so $T \in \mathcal{B}(H)$. By construction, $T_\lambda \to T$ strongly.

(2) The direction “ $\Longrightarrow$ ” follows from Proposition 4.6(6). For the converse, apply (1) to see that if $s(T_\lambda) \to_{\|\cdot\|_2} s(T)$, then $T_\lambda \to_{\mathrm{SOT}} S$ for

some $S \in \mathcal{M}$. Hence $s(T_\lambda) \to_{\|\cdot\|_2} s(S)$; therefore $s(S) = s(T)$ and so $S = T$. $\square$

The following argument is taken from the proof of [15, Corollary 2.4].

Lemma 4.9. *Let H be a separable Hilbert space and $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra. Suppose that $\Phi\colon \mathcal{M} \to \mathcal{M}$ is a bounded linear map which is strongly sequentially continuous on bounded sets, meaning that for every $r > 0$, whenever $X, X_1, X_2, X_3, \dots$ are operators in $\mathcal{M}$ with norm at most r with $X_n \to_{\mathrm{SOT}} X$ as $n \to \infty$, we have $\Phi(X_n) \to_{\mathrm{SOT}} \Phi(X)$. Then Φ is normal.*

Proof. For $\xi, \eta \in H$, let $\omega_{\xi,\eta}$ be the vector functional in $\mathcal{M}_*$ given by $\omega_{\xi,\eta}(X) = \langle X\xi, \eta\rangle$, $X \in \mathcal{M}$, and let

$$K = \ker \Phi^*(\omega_{\xi,\eta}) \quad \text{and} \quad K_r = K \cap \{X \in \mathcal{M}\colon \|X\| \le r\}, \quad \text{for } r > 0.$$

Let $r > 0$. Since H is separable, $\mathcal{M}_*$ is separable and so the strong operator topology is metrizable on the bounded set K_r. From the sequential strong continuity of Φ on $\{X \in \mathcal{M}\colon \|X\| \le r\}$, it follows that K_r is strongly closed. Since K_r is bounded and convex, each K_r is ultraweakly closed. By the Krein-Smulian theorem, K is ultraweakly closed, so $\Phi^*(\omega_{\xi,\eta})$ is ultraweakly continuous; that is, it lies in $\mathcal{M}_*$. The linear span of $\{\omega_{\xi,\eta}\colon \xi, \eta \in H\}$ is (norm) dense in $\mathcal{M}_*$, so this shows that $\Phi^*(\mathcal{M}_*) \subseteq \mathcal{M}_*$. Define $\Psi\colon \mathcal{M}_* \to \mathcal{M}_*$ by $\Psi(\omega) = \Phi^*(\omega)$. Then $\Phi = \Psi^*$, so Φ is normal. $\square$

Remark 4.10. Let R be a Feldman-Moore relation. It follows from the first part of the proof of [7, Theorem 1] that there is a countable family $\{f_j\colon \delta_j \to \rho_j\colon j \ge 0\} \subseteq \mathcal{I}(R)$ such that $\{\mathrm{Gr}\, f_j\colon j \ge 0\}$ is a partition of R. Indeed, it is shown there that there are Borel sets $\{D_j\colon j \ge 1\}$ which partition $R \setminus \Delta$ so that $D_j = \mathrm{Gr}\, f_j$, where $f_j\colon \pi_1(D_j) \to \pi_2(D_j)$ is a one-to-one map. Since $\mathrm{Gr}\, f_j$ and $\mathrm{Gr}(f_j^{-1})$ are both Borel sets, each f_j is in $\mathcal{I}(R)$, and we can take f_0 to be the identity mapping on X.

Theorem 4.11. *We have that $\{M(\varphi) : \varphi \in \mathfrak{S}\}$ coincides with the set of normal $\mathcal{A}$-bimodule maps on $\mathcal{M}$.*

Proof. Let $\varphi \in \mathfrak{S}$. If $a \in \Sigma$ and $\beta, \gamma \in L^\infty(X, \mu)$, then by Lemma 3.3,

$$\begin{aligned} M(\varphi)\big(D(\beta)L(a)D(\gamma)\big) &= M(\varphi)\big(L(c(\beta) \star a \star r(\gamma))\big) \\ &= L(c(\beta) \star \varphi \star a \star r(\gamma)) = D(\beta)M(\varphi)(L(a))D(\gamma) \end{aligned}$$

and $M(\varphi)$ is plainly linear, so $M(\varphi)$ is an $\mathcal{A}$-bimodule map.

Let $r > 0$ and let $T_n, T \in \mathcal{M}$ for $n \in \mathbb{N}$ with $\|T_n\|, \|T\| \le r$ and $T_n \to_{\mathrm{SOT}} T$. By Proposition 4.6(6), $s(T_n) \to_{\|\cdot\|_2} s(T)$, so by the $\|\cdot\|_2$ continuity of $m(\varphi)$,

$$m(\varphi)(s(T_n)) \to_{\|\cdot\|_2} m(\varphi)(s(T));$$

thus,

$$s(M(\varphi)(T_n)) \to_{\|\cdot\|_2} s(M(\varphi)(T)).$$

By Proposition 4.8,

$$M(\varphi)(T_n) \to_{\mathrm{SOT}} M(\varphi)(T).$$

Since $L^2(R,\nu)$ is separable, Proposition 4.6(3) and Lemma 4.9 show that $M(\varphi)$ is normal.

Now suppose that Φ is a normal $\mathcal{A}$-bimodule map on $\mathcal{M}$. By Remark 4.10, we may write R as a disjoint union $R = \bigcup_{k=1}^{\infty} F_k$, where $F_k = \operatorname{Gr} f_k$ and $f_k \in \mathcal{I}(R)$, $k \in \mathbb{N}$. Let

$$\varphi : R \to \mathbb{C}, \quad \varphi(x,y) = \sum_{k\geq 1} s(\Phi(V(f_k)))(x,y).$$

Note that φ is well-defined since the sets F_k are pairwise disjoint and, by Lemma 3.5 (1), $s(\Phi(V(f_k))) = s(\Phi(V(f_k))) \star \chi_{F_k}$. It now easily follows that φ is measurable. Moreover, since each $V(f_k)$ is a partial isometry (see Lemma 3.4(5)), by [8, Proposition 2.6] we have

$$\|\varphi\|_\infty = \sup_{k\geq 1} \|s(\Phi(V(f_k)))\|_\infty \leq \sup_{k\geq 1} \|\Phi(V(f_k))\| \leq \|\Phi\|;$$

thus, φ is essentially bounded.

