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A monoidal algebraic model for rational $SO(2)$-spectra

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Abstract

The category of rational $SO(2)$–equivariant spectra admits an algebraic model. That is, there is an abelian category $\mathcal{A}(SO(2))$ whose derived category is equivalent to the homotopy category of rational $SO(2)$–equivariant spectra. An important question is: does this algebraic model capture the smash product of spectra?

The category $\mathcal{A}(SO(2))$ is known as Greenlees' standard model, it is an abelian category that has no projective objects and is constructed from modules over a non–Noetherian ring. As a consequence, the standard techniques for constructing a monoidal model structure cannot be applied. In this paper a monoidal model structure on $\mathcal{A}(SO(2))$ is constructed and the derived tensor product on the homotopy category is shown to be compatible with the smash product of spectra. The method used is related to techniques developed by the author in earlier joint work with Roitzheim. That work constructed a monoidal model structure on Franke's exotic model for the $K_{(p)}$–local stable homotopy category.

A monoidal Quillen equivalence to a simpler monoidal model category $R^\bullet$–mod that has explicit generating sets is also given. Having monoidal model structures on $\mathcal{A}(SO(2))$ and $R^\bullet$–mod removes a serious obstruction to constructing a series of monoidal Quillen equivalences between the algebraic model and rational $SO(2)$–equivariant spectra.

Contents

1   Introduction

A particularly useful technique in algebraic topology is to construct algebraic models for stable homotopy categories. The first example is that the homotopy category of rational spectra is equivalent to the category of graded rational vector spaces. More interesting examples include the work of Franke modelling the $K_{(p)}$–local stable homotopy category (see Roitzheim [18]), work of Bousfield [7] on K-local spectra and work of Greenlees and others in the case of rational G-spectra:

[9], [8], [11] or [1]. In particular the work on rational G-spectra provides classifications of rational G-equivariant cohomology theories in terms of the algebraic models.

An important (and difficult) question is whether these algebraic models capture the monoidal products. That is, does the derived smash product in the topological setting correspond to a derived tensor product coming from the algebraic model? This is true in the case of rational spectra: Shipley [19] shows that commutative $H\mathbb{Q}$-algebras are rational commutative differential graded algebras. Conversely, the author and Roitzheim [2] show that this is false in the case of Franke's exotic model, even though the algebraic model does capture the Picard group. In this paper we focus on algebraic models for rational G-spectra where the group of equivariance is $\mathbb{T} = SO(2)$.

The homotopy category of rational $\mathbb{T}$–equivariant spectra is equivalent to (the derived category of) an abelian category $\mathcal{A}(\mathbb{T})$ known as Greenlees's standard model. This algebraic model is quite straightforward, we may (very roughly) describe the objects as morphisms of R-modules $\beta \colon N \to R[S^{-1}] \otimes U$, where R is a commutative ring, U is a $\mathbb{Q}$-module and β is an isomorphism after inverting the set $S \subset R$. For full details see Definition 2.7. It is easy to construct objects in $\mathcal{A}(\mathbb{T})$ and calculate maps between them. However this category exhibits some curious behaviours: it has no projectives, limits are complicated to construct and most functors in to the category are right adjoints. In particular the obvious evaluation functors (which send an object $\beta \colon N \to R[S^{-1}] \otimes U$ of $\mathcal{A}(\mathbb{T})$ to either the R-module N or the $\mathbb{Q}$-module U) are left adjoints, as is discussed at the end of Section 4. Furthermore, the ring R is not Noetherian and the condition that β be an isomorphism after inverting S makes it hard to relate $\mathcal{A}(\mathbb{T})$ to the category of R-modules. These problems make it very difficult to construct a derived monoidal product or a monoidal model structure where the weak equivalences are the homology isomorphisms. A model structure for $\mathcal{A}(\mathbb{T})$ is given in [9]. However, it is known that this model structure cannot be monoidal (see Example 4.12), leaving the important question of monoidality open.

In this paper we apply the methods of Barnes and Roitzheim [2] to resolve this problem and give a monoidal model structure for $\mathcal{A}(\mathbb{T})$. By extensively studying the dualisable objects of the category $\mathcal{A}(\mathbb{T})$ we show that they can be used to construct a new monoidal model structure. Furthermore, the weak equivalences of this new model structure are the homology isomorphisms, see Theorems 6.2 and 6.6. This model structure is Quillen equivalent to that of Greenlees, hence we have the correct homotopy category. Furthermore, the induced derived monoidal product on the homotopy category is the correct one, in the sense that it is compatible with the short exact sequence of Theorem 4.4.

While one could try to use the flat objects to make a monoidal model structure, these are harder to identify, as the ring R is poorly behaved. We also place a monoidal model structure on a larger category $\hat{\mathcal{A}}(\mathbb{T})$, whose objects are morphisms of R-modules $\beta \colon N \to R[S^{-1}] \otimes U$, where R is a commutative ring, U is a $\mathbb{Q}$-module and there is no isomorphism condition on β. In this category there aren't enough dualisable objects, so instead we use a set of flat objects that one cannot construct in $\mathcal{A}(\mathbb{T})$, see Remark 6.7.

We end the paper by producing a symmetric monoidal Quillen equivalence between $\mathcal{A}(\mathbb{T})$ and a related (but much larger) category $R^\bullet$–mod, see Theorem 7.7. While the weak equivalences of the model structure we put on this larger category are more complicated, we can give explicit generating sets for the model structure. It is easier still to construct objects in $R^\bullet$–mod as there are plenty of left adjoints into the category and limits are much simpler to construct. Moreover, $R^\bullet$–mod appears in the preprint of Greenlees and Shipley [11] as part of a series of Quillen equivalences between rational $\mathbb{T}$–equivariant spectra and $\mathcal{A}(\mathbb{T})$. Hence this paper fixes the primary obstruction to constructing a series of *monoidal* Quillen equivalences between rational $\mathbb{T}$–equivariant spectra and $\mathcal{A}(\mathbb{T})$. Such a series of Quillen equivalences would provide a classification of ring spectra (and modules over them) in terms of ring objects in $\mathcal{A}(\mathbb{T})$ (and modules over them). In terms of cohomology theories, this would provide a classification of rational $\mathbb{T}$–equivariant cohomology theories with a cup product. As a further application, we include a conjecture about extending the results of this paper to the case of the r-fold product of copies of $\mathbb{T}$.

Organisation In the first half we introduce the algebraic model $\mathcal{A}(\mathbb{T})$ and its properties. Section 2 has the formal definition of $\mathcal{A}(\mathbb{T})$ from [9]. In Section 3 we define limits in $\mathcal{A}(\mathbb{T})$, introduce the larger category $R^\bullet$–mod and give the adjunction between $R^\bullet$–mod and $\mathcal{A}(\mathbb{T})$. We then use this adjunction in Section 4 to show that $\mathcal{A}(\mathbb{T})$ has a closed symmetric monoidal product.

In the second half of the paper we construct the monoidal model structure. to do so, we need a class of objects which behave well with respect to the monoidal product. This class is introduced and studied in Section 5. We construct a monoidal model structure on $\mathcal{A}(\mathbb{T})$ in Section 6 and show that it is Quillen equivalent to the original model structure of Greenlees and has the correct monoidal behaviour. Finally in Section 7 we give the monoidal Quillen equivalence between $\mathcal{A}(\mathbb{T})$ and an explicitly defined model structure on $R^\bullet$–mod.

2 The model $\mathcal{A}(\mathbb{T})$

In this section we introduce Greenlees standard model $\mathcal{A}(\mathbb{T})$. This is the algebraic model for rational $\mathbb{T}$–equivariant spectra. We explain how to turn this into a differential graded category and define the injective model structure. The material of this section is taken from [9]. We begin with the ring $\mathcal{O}_\mathcal{F}$ and the set of "Euler classes". The category $\mathcal{A}(\mathbb{T})$ will be built from these constructions.

Definition 2.1 *Let $\mathcal{F}$ be the set of finite subgroups of $\mathbb{T}$ (the cyclic groups C_n for $n \geqslant 1$). Let $\mathcal{O}_\mathcal{F}$ be the* graded *ring $\prod_{H \in \mathcal{F}} \mathbb{Q}[c_H]$ (a countably infinite product of polynomial rings on one generator) with c_H of degree -2.*

Define $e_H \in \mathcal{O}_\mathcal{F}$ to be the idempotent arising from projection onto factor H. In general, if ϕ is a subset of $\mathcal{F}$ we define e_ϕ to be the idempotent coming from projection onto the factors in ϕ. We let c be the unique element of $\mathcal{O}_\mathcal{F}$ which in factor H is c_H. We can then write $c_H = e_H c$.

Definition 2.2 *Let $\nu \colon \mathcal{F} \to \mathbb{Z}_{\geqslant 0}$ be a function with finite support, Let n be the maximum value of ν and partition $\mathcal{F}$ into $n+1$ sets $\mathcal{F}_0$, $\mathcal{F}_1$, $\ldots \mathcal{F}_n$, where H is in $\mathcal{F}_i$ if and only if $\nu(H) = i$. If N is an $\mathcal{O}_\mathcal{F}$–module, then we define $\mathcal{O}_\mathcal{F}$–modules*

$$\Sigma^\nu N = \oplus_{i=0}^n \Sigma^{2i} e_{\mathcal{F}_i} N \quad and \quad \Sigma^{-\nu} N = \oplus_{i=0}^n \Sigma^{-2i} e_{\mathcal{F}_i} N.$$

Furthermore we have a map of $\mathcal{O}_\mathcal{F}$–modules

$$c^\nu \colon N \to \Sigma^\nu N.$$

We define this map differently on each of the subdivisions of $\mathcal{F}$. On part $\mathcal{F}_i$ we use the map $c^i \colon e_{\mathcal{F}_i} N \to \Sigma^{2i} e_{\mathcal{F}_i} N$.

For the sake of expediency, we shall sometimes pretend that c^ν is an element of $\mathcal{O}_\mathcal{F}$. Strictly speaking, this is false, as it does not have the same degree in each factor $\mathbb{Q}[c_H]$.

Definition 2.3 *For a function $\nu \colon \mathcal{F} \to \mathbb{Z}_{\geqslant 0}$ with finite support, define $c^\nu \in \mathcal{O}_\mathcal{F}$ to be the (inhomogeneous) element satisfying $e_H c^\nu = c_H^{\nu(H)}$. We define*

$$\mathcal{E} = \{c^\nu \mid \nu \colon \mathcal{F} \to \mathbb{Z}_{\geqslant 0} \ with \ finite \ support\} \subset \mathcal{O}_\mathcal{F}$$

*and call the elements of $\mathcal{E}$ **Euler classes**.*

Example 2.4 *The standard example of an element of $\mathcal{E}$ is given by the dimension function of a complex representation V of $\mathbb{T}$ with $V^{\mathbb{T}} = 0$. This function sends $H \in \mathcal{F}$ to to the dimension of V^H over $\mathbb{C}$. We call this element c^V. Note that $c^V c^{V'} = c^{V \oplus V'}$.*

For more details on Euler classes and representations, see [9, Section 4.6].

Definition 2.5 *Define a partial ordering on the functions $\mathcal{F} \to \mathbb{Z}_{\geqslant 0}$ with finite support by $\nu \geqslant \nu'$ if $\nu(H) \geqslant \nu'(H)$ for each $H \in \mathcal{F}$.*

For N an $\mathcal{O}_{\mathcal{F}}$–module define $\mathcal{E}^{-1}N$ as the colimit of terms $\Sigma^{\nu}N$ as ν runs over the partially ordered set of functions $\mathcal{F} \to \mathbb{Z}_{\geqslant 0}$ with finite support with $\nu \leqslant \nu + \nu'$ corresponding to $c^{\nu'} : \Sigma^{\nu}N \to \Sigma^{(\nu + \nu')}N$.

$$\mathcal{E}^{-1}N = \operatorname*{colim}_{\nu} \Sigma^{\nu}N$$

It follows that $c^{\nu} : \mathcal{E}^{-1}N \to \Sigma^{\nu}\mathcal{E}^{-1}N$ is an isomorphism with inverse $c^{-\nu}$. It is easily seen that $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is a ring. To illustrate its structure, we see that as a vector space, $(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}})_{2n}$ is $\prod_{H \in \mathcal{F}} \mathbb{Q}$ for $n \leqslant 0$ and is $\oplus_{H \in \mathcal{F}} \mathbb{Q}$ for $n > 0$. We also see that there is a natural isomorphism

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} N \cong \mathcal{E}^{-1}N.$$

Definition 2.6 *We say that an $\mathcal{O}_{\mathcal{F}}$–module N has no $\mathcal{E}$–**torsion** if the map $N \to \mathcal{E}^{-1}N$ is injective.*

We note here that a flat $\mathcal{O}_{\mathcal{F}}$-module has no $\mathcal{E}$–torsion: $\mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is injective, and hence remains so after tensoring with the flat module.

