# Group Contributions in TU-games : A class of k-lateral Shapley values

# Group Contributions in TU-games : A class of $k$-lateral Shapley values

Surajit Borkotokey* Dhrubajit Choudhury† Loyimee Gogoi‡ Rajnish Kumar§

**Abstract**

In this paper we introduce the notion of group contributions in TU-games and propose a new class of values which we call the class of $k$-lateral Shapley values. Most of the values for TU-games implicitly assume that players are independent in deciding to leave or join a coalition. However, in many real life situations players are bound by the decisions taken by their peers. This leads to the idea of group contributions where we consider the marginality of groups upto a certain size. We show that group contributions can play an important role in determining players' shares in the total resource they generate. The proposed value has the flavor of egalitarianism within group contributions. We provide two characterizations of our values.

**Keywords**: Game Theory; TU Cooperative game; the Shapley value; Group contributions; the $k$-lateral Shapley values.

**Mathematics Subject Classification** (2010) : 91A12

**JEL Classification**: C71, D63, D81

## 1 Introduction

A cooperative game with transferable utilities or simply a TU-game consists of a finite non-empty set of players $N$ and a characteristic function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. For each $S \subseteq N$, the real number $v(S)$ denotes the worth of the coalition $S$ generated by its members. The standard assumption of TU-games is that the grand coalition $N$ forms. The problem is then to find a suitable allocation rule to distribute the worth of the grand coalition among the players. A single point allocation rule is called a value. Among the values for TU-games the Shapley value [24], the equal division rule, the $\alpha$-egalitarian Shapley value[16] are some of the most popular values in the literature.

*Department of Mathematics, Dibrugarh University, Assam-786004, India; Email: surajitbor@yahoo.com
†Department of Mathematics, Dibrugarh University, Assam-786004, India; Email:dhrubajit@gmail.com
‡Department of Mathematics, Dibrugarh University, Assam-786004, India; Email: loyimeegogoi@gmail.com
§Economics Group, Queen's Management School, Belfast, UK; Email: rajnish.kumar@qub.ac.uk

In this paper, we propose a parametric class of values for TU-games: the $k$-lateral Shapley values with the parameter $k$ that ranges over the set $\{1, 2, ..., n\}$. The $k$-lateral Shapley value coincides respectively, with the Shapley value and the egalitarian Shapley value for the two extreme values of $k$, i.e., $k = 1$ and $k = n$. The approach we adopt here is based on a similar approach taken by Shapley [24] in his original paper and latter by Owen [21] and Kamijo [18], but-in contrast-it assumes the effects of the contributions of the groups of players of a maximal size given by the parameter $k$ within each coalition. The proposed notion of *group contribution* is compared to the standard individual or marginal contribution (productivity) of players. Our value provides incentives to those players who are productive in groups even if their individual productivities are less significant. Recall from [24] and also from [18, 21], that the Shapley value is the average of the marginal contributions of a player over all orderings of entrance that result into the formation of the grand coalition. Our approach, on the contrary, assumes that a whole clique of players must enter before the group contributions are built. These group contributions are then shared equally within the group. Such equal sharing within groups emphasizes solidarity within and only within the group. Therefore, our value inherits solidarity implicitly after their group contributions are computed. This is the case when the individual contributions of the players in a group cannot be distinguished as shown in the following example.

**Example 1** (Group Verses Individual Contribution)**.** Take for example, the game $(N, v)$ where $N = \{1, 2, 3, 4\}$ and $v$ is such that $v(S) = 0$ if $\{1, 2\} \not\subseteq S$, $v(1, 2) = 2$, $v(1, 2, 3) = 4$, $v(1, 2, 4) = 6$ and $v(1, 2, 3, 4) = 8$. None of the players is individually productive. The Shapley value for this game is $(3.0, 3.0, 0.6, 1.4)$ emphasizing more on the productivities of player 1 and 2. The definition of $v$ suggests that any coalition that contains 1 and 2 can only generate non-zero worths and therefore, these two players should get the most. However, the group contribution of players 3 and 4 in the grand coalition given by $v(1, 2, 3, 4) - v(1, 2) = 6$ is also significant. This implies that 3 and 4 are not individually productive but their contributions as a group is significant and therefore, they should be given due consideration while allocating their joint worth! The Shapley value is not sensitive to such finer contributions. We feel that there is a need to study these aspects under a more general framework.