We claim that $s(\Phi(T)) = \varphi \star s(T)$ for every $T \in \mathcal{M}$. First we consider the case $T = V(g)$ where $g \in \mathcal{I}(R)$. If we write $g_1 = g$, then for $m \geq 2$ we can find $g_m \in \mathcal{I}(R)$ with graph $G_m = \operatorname{Gr} g_m$ so that R is the disjoint union $R = \bigcup_{m\geq 1} G_m$. For example, we can define g_m to be the partial Borel isomorphism whose graph is $F_{m-1} \setminus G_1$. Now let $\psi(x,y) = \sum_{m\geq 1} s(\Phi(V(g_m)))(x,y)$, $(x,y) \in R$. By Proposition 3.5(2), we have $\varphi \star \chi_{F_k \cap G_m} = \psi \star \chi_{F_k\cap G_m}$ for every $k, m \geq 1$, so $\varphi = \psi$. In particular,

$$s(\Phi(V(g_1))) = \psi \star \chi_{G_1} = \varphi \star \chi_{G_1} = \varphi \star s(V(g_1)).$$

Hence if T is in the left $\mathcal{A}$-module $\mathcal{V}$ generated by $\{V(f)\colon f \in \mathcal{I}(R)\}$, then $s(\Phi(T)) = \varphi \star s(T)$. On the other hand, by [8, Proposition 2.3], $\mathcal{V} = \mathcal{M}_0(R,\sigma)$ and hence $\mathcal{V}$ is a strongly dense $*$-subalgebra of $\mathcal{M}$.

Now let $T \in \mathcal{M}$. By Kaplansky's Density Theorem, there exists a bounded net $\{T_\lambda\} \subseteq \mathcal{V}$ such that $T_\lambda \to T$ strongly. For every λ, we have that

$$s(\Phi(T_\lambda)) = \varphi \star s(T_\lambda).$$

By Proposition 4.6(6), $s(T_\lambda) \to_{\|\cdot\|_2} s(T)$ and, since $\varphi \in L^\infty(R)$, we have

$$\varphi \star s(T_\lambda) \to_{\|\cdot\|_2} \varphi \star s(T).$$

On the other hand, since Φ is normal, $\Phi(T_\lambda) \to \Phi(T)$ ultraweakly. Normal maps are bounded, so $\{\Phi(T_\lambda)\}$ is a bounded net in $\mathcal{M}$. By Proposition 4.8, $\Phi(T_\lambda)$ is strongly convergent. Thus, $\Phi(T_\lambda) \to \Phi(T)$ strongly. Since $\Phi(T) \in \mathcal{M}$, Proposition 4.8 yields

$$s(\Phi(T_\lambda)) \to_{\|\cdot\|_2} s(\Phi(T)).$$

By uniqueness of limits, $\varphi \star s(T) = s(\Phi(T))$. In particular, $\varphi \star s(T) \in \Sigma$ so φ is a Schur multiplier, and $\Phi(T) = L(\varphi \star s(T)) = M(\varphi)(T)$. It follows that $\Phi = M(\varphi)$. □

Remark 4.12. The authors are grateful to Adam Fuller and David Pitts for bringing the following to our attention. If $(\mathcal{M}, \mathcal{A})$ is a Cartan pair, then $\mathcal{A}$ is norming for $\mathcal{M}$ in the sense of [14], by [4, Corollary 1.4.9]. Hence by [14, Theorem 2.10], if φ is a Schur multiplier, then the map $M(\varphi)$ is competely bounded with $\|M(\varphi)\|_{\mathrm{cb}} = \|M(\varphi)\|$.

We now show that up to isomorphism, the set of Schur multipliers of a Cartan pair with respect to a Feldman-Moore coordinatisation R depends on $(\mathcal{M}, \mathcal{A})$, but not on R.

Proposition 4.13. *Let $(X_i, \mu_i, R_i, \sigma_i)$, $i = 1, 2$, be isomorphic Feldman-Moore relations and let $\rho : X_1 \to X_2$ be an isomorphism from R_1 onto R_2. Then $\tilde{\rho}\colon a \mapsto a \circ \rho^{-2}$ is a bijection from $\Sigma(R_1, \sigma_1)$ onto $\Sigma(R_2, \sigma_2)$, and an isometric isomorphism from $\mathfrak{S}(R_1, \sigma)$ onto $\mathfrak{S}(R_2, \sigma_2)$.*

Proof. It suffices to show that $\tilde{\rho}^{-1}(\Sigma(R_2, \sigma_2)) \subseteq \Sigma(R_1, \sigma_1)$. Indeed, by symmetry we would then have $\tilde{\rho}(\Sigma(R_1, \sigma_1)) \subseteq \Sigma(R_2, \sigma_2)$ and could conclude that these sets are equal. Since $\tilde{\rho}$ is an isomorphism for the pointwise product, it then follows easily that $\tilde{\rho}(\mathfrak{S}(R_1, \sigma_1)) = \mathfrak{S}(R_2, \sigma_2)$.

For $i = 1, 2$, let $s_i\colon \mathcal{M}(R_i, \sigma_i) \to \Sigma(R_i, \sigma_i)$ and $L_i = s_i^{-1}$ be the symbol map and the inverse symbol map for R_i, let ν_i be the right counting measure of R_i and let $H_i = L^2(R_i, \nu_i)$.

Let $a \in \Sigma(R_2, \sigma_2)$ and let $T = L_2(a)$. Since $T \in \mathcal{M}(R_2, \sigma_2)$, the Kaplansky density theorem gives a bounded net $\{T_\lambda\} \subseteq \mathcal{M}_0(R_2, \sigma_2)$ with $T_\lambda \to_{\mathrm{SOT}} T$. Let $a_\lambda = s_2(T_\lambda)$ and $a = s_2(T)$. By Proposition 4.6(6),

$$a_\lambda \to a \quad \text{in } H_2$$

so if $U\colon H_2 \to H_1$ is the unitary operator defined as in Remark 2.16, then

$$(a_\lambda \circ \rho^2) \star \eta = U a_\lambda \to U a = (a \circ \rho^2) \star \eta \quad \text{in } H_1$$

where $\eta(x, y) = h(\rho(y))^{-1/2}$ and $h = \frac{d(\rho_* \mu_1)}{d\mu_2}$. We can find a subnet, which can in fact be chosen to be a sequence $\{(a_n \circ \rho^2) \star \eta\}$, that converges almost everywhere. Hence

$$a_n \circ \rho^2 \to a \circ \rho^2 \quad \text{almost everywhere.}$$

On the other hand, since T_n converges to T in the strong operator topology, UT_nU^* converges to UTU^* strongly. Moreover, since $T_n \in \mathcal{M}_0(R_2, \sigma_2)$, Equation (4) gives $s_1(UT_nU^*) = a_n \circ \rho^2$. Therefore

$$a_n \circ \rho^2 = s_1(UT_nU^*) \to s_1(UTU^*) \quad \text{in } H_1.$$

So $\tilde{\rho}^{-1}(a) = a \circ \rho^2 = s_1(UTU^*) \in \Sigma(R_1, \sigma_1)$. □

5. A class of Schur multipliers

In this section, we examine a natural subclass of Schur multipliers on $\mathcal{M}(R)$ which coincides, by a classical result of A. Grothendieck, with the space of all Schur multipliers in the special case $\mathcal{M}(R) = \mathcal{B}(\ell^2)$. Throughout, we fix a Feldman-Moore relation (X, μ, R, σ), and we write $\mathcal{M}(R) = \mathcal{M}(R, \sigma)$. We first recall some measure theoretic concepts [1]. A measurable subset $E \subseteq X \times X$ is said to be *marginally null* if there exists a μ-null set $M \subseteq X$ such that $E \subseteq (M \times X) \cup (X \times M)$. Measurable sets $E, F \subseteq X \times X$ are called *marginally equivalent* if their symmetric difference is marginally null. The set E is called *ω-open* if E is marginally equivalent to a subset of the form $\cup_{k=1}^{\infty} \alpha_k \times \beta_k$, where $\alpha_k, \beta_k \subseteq X$ are measurable.