Definition 2.7 *We define the category $\mathcal{A} = \mathcal{A}(\mathbb{T})$ as follows. Its class of objects is the collection of $\mathcal{O}_{\mathcal{F}}$–module maps*

$$\beta : N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$$

*with N an $\mathcal{O}_{\mathcal{F}}$–module and U a graded rational vector space, such that $\mathcal{E}^{-1}\beta$ be an isomorphism. The $\mathcal{O}_{\mathcal{F}}$–module N is called the **nub** and U is called the **vertex**.*

A map (θ, ϕ) in $\mathcal{A}$ is a commutative square as below, where θ is a map of $\mathcal{O}_{\mathcal{F}}$–modules and ϕ is a map of graded rational vector spaces.

$$
\begin{array}{ccc}
N & \xrightarrow{\ \beta\ } & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U \\
\theta \downarrow & & \downarrow \mathrm{Id} \otimes \phi \\
N' & \xrightarrow{\ \beta'\ } & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U'
\end{array}
$$

The relation between this category and rational $\mathbb{T}$–equivariant stable homotopy theory is given by the following pair of theorems from [9]. We leave the definition of rational $\mathbb{T}$–equivariant spectra to the reference.

Theorem 2.8 (Greenlees) *The category of rational $\mathbb{T}$–equivariant spectra up to homotopy is equivalent to the derived category of $\mathcal{A}$.*

For a rational $\mathbb{T}$–equivariant spectrum X, there is an object $\pi_*^{\mathcal{A}}(X)$ of $\mathcal{A}$. It is constructed in terms of rational equivariant homotopy groups. We give the definition below, but leave explanations to the reference.

$$\pi_*^{\mathcal{A}}(X) = \left(\pi_*^{\mathbb{T}}(X \wedge DE\mathcal{F}_+) \otimes \mathbb{Q} \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (\pi_*(\Phi^{\mathbb{T}}X) \otimes \mathbb{Q})\right)$$

There is also an Adams short exact sequence which explains how to calculate maps in the homotopy category of rational $\mathbb{T}$–equivariant spectra.

Theorem 2.9 (Greenlees) *Let X and Y be $\mathbb{T}$–equivariant spectra. Then the sequence below is exact.*

$$0 \to \mathrm{Ext}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(\Sigma X), \pi_*^{\mathcal{A}}(Y)) \to [X, Y]_*^{\mathbb{T}} \otimes \mathbb{Q} \to \mathrm{Hom}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \to 0$$

In [9] a model structure is given for the category of objects in $\mathcal{A}$ that have a differential. We define what it means to have a differential and then introduce the model structure. We will leave the proof that $\mathcal{A}$ has all small limits and colimits to the next section.

If we think of $\mathcal{O}_{\mathcal{F}}$ as an object of $\mathrm{Ch}(\mathbb{Q})$ with trivial differential, then we can consider the category of $\mathcal{O}_{\mathcal{F}}$–modules in $\mathrm{Ch}(\mathbb{Q})$. Such an object N is an $\mathcal{O}_{\mathcal{F}}$–module in graded vector spaces along with maps $d_n \colon N_n \to N_{n-1}$. These maps satisfy the relations below.

$$d_{n-1} \circ d_n = 0 \quad c d_n = d_{n-2} c$$

Definition 2.10 *We define the category $d\mathcal{A}$. It has objects the collection of $\mathcal{O}_{\mathcal{F}}$–module maps in $\mathrm{Ch}(\mathbb{Q})$*

$$\beta \colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$$

where N is a rational chain complex with an action of $\mathcal{O}_{\mathcal{F}}$ and U is a rational chain complex. Furthermore, we ask that $\mathcal{E}^{-1}\beta$ be an isomorphism.

A map (θ, ϕ) in $d\mathcal{A}$ is a commutative square as for $\mathcal{A}$, such that θ is a map in the category of $\mathcal{O}_{\mathcal{F}}$–modules in $\mathrm{Ch}(\mathbb{Q})$ and ϕ is a map of $\mathrm{Ch}(\mathbb{Q})$.

The following result is the subject of [9, Appendix B]. Note that a map (θ, ϕ) in $d\mathcal{A}$ is a monomorphism if and only if both θ and ϕ and injective maps.

Proposition 2.11 (Greenlees) *The category $d\mathcal{A}$ has a model structure with cofibrations the monomorphisms and weak equivalences the quasi–isomorphisms. This is called the **injective model structure**. We write $d\mathcal{A}_i$ to denote this model structure. Moreover, $\mathrm{Ho}(d\mathcal{A}_i) = \mathcal{A}$.*

As we shall see shortly, the category $\mathcal{A}$ has a monoidal product which induces a monoidal product on $d\mathcal{A}$. However the injective model structure does not make $d\mathcal{A}$ into a monoidal model category. This failure occurs because in $d\mathcal{A}$ the nubs can have $\mathcal{E}$–torsion, see Example 4.12. This is analogous to how the injective model structure on $\mathrm{Ch}(\mathbb{Z})$ is not monoidal due to torsion. This is a serious defect, as we are unable to effectively compare this monoidal product to the smash product of $\mathbb{T}$–equivariant spectra. This defect is further complicated by the lack of projective objects of $\mathcal{A}$. Our primary aim is to find a cofibrantly generated monoidal model structure on $d\mathcal{A}$ which is Quillen equivalent to the injective model structure.

3 Limits

In this section we give the proof from [9] that $\mathcal{A}$ and $d\mathcal{A}$ have all small limits and colimits (a necessary condition for model categories). Along the way we will need to relate $\mathcal{A}$ to the larger categories $\hat{\mathcal{A}}$ and $R^\bullet$–mod of Greenlees [13], we define these new categories below. For the sake of exposition, we usually only refer to categories with differentials: $d\mathcal{A}$, $d\hat{\mathcal{A}}$ and $dR^\bullet$–mod. Analogues of the results also hold for the categories without differentials.

The motivation for the definition below is that we want to consider modules over a *diagram* of (graded) rings. In our case we call the diagram $R^\bullet$, its maps are the inclusions:

$$R^\bullet = (\mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \leftarrow \mathbb{Q}).$$

Definition 3.1 *We define $R^\bullet$–mod to be the category of quintuples $(N, \alpha, M, \gamma, U)$, where N is an $\mathcal{O}_{\mathcal{F}}$–module, M is an $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$–module, U is a (graded) $\mathbb{Q}$–module and α and β are morphisms of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$–modules*

$$\alpha \colon \mathcal{E}^{-1}N \to M \qquad \gamma \colon \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} U \to M.$$

A morphism of this category is a triple of maps

$$(f, g, h) \colon (N, \alpha, M, \gamma, U) \to (N', \alpha', M', \gamma', U')$$

where $f : N \to N'$ is a morphism of $\mathcal{O}_{\mathcal{F}}$–modules, $g : M \to M'$ is a morphism of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$–modules and $h : U \to U'$ is a morphism of $\mathbb{Q}$–modules, such that the following diagram commutes.

$$
\begin{array}{ccccc}
\mathcal{E}^{-1}N & \xrightarrow{\ \alpha\ } & M & \xleftarrow{\ \gamma\ } & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U \\
\downarrow{\scriptstyle \mathcal{E}^{-1}f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 1\otimes h} \\
\mathcal{E}^{-1}N' & \xrightarrow{\ \alpha'\ } & M' & \xleftarrow{\ \gamma'\ } & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U'
\end{array}
$$

We may also define $dR^{\bullet}$–mod just as we defined $d\mathcal{A}$.

Observe that limits and colimits in $dR^{\bullet}$–mod are defined objectwise. In order to define limits in $\mathcal{A}$ we will construct them in $dR^{\bullet}$–mod and then use an adjunction to move them to $d\mathcal{A}$.

Lemma 3.2 *There is an adjunction*

$$\mathrm{inc} : d\mathcal{A} \rightleftarrows dR^{\bullet}\text{–mod} : \Gamma.$$

where the left adjoint inc *sends an object* $(\beta : N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ *of* $d\mathcal{A}$ *to the quintuple*

$$(N, \beta, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U, \mathrm{Id}, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U).$$

The functor inc *is full and faithful. The right adjoint* Γ *is called the **torsion functor** and is defined below as the composite of two functors* Γ_v *and* Γ_h*, see Definitions 3.4 and 3.9. Moreover, the unit map* $A \to \Gamma \,\mathrm{inc}\, A$ *is an isomorphism for any object of* $d\mathcal{A}$*.*

Proof See [13, Sections 7 and 8] and [9, Section 20.2]. ∎

We need a category $d\hat{\mathcal{A}}$ that is half-way between $dR^{\bullet}$–mod and $d\mathcal{A}$. We define $d\hat{\mathcal{A}}$ as $d\mathcal{A}$ without the restriction that $\mathcal{E}^{-1}\beta$ be an isomorphism. Hence $d\mathcal{A}$ is a full subcategory of $d\hat{\mathcal{A}}$. An equivalent definition of $d\hat{\mathcal{A}}$ is given below as a full subcategory of $dR^{\bullet}$–mod. We will generally write an object of $d\hat{\mathcal{A}}$ as $(\beta : N \to \mathcal{O}_{\mathcal{F}} \otimes U)$ instead of a quintuple.

Definition 3.3 *The category* $d\hat{\mathcal{A}} = d\hat{\mathcal{A}}(\mathbb{T})$ *is the full subcategory of* $dR^{\bullet}$–mod *on objects of the form*

$$(N, \beta, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U, \mathrm{Id}, U).$$

To define the functor Γ it is easiest to describe it as the composite of two functors:

$$
\begin{array}{ccccc}
 & & \xrightarrow{\quad\Gamma\quad} & & \\
d\mathcal{A} & \xleftarrow{\ \Gamma_h\ } & d\hat{\mathcal{A}} & \xleftarrow{\ \Gamma_v\ } & dR^{\bullet}\text{–mod}.
\end{array}
$$

Definition 3.4 *For an object* $A = (N, \alpha, M, \gamma, U)$ *of* $dR^{\bullet}$–mod *We define* $\Gamma_v A \in d\hat{\mathcal{A}}$ *to be the map* β *in the pullback diagram below.*

$$
\begin{array}{ccc}
P & \xrightarrow{\ \beta\ } & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U \\
\downarrow & & \downarrow{\scriptstyle \gamma} \\
N & \to \ \mathcal{E}^{-1}N \ \xrightarrow{\ \alpha\ } & M
\end{array}
$$

It is easily seen that Γ_v is the right adjoint to the inclusion of $d\hat{\mathcal{A}}$ into $dR^{\bullet}$–mod. The functor Γ_h is much more complicated to define. In order to do so, we introduce algebraic spheres and suspensions.

Definition 3.5 *If* $A = (\beta : N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ *is an element of* $d\hat{\mathcal{A}}$ *and* $\nu : \mathcal{F} \to \mathbb{Z}_{\geqslant 0}$ *is a function with finite support then we define objects of* $d\hat{\mathcal{A}}$

$$\Sigma^{\nu} A = \left((c^{-\nu} \otimes \mathrm{Id}_U) \circ \beta : \Sigma^{\nu} N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U \right)$$
$$\Sigma^{-\nu} A = \left((c^{\nu} \otimes \mathrm{Id}_U) \circ \beta : \Sigma^{-\nu} N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U \right)$$

where $c^\nu \colon \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \xrightarrow{\cong} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$. We call the functor $\Sigma^\nu \colon d\hat{\mathcal{A}} \to d\hat{\mathcal{A}}$ **suspension by the function** ν and $\Sigma^{-\nu}$ **desuspension by the function** ν. It is readily seen that if A is in $d\mathcal{A}$, then so are $\Sigma^\nu A$ and $\Sigma^{-\nu} A$.

