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# A New Approximation to the Distribution of Positive RVs applied to Gaussian Quadratic Forms

Pablo Ramírez-Espinosa, David Morales-Jimenez, José A. Cortés, José F. Paris, Eduardo Martos-Naya

***Abstract*—This letter introduces a new approach to the problem of approximating the probability density function (PDF) and the cumulative distribution function (CDF) of a positive random variable. The novel approximation strategy is based on the analysis of a suitably defined sequence of auxiliary variables which converges in distribution to the target variable. By leveraging such convergence, simple approximations for both the CDF and PDF of the target variable are given in terms of the derivatives of its moment generating function (MGF). In contrast to classical approximation methods based on truncated series of moments or cumulants, our approximations always represent a valid distribution and the relative error between variables is independent of the variable under analysis. The derived results are then used to approximate the statistics of positive-definite real Gaussian quadratic forms, comparing our proposed approach with other existing approximations in the literature.**

## I. Introduction

Approximating the distribution of a random variable (RV) is a classical problem in applied statistics, central to a vast number of applications in signal processing and communication theory, to mention but a couple of relevant fields. In many cases, closed-form expressions for the moment generating function (MGF) of the RV or its moments are known, but the probability density function (PDF) and cumulative distribution function (CDF) remain intractable or given in complicated integral forms. A natural approach is then to approximate the PDF and CDF of a RV from its cumulants or moments, and significant efforts have been made in this direction: Pearson curves [1], saddle point techniques [2], Gram-Charlier and Edgeworth series [3] or orthogonal polynomial series expansions [4] are some well-known examples of these (classical) approximations. These have been extensively used as they are typically simple and provide accurate representations in many cases. Similar techniques have been recently applied in [5], where approximations for the distribution of the sum of independent RVs are investigated.

However, a major drawback of these classical results is that the resulting expressions are not guaranteed to be proper PDFs (in fact, they typically are not); that is, the approximated PDF does not necessarily integrate to one and the probabilities may be negative [4, p. 731][6, sec. 2]. Another important issue is their accuracy, which typically depends on the parametrization of the target RV, leading in some cases to poor approximations.

We here aim to provide an alternative approximation to the distribution of positive RVs which circumvents the aforementioned limitations (of classical approaches). Instead of directly expanding the PDF (or CDF) of the target variable (which typically results in series involving functions of the moments/cumulants), we focus on a sequence of auxiliary RVs, defined so that the chief probability functions (PDF and CDF) converge to those of the target RV under certain conditions. The proposed approach has a double benefit:

- Since the PDF and CDF of the target variable are approximated by those of an auxiliary variable, the resulting expressions always represent a valid distribution.
- The relative error between the target variable and the auxiliary one can be characterized in simple closed form, and it is independent of the variable under analysis.

The derived expressions are then used to approximate the distribution of positive-definite quadratic forms (QFs) in real Gaussian variables, which has been subject of study for decades due to their numerous applications in statistics, signal processing and communications [7–15]. Gaussian QFs appear, for instance, in the detection of signals in Gaussian noise [16, sec. 13.5], the spectral detection of normally distributed stationary processes, $\chi^2$ tests [7, sec. 7] and analysis of variance [11]. Complex Gaussian QFs, which can be regarded as a particular case of the real one [17], are also relevant in many signal processing problems such as the performance analysis of adaptive filter algorithms [18], energy detection [19] or performance analysis of maximum likelihood (ML) estimators [20]. Also, in wireless communications, QFs naturally arise when analyzing differential modulation schemes [21] or diversity techniques [15, 22].

## II. Approximating the Distribution of Positive RVs

Consider a positive RV $X$ with continuous distribution, and assume that the MGF of $X$ is infinitely differentiable. In order to approximate its CDF, we define the auxiliary variable

$$X_m = X/\xi_m \tag{1}$$

where $\xi_m$ is a Gamma-distributed RV, independent of $X$, with PDF:

$$f_{\xi_m}(u) = \frac{(m-1)^m}{\Gamma(m)} u^{m-1} e^{-(m-1)u}. \tag{2}$$

Since the sequence $\{\xi_m : m \in \mathbb{N}^+\}$ converges to one in distribution, it follows from Slutsky's Theorem [23] that the sequence $\{X_m : m \in \mathbb{N}^+\}$ converges to $X$ in distribution.