In the sequel, we will use some notions from Operator Space Theory; we refer the reader to [2] and [13] for background material. Recall that every element u of the extended Haagerup tensor product $\mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$ can be identified with a series

$$u = \sum_{i=1}^{\infty} A_i \otimes B_i,$$

where $A_i, B_i \in \mathcal{A}$ and, for some constant $C > 0$, we have

$$\left\| \sum_{i=1}^{\infty} A_i A_i^* \right\| \leq C \quad \text{and} \quad \left\| \sum_{i=1}^{\infty} B_i^* B_i \right\| \leq C$$

(the series being convergent in the weak* topology). Let $\mathcal{A} = \mathcal{A}(R)$. The element u gives rise to a completely bounded $\mathcal{A}'$-bimodule map Ψ_u on $\mathcal{B}(L^2(R, \nu))$ defined by

$$\Psi_u(T) = \sum_{i=1}^{\infty} A_i T B_i, \quad T \in \mathcal{B}(L^2(R, \nu)).$$

For each T, this series is w^*-convergent. Moreover, this element $u \in \mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$ also gives rise to a function $f_u \colon X \times X \to \mathbb{C}$, given by

$$f_u(x, y) = \sum_{i=1}^{\infty} a_i(x) b_i(y),$$

where a_i (resp. b_i) is the function in $L^\infty(X, \mu)$ such that $D(a_i) = A_i$ (resp. $D(b_i) = B_i$), $i \in \mathbb{N}$. We write $u \sim \sum_{i=1}^{\infty} a_i \otimes b_i$. Since

$$\left\| \sum_{i=1}^{\infty} |a_i|^2 \right\|_\infty \leq C \quad \text{and} \quad \left\| \sum_{i=1}^{\infty} |b_i|^2 \right\|_\infty \leq C, \tag{5}$$

the function f_u is well-defined up to a marginally null set. Moreover, f_u is *ω-continuous* in the sense that $f_u^{-1}(U)$ is an ω-open subset of $X \times X$ for every open set $U \subseteq \mathbb{C}$, and f_u determines uniquely the corresponding element $u \in \mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$ (see [11]).

Definition 5.1. Given $u \in \mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$, we write

$$\varphi_u\colon R \to \mathbb{C}$$

for the restriction of f_u to R.

In what follows, we identify $u \in \mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$ with the corresponding function f_u, and write $\|\cdot\|_{\mathrm{eh}}$ for the norm of $\mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$.

Lemma 5.2. *If $E \subseteq X \times X$ is a marginally null set, then $E \cap R$ is ν-null. Thus, given $u \in \mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$, the function φ_u is well-defined as an element of $L^\infty(R, \nu)$. Moreover, $\|\varphi_u\|_\infty \leq \|u\|_{\mathrm{eh}}$.*

Proof. If $E \subseteq X \times M$, where $M \subseteq X$ is μ-null, then $(E \cap R)_y = \emptyset$ if $y \notin M$, and hence $\nu(E \cap R) = 0$. Recall from Proposition 4.7 that ν has the same null sets as the measure ν^{-1}; so if $E \subseteq M \times X$, then $\nu(E \cap R) = 0$. Hence any marginally null set is ν-null.

Since $\|u\|_{\mathrm{eh}}$ is the least possible constant C so that (5) holds, the set $\{(x, y) \in X \times X\colon |u(x, y)| > \|u\|_{\mathrm{eh}}\}$ is marginally null with respect to μ, so its intersection with R is ν-null. Hence $\|\varphi_u\|_\infty \leq \|u\|_{\mathrm{eh}}$. □

Definition 5.3. Let

$$\mathfrak{A}(R) = \{\varphi_u : u \in \mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}\}.$$

By virtue of Lemma 5.2, $\mathfrak{A}(R) \subseteq L^\infty(R, \nu)$.

Lemma 5.4. *If $a, b \in L^\infty(X, \mu)$ and $u = a \otimes b$, then for $T \in \mathcal{M}(R, \sigma)$ we have*

$$M(\varphi_u)(T) = D(a)TD(b).$$

In particular, $\varphi_u \in \mathfrak{S}(R, \sigma)$.

Proof. By Lemma 3.3,

$$s(D(a)TD(b))(x, y) = a(x)s(T)(x, y)b(y), \quad (x, y) \in R.$$

The claim is now immediate. □

Lemma 5.5. *Let (Z, θ) be a σ-finite measure space and let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $L^2(Z, \theta)$ such that*

(i) f_k *converges weakly to* $f \in L^2(Z, \theta)$*,*
(ii) f_k *converges (pointwise) almost everywhere to* $g \in L^2(Z, \theta)$*, and*
(iii) $\sup_{k\geq 1} \|f_k\|_\infty < \infty$.

Then $f = g$.

Proof. Let $\xi \in L^2(Z, \theta)$. As f_k converges weakly, $\{\|f_k\|_2\}$ is bounded. Let $Y \subseteq Z$ be measurable with $\theta(Y) < \infty$. If we write $B = \sup_{k\geq 1} \|f_k\|_\infty$, then

$$|f_k\bar{\xi}\chi_Y| \leq B|\xi|\chi_Y.$$

Since $B|\xi|\chi_Y$ is integrable,

$$\begin{aligned}\langle f\chi_Y, \xi\rangle &= \langle f, \chi_Y\xi\rangle = \lim_{k\to\infty} \langle f_k, \chi_Y\xi\rangle = \lim_{k\to\infty} \int f_k\bar{\xi}\chi_Y d\mu \\ &= \int g\bar{\xi}\chi_Y\, d\mu = \langle g\chi_Y, \xi\rangle\end{aligned}$$

by the Lebesgue Dominated Convergence Theorem. So $f\chi_Y = g\chi_Y$. Since Z is σ-finite, this yields $f = g$. $\square$