Definition 3.6 *Define* $\mathcal{O}_{\mathcal{F}}(\nu)$ *to be the submodule of* $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ *given by*

$$\mathcal{O}_{\mathcal{F}}(\nu) = \{ x \in \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \mid c^\nu x \in \mathcal{O}_{\mathcal{F}} \}.$$

That is $\mathcal{O}_{\mathcal{F}}(\nu)$ *is generated by the (finite collection) of elements* $e_{\mathcal{F}_i} c^{-i}$, *where* $\mathcal{F}_i$ *is the set of subgroups* H *of* $\mathbb{T}$ *where* $\nu(H) = i$, *for* $i \geqslant 0$.

We define $S^\nu \in d\mathcal{A}$, *an* **algebraic sphere**, *to be the inclusion map*

$$S^\nu = (\mathcal{O}_{\mathcal{F}}(\nu) \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}})$$

equipped with trivial differential. We can also define $S^{-\nu} \in d\mathcal{A}$. *The nub* $\mathcal{O}_{\mathcal{F}}(-\nu)$ *is the set of those* $x \in \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ *such that* $c^{-\nu} x$ *is in* $\mathcal{O}_{\mathcal{F}}$.

Example 3.7 *If* V *is a complex representation of* $\mathbb{T}$ *with* $V^{\mathbb{T}} = 0$, *then there is a* $\mathbb{T}$–*equivariant spectrum* $\Sigma^\infty S^V$ *and* $\pi_*^{\mathcal{A}}(\Sigma^\infty S^V) = S^\nu$ *where* $\nu(H) = \dim_{\mathbb{C}}(V^H)$. *We call this a* **representation sphere**.

Lemma 3.8 *For* $\nu \colon \mathcal{F} \to \mathbb{Z}_{\geqslant 0}$ *of finite support, multiplication by* $c^{-\nu}$ *induces an isomorphism in* $d\mathcal{A}$

$$c^{-\nu} \colon \Sigma^\nu S^0 \longrightarrow S^\nu.$$

Proof This isomorphism is a more complicated version of the following simple observation. Let d have degree -2. Define $\mathbb{Q}\langle d^{-n} \rangle$ to be the $\mathbb{Q}[d]$ submodule of $\mathbb{Q}[d, d^{-1}]$ generated by d^{-n}. So as a graded $\mathbb{Q}$ module, $\mathbb{Q}\langle d^{-n} \rangle$ has a copy of $\mathbb{Q}$ in every degree $2k$ for $k \leqslant n$. Multiplication by d^{-n} is an isomorphism

$$d^{-n} \colon \Sigma^{2n}\mathbb{Q}[d] \to \mathbb{Q}\langle d^{-n} \rangle. \qquad \blacksquare$$

For A and B in $d\mathcal{A}$, we define $\mathcal{A}(A, B)_*$ to be the graded set of maps from the underlying object of A in $\mathcal{A}$ to the underlying object of B in $\mathcal{A}$. We equip this graded $\mathbb{Q}$–module with the differential induced by the convention $df_n = d_B f_n + (-1)^{n+1} f_n d_A$. By considering an object of $\mathcal{A}$ as an object of $d\mathcal{A}$ with no differential, we can extend the definition of $\mathcal{A}(A, B)_*$ to allow for the case where A is in $\mathcal{A}$.

Let $C = (N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ be an object of $d\hat{\mathcal{A}}$. We define a rational chain complex from C by

$$\mathcal{E}^{-1}N(c^0) = \operatorname*{colim}_\nu \hat{\mathcal{A}}(S^{-\nu}, C)_*.$$

where the colimit runs over the partially ordered set of functions $\mathcal{F} \to \mathbb{Z}_{\geqslant 0}$ of finite support and uses the inclusion map in $d\hat{\mathcal{A}}$: $S^{-\nu'} \to S^{-\nu}$ for $\nu \geqslant \nu'$. The differential on the graded $\mathbb{Q}$–module $\hat{\mathcal{A}}(S^{-\nu}, C)_*$ is $df_n = d_C f_n$.

Definition 3.9 *Let* $C = (N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ *be an object of* $d\hat{\mathcal{A}}$. *We define* $\Gamma_h C \in d\mathcal{A}$ *to be the left–hand vertical arrow of the following diagram. The lower horizontal map is induced by evaluation:* $c^{-\omega} x \otimes (\theta, \phi) \mapsto c^{-\omega}\theta(x)$, *where* $(\theta, \phi) \colon S^{-\nu} \to C$.

$$
\begin{array}{ccc}
N' & \longrightarrow & N \\
\downarrow & & \downarrow \\
\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathcal{E}^{-1}N(c^0) & \longrightarrow & \mathcal{E}^{-1}N
\end{array}
$$

Now that we understand Γ_h and Γ_v, we may construct limits and colimits in $d\mathcal{A}$ and $d\hat{\mathcal{A}}$. We leave it as an exercise to the interested reader to verify that these are actually constructions of colimits and limits.

Definition 3.10 *Let I be some small category and let $\{N_i \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U_i\}$ be the objects of some I–shaped diagram in $d\mathcal{A}$ (or $d\hat{\mathcal{A}}$). The colimit over I is*

$$\operatorname*{colim}_i N_i \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (\operatorname*{colim}_i U_i).$$

The limit in $d\hat{\mathcal{A}}$ is formed by first including the objects into $dR^\bullet$–mod, taking limits in this larger category and then applying Γ_v. Similarly, the limit in $d\mathcal{A}$ is formed by first including the objects into $dR^\bullet$–mod, taking limits in this larger category and then applying Γ.

To rephrase the above, the limit of the I–shaped diagram $\{N_i \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U_i\}$ in $d\hat{\mathcal{A}}$ is the map f in the following pullback square. Equally, the limit of the I–shaped diagram $\{N_i \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U_i\}$ in $d\mathcal{A}$ is $\Gamma_h f$.

$$
\begin{array}{ccc}
M & \xrightarrow{\ f\ } & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \lim(U_i) \\
\downarrow & & \downarrow \\
\lim(N_i) & \longrightarrow & \lim(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U_i)
\end{array}
$$

We reiterate that the above constructions restrict to categories without differentials, since these constructions preserve objects whose differential is zero.

4 Monoidal products

In this section we give the definitions of the monoidal product and function object from [9]. We also introduce useful adjoint pairs with the categories of $\mathbb{Q}$–modules and $\mathcal{O}_{\mathcal{F}}$–modules. Again we use the notation of categories with differentials, but the obvious analogues hold for categories without differentials.

Definition 4.1 *For $\beta\colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$ and $\beta'\colon N' \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U'$ in $d\mathcal{A}$ (or $d\hat{\mathcal{A}}$), their **tensor product** is*

$$\beta \otimes \beta'\colon N \otimes_{\mathcal{O}_{\mathcal{F}}} N' \to (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U) \otimes_{\mathcal{O}_{\mathcal{F}}} (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U') \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (U \otimes_{\mathbb{Q}} U')$$

The unit of this monoidal product is the object $S^0 = (i\colon \mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q})$. The differential is given by the usual rule: $d_{n,m} = d_n \otimes 1 + (-1)^n 1 \otimes d_m$.

Similarly the tensor product in $dR^\bullet$–mod is defined objectwise, with unit $(\mathcal{O}_{\mathcal{F}}, \mathrm{Id}, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}, \mathrm{Id}, \mathbb{Q})$.

Example 4.2 *The tensor product of two algebraic spheres S^ν and $S^{\nu'}$ is the algebraic sphere $S^{\nu+\nu'}$.*

Lemma 4.3 *The functor inc is a symmetric monoidal functor from $d\mathcal{A}$ to $R^\bullet$–mod.*

The monoidal product on $\mathcal{A}$ is a model for the smash product of rational $\mathbb{T}$–equivariant spectra, in the sense of the following result, which is [9, Theorem 24.1.2].

Theorem 4.4 (Greenlees) *For rational $\mathbb{T}$-spectra X and Y there is a short exact sequence in $\mathcal{A}$*

$$0 \to \pi_*^{\mathcal{A}}(X) \otimes \pi_*^{\mathcal{A}}(Y) \to \pi_*^{\mathcal{A}}(X \wedge Y) \to \Sigma \operatorname{Tor}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \to 0.$$

The monoidal structure on each of $R^\bullet$–mod, $\hat{\mathcal{A}}$ and $d\mathcal{A}$ is closed, that is, each category has an internal function object. We deal with $R^\bullet$–mod first, as the other two function objects are built from this.

Definition 4.5 *In $R^\bullet$–mod the internal function object is defined as*

$$F_{R^\bullet}\big((N, \alpha, M, \gamma, U), (N', \alpha', M', \gamma', U')\big) = (D, \theta, \operatorname{Hom}_{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}}(M, M'), \phi, E)$$

8

with D, E, θ and ϕ defined in the pullback diagrams below. In the following, we let i^* denote the context-appropriate forgetful functor and let $\bar{\alpha}\colon N \to i^*M$ be the adjoint to α and $\bar{\gamma}\colon U \to i^*M$. be the adjoint to γ.

$$
\begin{array}{ccc}
D & \xrightarrow{\ \bar{\theta}\ } & i^*\operatorname{Hom}_{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}}(M, M') \\
\downarrow & & \downarrow \\
& & \operatorname{Hom}_{\mathcal{O}_{\mathcal{F}}}(i^*M, i^*M') \\
\downarrow & & \downarrow{\scriptstyle \bar{\alpha}^*} \\
\operatorname{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, N') & \xrightarrow{\ \bar{\alpha}'_*\ } & \operatorname{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, i^*M')
\end{array}
\qquad
\begin{array}{ccc}
i^*\operatorname{Hom}_{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}}(M, M') & \xleftarrow{\ \bar{\phi}\ } & E \\
\downarrow & & \downarrow \\
\operatorname{Hom}_{\mathbb{Q}}(i^*M, i^*M') & & \\
\downarrow{\scriptstyle \bar{\gamma}^*} & & \downarrow \\
\operatorname{Hom}_{\mathbb{Q}}(U, i^*M') & \xleftarrow{\ \bar{\gamma}'_*\ } & \operatorname{Hom}_{\mathbb{Q}}(U, U')
\end{array}
$$

Definition 4.6 *Letting* inc' *denote the inclusion of* $\hat{\mathcal{A}}$ *into* $R^\bullet$*–mod, we define the internal function object of* $d\hat{\mathcal{A}}$ *to be:*

$$ F_{\hat{\mathcal{A}}}(A, B) = \Gamma_v F_{R^\bullet}(\mathrm{inc}'\, A, \mathrm{inc}'\, B). $$

Similarly, using the functor inc *of Lemma 3.2, we define the internal function object of* $d\mathcal{A}$ *to be:*

$$ F_{\mathcal{A}}(A, B) = \Gamma F_{R^\bullet}(\mathrm{inc}\, A, \mathrm{inc}\, B). $$

Lemma 4.7 *The categories* $dR^\bullet$*–mod,* $d\hat{\mathcal{A}}$ *and* $d\mathcal{A}$ *are all closed monoidal categories.*

Proof We leave the first two cases as an exercise and concentrate on the third. Since inc is full, faithful and monoidal,

$$ d\mathcal{A}(A \otimes B, C) \cong dR^\bullet\text{–mod}(\mathrm{inc}\, A \otimes \mathrm{inc}\, B, \mathrm{inc}\, C). $$

Hence $d\mathcal{A}(A \otimes B, C)$ is naturally isomorphic to $d\mathcal{A}(A, \Gamma F_{R^\bullet}(\mathrm{inc}\, B, \mathrm{inc}\, C))$. $\blacksquare$

One reason for the complicated form of the above definition is that we need to make sure that our structure maps have the correct form. For example, in Definition 4.5, E must be a $\mathbb{Q}$–module and ϕ be of the form

$$ \phi\colon \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes E \to \operatorname{Hom}_{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}}(N, N'). $$

Using the extra restrictions on objects of $d\hat{\mathcal{A}}$ and $d\mathcal{A}$ we can give a more direct construction of $F_{\mathcal{A}}(-, -)$ and $F_{\hat{\mathcal{A}}}(-, -)$. In particular, the pullback squares below are essentially the 'adjoints' of those above. We leave it to the reader to verify that the constructions in the following example agree with the definitions above.