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This is stated in the following lemma, where the CDF of $X_m$ is derived, allowing us to approximate the CDF of $X$ by that of $X_m$, with sufficiently large $m$.

*Lemma 1:* Let $X$ be a positive RV with infinitely differentiable MGF given by $M_X(s)$, and $X_m$ as in (1). Then, the CDF of $X$ satisfies

$$F_X(x) = \lim_{m\to\infty} F_{X_m}(x) = \lim_{m\to\infty} \sum_{k=0}^{m-1} \frac{(m-1)^k}{x^k k!} \left. \frac{d^k}{ds^k} M_X(s) \right|_{s=(1-m)/x}. \quad (3)$$

*Proof:* The CDF of $X_m$ can be calculated as

$$F_{X_m}(x) = \int_0^\infty F_X(ux) f_{\xi_m}(u) du. \quad (4)$$

Performing the change of variables $y = ux$ and integrating by parts we obtain

$$F_{X_m}(x) = 1 - \int_0^\infty f_X(y) F_{\xi_m}(y/x) dy \quad (5)$$

where $f_X(\cdot)$ is the PDF of $X$ and $F_{\xi_m}(\cdot)$ is the CDF of $\xi_m$, which for integer $m$ is given by

$$F_{\xi_m}(u) = 1 - \sum_{k=0}^{m-1} \frac{(m-1)^k}{k!} u^k e^{-(m-1)u}. \quad (6)$$

Substituting (6) in (5) and rewriting the integral as the expectation in $X$ lead to

$$F_{X_m}(x) = \sum_{k=0}^{m-1} \frac{(m-1)^k}{k! x^k} \mathbb{E}_X \left[ X^k e^{-(m-1)X/x} \right], \quad (7)$$

where the expectation can be seen as that of the $k$-th order derivative of $e^{sX}$ evaluated at $s = (1-m)/x$. By Leibniz's integral rule, we can interchange the derivative and the expectation, yielding (3). ■

Generally, convergence in distribution does not imply convergence of the PDFs. However, the implication holds under certain conditions. Specifically, if the densities of $X$ and $X_m$, namely $f_X(x)$ and $f_{X_m}(x)$, are continuous functions and $f_{X_m}(x)$ is bounded and equicontinuous, then the convergence of the CDFs implies the convergence of the density functions [24]. In our case, $f_{X_m}(x)$ is calculated as

$$f_{X_m}(x) = \int_0^\infty u f_X(ux) f_{\xi_m}(u) du \quad (8)$$

Note that, if $f_X(x)$ is uniformly continuous, it can be easily proved that $f_{X_m}(x)$ satisfies the aforementioned conditions, so that $f_{X_m}(x) \to f_X(x)$ as $m \to \infty$. This allows to approximate the PDF of $X$ by that of $X_m$, as stated next.

*Corollary 1:* Let $X$ be a positive RV with infinitely differentiable MGF $M_X(s)$ and uniformly continuous PDF, and $X_m$ as in (1). Then, the PDF of $X$ satisfies

$$f_X(x) = \lim_{m\to\infty} f_{X_m}(x) = \lim_{m\to\infty} \frac{(m-1)^m}{x^{m+1}\Gamma(m)} \left. \frac{d^m}{ds^m} M_X(s) \right|_{s=(1-m)/x}. \quad (9)$$

*Proof:* Similar to the CDF, $f_{X_m}(x)$ is obtained from (8) by substituting $f_{\xi_m}(u)$ and rewriting the resulting integral as the expectation of the $k$-th order derivative of the exponential function. ■

Finally, we formulate the relative error between $X_m$ and $X$:

*Lemma 2:* Let $X$ be a positive RV, and $X_m$ as in (1). For $m > 2$, the normalized mean square error (NMSE) admits the following compact expression:

$$\overline{\epsilon^2} = \mathbb{E}\left[(X - X_m)^2\right] / \mathbb{E}\left[X^2\right] = (m-2)^{-1}. \quad (10)$$