Theorem 5.6. *If $u \in \mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$, then $M(\varphi_u)$ is the restriction of Ψ_u to $\mathcal{M}(R,\sigma)$ and $\|M(\varphi_u)\| \le \|u\|_{\mathrm{eh}}$. Hence*

$$\mathfrak{A}(R) \subseteq \mathfrak{S}(R,\sigma).$$

Proof. Let $H = L^2(R,\nu)$, let $u \in \mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$ and let $\Psi = \Psi_u$; thus, Ψ is a completely bounded map on $\mathcal{B}(H)$. It is well-known that $\|\Psi\|_{\mathrm{cb}} = \|u\|_{\mathrm{eh}}$. We have $u \sim \sum_{i=1}^\infty a_i \otimes b_i$, for some $a_i, b_i \in \mathcal{A}$ with

$$C = \max\left\{ \Big\| \sum_{i=1}^\infty |a_i|^2 \Big\|_\infty, \ \Big\| \sum_{i=1}^\infty |b_i|^2 \Big\|_\infty \right\} < \infty.$$

For $k \in \mathbb{N}$, set $u_k = \sum_{i=1}^k a_i \otimes b_i$ and $\Psi_k = \Psi_{u_k}$. By Lemma 5.4, Ψ_k leaves $\mathcal{M}(R,\sigma)$ invariant. Since $\Psi_k(T) \to_{w^*} \Psi(T)$ for each $T \in \mathcal{B}(H)$, it follows that Ψ also leaves $\mathcal{M}(R,\sigma)$ invariant.

Let Φ and Φ_k be the restrictions of Ψ and Ψ_k, respectively, to $\mathcal{M}(R,\sigma)$. Set $\varphi_k = \varphi_{u_k}$ for each $k \in \mathbb{N}$. Let $c \in \Sigma(R,\sigma)$ and let $T = L(c)$. By Lemma 5.4, $\varphi_k \in \mathfrak{S}(R,\sigma)$, so $\varphi_k \star c \in \Sigma(R,\sigma)$ and

$$L(\varphi_k \star c) = \Phi_k(T) \to_{w^*} \Phi(T) \quad \text{as } k \to \infty.$$

Hence for every $\eta \in H$, we have

$$\langle \varphi_k \star c, \eta\rangle = \langle L(\varphi_k \star c)(\chi_\Delta), \eta\rangle \to \langle \Phi(T)(\chi_\Delta), \eta\rangle = \langle s(\Phi(T)), \eta\rangle.$$

So

$$\varphi_k \star c \to s(\Phi(T)) \qquad \text{weakly in } L^2(R,\nu).$$

However, $u_k \to u$ marginally almost everywhere, so by Lemma 5.2, $\varphi_k \to \varphi_u$ almost everywhere, and thus

$$\varphi_k \star c \to \varphi_u \star c \quad \text{almost everywhere.}$$

Since

$$\sup_{k\ge 1} \|\varphi_k \star c\|_\infty \le C\|c\|_\infty < \infty,$$

Lemma 5.5 shows that $\varphi_u \star c = s(\Phi(T))$. Hence

$$L(\varphi_u \star s(T)) = \Phi(T) \in \mathcal{M}(R,\sigma)$$

for every $T \in \mathcal{M}(R,\sigma)$, so φ_u is a Schur multiplier and $M(\varphi_u) = \Phi = \Psi|_{\mathcal{M}(R,\sigma)}$. Since $\|M(\varphi_u)\| \le \|M(\varphi_u)\|_{\mathrm{cb}}$ (and in fact we have equality by Remark 4.12), this shows that $\|M(\varphi_u)\| \le \|\Psi\|_{\mathrm{cb}} = \|u\|_{\mathrm{eh}}$. $\square$

6. Schur multipliers of the hyperfinite II_1-factor

Recall the following properties of the classical Schur multipliers of $B(\ell^2)$.

(1) Every symbol function is a Schur multiplier.
(2) Every Schur multiplier is in $\mathfrak{A}(R)$.

In this section, we consider a specific Feldman-Moore coordinatisation of the hyperfinite II_1 factor, and show that in this context the first property is satisfied but the second is not.

The coordinatisation we will work with is defined as follows. Let (X, μ) be the probability space $X = [0, 1)$ with Lebesgue measure μ, and equip X with the commutative group operation of addition modulo 1. For $n \in \mathbb{N}$, let $\mathbb{D}_n$ be the finite subgroup of X given by

$$\mathbb{D}_n = \{\tfrac{i}{2^n} : 0 \leq i \leq 2^n - 1\},$$

and let

$$\mathbb{D} = \bigcup_{n=0}^{\infty} \mathbb{D}_n.$$

The countable subgroup $\mathbb{D}$ acts on X by translation; let $R \subseteq X \times X$ be the corresponding orbit equivalence relation:

$$R = \{(x, x + r) : x \in X,\ r \in \mathbb{D}\}.$$

For $r \in \mathbb{D}$, define

$$\Delta_r = \{(x, x + r) \colon x \in X\}$$

and note that $\{\Delta_r \colon r \in \mathbb{D}\}$ is a partition of R.

Let $\mathbf{1}$ be the 2-cocycle on R taking the constant value 1; then $(X, \mu, R, \mathbf{1})$ is a Feldman-Moore relation. Let ν be the corresponding right counting measure. Clearly, if $E_r \subseteq \Delta_r$ is measurable, then $\nu(E) = \mu(\pi_1(E_r)) = \mu(\pi_2(E_r))$. Hence if E is a measurable subset of R, then for $j = 1, 2$ we have

$$\nu(E) = \sum_{r \in \mathbb{D}} \nu(E \cap \Delta_r) = \sum_{r \in \mathbb{D}} \mu(\pi_j(E \cap \Delta_r)). \tag{6}$$

It is well-known (see e.g., [10]) that $\mathcal{R} = \mathcal{M}(R, \mathbf{1})$ is (*-isomorphic to) the hyperfinite II_1-factor.

For $1 \leq i, j \leq 2^n$, define

$$\Delta_{ij}^n = \left\{ \left(x, x + \frac{j - i}{2^n} \right) : \frac{i - 1}{2^n} \leq x < \frac{i}{2^n} \right\}.$$

Let χ_{ij}^n be the characteristic function of Δ_{ij}^n, and write

$$\Sigma_n = \operatorname{span}\{\chi_{ij}^n \colon 1 \leq i, j \leq 2^n\}.$$

Writing L for the inverse symbol map of R, let $\mathcal{R}_n \subseteq \mathcal{R}$ be given by

$$\mathcal{R}_n = \{L(a) \colon a \in \Sigma_n\}.$$

We also write

$$\iota_n \colon \mathcal{R}_n \to M_{2^n}, \quad \sum_{i,j} \alpha_{ij} L(\chi_{ij}^n) \mapsto (\alpha_{ij}).$$

Recall that $\star$ denotes pointwise multiplication of symbols. We write $A \odot B$ for the Schur product of matrices $A, B \in M_k$ for some $k \in \mathbb{N}$.