Example 4.8 *Consider two elements of* $d\hat{\mathcal{A}}$,

$$ A = (\beta\colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U) \quad and \quad B = (\beta'\colon N' \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U'). $$

The **function object** $F_{\hat{\mathcal{A}}}(A, B) \in d\hat{\mathcal{A}}$ *is the map* δ, *as defined by the pullback square below.*

$$
\begin{array}{ccc}
Q & \xrightarrow{\ \delta\ } & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \operatorname{Hom}_{\mathbb{Q}}(U, U') \\
\downarrow & & \downarrow \\
& & \operatorname{Hom}_{\mathcal{O}_{\mathcal{F}}}(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U') \\
\downarrow & & \downarrow{\scriptstyle \beta^*} \\
\operatorname{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, N') & \xrightarrow[\ \beta'_*\ ]{} & \operatorname{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U')
\end{array}
$$

Now assume that A *and* B *are in* $d\mathcal{A}$. *Then we may construct the map* δ *as above and we see that* $F_{\mathcal{A}}(A, B) \in d\mathcal{A}$ *is* $\Gamma_h \delta$.

For the sake of notation we often just write F for any of these internal function objects. We can use the monoidal product and internal function object to show that $d\mathcal{A}$ is enriched, tensored and cotensored over $\mathrm{Ch}(\mathbb{Q})$.

Definition 4.9 *For $K \in \mathrm{Ch}(\mathbb{Q})$ we define $LK \in d\mathcal{A}$ as*

$$LK = (i \otimes \mathrm{Id}_K : \mathcal{O}_{\mathcal{F}} \otimes K \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes K)$$

For A and B in $d\mathcal{A}$, we define $\mathcal{A}(A,B)_*$ to be the graded set of maps of $\mathcal{A}$ (ignoring the differential). We then equip this graded $\mathbb{Q}$–module with the differential induced by the convention $df_n = d_B f_n + (-1)^{n+1} f_n d_A$. This construction gives a functor as below.

$$R : d\mathcal{A} \to \mathrm{Ch}(\mathbb{Q}) \qquad RA := \mathcal{A}(S^0, A)_*$$

The functors L and R form an adjoint pair between $\mathrm{Ch}(\mathbb{Q})$ and $d\mathcal{A}$. Furthermore, they give $d\mathcal{A}$ the structure of a closed $\mathrm{Ch}(\mathbb{Q})$–module in the sense of Hovey [15, Section 4.1].

This module structure and the closed monoidal product interact to give $d\mathcal{A}$ a tensor product, a cotensor product and an enrichment over $\mathrm{Ch}(\mathbb{Q})$. Let $K \in \mathrm{Ch}(\mathbb{Q})$ and $A = (\beta : N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ in $d\mathcal{A}$. Their **tensor product** $A \otimes K$ is defined to be $A \otimes LK$. Thus $A \otimes K$ is given by

$$\beta \otimes \mathrm{Id}_K : N \otimes_{\mathbb{Q}} K \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (U \otimes_{\mathbb{Q}} K).$$

The **cotensor product** A^K is defined as $F(LK, A)$. The **enrichment** is given by $RF(A,B)$ for A and B in $d\mathcal{A}$. The enrichment, tensor and cotensor are related by the natural isomorphisms below.

$$d\mathcal{A}(A, B^K) \cong d\mathcal{A}(A \otimes K, B) \cong \mathrm{Ch}(\mathbb{Q})(K, RF(A,B))$$

We also need to relate $d\mathcal{A}$ to the category of $d\mathcal{O}_{\mathcal{F}}$–modules. In particular, the following construction will be essential to Proposition 5.7.

Definition 4.10 *There is an adjunction*

$$g_* : d\mathcal{A} \rightleftarrows d\mathcal{O}_{\mathcal{F}}\text{–mod} : g^*$$

The left adjoint g_* sends an object of $d\mathcal{A}$ to its nub. For N in $d\mathcal{O}_{\mathcal{F}}$–mod we define the right adjoint by $g^*N = \Gamma(N, 0, 0, 0, 0) = \Gamma_h(N \to 0)$, where $(N, 0, 0, 0, 0) \in dR^\bullet$–mod and $(N \to 0) \in d\hat{\mathcal{A}}$.

More specifically, we may also describe the object $g^*N \in d\mathcal{A}$ as the left hand vertical in the pullback diagram below, where $\mathcal{E}^{-1}N$ is considered an object of $\mathrm{Ch}(\mathbb{Q})$.

$$
\begin{array}{ccc}
P & \longrightarrow & N \\
\downarrow & & \downarrow \\
\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} \mathcal{E}^{-1}N & \longrightarrow & \mathcal{E}^{-1}N
\end{array}
$$

Lemma 4.11 *The functor g^* from the category of $\mathcal{O}_{\mathcal{F}}$–modules to $\mathcal{A}$ is exact and commutes with filtered colimits.*

Proof The functor sending an $\mathcal{O}_{\mathcal{F}}$–module N to $N \to 0$ in $\hat{\mathcal{A}}$ is clearly exact. By [9, Proposition 20.3.4], we see that the right derived functors of Γ_h are all zero on such an object. Hence Γ_h is exact and g^* is also exact. Since g^* is defined in terms of a forgetful functors, tensor products and a pullback, it must commute with filtered colimits. ∎

Notice that the evaluation functor which sends an object of $d\mathcal{A}$ to its vertex is also a *left* adjoint. The right adjoint to evaluation at the vertex is the functor which sends $V \in \mathrm{Ch}(\mathbb{Q})$ to the object $\mathrm{Id} : \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$.

Another curious feature about the category $d\mathcal{A}$ is that the nub of every object is a **Hausdorff module**, that is the natural map of $\mathcal{O}_{\mathcal{F}}$-modules

$$N \to \prod_H e_H N$$

is injective, see [9, Section 5.10]. Hence the cokernel of the natural map $\oplus_H \mathbb{Q}[c_H] \to \mathcal{O}_{\mathcal{F}}$ cannot occur as the nub of an element of $\mathcal{A}$.

In general it is hard to construct left adjoints in to $d\mathcal{A}$, as the nubs must be Hausdorff and the structure map must be an isomorphism after inverting β. Conversely, right adjoints into $d\mathcal{A}$ are easier to construct as we can land in the simpler categories $d\hat{\mathcal{A}}$ and $dR^\bullet$–mod and then apply Γ_h or Γ. The reader should compare this behaviour with $dR^\bullet$–mod, where the evaluation functors are all *right* adjoints. That is, the functor which sends $X = (M, \alpha, N, \gamma, U)$ to $M \in d\mathcal{O}_{\mathcal{F}}$–mod is a right adjoint, as is the functor which sends X to $N \in d\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$–mod or the functor which sends X to $U \in \text{Ch}(\mathbb{Q})$.

We finish with the promised example that shows the injective model structure on $\mathcal{A}$ is not monoidal.

Example 4.12 *Let $N = \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}/\mathcal{O}_{\mathcal{F}}$, so that $\mathcal{E}^{-1}N = 0$. Let g be the map $(0 \to 0) \to (N \to 0)$ in $d\mathcal{A}$, where we equip N with the trivial differential. Let f be the inclusion*

$$(\mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}) \longrightarrow (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q})$$

in $d\mathcal{A}$. Then f and g are monomorphisms, hence cofibrations in the injective model structure on $d\mathcal{A}$. Their pushout product, $f \square g$ is the map $(N \to 0) \to (0 \to 0)$, which is not a monomorphism. Hence the injective model structure is not monoidal.

5 Dualisable objects of $\mathcal{A}(\mathbb{T})$

In this section we introduce the class of dualisable objects of our category $\mathcal{A}$ and characterise them as objects whose nub is finitely generated and projective, see Proposition 5.7. We then use this characterisation to show that there is only a set of isomorphism classes of dualisable objects, see Corollary 5.8. We construct an important collection of dualisable objects called the wide spheres. We will use the dualisable objects and wide spheres in the next section to construct the desired monoidal model structure on $d\mathcal{A}$. The results of this section are stated in terms of $\mathcal{A}$. They can all be extended to categories with differential.

Definition 5.1 *A object A of $\mathcal{A}$ is said to be **dualisable** if for any $B \in \mathcal{A}$ the canonical map $F(A, S^0) \otimes B \to F(A, B)$ is an isomorphism. The **functional dual** of an object B is the object $DB := F(B, S^0)$.*

We may also define dualisable objects in $\mathcal{O}_{\mathcal{F}}$–modules and (graded) $\mathbb{Q}$–modules. Recall that a graded $\mathbb{Q}$–module is dualisable if and only if it is finite dimensional. Equally an $\mathcal{O}_{\mathcal{F}}$–module is dualisable if it is finitely generated and projective (such objects are retracts of finite products of $\mathcal{O}_{\mathcal{F}}$). Dualisable objects satisfy a number of useful properties, we state some below for $\mathcal{A}$, but the obvious analogues hold for $\mathcal{O}_{\mathcal{F}}$–modules and (graded) $\mathbb{Q}$–modules.

Lemma 5.2 *Let A be a dualisable object of $\mathcal{A}$. Then DA is dualisable, $D(DA) \cong A$ and A is flat (that is, $- \otimes A$ is exact). For any B and C in $\mathcal{A}$ we have a natural isomorphism*

$$F(B, A \otimes C) \cong F(B \otimes DA, C)$$

Proof Most of these statements are proven in Lewis, May and Steinburger [17, Section III.1]. To see that dualisable implies flat, we must prove that $A \otimes -$ is an exact functor. It is always right exact and it is isomorphic to $F(DA, -)$, which is always left exact. ∎

Following Hovey [16], we make the following definition. Other sources call modules satisfying this condition small or compact.

Definition 5.3 *We say that an object X of a category $\mathcal{C}$ is **finitely presented** if the functor $\mathcal{C}(X, -)$ commutes with filtered colimits.*

It is a standard result that a module over a ring R is finitely presented if and only if it is the cokernel of a map of free R-modules of finite rank. In particular a $\mathbb{Q}$–module is finitely presented if and only if it has finite dimension.

Example 5.4 *The algebraic sphere S^ν is finitely presented. In particular, $S^0 = (\mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}})$ is finitely presented.*

The ring $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is the filtered colimit over functions $\nu \colon \mathcal{F} \to \mathbb{Z}_{\geqslant 0}$ with finite support of $\Sigma^\nu \mathcal{O}_{\mathcal{F}}$. Indeed, for any $\mathcal{O}_{\mathcal{F}}$–module M, $\mathcal{E}^{-1}M$ is the filtered colimit over ν of $\Sigma^\nu M$. Hence, if N is a finitely presented $\mathcal{O}_{\mathcal{F}}$–module, then

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, \mathcal{E}^{-1}M) \cong \mathrm{colim}_\nu \Sigma^\nu \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, M) \cong \mathcal{E}^{-1}\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, M)$$

Recall that a finitely generated projective module is finitely presented. These two facts allow us to prove the following analogue of [16, Propositions 1.3.2, 1.3.3 and 1.3.4]. The first states that if $A \in \mathcal{A}$ is nice, then $F(A, -)$ is much simpler to describe.

Proposition 5.5 *Let $A = (\beta \colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ and $B = (\beta' \colon N' \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U')$ be objects of $\mathcal{A}$ and assume that the nub of A is finitely presented and has no $\mathcal{E}$–torsion. Then $F(A, B)$ is isomorphic to*

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, N') \longrightarrow \mathcal{E}^{-1}\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, N') \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathrm{Hom}_{\mathbb{Q}}(U, U')$$

Proof Since N is finitely presented, there is a surjection from some finite sum of copies of $\mathcal{O}_{\mathcal{F}}$ to N. Hence we have a surjection from some finite sum of copies of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ to $\mathcal{E}^{-1}N$ and thus a surjection to $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$. It follows that U must be finite dimensional. Hence the diagonal map

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathrm{Hom}_{\mathbb{Q}}(U, U') \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U')$$

is an isomorphism. The target of this map is isomorphic (using β and β') to the domain of the map below, which is induced by $\alpha \colon N \to \mathcal{E}^{-1}N$.

$$\alpha^* \colon \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(\mathcal{E}^{-1}N, \mathcal{E}^{-1}N') \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, \mathcal{E}^{-1}N')$$

Since N has no $\mathcal{E}$–torsion, the above map is in fact an isomorphism. The module N is also finitely presented, so we see that the codomain of the above is naturally isomorphic to $\mathcal{E}^{-1}\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, N')$. Hence we have specified an isomorphism

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathrm{Hom}_{\mathbb{Q}}(U, U') \longrightarrow \mathcal{E}^{-1}\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, N').$$

Recall that the definition of $F(A, B)$ is given in terms of a pullback over the diagonal map and the map α^*. It follows that $F(A, B)$ is given by applying the torsion functor to the map

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, N') \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, \mathcal{E}^{-1}N') \longrightarrow \mathcal{E}^{-1}\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, N').$$

The codomain of the above map is isomorphic to $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathrm{Hom}_{\mathbb{Q}}(U, U')$, hence the torsion functor has no effect and the result is proven. $\blacksquare$

The next result relates the notion of being finitely presented in the category of $\mathcal{O}_{\mathcal{F}}$–modules to the notion of being finitely presented in $\mathcal{A}$.