The implicit idea of group contributions is, however, not new in the literature of TU-games. Grabisch [14] proposed a model where the players in a coalition interact with each other to form groups based on the similar interests. Alternatively, in TU-games with coalition structures (see for example [1, 15, 17, 21] etc.) the grand coalition is partitioned into groups or union structures. The value is then computed in two stages: first, among the groups of the coalition structure and next, among the coalition members. All these models however, assume that the coalition structure is given exogenously and therefore, the group sizes are also fixed *a priori*. On the other hand, the Equal Division rule is considered to be the most widely used allocation rule to share the joint costs or the joint surplus in smaller groups. This rule seems prevalent even when there are obvious differences in the individual contributions by the members in a group. Examples include the profit sharing of law firms where the member lawyers of the firm get equal shares of the profit irrespective of how they differ in their abilities in various dimensions. Another example is that of sharing the resources in a family. In deciding the family laws, e.g., the Hindu Undivided Family (HUF) inheritance law, the equality principle is the main underlying idea. All the siblings in an HUF, which can include up to several generations, have equal inheritance rights on the property of the family. Their rights do not depend on their individual contributions in the family wealth. An interesting example discussed in [13] is that of the sharing of the profits by the salmon fishermen in the Pacific Northwest. There are fishing groups who share the information on the whereabouts of the hunts within the group. It is a common knowledge within the group about who is good at finding the schools of Salmon, but there is no provision of side payments. Many times the groups of limited size tend to be formed amongst homogeneous agents who are similar in some attributes viz., their abilities (see e.g., [13]). In other words, there is an ordering of the agents based on, say, their productivity. Groups are formed as intervals[1] on that ordering. However, when there are complementarities among the agents, which is inherently the case in characteristic function form TU-games, such orderings cannot be made.

It follows that under the present framework, the Shapley value considers the contributions of all groups of size 1 and therefore, our value recovers the Shapley value under the special case $k = 1$. Consequently the interactions among the players responsible for generating group contributions of group size 1 can be termed as the individual interactions and the Shapley value builds on this notion of individual interactions. A $k \geq 1$ signifies the maximum allowable level of group interactions within a coalition: call it the $k$-lateral interaction. We call our value the $k$-lateral Shapley value to highlight this interaction on one hand and its generalization of the Shapley value on the other hand. Joosten [16] introduced the $\alpha$-Egalitarian Shapley value which is a convex combination of the Egalitarian value and the Shapley value determined by the convexity parameter $\alpha \in [0, 1]$. It was further characterized by van den Brink et al. [28] and Casajus and Huettner[7]. Note that when $k = n$, the $k$-lateral Shapley value is the $\frac{1}{2}$-Egalitarian Shapley value (i.e., $\alpha = \frac{1}{2}$). Thus our value takes the Shapley value on one extreme ($k = 1$) and the $\frac{1}{2}$-Egalitarian Shapley value ($k = n$) on the other extreme.

Recall that the Shapley value is characterized by efficiency, symmetry, linearity and the null player property. The departure from Shapley like values have been studied by looking at the alternatives of these axioms. This paper is in the line of papers that look at the alternative to the null player property, i.e., the player who obtains a zero payoff (see [18, 27, 23]). In [26], the null player axiom, where players with zero productivity get zero payoff is replaced by the nullifying player axiom. According to this axiom, players having the property that their inclusion in a coalition makes the coalition non-productive, get zero payoff. The nullifying player axiom leads to the characterization of the Equal division. Similarly in the

[1] By an interval we mean a subset of the agents who are consecutive in the ordering.

characterization of the solidarity value in [20], the null player axiom is replaced by the A-null player axiom where players show solidarity to the non-productive players in the game by sharing some of their marginal contributions. Alternative characterizations of the Shapley and other values that follow similar arguments can be found in [2, 6, 19, 25] etc.

In our characterizations, we consider two types of null players, we call them the $k$-coalitional null players of type I and type II or simply the $k^1$ and $k^2$-coalitional null players. Both these $k$-coalitional null players may contribute huge positive values in some groups and huge negative values on some other groups as long as they are null on an average (type I) or expectation (type II) and our value awards them zero payoffs. It follows that the axioms on these two types of $k$-null players are therefore, less extreme than both the null player and the nullifying player. Consequently, our value is less marginalistic than the Shapley value and also less egalitarian than the Equal Division.

The rest of the paper proceeds as follows. In Section 2 we present the preliminary concepts. Section 3 describes a procedure to compute the $k$-lateral Shapley value followed by its characterization using some standard axioms in Section 4. Section 5 discusses an example and finally Section 6 concludes.

## 2 Preliminaries

Let $N \subset \mathbb{N}$ be given. Recall from Section 1 that a TU game is a pair $(N, v)$ consisting of a set $N$ of players and the coalition function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. With some abuse of notations we denote the singleton sets without braces. Thus we write $S \cup i$ for $S \cup \{i\}$, $S \setminus i$ for $S \setminus \{i\}$ etc. The size (cardinality) of coalition $S$ is denoted by the corresponding lower case letter $s$. Let $\mathcal{G}(N)$ denote the class of all TU-games with player set $N$. $\mathcal{G}(N)$ forms a vector space of dimension $2^n - 1$ under the standard addition and scalar multiplication of set functions. If no ambiguity about $N$ arises, we denote the TU-game $(N, v)$ simply by $v$.