*Proof:* The result is straightforwardly obtained from the definition of the NMSE by noticing that, since $\xi_m \sim \Gamma(m, 1/(m-1))$, then $1/\xi_m$ is inverse Gamma distributed with $\mathbb{E}[1/\xi_m] = 1$ and $\mathbb{E}\left[(1 - 1/\xi_m)^2\right] = 1/(m-2)$. ■

Lemma 1 and Corollary 1 provide a general result to approximate the distribution of a positive RV in terms of the derivatives of its MGF. The main benefits of this approach are: (i) we approximate the distribution of a RV $X$ by that of another RV $X_m$, so that the approximation (for any $m$) is always a valid distribution, and (ii) the NMSE between $X$ and $X_m$ is under control, and easily characterized. Note however that, although informative on the closeness between $X_m$ and $X$, the NMSE does not directly translate into the error between the approximated CDF and the true one.

As with classical series expansions, where the moments are required, the calculation of (3) and (9) may be challenging if the derivatives of $M_X(s)$ are difficult to compute or not expressible in closed form. Moreover, the convergence of the proposed series may be slow and a relatively large $m$ may be needed, implying the computation of high-order derivatives of $M_X(s)$, which may be tedious in some cases.

However, there are cases where these derivatives can be readily obtained. That is, for instance, the case of Gaussian QFs, where the moments—equivalently, the derivatives of the MGF—can be easily computed recursively. The approach would further appeal to other cases, e.g., those where the MGF is given as a product of elementary functions [7, eq. (3.2b.3)].

## III. Distribution of Real Quadratic Forms

We aim to characterize the random variable

$$Q = (\mathbf{x} + \boldsymbol{\mu})^T \mathbf{A} (\mathbf{x} + \boldsymbol{\mu}) \quad (11)$$

where $\boldsymbol{\mu} \in \mathbb{R}^{n\times 1}$ is a constant vector, $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix and $\mathbf{x} \in \mathbb{R}^{n\times 1}$ is a random vector following a multivariate normal distribution with zero mean and covariance matrix $\mathbf{\Sigma}$, i.e. $\mathbf{x} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{\Sigma})$.

In this case, the MGF of $Q$ is given by [7, eq. (3.2a.5)]

$$M_Q(s) = \prod_{j=1}^n \exp\left(\frac{b_j^2 \lambda_j s}{1 - 2\lambda_j s}\right) (1 - 2\lambda_j s)^{-1/2} \quad (12)$$

where $\lambda_j$, $j = 1, \ldots, n$, are the eigenvalues of $\mathbf{C} = \mathbf{\Sigma}^{1/2}\mathbf{A}\mathbf{\Sigma}^{1/2}$ and $b_j$ are the entries of $\mathbf{b} = \mathbf{U}^T \mathbf{\Sigma}^{-1/2} \boldsymbol{\mu}$ with $\mathbf{U}$ the unitary matrix resulting from the diagonalization of $\mathbf{C}$.

### A. Literature Review

Closed form expressions for the PDF and CDF of real QFs can only be obtained in certain simple cases (e.g., $\mathbf{A} = \mathbf{\Sigma} = \mathbf{I}$, the identity matrix), remaining in complicated integral form for the general case, even when the involved Gaussian variables have zero mean. The ubiquitous relevance of Gaussian QFs has motivated a rich body of work over the last few decades, aiming at approximating the chief probability functions of $Q$. Approximations for both the PDF and CDF are given in terms of infinite series of powers (Maclaurin series) [7–9], Laguerre polynomials [7, 8, 10], central $\chi^2$ distributions [7, 8, 11], non-central $\chi^2$ densities [8, 13] and Hermite polynomials [14].