Proposition 5.6 *If $F(A, -)$ preserves filtered colimits in $\mathcal{A}$, then A is finitely presented. If $A \in \mathcal{A}$ is finitely presented in $\mathcal{A}$, then its nub is finitely presented in the category of $\mathcal{O}_{\mathcal{F}}$–modules. Conversely, if the nub of A is finitely presented in the category of $\mathcal{O}_{\mathcal{F}}$–modules and has no $\mathcal{E}$–torsion, then A is finitely presented in $\mathcal{A}$.*

Proof To prove the first statement we take some filtered colimit diagram B_i in $\mathcal{A}$. Then we have isomorphisms as below.

$$\begin{array}{rclcl}
\mathrm{colim}_i\, \mathcal{A}(A, B_i) & \cong & \mathrm{colim}_i\, \mathcal{A}(S^0, F(A, B_i)) & \cong & \mathcal{A}(S^0, \mathrm{colim}_i\, F(A, B_i)) \\
& \cong & \mathcal{A}(S^0, F(A, \mathrm{colim}_i\, B_i)) & \cong & \mathcal{A}(A, \mathrm{colim}_i\, B_i)
\end{array}$$

The second statement follows from the fact that the functor g^* from $\mathcal{O}_{\mathcal{F}}$–mod to $\mathcal{A}$ commutes with filtered colimits.

For the converse, let $A = (\beta\colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ with N finitely presented as an $\mathcal{O}_{\mathcal{F}}$–module and with no $\mathcal{E}$–torsion. As with the proof of the previous result, we see that U is finite–dimensional. Let B_i be a filtered system of elements of $\mathcal{A}$ with nubs N_i and vertices U_i. We must prove that the canonical map

$$\mathrm{colim}_i F(A, B_i) \longrightarrow F(A, \mathrm{colim}_i B_i)$$

is an isomorphism in $\mathcal{A}$. This follows immediately from the description of $F(A, B_i)$ in Proposition 5.5 and the facts below.

$$\begin{array}{rcl}
\mathrm{colim}_i\, \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, N_i) & \cong & \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, \mathrm{colim}_i N_i) \\
\mathrm{colim}_i\, \mathrm{Hom}_{\mathbb{Q}}(U, U_i) & \cong & \mathrm{Hom}_{\mathbb{Q}}(U, \mathrm{colim}_i U_i)
\end{array}$$
∎

Using the previous two propositions, we may give a characterisation of the dualisable objects of $\mathcal{A}$. Note that by the proof of Proposition 5.5, the vertex of an object of $\mathcal{A}$ whose nub is finitely generated is finite–dimensional.

Proposition 5.7 *An object A of $\mathcal{A}$ is dualisable if and only if its nub is finitely generated and projective as an $\mathcal{O}_{\mathcal{F}}$–module.*

Proof Let $A = (\beta\colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ and $B = (\beta'\colon N' \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U')$. Assume that N is finitely generated and projective as an $\mathcal{O}_{\mathcal{F}}$–module. Then N is finitely presented and has no $\mathcal{E}$–torsion, hence

$$F(A, B) = (\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, N') \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathrm{Hom}_{\mathbb{Q}}(U, U'))$$

Since N and U are finitely generated and projective, it follows that they are dualisable. Hence $F(A, B)$ is isomorphic to the map below. But that is simply $DA \otimes B$, so A is dualisable.

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, \mathcal{O}_{\mathcal{F}}) \otimes_{\mathcal{O}_{\mathcal{F}}} N' \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathrm{Hom}_{\mathbb{Q}}(U, \mathbb{Q}) \otimes U'$$

For the converse, assume that A is dualisable. The functor $F(A, -)$ commutes with colimits as it is isomorphic to $DA \otimes -$. Hence the nub of A is finitely presented. Furthermore, A is flat, so the nub of A cannot have any $\mathcal{E}$–torsion. We must now prove that the nub of A is projective.

Let E be an exact sequence in $\mathcal{O}_{\mathcal{F}}$–modules. Recall the functor $g^*\colon \mathcal{O}_{\mathcal{F}}$–mod $\to \mathcal{A}$ from the previous section. We have shown that this functor is exact, so g^*E is an exact sequence in $\mathcal{A}$. The sequence $F(A, g^*E)$ is isomorphic to $DA \otimes g^*E$, hence both these sequences are exact. We also see that $F(A, g^*E)$ is isomorphic to $g^* \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, E)$ since the left adjoint to g^* is monoidal. Now we must show that $\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, E)$ is an exact sequence. This amounts to proving that if $\alpha\colon X \to Y$ is a map of $\mathcal{O}_{\mathcal{F}}$–modules, such that $g^*\alpha$ is a surjection, then α itself is a surjection. But this follows by looking at the pullback diagrams defining $g^*\alpha$ and noting that the map from the nub of g^*Y to Y is a surjection. Thus we conclude that the sequence $\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, E)$ is exact and hence N is projective. ∎

Note further that the structure map of any dualisable object is injective as the nub has no $\mathcal{E}$–torsion.

Corollary 5.8 *The collection of isomorphism classes of dualisable objects is a set.*

Proof Consider some dualisable object

$$\beta\colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$$

of $\mathcal{A}$. We know that N has no $\mathcal{E}$-torsion, so β is a monomorphism. It is simple to check that this object is isomorphic to an inclusion of $\mathcal{O}_{\mathcal{F}}$-modules:

$$\beta' : N' \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$$

For fixed U, there is only a set of such inclusions. Hence, up to isomorphism in $\mathcal{A}$, there is only a set of objects of $\mathcal{A}$ with vertex U.

Now note that if $\phi : U \to U'$ is an isomorphism of graded $\mathbb{Q}$-modules, then β' and $(\mathrm{Id} \otimes \phi) \circ \beta'$ are isomorphic objects of $\mathcal{A}$. The collection of isomorphism classes of graded $\mathbb{Q}$-modules forms a set. Hence the collection of isomorphism classes of objects of $\mathcal{A}$ forms a set. ∎

Definition 5.9 *We let $\mathcal{P}$ denote a set of representatives for the isomorphisms classes of dualisable objects.*

We need to introduce a special and useful collection of dualisable objects of $\mathcal{A}$, which are only slightly more complicated than the algebraic spheres. We will use this class in the proof of Theorem 6.2.

Definition 5.10 *A **wide sphere** is an object $S \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes T$ with T a finitely generated vector space on elements $t_1, \ldots, t_d$. The nub S is the $\mathcal{O}_{\mathcal{F}}$ submodule of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$ generated by elements $c^{a_1} \otimes t_1, \ldots, c^{a_d} \otimes t_d$ for Euler classes c^{a_i} and an element $\Sigma_{i=1}^{d} \sigma_i \otimes t_i$ of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes T$.*

The reason we study wide spheres is that they are flat and there are enough wide spheres in the sense that any object in $\mathcal{A}$ admits a surjection from a coproduct of wide spheres, see [9, Lemma 22.3.4]. It follows that they can be used to define a derived monoidal product. We reproduce the proof that there are enough wide spheres.

Proposition 5.11 *Given any $A \in \mathcal{A}$ there is a surjection from a direct sum of wide spheres to the object A.*

Proof Take an object $A = (\beta : N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$. We want to show that for any $n \in N$ or any $u \in U$ there is a wide sphere and a map to A such that n or u is in the image of this map. Since $\mathcal{E}^{-1}\beta$ is an isomorphism, it suffices to only consider elements of the nub.

So consider $n \in N$ with $\beta(n) = \Sigma_{i=1}^{d} \sigma_i \otimes u_i$. For each i there is an element $p_i \in N$ with $\beta(p_i) = c^{b_i} \otimes u_i$ for b_i a function with finite support. We may assume that we have chosen the b_i so that $\sigma_i c^{b_1 + \cdots + b_d} c^{-b_i} \in \mathcal{O}_{\mathcal{F}}$. We can always multiply the c^{b_i} by some Euler class so that this holds. Now we must find another Euler class. We know that

$$\beta\left(\Sigma_{i=1}^{d} \sigma_i c^{b_1 + \cdots + b_d} c^{-b_i} \cdot p_i\right) = \Sigma_{i=1}^{d} \sigma_i c^{b_1 + \cdots + b_d} \otimes u_i = \beta\left(c^{b_1 + \cdots + b_d} \cdot n\right)$$

Since $\mathcal{E}^{-1}\beta$ is an isomorphism, there must be some Euler class c^b such that

$$\Sigma_{i=1}^{d} c^b \sigma_i c^{b_1 + \cdots + b_d} c^{-b_i} \cdot p_i = c^b c^{b_1 + \cdots + b_d} \cdot n.$$

Now we can define our wide sphere. Let T be the subspace of U generated by the elements u_i. Let S be the $\mathcal{O}_{\mathcal{F}}$–submodule of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ generated by the elements $\Sigma_{i=1}^{d} \sigma_i \otimes u_i$ and $c^{b+b_1+\cdots+b_d} \otimes u_i$. The structure map is the inclusion and it is clearly an isomorphism after inverting $\mathcal{E}$.

We are ready to describe our desired map from this wide sphere to A. On the nub it sends $\Sigma_{i=1}^{d} \sigma_i \otimes u_i$ to n and $c^{b+b_1+\cdots+b_d} \otimes u_i$ to $c^{b+b_1+\cdots+b_d} c^{-b_i} \cdot p_i$. On the vertex it is the inclusion. It is a useful exercise to check that this defines a map in $\mathcal{A}$ from the wide sphere to A. The Euler classes c^{b_i} and c^b are needed to ensure that the non–trivial relation between $\Sigma_{i=1}^{d} \sigma_i \otimes u_i$ and the terms $c^{b+b_1+\cdots+b_d} \otimes u_i$ in the nub of the wide sphere is replicated by their images in N. ∎

Lemma 5.12 *Any wide sphere is dualisable.*

Proof The nub of a wide sphere is finitely generated by definition. The nubs are projective by [9, Lemma 23.3.3], hence the wide spheres are dualisable by Proposition 5.7. ∎

Note that any algebraic sphere S^ν is in particular a wide sphere (and is dualisable).

6 The dualisable model structure

In this section we complete our construction of a monoidal model structure on $d\mathcal{A}$, the main results are Theorems 6.2 and 6.6. The inspiration for the method used comes from Barnes and Roitzheim [2, Remark 6.11]. The key facts are that we need enough dualisable objects (there is a surjection from a coproduct of wide spheres to any object of $\mathcal{A}$) and that the collection of isomorphism classes of dualisable objects forms a set. Both of these statements have been proven in the previous section, so we are ready to construct our model structure. Our starting point is to prove a general result on the existence of model structures on $d\mathcal{A}$ whose weak equivalences are the homology isomorphisms.

Proposition 6.1 *Let I be a set of monomorphisms such that the maps with the right lifting property with respect to I are homology isomorphisms. Then there is a cofibrantly generated model structure on $d\mathcal{A}$ with weak equivalences the homology isomorphisms and I as the set of generating cofibrations.*

Proof We wish to use Smith's theorem, which appears as Beke [5, Theorem 1.7], to construct model structures on $d\mathcal{A}$. That theorem uses some technical set–theoretic terms that we need to mention, but do not want to define. They are the **solution set condition** as introduced in [5, Definition 1.5] and the notions of **locally presentable categories** and **locally accessible categories** as defined in Borceux [6, Sections 5.2 and 5.3]. Given our assumptions, we must prove that $d\mathcal{A}$ is a locally presentable category, and that the set of homology isomorphisms of $d\mathcal{A}$ satisfies the solution set condition.

By [5, Proposition 3.10] we see that $\mathcal{A}$ is locally presentable if and only if it has a set of objects G_i such that $\mathcal{A}(\oplus_i G_i, -)$ is a faithful functor from $\mathcal{A}$ to sets. Such a set exists by [9, Lemma 22.3.4] which says that there are enough wide spheres (see Proposition 5.11). We can extend this result to $d\mathcal{A}$ by taking the set to be the collection of wide spheres tensored with $D^1 \in \mathrm{Ch}(\mathbb{Q})$. Hence $d\mathcal{A}$ is locally presentable.