The increase or decrease in worth when player $i \in S \subseteq N$ leaves coalition $S$ is called the marginal contribution of player $i$ in the coalition $S$ which is denoted by $m_i^v(S)$ and is given by

$$m_i^v(S) = v(S) - v(S \setminus i). \tag{2.1}$$

The unanimity games $u_T : 2^N \to \mathbb{R}$ and the identity games $\tilde{u}_T : 2^N \to \mathbb{R}$, $T \subseteq N$ are respectively defined as follows.

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

$$\tilde{u}_T(S) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

Both the classes of unanimity games and identity games are bases for the linear space $\mathcal{G}(N)$.

A value on $\mathcal{G}(N)$ is a function that assigns a single payoff vector $\Phi(v) = (\Phi_i(v))_{i\in N} \in \mathbb{R}^n$ to every game $v \in \mathcal{G}(N)$. Different values have been proposed in the literature since the introduction of the Shapley value (see, e.g., [2, 5, 6, 8, 19]). Here we mention briefly about the Shapley value, the Equal Division and the $\alpha$-egalitarian Shapley value as they are closely related to our proposed value. Recall Shapley's interpretation of the Shapley value from Section 1 (also see [4]) that says that suppose the "grand coalition" $N = \{1, 2, ..., n\}$ forms in a way such that the players enter the coalition one by one. This order of entrance can be expressed by a permutation $\pi : N \to N$ of the players. Let the collection of all permutations on $N$ be denoted by $\Pi(N)$. For every $\pi \in \Pi(N)$, let $P(\pi, i) = \{j \in N | \pi(j) < \pi(i)\}$ be the set of players that enter before player $i$ in the order $\pi$. The Shapley value [24] is the solution $\Phi^{Sh} : \mathcal{G}(N) \to \mathbb{R}^n$ that assigns to every player $i$ her expected marginal contribution in $P(\pi, i) \cup i$, given that every order of entrance $\pi$ has equal probability of $\frac{1}{n!}$ to occur and is given by,

$$\Phi_i^{Sh}(v) = \frac{1}{n!} \sum_{\pi\in\Pi(N)} [v(P(\pi, i) \cup i) - v(P(\pi, i))] \tag{2.4}$$

After simplifications Eq.(2.4) becomes,

$$\Phi_i^{Sh}(v) = \sum_{S\subseteq N\,:\,i\in S} \frac{(s-1)!\,(n-s)!}{n!} [v(S) - v(S \setminus i)] \tag{2.5}$$

or

$$\Phi_i^{Sh}(v) = \sum_{S\subseteq N\setminus i} \frac{s!\,(n-s-1)!}{n!} [v(S \cup i) - v(S)], \quad \forall v \in \mathcal{G}(N). \tag{2.6}$$

The Equal division rule is a solution $\Phi^{ED} : \mathcal{G}(N) \to \mathbb{R}^n$ that distributes the worth $v(N)$ of the grand coalition equally among all players in any game, i.e.,

$$\Phi_i^{ED}(v) = \frac{v(N)}{n}, \quad \forall v \in \mathcal{G}(N). \tag{2.7}$$

For $\alpha \in [0, 1]$, the $\alpha$-egalitarian Shapley value $\Phi^{\alpha-ES}$ due to [16] is a convex combination of $\Phi^{ED}$ and $\Phi^{Sh}$ which has the following form:

$$\Phi_i^{\alpha-ES}(v) = \alpha\Phi_i^{ED}(v) + (1-\alpha)\Phi_i^{Sh}(v), \quad \forall v \in \mathcal{G}(N). \tag{2.8}$$

It follows from Eq.(2.8), that the parameter $\alpha$ in $\Phi^{\alpha-ES}$ determines the amount of solidarity that is shown among the players in sharing the wealth.

For the game $v \in \mathcal{G}(N)$, a player $i \in N$ is called a null player if for every coalition $S \subseteq N$, we have $v(S) = v(S \setminus i)$. A player $i \in N$ is called a nullifying player if $v(S) = 0$ for all coalitions $S$ such that $i \in S$. There has been a number of characterizations of the Shapley value, the Equal division rule and the $\alpha$-egalitarian Shapley value in the literature (see, e.g., [8, 9, 11, 12, 31, 32]). The following four axioms are standard to characterize the Shapley value.