Lemma 6.1.
1. *The set* $\{L(\chi^n_{ij})\colon 1 \le i, j \le 2^n\}$ *is a matrix unit system in* $\mathcal{R}$.
2. *The map* ι_n *is a* $*$*-isomorphism. In particular,* ι_n *is an isometry.*
3. *For* $a, b \in \Sigma_n$*, we have*
   (a) $a \star b \in \Sigma_n$;
   (b) $\iota_n(L(a \star b)) = \iota_n(L(a)) \odot \iota_n(L(b))$*; and*
   (c) $\|L(a \star b)\| \le \|L(a)\|\,\|L(b)\|$.

Proof. Checking (1) is an easy calculation, and (2) is then immediate. Statement (3a) is obvious, and (3b) is plain from the definition of ι_n. It is a classical result of matrix theory that if $A, B \in M_k$, then $\|A \odot B\| \le \|A\|\,\|B\|$. Statement (3c) then follows from (2) and (3b). □

Let $\tau\colon \mathcal{R} \to \mathbb{C}$ be given by

$$\tau(L(a)) = \int_X a(x, x)\, d\mu(x).$$

Since $\nu = \nu^{-1}$, an easy calculation shows that τ is a trace on $\mathcal{R}$.

For $a \in L^\infty(R, \nu)$, let

$$\lambda^n_{ij}(a) = 2^n \int_{(i-1)/2^n}^{i/2^n} a(x, x + (j - i)/2^n)\, d\mu(x)$$

be the average value of a on Δ^n_{ij}, and define

$$E_n\colon \Sigma(R, \mathbf{1}) \to \Sigma_n, \quad a \mapsto \sum_{i,j} \lambda^n_{ij}(a)\chi^n_{ij}$$

and

$$\mathbb{E}_n\colon \mathcal{R} \to \mathcal{R}_n, \quad L(a) \mapsto L(E_n(a)).$$

Lemma 6.2. $\mathbb{E}_n$ *is the* τ*-preserving conditional expectation of* $\mathcal{R}$ *onto* $\mathcal{R}_n$*. In particular,* $\mathbb{E}_n$ *is norm-reducing.*

Proof. By [19, Lemma 3.6.2], it suffices to show that $\mathbb{E}_n$ is a τ-preserving $\mathcal{R}_n$-bimodule map. For $a \in \Sigma(R, \mathbf{1})$, we have

$$\begin{aligned}\tau(\mathbb{E}_n(L(a))) &= \tau(L(E_n(a))) \\ &= \int E_n(a)(x, x)\, d\mu(x) \\ &= \sum_{i=1}^{2^n} \lambda^n_{ii}(a)\mu([(i-1)/2^n, i/2^n)) \\ &= \tau(L(a)),\end{aligned}$$

so $\mathbb{E}_n$ is τ-preserving. For $b, c \in \Sigma_n$, a calculation gives

$$E_n(b *_{\mathbf{1}} a *_{\mathbf{1}} c) = b *_{\mathbf{1}} E_n(a) *_{\mathbf{1}} c,$$

hence $\mathbb{E}_n(BTC) = B\mathbb{E}_n(T)C$ for $B, C \in \mathcal{R}_n$ and $T \in \mathcal{R}$. □

Lemma 6.3. *Let $a \in \Sigma(R, \mathbf{1})$.*

(1) $\|E_n(a)\|_\infty \le \|a\|_\infty$.
(2) $E_n(a) \to_{\|\cdot\|_2} a$ *as* $n \to \infty$.

Proof. (1) follows directly from the definition of E_n.

(2) For $T \in \mathcal{R}$, we have $\mathbb{E}_n(T) \to_{\text{SOT}} T$ as $n \to \infty$ (see e.g., [15]). By Proposition 4.6(6),

$$E_n(a) = s(\mathbb{E}_n(L(a))) \to_{\|\cdot\|_2} s(L(a)) = a. \qquad \square$$

Theorem 6.4. *We have $\Sigma(R, \mathbf{1}) \subseteq \mathfrak{S}(R, \mathbf{1})$. Moreover, if $a, b \in \Sigma(R, \mathbf{1})$, then $\|L(a \star b)\| \le \|L(a)\|\|L(b)\|$.*

Proof. Let $a, b \in \Sigma(R, \mathbf{1})$, and for $n \in \mathbb{N}$, let $a_n = E_n(a)$ and $b_n = E_n(b)$. Lemmas 6.1 and 6.2 give

(7)
$$\|L(a_n \star b_n)\| \le \|L(a_n)\|\,\|L(b_n)\| = \|\mathbb{E}_n(L(a))\|\,\|\mathbb{E}_n(L(b))\| \le \|L(a)\|\|L(b)\|.$$

On the other hand,

$$\begin{aligned} \|a_n \star b_n - a \star b\|_2 &\le \|a_n \star (b_n - b)\|_2 + \|b \star (a_n - a)\|_2 \\ &\le \|a_n\|_\infty \|(b_n - b)\|_2 + \|b\|_\infty \|(a_n - a)\|_2 \end{aligned}$$

so by Lemma 6.3,

$$a_n \star b_n \to_{\|\cdot\|_2} a \star b.$$

Let $T_n = L(a_n \star b_n)$. Since (T_n) is bounded by (7), Proposition 4.8 shows that (T_n) converges in the strong operator topology, say to $T \in \mathcal{R}$, and

$$a_n \star b_n = s(T_n) \to_{\|\cdot\|_2} s(T).$$

Hence $a \star b = s(T) \in \Sigma(R, \mathbf{1})$, so $a \in \mathfrak{S}(R, \mathbf{1})$.

Since $T_n \to_{\text{SOT}} T$, we have $\|T\| \le \limsup_{n\to\infty} \|T_n\|$. Hence by (7),

$$\|L(a \star b)\| \le \limsup_{n\to\infty} \|L(a_n \star b_n)\| \le \|L(a)\|\|L(b)\|. \qquad \square$$

Remark 6.5. For each masa $\mathcal{A} \subseteq \mathcal{R}$, Pop and Smith define a Schur product $\star_{\mathcal{A}} \colon \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ in [15]. The proof of Theorem 6.4 shows that for the specific Feldman-Moore coordinatisation $(X, \mu, R, \mathbf{1})$ described above and the masa $\mathcal{A} = \mathcal{A}(R) \subseteq \mathcal{R} = \mathcal{M}(R, \mathbf{1})$, if we identify operators in $\mathcal{R}$ with their symbols, then Definition 4.1 extends $\star_{\mathcal{A}}$ to a map $\mathfrak{S}(R, \mathbf{1}) \times \mathcal{R} \to \mathcal{R}$. It is easy to see that this is a proper extension: the constant function $\varphi(x, y) = 1$ is plainly in $\mathfrak{S}(R, \mathbf{1})$, but φ is not the symbol of an operator in $\mathcal{R}$ ([15, Remark 3.3]).