We need to know that the set of homology isomorphisms of $d\mathcal{A}$ satisfies the solution set condition. By [5, Propositions 1.15 and 3.13] we must prove that the homology isomorphisms are an accessible category. This follows from the following facts: the homology functor $\mathrm{H}_* : d\mathcal{A} \to \mathcal{A}$ commutes with filtered colimits, the isomorphisms of $\mathcal{A}$ are accessible (they are so in any locally presentable category) and [5, Proposition 1.18]. ∎

Now we are ready to use the dualisable objects to make our desired monoidal model structure on $\mathcal{A}(\mathbb{T})$. As is standard, we write S^{n-1} for that object of $\mathrm{Ch}(\mathbb{Q})$ which is $\mathbb{Q}$ concentrated in degree $n-1$ and D^n for the chain complex with $\mathbb{Q}$ in degrees n and $n-1$ (with the identity as the differential). We let i_n denote the inclusion map from S^{n-1} to D^n. Recall $\mathcal{P}$ from Definition 5.9, a set or representatives of the isomorphism classes of dualisable objects.

Theorem 6.2 *There is a cofibrantly generated model structure on $d\mathcal{A}$ with weak equivalences the homology isomorphisms. The generating cofibrations have the form*

$$i_n \otimes P : S^{n-1} \otimes P \longrightarrow D^n \otimes P$$

*for $P \in \mathcal{P}$ and $n \in \mathbb{Z}$. We call this model structure the **dualisable model structure** and denote it $d\mathcal{A}_{dual}$.*

Proof We have a set of generating cofibrations I and we must show that any map $f : A \to B$ with the right lifting property with respect to I is a homology isomorphism. We see that such a map must have the property that for any dualisable object P, the induced map of chain complexes

$$f_* : \mathcal{A}(P, X)_* \to \mathcal{A}(P, Y)_*$$

is a homology isomorphism. In particular f_* has the right lifting property with respect to $0 \to D^n \otimes \mathbb{Q}$ for $n \in \mathbb{Z}$. In turn, f has the right lifting property with respect to any map of the form

$$0 \to D^n \otimes P$$

for $P \in \mathcal{P}$ and $n \in \mathbb{Z}$. By Proposition 5.11, it follows that f must be surjective. It follows that the homotopy fibre Z of $f = (\theta, \phi)$ is given by

$$(\ker \theta \to \mathcal{E}^{-1} \mathcal{O}_{\mathcal{F}} \otimes \ker \phi) \in d\mathcal{A}.$$

The chain complex $\mathcal{A}(P, Z)_*$ is precisely the homotopy fibre of f_* and hence is acyclic. We must show that this implies that Z has trivial homology. It suffices to show that any cycle n in the nub of Z is also a boundary. By Proposition 5.11 there is a map α from a wide sphere P to Z, with n in its image. We adapt the proof to ensure that α is a cycle in $\mathcal{A}(P, Z)_* \simeq 0$.

Let $n \in N$ with $\beta(n) = \Sigma_{i=1}^d \sigma_i \otimes u_i$. Since n is a cycle, so is each u_i. For each i there is an element $q_i \in N$ with $\beta(q_i) = c^{h_i} \otimes u_i$. For each i, dq_i must be $\mathcal{E}$–torsion, as each u_i is a cycle. So there is some Euler class c^{r_i} such that $c^{r_i} q_i$ is a cycle. We define $p_i = c^{r_i} q_i$ and follow the rest of the proof of Proposition 5.11 to obtain a map α from a wide sphere P to Z. With these choices, α is a cycle. Since $\mathcal{A}(P, Z)_*$ has no homology, α is a boundary, hence so is n. ∎

There is also a relative projective model structure on $d\mathcal{A}$, which has the same cofibrations as the dualisable model structure, but has generating acyclic cofibrations given by $0 \to D^n P$ for $n \in \mathbb{Z}$ and $P \in \mathcal{P}$, see Definition 5.9. unfortunately the weak equivalences are not the homology isomorphisms, so we will not use this model structure. We note that one alternative approach to the above theorem would be to left Bousfield localise the relative projective model structure at the class of homology isomorphisms. The key step for that approach would be to find a set of maps S such that the S-equivalences are the homology isomorphisms. Since this is essentially done for us by Smith's theorem, the alternate approach is unlikely to be quicker.

Recall the injective model structure of [9] on $d\mathcal{A}$ which we write as $d\mathcal{A}_i$. The cofibrations are the monomorphisms and the weak equivalences are the quasi-isomorphisms.

Corollary 6.3 *The identity functor from $d\mathcal{A}_{dual}$ to $d\mathcal{A}_i$ is the left adjoint of a Quillen equivalence.*

$$\mathrm{Id} : d\mathcal{A}_{dual} \rightleftarrows d\mathcal{A}_i : \mathrm{Id}$$

Proof The generating cofibrations of $d\mathcal{A}_{dual}$ are monomorphisms and the weak equivalences are exactly the homology isomorphisms. ∎

Lemma 6.4 *There is a symmetric monoidal Quillen pair*

$$L : \mathrm{Ch}(\mathbb{Q}) \rightleftarrows d\mathcal{A}_{dual} : R$$

where $LV = S^0 \otimes V$ and $RA = \mathcal{A}(S^0, A)_*$. Thus, $d\mathcal{A}_{dual}$ is a closed $\mathrm{Ch}(\mathbb{Q})$–model category. Moreover, if we let $[-, -]_*^{\mathcal{A}}$ denote maps in the homotopy category of $d\mathcal{A}_{dual}$ and assume that A is cofibrant and B is fibrant, then we have a natural isomorphism

$$[A, B]_*^{\mathcal{A}} \cong \mathrm{H}_*(\mathcal{A}(A, B)_*).$$

Proof It is routine to check that (L, R) are a symmetric monoidal Quillen pair. The statement about maps in the homotopy category follows from the enrichment in $\mathrm{Ch}(\mathbb{Q})$. ∎

More generally, if $i : V \to V'$ is a cofibration in $\mathrm{Ch}(\mathbb{Q})$ and P is a dualisable object of $\mathcal{A}$, then $i \otimes P$ is a cofibration of $d\mathcal{A}_{dual}$. While we do not have explicit generating sets for the dualisable model structure, it is quite well–behaved, as the following two results show.

Lemma 6.5 *If (θ, ϕ) is a cofibration of $d\mathcal{A}_{dual}$, then θ (the map on the nubs) is a cofibration of the projective model structure on $d\mathcal{O}_{\mathcal{F}}$–mod. If $A \in d\mathcal{A}_{dual}$ is cofibrant, then $A \otimes -$ preserves weak equivalences. If (θ, ϕ) is a fibration in $d\mathcal{A}_{dual}$ then θ and ϕ are surjective.*

Proof We claim that if P is the nub of a dualisable object of $\mathcal{A}$, then it is cofibrant as an object of $d\mathcal{O}_{\mathcal{F}}$–mod (with the projective model structure). Consider a lifting problem comparing $0 \to P$ with an acyclic fibration f of $d\mathcal{O}_{\mathcal{F}}$–mod (that is, f is a surjection and a homology isomorphism). We know that $\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(P, -) \cong DP \otimes -$ is an exact functor, so $\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(P, f)$ is also an acyclic fibration. Thus we can solve the lifting problem and our claim is true. Let

$i_n \otimes P \colon S^{n-1} \otimes P \to D^n \otimes P$, be a generating cofibration of $d\mathcal{A}_{dual}$. Then since the nubs of the domain and codomain are cofibrant $\mathcal{O}_{\mathcal{F}}$–modules, the map induced by $i_n \otimes P$ on nubs is a cofibration of $\mathcal{O}_{\mathcal{F}}$–modules. It follows that every cofibration of $d\mathcal{A}_{dual}$ is a cofibration on the nubs.

The second statement follows from the fact that if N is a cofibrant object of $d\mathcal{O}_{\mathcal{F}}$–mod, then $- \otimes N$ preserves homology isomorphisms.

The last statement follows from the fact that for any dualisable P, the map $0 \to D^n \otimes P$ is a cofibration of $d\mathcal{A}_{dual}$ and is a homology isomorphism. Hence any fibration f has the right lifting property with respect to this map. By Proposition 5.11, it follows that the components of f must be surjective. ∎

Theorem 6.6 *The category $d\mathcal{A}$, equipped with the dualisable model structure, is a proper symmetric monoidal model category that satisfies the monoid axiom. Moreover, the monoidal product on $\mathrm{Ho}(d\mathcal{A}_{dual})$ is a model (in the sense of Theorem 4.4) for the smash product of rational $\mathbb{T}$–equivariant spectra.*

Proof Since the cofibrations are contained in the monomorphisms, left properness follows from the left properness of $\mathrm{Ch}(\mathbb{Q})$ and $\mathcal{O}_{\mathcal{F}}$–mod. For right properness take some pullback diagram of a quasi-isomorphism along a fibration in $d\mathcal{A}$:

$$ A \xrightarrow{\;\simeq\;} B \xleftarrow{\;\;f\;\;} C \;. $$

The component maps of f are surjections by Lemma 6.5. Hence the pullback of this diagram in $d\mathcal{A}$ is given by the objectwise pullback. Right properness of $d\mathcal{A}$ then follows from the right properness of $\mathrm{Ch}(\mathbb{Q})$ and $\mathcal{O}_{\mathcal{F}}$–mod.

To prove the pushout product axiom we note that the unit is cofibrant and the pushout of two cofibrations is again a cofibration by the pushout product axiom for $\mathrm{Ch}(\mathbb{Q})$. Now consider the pushout of an acyclic cofibration and a generating cofibration. It is routine to check that the domain and codomain of the pushout product both have trivial homology, hence the map is a weak equivalence.

To prove the monoid axiom, note that for any generating cofibration i and any $A \in d\mathcal{A}$ the map $i \otimes A$ is a monomorphism. It follows that for an acyclic cofibration j, $j \otimes A$ is a monomorphism. By Barwick [4, Corollary 2.7] we may assume that the domains of the generating acyclic cofibrations are cofibrant. Hence the cofibre Cj of a generating acyclic cofibration j is both cofibrant and acyclic. Since Cj is cofibrant, $Cj \otimes A$ is weakly equivalent to $Cj \otimes QA$ for QA a cofibrant replacement of A. But Cj is acyclic, so $Cj \otimes QA$ and hence $Cj \otimes A$ are acyclic. Thus any map of the form $j \otimes A$ is a monomorphism and a quasi–isomorphism. Such maps are closed under pushouts and transfinite compositions, so $d\mathcal{A}_{dual}$ satisfies the monoid axiom.

To prove the last statement, recall Greenlees's short exact sequence from Theorem 4.4. That results relates the smash product of rational $\mathbb{T}$–spectra to the tensor product in $\mathcal{A}$. The ad-hoc construction of Tor (and hence the derived monoidal product) in the reference is defined using the wide spheres. Since the wide spheres are dualisable, the construction of the derived monoidal product of $\mathrm{Ho}(d\mathcal{A}_{dual})$ agrees with that of the reference. ∎

Thus we have completed our task of finding a monoidal model structure on $d\mathcal{A}$ which is Quillen equivalent to the injective model structure of [9] and has the correct monoidal product.

We can also use the above results to put a monoidal model structure on $d\hat{\mathcal{A}}$. This is in fact quite instructive, as we seem to need a larger class than the dualisable objects in this case. Indeed, we actually construct a model structure using flat objects of $d\hat{\mathcal{A}}$ which are not dualisable. This shows that even in quite reasonable categories it is not always possible to construct a dualisable model structure. See [2, Remark 6.11] for a discussion of when a dualisable model structure is likely to exist.