However, these approximations suffer from several drawbacks. The series in [10, 11] are restricted to the central case, and the series coefficients in [9] involve complicated sums, being inefficient for numerical computation. The same happens with the result in [14], where the number of terms to be computed depends on a combinatorial, increasing with the order of the QF. Numerical issues are also relevant in the case of non-central $\chi^2$ expansions [8, 13], where several Marcum-$Q$ functions need to be computed. The approximations based on Maclaurin series and the Laguerre polynomials and central $\chi^2$ expansions in [7, 8] render more useful results, but they fail under a wide range of conditions as we will see later.

Another approximation to the CDF of real QFs was recently given in [15], where the saddle-point technique was applied. Unfortunately, the non-central case was not analyzed.

### *B. Approximated PDF and CDF*

By applying Lemma 1 and Corollary 1 we can approximate the PDF and CDF of $Q$ by a linear combination of derivatives of the MGF of $Q$, which in this case can be written in closed form in an efficient recursive way [7, pp. 51-53]. The $k$-th order derivative of $M_Q(s)$ admits

$$\frac{d^k}{ds^k} M_Q(s) = M_Q(s) D_k(s) \tag{13}$$

where $D_0(s) = 1$ and, for $k \geq 1$,

$$D_k(s) = \sum_{j=0}^{k-1} \binom{k-1}{j} g_{k-1-j}(s) D_j(s), \tag{14}$$

$$g_i(s) = 2^i \, i! \sum_{t=1}^{n} \frac{\lambda_t^{i+1} \left[(i+1) b_t^2 + 1 - 2\lambda_t s\right]}{(1 - 2\lambda_t s)^{i+2}}. \tag{15}$$

Using (13) in (9) and (3) we immediately have the approximated expressions for the PDF and CDF of $Q$, respectively:

$$f_Q(x) \approx \frac{(m-1)^m}{x^{m+1}\Gamma(m)} M_Q\left(\frac{1-m}{x}\right) D_m\left(\frac{1-m}{x}\right), \tag{16}$$

$$F_Q(x) \approx M_Q\left(\frac{1-m}{x}\right) \sum_{k=0}^{m-1} \frac{(m-1)^k}{x^k k!} D_k\left(\frac{1-m}{x}\right). \tag{17}$$

The recursion in the calculation of $D_k(\cdot)$ and the fact that the derivatives of $M_Q(s)$ are positive—consequently, also $D_k(\cdot)$—allow for an efficient and numerically stable computation of (16) and (17), with independence of the QF parameters. This is in stark contrast to previous (existing) methods, as will be illustrated in the next section.

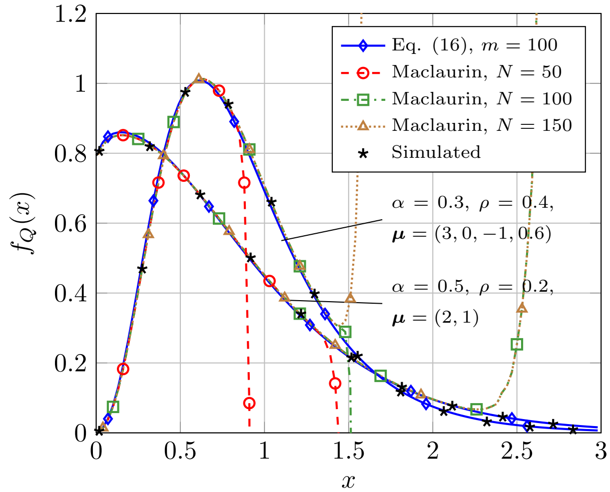


Fig. 1. PDF of $Q$ for different values of $\alpha$, $\rho$ and $\boldsymbol{\mu}$. The proposed method is compared with the Maclaurin expansion in [7, 8] and with MC simulations.