Corollary 6.6. *Let $\mathcal{R}$ be the hyperfinite II_1 factor, and let $\tilde{\mathcal{A}}$ be any masa in $\mathcal{R}$. For any Feldman-Moore coordinatination $(\tilde{X}, \tilde{\mu}, \tilde{R}, \tilde{\sigma})$ of the Cartan pair $(\mathcal{R}, \tilde{\mathcal{A}})$, we have $\Sigma(\tilde{R}, \tilde{\sigma}) \subseteq \mathfrak{S}(\tilde{R}, \tilde{\sigma})$.*

Proof. By [5], we have $(\mathcal{R}, \tilde{\mathcal{A}}) \cong (\mathcal{R}, \mathcal{A})$. Hence by Theorem 2.15,

$$(\tilde{X}, \tilde{\mu}, \tilde{R}, \tilde{\sigma}) \cong (X, \mu, R, \mathbf{1})$$

via an isomorphism $\rho \colon \tilde{X} \to X$. Consider the map $\tilde{\rho} \colon a \mapsto a \circ \rho^{-2}$ as in Proposition 4.13. By Theorem 6.4,

$$\Sigma(\tilde{R}, \tilde{\sigma}) = \tilde{\rho}(\Sigma(R, \mathbf{1})) \subseteq \tilde{\rho}(\mathfrak{S}(R, \mathbf{1})) = \mathfrak{S}(\tilde{R}, \tilde{\sigma}). \qquad \square$$

In view of Theorem 6.4 and Proposition 4.4, it is natural to ask the following question.

Question 6.7. Does the inclusion $\Sigma(R, \sigma) \subseteq \mathfrak{S}(R, \sigma)$ hold for an arbitrary Feldman-Moore relation (X, μ, R, σ)?

We now turn to the inclusion

$$\mathfrak{A}(R) \subseteq \mathfrak{S}(R, \sigma)$$

established in Section 5. While these sets are equal in the classical case, we will show that in the current context this inclusion is proper.

For $D \subseteq \mathbb{D}$, we define

$$\Delta(D) = \bigcup_{r \in D} \Delta_r.$$

Note that $\Delta(D)$ is marginally null only if $D = \emptyset$, and its characteristic function $\chi_{\Delta(D)}$ is a "Toeplitz" idempotent element of $L^\infty(R, \nu)$.

Proposition 6.8.

(1) *If* $\emptyset \neq D \subsetneq \mathbb{D}$ *and either* D *or* $\mathbb{D} \setminus D$ *is dense in* $[0, 1)$*, then the characteristic function* $\chi_{\Delta(D)}$ *is not in* $\mathfrak{A}(R)$*.*

(2) *Let* $0 \neq \varphi \in L^\infty(R)$ *and*

$$E = \{r \in \mathbb{D} \colon \varphi|_{\Delta_r} = 0 \ \mu\text{-a.e.}\}.$$

If E *is dense in* $[0, 1)$*, then* $\varphi \notin \mathfrak{A}(R)$*.*

Proof. (1) Suppose first that $\mathbb{D} \setminus D$ is dense in $[0, 1)$ and, by way of contradiction, that $\chi_{\Delta(D)} \in \mathfrak{A}(R)$. There is an element $\sum_{i=1}^\infty a_i \otimes b_i \in \mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$ and a ν-null set $N \subseteq R$ such that

$$\chi_{\Delta(D)}(x, y) = \sum_{i=1}^\infty a_i(x) b_i(y) \text{ for all } (x, y) \in R \setminus N.$$

Let $f \colon X \times X \to \mathbb{C}$ be the extension of $\chi_{\Delta(D)}$ which is defined (up to a marginally null set) by

$$f(x, y) = \sum_{i=1}^\infty a_i(x) b_i(y) \text{ for marginally almost every } (x, y) \in X \times X.$$

By [6, Theorem 6.5], f is ω-continuous. Hence the set

$$F = f^{-1}(\mathbb{C} \setminus \{0\})$$

is ω-open. Since $D \neq \emptyset$ and $\Delta(D) \subseteq F$, the set F is not marginally null. So there exist Borel sets $\alpha, \beta \subseteq [0,1)$ with non-zero Lebesgue measure so that $\alpha \times \beta \subseteq F$. For $j = 1, 2$, let $N_j = \pi_j(N)$. By equation (6), $\mu(N_j) = 0$. Let $\alpha' = \alpha \setminus N_1$ and $\beta' = \beta \setminus N_2$; then α' and β' have non-zero Lebesgue measure, and hence the set

$$\beta' - \alpha' = \{y - x \colon x \in \alpha',\, y \in \beta'\}$$

contains an open interval by Steinhaus' theorem, so it intersects the dense set $\mathbb{D} \setminus D$. So there exist $r \in \mathbb{D} \setminus D$ and $x \in \alpha'$ with $x + r \in \beta'$. Now

$$(x, x + r) \in F \setminus \Delta(D),$$

so

$$0 \neq f(x, x + r) = \chi_{\Delta(D)}(x, x + r) = 0,$$

a contradiction. So $\chi_{\Delta(D)} \notin \mathfrak{A}(R)$ if $D \neq \emptyset$ and $\mathbb{D} \setminus D$ is dense in $[0, 1)$.

If $D \neq \mathbb{D}$ and D is dense in $[0, 1)$ then $\chi_{\Delta(\mathbb{D}\setminus D)} \notin \mathfrak{A}(R)$; since $\mathfrak{A}(R)$ is a linear space containing the constant function 1, this shows that $1 - \chi_{\Delta(\mathbb{D}\setminus D)} = \chi_{\Delta(D)} \notin \mathfrak{A}(R)$.

(2) The argument is similar. If $\varphi \in \mathfrak{A}(R)$ then there is a ν-null set $N \subseteq R$ such that $\varphi(x, y) = \sum_{i=1}^{\infty} a_i(x) b_i(y)$ for all $(x, y) \in R \setminus N$ where $\sum_{i=1}^{\infty} a_i \otimes b_i \in \mathcal{A} \otimes_{\mathrm{eh}} \mathcal{A}$, and $\varphi(x, y) = 0$ for all $(x, y) \in R \setminus N$ with the property $y - x \in E$. Let $f \colon [0, 1)^2 \to \mathbb{C}$, $f(x, y) = \sum_{i=1}^{\infty} a_i(x) b_i(y)$, $x, y \in [0, 1)$. Then f is non-zero and ω-continuous, so $f^{-1}(\mathbb{C} \setminus \{0\})$ contains $\alpha' \times \beta'$ where α', β' are sets of non-zero measure so that $(\alpha' \times \beta') \cap N = \emptyset$. Hence $\beta' - \alpha'$ contains an open interval of $[0, 1)$, and intersects the dense set E in at least one point $r \in \mathbb{D}$; so there is $x \in [0, 1)$ such that $(x, x + r) \in (\alpha' \times \beta') \cap (R \setminus N)$. Then $0 = \varphi(x, x + r) = f(x, x + r) \neq 0$, a contradiction. ☐