Remark 6.7 *There is a cofibrantly generated monoidal model structure on $d\hat{\mathcal{A}}$ whose weak equivalences are the homology isomorphisms. To prove this, we apply arguments analogous to Proposition 6.1 and Theorem 6.2. To adapt these arguments we first show that $\hat{\mathcal{A}}$ has a set of generators, so*

that $d\hat{\mathcal{A}}$ is a locally presentable category. A set of generators for $\hat{\mathcal{A}}$ is given by the set of objects of the form

$$0 \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V \quad and \quad \Sigma^k \mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$$

where is V a finite dimensional graded vector space without differential and $k \in \mathbb{Z}$. The first kind allows us to 'hit' any element of the vertex. To 'hit' an element n (of degree k) in the nub of $A = (N \to \mathcal{O}_{\mathcal{F}} \otimes U)$, let U' be the vector subspace of U generated by the u_i that occur in $\beta(n) = \sum_i \sigma_i \otimes u_i$. We define $B = (\Sigma^k \mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U')$ by sending the element $1 \in \Sigma^k \mathcal{O}_{\mathcal{F}}$ to $\sum_i \sigma_i \otimes u_i$ in $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U'$. We then have a map from A to B defined by sending 1 to n on the nubs and using the inclusion for the vertices. If we enlarge this set of generators to include the set $\mathcal{P}$ of representatives of isomorphism classes of dualisable objects of $\mathcal{A}$, then the identity is a left Quillen functor from $d\mathcal{A}_{dual}$ to this model structure on $d\hat{\mathcal{A}}$.

Note that the wide spheres are not sufficient, as there are objects of $d\hat{\mathcal{A}}$ whose structure map is not surjective (even after inverting $\mathcal{E}$). Indeed, we seem to need generators of the form $0 \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ to hit every element of the vertex (and these generators are not in $\mathcal{A}$ and are not dualisable in $\hat{\mathcal{A}}$). That is, there are not enough dualisable objects in this category. Since the nub and vertex of these generators are projective, it follows that this model structure on $d\hat{\mathcal{A}}$ is monoidal.

7 The Quillen equivalence

In this section we to construct a symmetric monoidal Quillen equivalence between $d\mathcal{A}_{dual}$ and a suitable model structure on $dR^\bullet$–mod, see Theorem 7.7. This gives a choice of algebraic model for rational $\mathbb{T}$–spectra, with both categories having some advantages, as discussed in the introduction. We note that a Quillen equivalence between $d\mathcal{A}_{dual}$ and $dR^\bullet$–mod is given by Greenlees and Shipley in [11, Propositions 16.5 and 17.8]. However our proofs take into account the monoidal structure, are more explicit and are more algebraic in nature.

Proposition 7.1 *There is a proper and cellular model structure on $dR^\bullet$–mod with cofibrations and weak equivalences defined objectwise (using the projective model structures of $d\mathcal{O}_{\mathcal{F}}$–mod and $\mathrm{Ch}(\mathbb{Q})$). If we equip $dR^\bullet$–mod with this model structure there is a Quillen pair (see Lemma 3.2)*

$$\mathrm{inc} : d\mathcal{A}_{dual} \rightleftarrows dR^\bullet\text{–mod} : \Gamma.$$

Furthermore, the model category $dR^\bullet$–mod is a symmetric monoidal model category and inc *is a monoidal functor. The fibrant objects of this model category are exactly those objects $(N, \alpha, M, \gamma, U)$ such that the adjoints of the structure maps $\bar{\alpha}: N \to i^*M$ and $\bar{\gamma}: U \to i^*M$ are surjective.*

Proof It is a standard task to check that the model structure on $dR^\bullet$–mod exists, is proper and is cellular. Full details in a much more general setting are given by Greenlees and Shipley in [12, Section 3]. The adjunction is a Quillen pair since inc preserves cofibrations and homology isomorphisms. ∎

We can give the generating sets for this model structure. For each $n \in \mathbb{Z}$, let $i_n: S^{n-1} \to D^n$ in $\mathrm{Ch}(\mathbb{Q})$ be the inclusion. We also temporarily let $R = \mathcal{O}_{\mathcal{F}}$ and $T = \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$. The generating cofibrations, denoted $I_{R^\bullet}$, are given by the following maps, for each $n \in \mathbb{Z}$.

$$
\begin{array}{rccc}
(i_n \otimes R, \mathrm{Id}, \mathrm{Id}) & : & (S^{n-1} \otimes R, i_n \otimes T, D^n \otimes T, 0, 0) & \longrightarrow & (D^n \otimes R, \mathrm{Id}, D^n \otimes T, 0, 0) \\
(\mathrm{Id}, i_n \otimes T, \mathrm{Id}) & : & (0, 0, S^{n-1} \otimes T, 0, 0) & \longrightarrow & (0, 0, D^n \otimes T, 0, 0) \\
(\mathrm{Id}, \mathrm{Id}, i_n) & : & (0, 0, D^n \otimes T, i_n \otimes T, S^{n-1}) & \longrightarrow & (0, 0, D^n \otimes T, \mathrm{Id}, D^n).
\end{array}
$$

The set of generating acyclic cofibrations, denoted $J_{R^\bullet}$, is given by the following collection of maps for $n \in \mathbb{Z}$.

$$
\begin{array}{rccc}
(0, \mathrm{Id}, \mathrm{Id}) & : & (0, 0, D^n \otimes T, 0, 0) & \longrightarrow & (D^n \otimes R, \mathrm{Id}, D^n \otimes T, 0, 0) \\
(\mathrm{Id}, 0, \mathrm{Id}) & : & (0, 0, 0, 0, 0) & \longrightarrow & (0, 0, D^n \otimes T, 0, 0) \\
(\mathrm{Id}, \mathrm{Id}, 0) & : & (0, 0, D^n \otimes T, 0, 0) & \longrightarrow & (0, 0, D^n \otimes T, \mathrm{Id}, D^n).
\end{array}
$$

Lemma 7.2 *Let $A \in d\mathcal{A}_{dual}$ and let $\widehat{f}$ be the fibrant replacement functor of $dR^{\bullet}$–mod, then the induced map $\Gamma \operatorname{inc} A \to \Gamma \widehat{f} \operatorname{inc} A$ is a quasi-isomorphism. Hence the derived functor of inc from $\operatorname{Ho}(d\mathcal{A}_{dual})$ to $\operatorname{Ho}(dR^{\bullet}$–mod$)$ is fully faithful.*

Proof Let $A = (\beta \colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$, the fibrant replacement, in $dR^{\bullet}$–mod, of $\operatorname{inc} A = (N, \mathcal{E}^{-1}\beta, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U, \operatorname{Id}, U)$ is constructed by factoring β and Id as acyclic cofibrations followed by a surjections (using the projective model structures of $d\mathcal{O}_{\mathcal{F}}$–mod and $\operatorname{Ch}(\mathbb{Q})$)

$$N \longrightarrow N' \xrightarrow{\beta'} \mathcal{O}_{\mathcal{F}} \otimes U \qquad U \xrightarrow{i} V \xrightarrow{j} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U.$$

The cofibrations induce a quasi-isomorphism $\alpha \colon \operatorname{inc} A \to \widehat{f} \operatorname{inc} A$. Moreover the map $j \in \operatorname{Ch}(\mathbb{Q})$ induces a surjective quasi-isomorphism

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V \xrightarrow{j'} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U.$$

One can check that in the pullback square defining $\Gamma_v \widehat{f} \operatorname{inc} A = (\beta'' \colon P \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ (see Definition 3.4) the lower horizontal map is given by β' and the right hand vertical map is given by j'. This implies that $P \to N'$ is a quasi-isomorphism and hence the map $\Gamma_v \alpha$ is a quasi-isomorphism. Furthermore, β'' is surjective and the structure map of $\Gamma_v \operatorname{inc} A$ (which is simply A as an object of $d\hat{\mathcal{A}}$) is surjective after inverting $\mathcal{E}$.

By [9, Proposition 20.3.4], Γ_h is exact on objects with structure maps that are surjective after inverting $\mathcal{E}$. Both objects have this property, so we have a commutative square

$$
\begin{array}{ccc}
\operatorname{H}_*(\Gamma_h \Gamma_v \operatorname{inc} A) & \xrightarrow{(\Gamma_h \Gamma_v \alpha)_*} & \operatorname{H}_*(\Gamma_h \Gamma_v \widehat{f} \operatorname{inc} A) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\Gamma_h \operatorname{H}_*(\Gamma_v \operatorname{inc} A) & \xrightarrow[\cong]{\Gamma_h (\Gamma_v \alpha)_*} & \Gamma_h \operatorname{H}_*(\Gamma_v \widehat{f} \operatorname{inc} A).
\end{array}
$$

Thus the map $\Gamma \alpha = \Gamma_h \Gamma_v \alpha \colon \Gamma \operatorname{inc} A \to \Gamma \widehat{f} \operatorname{inc} A$ is a quasi-isomorphism. For the second statement, the derived unit is the composite

$$A \xrightarrow{\cong} \Gamma \operatorname{inc} A \longrightarrow \Gamma \widehat{f} \operatorname{inc} A.$$

The first map is an isomorphism and the second a quasi-isomorphism, so the derived unit is an isomorphism on homotopy categories. ∎

We would like to make the Quillen pair between $d\mathcal{A}$ and $dR^{\bullet}$–mod into a Quillen equivalence. We do so by right Bousfield localising $dR^{\bullet}$–mod. A comprehensive account of right Bousfield localisations can be found in Hirschhorn [14]. However we will primarily use work of the author and Roitzheim [3] since we are in a stable setting and are interested in the monoidal properties of the localisation. We first need to give a set of cells, these will determine the new weak equivalences of $dR^{\bullet}$–mod. Moreover, every object of the new homotopy category will be built from these cells via homotopy colimits [14, Theorem 5.1.5].

As we want to make $(\operatorname{inc}, \Gamma)$ into a Quillen equivalence we look for our cells in $d\mathcal{A}$. Since $\mathcal{A}$ models rational $\mathbb{T}$–spectra, we know a set of objects that detects weak equivalences in $d\mathcal{A}_{dual}$: the objects $\pi_*^{\mathcal{A}}(\mathbb{T}/H_+)$ for H a closed subgroup of $\mathbb{T}$. By [11, Lemma 13.6] we can construct each such object from S^0 using cofibre sequences and suspensions by functions with finite support. Hence we have the following definition, where we omit the functor $\operatorname{inc} \colon d\mathcal{A} \to dR^{\bullet}$–mod from our notation.

Definition 7.3 *We define the set of **cells** $\{S^{\nu}\}$ to be the set of all shifts of algebraic spheres S^{ν} and $S^{-\nu}$ for $\nu \colon \mathcal{F} \to \mathbb{Z}_{\geqslant 0}$ of finite support, see Definition 3.6.*

$$\{S^{\nu}\} = \{S^{n+\nu}, S^{n-\nu} \mid n \in \mathbb{Z}, \ \nu \colon \mathcal{F} \to \mathbb{Z}_{\geqslant 0} \text{ with finite support}\} \subset dR^{\bullet}\text{–mod}$$

Note that the set $\{S^{\nu}\}$ consists of cofibrant objects of $dR^{\bullet}$–mod and that $\{S^{\nu}\}$ is closed under the tensor product. In the language of Barnes and Roitzheim [3] this set is a monoidal and stable set of cells. We give a proof that this set detects weak equivalences in $d\mathcal{A}_{dual}$ that does not require spectra. In the language of triangulated categories, the following result says that the algebraic spheres generate $\operatorname{Ho}(d\mathcal{A}_{dual})$.

Proposition 7.4 *An object A of $d\mathcal{A}_{dual}$ is weakly equivalent to 0 if and only if $[S^\nu, A]^{\mathcal{A}}_* = 0$ for all algebraic spheres S^ν.*

Proof The only if direction is immediate, so we consider the converse. Assume that we have an A with $[S^\nu, A]^{\mathcal{A}}_* = 0$ for all algebraic spheres S^ν. The algebraic spheres S^ν are dualisable in $\mathcal{A}$ and hence are cofibrant in $d\mathcal{A}_{dual}$. Thus the adjunction (L, R) of Definition 4.9 tells us that $[S^\nu, A]^{\mathcal{A}}_* = 0$ if and only if $\mathcal{A}(S^\nu, \widehat{f}A)_*$ is an acyclic chain complex, where $\widehat{f}$ denotes fibrant replacement in the dualisable model structure.

Let $\widehat{f}A = (\beta \colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$, we want to show that $\mathrm{H}_*(U) = 0$. Let u be a cycle of degree k in U. Since $\mathcal{E}^{-1}\beta$ is an isomorphism, there is an Euler class c^ν such that $1 \otimes u$ is the image of some cycle x under

$$c^{-\nu} \circ \beta \colon \Sigma^\nu N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U.$$

Define a map $\Sigma^k \mathcal{O}_{\mathcal{F}} \to \Sigma^\nu N$ by sending 1 to x and define a map $\Sigma^k \mathbb{Q} \to U$ by sending the generator to u. This gives a cycle map $\Sigma^k S^0 \to \Sigma^\nu \widehat{f}A$. Hence we have a cycle map $\Sigma^k S^{-\nu} \to \widehat{f}A$. This cycle map is a boundary as $\mathcal{A}(S^V, \widehat{f}A)_* \simeq 0$. So u is a boundary and $\mathrm{H}_*(U) = 0$.