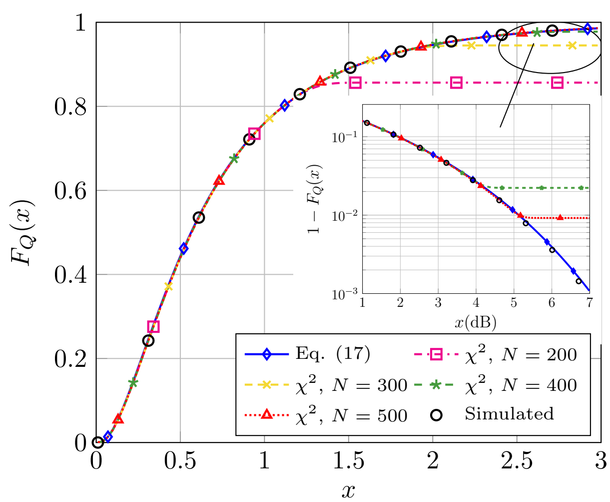


Fig. 2. CDF of $Q$ for $\alpha = 0.7$, $\rho = 0.5$, $\boldsymbol{\mu} = (2, 1, -1, 0.6, -0.9)$. Our proposed method with $m = 100$ is compared with the $\chi^2$ expansion in [7] and with MC simulations.

## IV. Numerical Results and Discussion

We now compare the proposed approximation for real QFs in (16) and (17) with the resuls in terms of Maclaurin series, Laguerre polynomials series and $\chi^2$ densities expansion in [7, 8]. The other classical approaches and the approximation given in [15] are disregarded due to the drawbacks listed in section III-A. All the calculations haven been done using Mathematica software and validated through Monte Carlo (MC) simulations.

For the sake of simplicity, we consider $(\mathbf{A})_{i,j} = \alpha^{|i-j|}$ and $(\boldsymbol{\Sigma})_{i,j} = \rho^{|i-j|}$, with $0 < \alpha, \rho < 1$, and compare our proposed approximation with previous methods in Figs. 1-4. We use different values of $n$, $\alpha$, $\rho$ and $\boldsymbol{\mu}$ in the distinct representations in order to show the generality and robustness (in the parameters range) of our proposed approximation.

Fig. 1 focuses on the comparison between our approximation and the Maclaurin series by plotting the approximated PDF of $Q$. It can be observed that the Maclaurin series

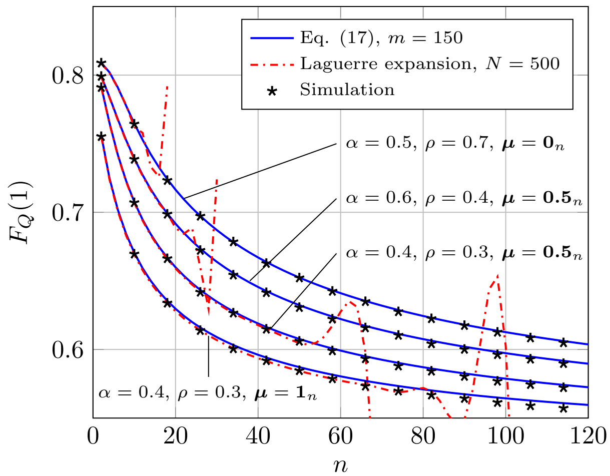


Fig. 3. CDF of $Q$ at $x = 1$ for distinct QF order, $n$. Our proposed method is compared with the Laplace approximation with $\beta_L = 10$ in [7, 8] and with MC simulations.

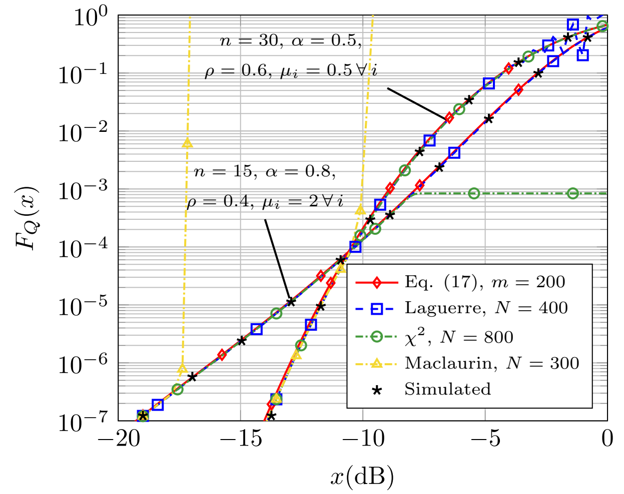


Fig. 4. CDF of $Q$ for different values of $n$, $\alpha$, $\rho$, $\mu_i$. The proposed method is compared with the Maclaurin series, the Laguerre expansion and the $\chi^2$ expansion in [7, 8] and contrasted with MC simulations. $\beta_L = 9$ for $n = 15$ and $\beta_L = 12$ for $n = 30$.