Corollary 6.9. *The inclusion* $\mathfrak{A}(R) \subseteq \mathfrak{S}(R, \mathbf{1})$ *is proper.*

Proof. Since $\Delta = \Delta(\{0\})$, Proposition 6.8 shows that $\chi_\Delta \notin \mathfrak{A}(R)$. It is easy to check (as in Lemma 6.2) that the Schur multiplication map $M(\chi_\Delta)$ is the conditional expectation of $\mathcal{R}$ onto $\mathcal{A}$, so $\chi_\Delta \in \mathfrak{S}(R, \mathbf{1})$. ☐

Corollary 6.10. *Let* $(\tilde{X}, \tilde{\mu}, \tilde{R}, \tilde{\sigma})$ *be a Feldman-Moore relation and suppose that* $\mathcal{M}(\tilde{R}, \tilde{\sigma})$ *contains a direct summand isomorphic to the hyperfinite* II_1 *factor. Then the inclusion* $\mathfrak{A}(\tilde{R}) \subseteq \mathfrak{S}(\tilde{R}, \tilde{\sigma})$ *is proper.*

Proof. Let P be a central projection in $\mathcal{M}(\tilde{R}, \tilde{\sigma})$ so that $P\mathcal{M}(\tilde{R}, \tilde{\sigma})$ is (isomorphic to) the hyperfinite II_1 factor $\mathcal{R}$. It is not difficult to verify that $\mathcal{A}_P = P\mathcal{A}(\tilde{R})$ is a Cartan masa in $\mathcal{R}$ (see the arguments in the proof of [8, Theorem 1]). By [5], the Cartan pair $(\mathcal{R}, \mathcal{A}_P)$ is isomorphic to the Cartan pair $(\mathcal{R}, \mathcal{A})$ considered throughout this section. It follows from Theorem 2.15 that there is a Borel isomorphism $\rho : \tilde{X} \to X_0 \cup [0, 1)$ (a disjoint union) with $\rho^2(\tilde{R}) = R_0 \cup R$ (again, a disjoint union), where $R_0 \subseteq X_0 \times X_0$ is a standard equivalence relation and R is the equivalence relation defined at the start of the present section. It is easy to check that $\rho^2(\mathfrak{A}(\tilde{R})) = \mathfrak{A}(R_0 \cup R)$. We may thus assume that $\tilde{X} = X_0 \cup [0,1)$ and $\tilde{R} = R_0 \cup R$.

Now suppose that $\mathfrak{S}(\tilde{R}, \tilde{\sigma}) = \mathfrak{A}(\tilde{R})$. Let $P = P([0,1))$. Given $\varphi \in \mathfrak{S}(R)$, let $\psi : \tilde{R} \to \mathbb{C}$ be its extension defined by letting $\psi(x,y) = 0$ if $(x,y) \in R_0$. Then

$$M(\psi)(T \oplus S) = PM(\psi)(T \oplus S)P = M(\varphi)(T) \oplus 0, \quad T \in \mathcal{M}(R).$$

So $\psi \in \mathfrak{S}(R, \mathbf{1})$ and hence $\psi \in \mathfrak{A}(\tilde{R})$. It now easily follows that $\varphi \in \mathfrak{A}(R)$, contradicting Corollary 6.9. □

In fact, the only Toeplitz idempotent elements of $\mathfrak{S}(R) := \mathfrak{S}(R, \mathbf{1})$ are trivial. To see this, we first explain how $\mathfrak{S}(R)$ can be obtained from multipliers of the Fourier algebra of a measured groupoid. We refer the reader to [16, 17] for basic notions and results about groupoids.

The set $\mathcal{G} = X \times \mathbb{D}$ becomes a groupoid under the partial product

$$(x, r_1) \cdot (x + r_1, r_2) = (x, r_1 + r_2) \quad \text{for } x \in X,\ r_1, r_2 \in \mathbb{D}$$

where the set of composable pairs is

$$\mathcal{G}^2 = \{\big((x_1, r_1), (x_2, r_2)\big) : x_2 = x_1 + r_1\}$$

and inversion is given by

$$(x,t)^{-1} = (x + t, -t).$$

The domain and range maps in this case are $d(x,t) = (x,t)^{-1} \cdot (x,t) = (x+t, 0)$ and $r(x,t) = (x,t) \cdot (x,t)^{-1} = (x, 0)$, so the unit space, $\mathcal{G}_0$, of this groupoid, which is the common image of d and r, can be identified with X. Let λ be the Haar, that is, the counting, measure on $\mathbb{D}$. The groupoid $\mathcal{G}$ can be equipped with the Haar system $\{\lambda^x : x \in X\}$, where $\lambda^x = \delta_x \times \lambda$ and δ_x is the point mass at x.

Recall that μ is Lebesgue measure on X. Consider the measure $\nu_{\mathcal{G}}$ on $\mathcal{G}$ given by $\nu_{\mathcal{G}} = \mu \times \lambda = \int \lambda^x d\mu(x)$. Since μ is translation invariant and λ is invariant under the transformation $t \mapsto -t$, it is easy to see that $\nu_{\mathcal{G}}^{-1} = \nu_{\mathcal{G}}$, where $\nu_{\mathcal{G}}^{-1}(E) = \nu_{\mathcal{G}}(\{e^{-1} \colon e \in E\})$.Therefore $\mathcal{G}$ with the above Haar system and the measure μ becomes a measured groupoid.

Consider the map

$$\theta : R \to X \times \mathbb{D}, \quad \theta(x, x+r) = (x, r), \quad x \in X,\ r \in \mathbb{D}.$$

Clearly θ is a continuous bijection (here $\mathbb{D}$ is equipped with the discrete topology). We claim the measure $\theta_*\nu \colon E \mapsto \nu(\theta^{-1}(E))$ is equal to $\nu_{\mathcal{G}}$, where, as before, ν is the right counting measure for the Feldman-Moore relation $(X, \mu, R, \mathbf{1})$. Indeed, for $E \subseteq \mathcal{G}$, we have

$$\begin{aligned}(\theta_*\nu)(E) &= \nu(\theta^{-1}(E)) \\ &= \sum_{r\in\mathbb{D}} \mu(\pi_1(\theta^{-1}(E) \cap \Delta_r)) \quad \text{by equation (6)} \\ &= \sum_{r\in\mathbb{D}} \mu(\pi_1(E \cap (X \times \{r\}))) = (\mu \times \lambda)(E) = \nu_{\mathcal{G}}(E)\end{aligned}$$

since it is easily seen that $\pi_1(\theta^{-1}(E) \cap \Delta_r) = \{x \in X : (x, r) \in E\}$. It follows that the operator

$$U : L^2(R, \nu) \to L^2(\mathcal{G}, \nu_{\mathcal{G}}), \quad \xi \mapsto \xi \circ \theta^{-1}$$

is unitary.