Now we want to show that $\mathrm{H}_*(N) = 0$. Let $Z = (N, 0, 0, 0, 0)$ in $S^\bullet\text{–mod}$, which is a fibrant object. Since $\mathrm{H}_*(U) = 0$, the map inc $\widehat{f}A \to Z$ (which is the identity on the first component) is a quasi-isomorphism. There are isomorphisms (of graded groups):

$$\mathrm{H}_*(N) = [\mathcal{O}_{\mathcal{F}}, N]^{\mathcal{O}_{\mathcal{F}}}_* \cong [S^0, Z]^{R^\bullet}_* \cong [S^0, \mathrm{inc}\,\widehat{f}A]^{R^\bullet}_* \cong [S^0, A]^{\mathcal{A}}_*$$

where the last isomorphism follows from Lemma 7.2 (recall that $S^0 \in dR^\bullet\text{–mod}$ is in the image of inc). The last term is trivial by assumption, so the result follows. $\blacksquare$

We can now describe the right Bousfield localisation of $dR^\bullet\text{–mod}$ at the set of cells $\{S^\nu\}$. Recall the set $J_{R^\bullet}$ of generating acyclic cofibrations for the model structure of Proposition 7.1 (described after the proof of that result).

Theorem 7.5 *There is a stable monoidal model structure on $dR^\bullet\text{–mod}$ whose weak equivalences are those maps $f \colon A \to B$ such that*

$$[S^\nu, f]^{R^\bullet}_* \colon [S^\nu, A]^{R^\bullet}_* \longrightarrow [S^\nu, B]^{R^\bullet}_*$$

is an isomorphism (of sets of maps in the homotopy category of $dR^\bullet\text{–mod}$) for all algebraic spheres S^ν. Furthermore, this model structure is proper and cellular. The set of generating acyclic cofibrations is given by $J_{R^\bullet}$. The set of generating cofibrations is given by the union of $J_{R^\bullet}$ with the set of maps $i_n \otimes S^\nu \colon S^{n-1} \otimes S^\nu \to D^n \otimes S^\nu$ where $n \in \mathbb{Z}$ and $\nu \colon \mathcal{F} \to \mathbb{Z}_{\mathbb{Z} \geqslant 0}$ with finite support. We write $\{S^\nu\}\text{–cell–}dR^\bullet\text{–mod}$ for this model structure.

Proof See [3, Theorems 5.9 and 7.2]. $\blacksquare$

Our next task is to examine the weak equivalences of $\{S^\nu\}\text{–cell–}dR^\bullet\text{–mod}$ a bit more carefully. Given a map $g \colon X \to Y$ in $dR^\bullet\text{–mod}$ we choose $\widehat{f}g$ to be a map which makes the following square commute, where $X \to \widehat{f}X$ and $Y \to \widehat{f}Y$ are fibrant replacements of X and Y.

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\
\widehat{f}X & \xrightarrow{\ \widehat{f}g\ } & \widehat{f}Y
\end{array}
$$

Recall the functor Γ from Lemma 3.2, the torsion functor from $dR^\bullet\text{–mod}$ to $d\mathcal{A}$.

Lemma 7.6 *A map g is a weak equivalence in $\{S^\nu\}\text{–cell–}dR^\bullet\text{–mod}$ if and only if $\Gamma\widehat{f}g$ is a quasi-isomorphism in $d\mathcal{A}$. We call such maps Γ-**equivalences**.*

Proof Since each cell S^ν is in the image of the functor inc, it follows that a map g is a weak equivalence if and only if

$$[S^\nu, \Gamma\widehat{f}g]_*^{\mathcal{A}} : [S^\nu, \Gamma\widehat{f}A]_*^{\mathcal{A}} \longrightarrow [S^\nu, \Gamma\widehat{f}B]_*^{\mathcal{A}}$$

is an isomorphism (of maps in the homotopy category of $d\mathcal{A}_{dual}$). Let Z be the homotopy fibre of $\Gamma\widehat{f}g$ (which is fibrant). Then, by Proposition 7.4, Z is quasi-isomorphic to 0 if and only if g is a weak equivalence of $\{S^\nu\}$–cell–$dR^\bullet$–mod. The result follows immediately. ∎

We can now give the main theorem of this section. Note that while the algebraic spheres are the correct set of cells to use at the level of homotopy, we do not believe they would be sufficient for the purposes of Theorem 6.2.

Theorem 7.7 *There is a commutative diagram of Quillen pairs as below, with left adjoints on top.*

Furthermore, the Quillen adjunction (inc, Γ) *between* $d\mathcal{A}_{dual}$ *and* $\{S^\nu\}$–cell–$dR^\bullet$–mod *is a symmetric monoidal Quillen equivalence. Hence the derived adjunction*

$$\mathrm{Ho}(d\mathcal{A}_{dual}) \rightleftarrows \mathrm{Ho}\left(\{S^\nu\}\text{–cell–}dR^\bullet\text{–mod}\right)$$

is symmetric monoidal equivalence of symmetric monoidal categories.

Proof We first need to show that inc is a left Quillen functor from $d\mathcal{A}$ with the dualisable model structure to $\{S^\nu\}$–cell–$dR^\bullet$–mod. By [14, Theorem 3.1.6] it suffices to show that for any cofibrant $A \in d\mathcal{A}_{dual}$ and any weak-equivalence g of $\{S^\nu\}$–cell–$dR^\bullet$–mod, the map $[\mathrm{inc}\, A, g]_*^{R^\bullet}$ is an isomorphism. By adjointness, it suffices to show that $[A, \Gamma\widehat{f}g]_*^{\mathcal{A}}$ is an isomorphism. The morphism $\Gamma\widehat{f}g$ is a quasi-isomorphism of $d\mathcal{A}_{dual}$ by Lemma 7.6, hence $[\mathrm{inc}\, A, g]_*^{R^\bullet}$ is an isomorphism as desired. Furthermore, the functor inc is clearly symmetric monoidal.

To show that (inc, Γ) is a Quillen equivalence, we show that the derived unit and counit are weak equivalences. The derived unit map is a quasi-isomorphism by Lemma 7.2. For the derived counit, let C be a fibrant object of $\{S^\nu\}$–cell–$dR^\bullet$–mod, let $\widehat{c}$ denote a cofibrant replacement in $d\mathcal{A}_{dual}$ and let $\widehat{f}$ be fibrant replacement in $dR^\bullet$–mod. We prove that the derived counit map

$$\mathrm{inc}\,\widehat{c}\,\Gamma C \to \mathrm{inc}\,\Gamma C \to C$$

is a Γ–equivalence. We first note that since inc preserves quasi-isomorphisms and every quasi-isomorphism is weak equivalence of $\{S^\nu\}$–cell–$dR^\bullet$–mod, the first map of the above composite is a weak equivalence. Now we show that $\mathrm{inc}\,\Gamma C \to C$ is a Γ–equivalence. There is a commutative diagram as below.

The left vertical map is a weak equivalence by Lemma 7.2. The right hand vertical is a weak equivalence as C is already fibrant. Finally the lower horizontal map is an isomorphism as $\Gamma\,\mathrm{inc} \cong \mathrm{Id}$. Hence the derived counit map is a weak equivalence. ∎

We can also phrase the Quillen equivalence of the previous theorem in terms of an inclusion of triangulated subcategories. That is, $\mathcal{A} = \mathrm{Ho}(d\mathcal{A}_{dual})$ is the smallest full triangulated subcategory of $\mathrm{Ho}(R^\bullet$–mod) that is closed under coproducts and contains the cells $\{S^\nu\}$. This claim follows

from combining the following proposition, Proposition 7.4 and [3, Theorem 9.3] (with spectra replaced by $\mathrm{Ch}(\mathbb{Q})$). Hence we are fully justified in calling $R^\bullet$–mod a larger category than $d\mathcal{A}_{dual}$.

We say that an object A of a stable model category $\mathcal{C}$ is **homotopically compact** if given any collection of objects B_i the natural map

$$\oplus_i [A, B_i]^{\mathcal{C}}_* \longrightarrow [A, \coprod_i B_i]^{\mathcal{C}}_*$$

is an isomorphism of sets of maps in the homotopy category of $\mathcal{C}$.

Proposition 7.8 *The shifted algebraic spheres $S^{n+\nu}$ are homotopically compact in $d\mathcal{A}_{dual}$ and $\{S^\nu\}$–cell–$dR^\bullet$–mod.*

Proof Since these two model categories are Quillen equivalent, it suffices to show the cells of $\{S^\nu\}$–cell–$dR^\bullet$–mod are homotopically compact. An object $(N, \alpha, M, \gamma, U)$ of this model category is fibrant if and only if the adjoints of the structure maps are surjective. It follows that if B_i is a collection of fibrant objects of $\{S^\nu\}$–cell–$dR^\bullet$–mod then $\oplus_i B_i$ is fibrant. Hence we have an isomorphism

$$[S^V, \oplus_i B_i]^{\{S^\nu\}-\mathrm{cell}-R^\bullet}_* \longrightarrow \mathrm{H}_* \left(\mathcal{A}(S^V, \oplus_i B_i)_* \right)$$

It is easily seen that an $\mathcal{O}_{\mathcal{F}}$–module map from $\mathcal{O}_{\mathcal{F}}(V)$ into an infinite direct sum lands in some finite sum. The same is true for $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ in $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$–modules and $\mathbb{Q}$ in $\mathrm{Ch}(\mathbb{Q})$. So it follows that the natural map

$$\oplus_i \mathcal{A}(S^V, B_i)_* \longrightarrow \mathcal{A}(S^V, \oplus_i B_i)_*$$

is an isomorphism and the result follows immediately. ∎

As well as $\{S^\nu\}$–cell–$R^\bullet$–mod and $d\mathcal{A}_{dual}$ we would also like a model structure on $d\hat{\mathcal{A}}$ and a Quillen equivalence to either of these two model categories. An adaptation of our earlier work provides such a model structure. Recall the functor Γ_h from $d\hat{\mathcal{A}}$ to $d\mathcal{A}$ of Definition 3.9. Let inc'' be its left adjoint.

Theorem 7.9 *There is a model structure on $d\hat{\mathcal{A}}$ where the weak equivalences are those maps f such that $\mathrm{H}_*(\Gamma_h f)$ is an isomorphism and the generating cofibrations are the maps $\mathrm{inc}''(i_n \otimes P)$ for $P \in \mathcal{P}$ and $n \in \mathbb{Z}$. Moreover the Quillen pair $(\mathrm{inc}'', \Gamma_h)$ is a Quillen equivalence between this model structure on $d\hat{\mathcal{A}}$ and $d\mathcal{A}_{dual}$.*

Proof By Remark 6.7 the technical conditions in the proof of Proposition 6.1 hold for the category $d\hat{\mathcal{A}}$. The existence of the model structure then follows the same method as Theorem 6.2, except that the weak equivalences are those maps f such that $\mathrm{H}_*(\Gamma_h f)$ is an isomorphism. This is possible since Γ_h commutes with filtered colimits. An analogous argument to Theorem 7.7 provides the Quillen equivalence. ∎

Conjecture 7.10 *These results extend to the case of the product of r copies of $\mathbb{T}$, $r > 1$. That is, there is a monoidal model structure on $\mathcal{A}(\mathbb{T}^r)$ and a monoidal Quillen equivalence between this model category and a right Bousfield localisation of a category of modules over a diagram of rings.*

The algebraic model for $\mathbb{T}^r$–equivariant rational spectra is defined in [10]. In [11] the algebraic model is given an injective model structure where the cofibrations are the monomorphisms and the weak equivalences are the homology isomorphisms. This model structure is not monoidal for the same reason that $d\mathcal{A}_i$ is not. Thus if we want to study the monoidal properties of rational $\mathbb{T}^r$–spectra we need an analogue of the dualisable model structure.

The key steps to generalising this section to $\mathbb{T}^r$ are showing that one has the analogue of wide spheres (which form a set of categorical generators) and algebraic spheres (which generate the homotopy category). We leave this for future work since the algebraic model for rational $\mathbb{T}^r$–spectra is much more complicated to define and constructing these two collections of spheres would be a very substantial task.

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