renders a valid approximation only up to a certain point in $x$, remarkably failing thereafter; the value of $x$ from which the series diverges depends on the parameters of the QF. As illustrated, increasing the number of terms in the series from 100 to 150 does not seem to solve the problem. In contrast, our proposed solution shows an excellent agreement with simulations for all parameter choices and all $x$.

Fig. 2 compares our results with the $\chi^2$ densities expansion by plotting the CDF of $Q$ (similar trends are observed for the PDF). The $\chi^2$ approximation artificially introduces an additional parameter $\beta_\chi < \min_j\{\lambda_j\}$ that controls the series convergence [8]. It has been observed that a high value of $\beta_\chi$ in that range facilitates the convergence, so we fixed $\beta_\chi = 0.9\min_j\{\lambda_j\}$. We observe that the number of terms required for the $\chi^2$ expansion is much larger than in our method ($m$) to accurately approximate the distribution of $Q$. Furthermore, it is important to note that the $\chi^2$ expansion does not yield a valid CDF, as observed in the right-hand tail.

A comparison between (17) and the Laguerre series expansion is shown in Fig. 3. This approximation also introduces a parameter $\beta_L > \max_j\{\lambda_j\}$ that controls the convergence of the series. However, we noticed that the recommended value of $\beta_L$ in [8] renders considerable numerical errors that prevent the series convergence. Hence, finding the value of $\beta_L$ that yields the best approximation in each case is a difficult task. For the calculation in Fig. 3, a value of $\beta_L = 10$ has been chosen, which seems to minimize such numerical issues. as shown in Fig. 3 the series diverges when the order of $Q$ increases, with independence of the QF parameters. Note that the considered values of $n$ are not particularly extreme, but rather moderate, particularly in applications such as signal detection where large numbers of samples are typically employed.

The last comparison is shown in Fig. 4, where the CDF of $Q$ is plotted in logarithmic scale in order to analyze the accuracy of the distinct approaches to approximate the left tail of the distribution. As observed, the aforementioned drawbacks of the alternative methods are also relevant when focusing on the left tail of the distribution. In turn, our proposed technique renders again an excellent approximation with independence of the parameters of $Q$.

The goodness of the approximations in the previous cases has been quantified using the Kolmogorov-Smirnov (K-S) test, showing that the proposed method provides the same (or even better) accuracy than the alternative ones by computing much less terms ($m$) with independence of the QF parameters. For instance, for data in Fig. 2, our technique achieves a K-S test result of $D = 0.036$ for $m = 100$ while the $\chi^2$ expansion needs $N = 500$ to get $D = 0.035$.

Moreover, the complexity in terms of the computation time has also been investigated, observing that, for a given accuracy, the proposed approximation is faster than the alternative ones (since it needs much less computed terms).

## V. Conclusion

This letter addresses the problem of approximating the distribution of positive RVs whose closed-form expression of the MGF is known. A new approach based on a sequence of auxiliary variables that converges in distribution to the target one is proposed, rendering approximations for the PDF and the CDF in terms of the derivatives of the MGF of the RV under analysis. Compared to previous techniques found in the literature, the proposed method offers a twofold benefit: (i) the resulting expressions for the chief probability functions always represent a valid distribution, and (ii) the NMSE between the target variable and the auxiliary one admits a simple formulation which is independent of the considered RV.

The presented approach is then applied to the study of positive-definite real Gaussian quadratic forms, for which we provide novel approximations for the PDF and CDF that can be easily computed and that outperform previous ones given in the literature in a wide range of conditions. The derived expressions provide an accurate approximation using a moderate number of terms even when other approaches fail (e.g., yielding CDF values larger than one).

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