Let $C_c(\mathcal{G})$ be the space of compactly supported continuous functions on $\mathcal{G}$. This becomes a $*$-algebra with respect to the convolution given by

$$(f * g)(x, t) = \sum_{r\in\mathbb{D}} f(x, r)g(x + r, t - r),$$

and involution given by $f^*(x, t) = \overline{f(x + t, -t)}$.

Let Reg be the representation of $C_c(\mathcal{G})$ on the Hilbert space $L^2(\mathcal{G}, \nu_{\mathcal{G}})$ given for $\xi, \eta \in L^2(\mathcal{G}, \nu_{\mathcal{G}})$ by

$$\begin{aligned}\langle \mathrm{Reg}(f)\xi, \eta\rangle &= \int f(x, t)\xi((x, t)^{-1}(y, s))\overline{\eta(y, s)}d\lambda^{r(x,t)}(y, s)d\lambda^u(x, t)d\mu(u) \\ &= \int f(x, t)\xi(x + t, s - t)\overline{\eta(x, s)}d\lambda(s)d\lambda(t)d\mu(x) \\ &= \int f(x, t)\xi(x + t, s - t)\overline{\eta(x, s)}d\lambda(t)d\nu_{\mathcal{G}}(x, s)\end{aligned}$$

hence

$$(\mathrm{Reg}(f)\xi)(x, s) = \int f(x, t)\xi(x + t, s - t)d\lambda(t) = \sum_t f(x, t)\xi(x + t, s - t).$$

In [17, Section 2.1], the von Neumann algebra $\mathrm{VN}(\mathcal{G})$ of $\mathcal{G}$ is defined to be the bicommutant $\mathrm{Reg}(C_c(\mathcal{G}))''$.

If $f \in C_c(\mathcal{G})$, then $f \circ \theta$ has a band limited support and for $\xi \in L^2(R, \nu)$, we have

$$\begin{aligned}(U^* \mathrm{Reg}(f)U\xi)(x, x + t) &= \sum_s f(x, s)\xi(x + s, x + t) \\ &= \sum_s f(\theta(x, x + s))\xi(x + s, x + t) \\ &= (L(f \circ \theta)\xi)(x, x + t).\end{aligned}$$

Hence

$$U^* \mathrm{Reg}(f)U = L(f \circ \theta) \tag{8}$$

and so $\mathrm{VN}(\mathcal{G})$ is spatially isomorphic to $\mathcal{M}(R)$.

The von Neumann algebra $\mathrm{VN}(\mathcal{G})$ is the dual of the Fourier algebra $A(\mathcal{G})$ of the measured groupoid $\mathcal{G}$, which is a Banach algebra of complex-valued functions on $\mathcal{G}$. If the operator M_φ on $A(\mathcal{G})$ of multiplication by the function $\varphi \in L^\infty(\mathcal{G})$ is bounded, then its adjoint M_φ^* is a bounded linear map on $\mathrm{VN}(\mathcal{G})$. Moreover, in this case we have $M_\varphi^* \mathrm{Reg}(f) = \mathrm{Reg}(\varphi f)$, for $f \in C_c(\mathcal{G})$. The function φ is then called a multiplier of $A(\mathcal{G})$ [17] and we write $\varphi \in MA(\mathcal{G})$. If the map M_φ is also completely bounded then φ is called a completely bounded multiplier of $A(\mathcal{G})$ and we write $\varphi \in M_0A(\mathcal{G})$. By equation (8) and Remark 4.12, we have

$$\varphi \in M_0A(\mathcal{G}) \iff \varphi \circ \theta \in \mathfrak{S}(R, \mathbf{1}). \tag{9}$$

We are now ready to prove the following statement:

Proposition 6.11. *If $D \subseteq \mathbb{D}$, then the following are equivalent:*

1. *The function $\chi_{\Delta(D)} \in L^\infty(R, \nu)$ is in $\mathfrak{S}(R)$.*
2. *The function $\chi_D \in \ell^\infty(\mathbb{D})$ is in the Fourier-Stieltjes algebra $B(\mathbb{D})$ of $\mathbb{D}$.*
3. *D is in the coset ring of $\mathbb{D}$.*

Proof. To see that (1) and (2) are equivalent, observe that if $\pi : \mathcal{G} \to \mathbb{D}$, $(x, t) \mapsto t$ is the projection homomorphism of $\mathcal{G}$ onto $\mathbb{D}$, then

$$\chi_{\Delta(D)} = \chi_D \circ \pi \circ \theta.$$

Moreover, since $\mathbb{D}$ is commutative, we have $B(\mathbb{D}) = M_0A(\mathbb{D})$. So

$$\begin{aligned} \chi_D \in B(\mathbb{D}) &\iff \chi_D \in M_0A(\mathbb{D}) \\ &\iff \chi_D \circ \pi \in M_0A(\mathcal{G}) \text{ by [17, Proposition 3.8]} \\ &\iff \chi_{\Delta(D)} = \chi_D \circ \pi \circ \theta \in \mathfrak{S}(R, \mathbf{1}) \text{ by (9).} \end{aligned}$$

The equivalence of (2) and (3) follows from [18, Chapter 3]. □

Theorem 6.12. *The only elements of $\mathfrak{A}(R)$ of the form $\chi_{\Delta(D)}$ for some $D \subseteq \mathbb{D}$ are 0 and 1.*

Proof. If $\chi_{\Delta(D)} \in \mathfrak{A}(R)$ then $\chi_{\Delta(D)} \in \mathfrak{S}(R)$ by Proposition 5.6, so D is in the coset ring of $\mathbb{D}$ by Proposition 6.11. All proper subgroups of $\mathbb{D}$ are finite, so D is in the ring of finite or cofinite subsets of $\mathbb{D}$. Hence either $\mathbb{D} \setminus D$ or D is dense in $[0, 1)$, so either $D = \emptyset$ or $D = \mathbb{D}$ by Proposition 6.8. □

Remark 6.13. We note that there exist non-trivial idempotent elements of $\mathfrak{A}(R)$. For example, if α, β are measurable subsets of X, then the characteristic function of $(\alpha \times \beta) \cap R$ is always idempotent. Note that the sets of the form $(\alpha \times \beta) \cap R$ are not unions of full diagonals unless they are equivalent to either R or the empty set.

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School of Mathematical Sciences, University College Dublin, Belfield, Dublin 4, Ireland

E-mail address: rupert.levene@ucd.ie

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

E-mail address: nspronk@uwaterloo.ca

Pure Mathematics Research Centre, Queen's University Belfast, Belfast BT7 1NN, United Kingdom

E-mail address: i.todorov@qub.ac.uk

Department of Mathematics, Chalmers University of Technology and the University of Gothenburg, Sweden
E-mail address: turowska@chalmers.se