**Axiom** 1. Efficiency (*Eff*): A value $\Phi : \mathcal{G}(N) \to \mathbb{R}^n$ is efficient if for each game $v \in \mathcal{G}(N)$ :

$$\sum_{i \in N} \Phi_i(v) = v(N)$$

**Axiom** 2. Null Player (*NP*): A value $\Phi : \mathcal{G}(N) \to \mathbb{R}^n$ satisfies the null player axiom if for every game $v \in \mathcal{G}(N)$ it holds that $\Phi_i(v) = 0$ for every null player $i \in N$.

**Axiom** 3. Symmetry (*Sym*): A value $\Phi : \mathcal{G}(N) \to \mathbb{R}^n$ satisfies Symmetry if for $i, j \in N$ such that $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, then $\Phi_i(v) = \Phi_j(v)$.

**Axiom** 4. Linearity (*Lin*): A value $\Phi : \mathcal{G}(N) \to \mathbb{R}^n$ is linear if for all games $u, w \in \mathcal{G}(N)$ every pair of $\alpha, \beta \in \mathbb{R}$ and every player $i \in N$:

$$\Phi_i(\alpha u + \beta w) = \alpha \Phi_i(u) + \beta \Phi_i(w).$$

Replacing the null player axiom *NP* by the axiom of nullifying player namely, the nullifying player gets zero payoff, the Equal division rule can be characterized [26]. The axiom of null player in a productive environment (*NPE*) states that for all $v \in \mathcal{G}(N)$ and $i \in N$ such that $i$ is a null player in $v$ and $v(N) \geq 0$ then $\Phi_i(v) \geq 0$. The *NPE* along with *Eff*, *Sym* and *Lin* characterize the class of $\alpha$-egalitarian Shapley values [8].

A value that satisfies *Eff*, *Sym* and *Lin* is called an *ESL* value [23]. We will use the following proposition from [23] for characterization of our $k$-lateral Shapley value at a latter stage.

**Proposition 1.** *(Proposition 2 in [23], pp 184) A value $\Phi^{ESL}$ on $\mathcal{G}(N)$ is an* ESL *value if and only if there exists a unique collection of real constants $B = \{b_s : s \in \{0, 1, 2, 3, ..., n\}\}$ with $b_0 = 0$ and $b_n = 1$ such that for every $v \in \mathcal{G}(N)$,*

$$\Phi_i^{ESL}(v) = \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} \left\{ b_{s+1} v(S \cup i) - b_s v(S) \right\} \tag{2.9}$$

*or equivalently,*

$$\Phi_i^{ESL}(v) = \Phi_i^{Sh}(Bv) \tag{2.10}$$

*where $(Bv)(S) = b_s v(S)$ for each coalition of size $s$.*

## 3 The class of $k$-lateral Shapley values

In this section we introduce a new class of values for TU Cooperative games: the class of $k$-lateral Shapley values. As mentioned in Section 1, our approach resembles with Shapley's [24] approach where the players are allowed to enter into a coalition prescribed by a particular order assuming that all possible orders of entrance have equal probabilities. Motivated by [18, 21], we compute the group contributions of the players over all orders of entrance into forming the grand coalition, and allow each member in this group to receive equal shares from their group contributions.

Let $N = \{1, 2, 3, ..., n\}$ be given. In Shapley's procedure, the marginal contributions of each player are computed immediately after she joins the other players who have entered before her. On the contrary, Owen [21] and Kamijo[17] adopted a procedure where the players join a coalition one by one following an order but their contributions are computed from the components of the fixed coalition structure. In our counting process also, the players are allowed to enter according to the same order $\pi$ one by one but we wait till they form groups of a particular size. Fix a $k$: $1 \leq k \leq n$, such that $k$ is the maximum allowable size of these groups. For a fixed order $\pi$ and a fixed ordered sequence $c \in \{(c_1, c_2, ...) : \sum c_i = n, 0 < c_i \leq k\}$ of positive integers, players join a coalition one by one by the order $\pi$ and form the ordered group of players $C_1^\pi, C_2^\pi, ...$ of sizes $c_1$, $c_2, ...$ within the coalition. The group contribution of an ordered group $C_r^\pi$ of players is $v(\cup_{j=1}^{j=r} C_j^\pi) - v(\cup_{j=1}^{j=r-1} C_j^\pi)$. Denote by $A_i^v(\cup_{j=1}^{j=r} C_j^\pi)$ the equal share of each player $i \in C_r^\pi$ from her group contributions in $C_r^\pi$. Thus, we have

$$A_i^v(\cup_{j=1}^{j=r} C_j^\pi) = \frac{v(\cup_{j=1}^{j=r} C_j^\pi) - v(\cup_{j=1}^{j=r-1} C_j^\pi)}{c_r}, \;\; \forall i \in C_r^\pi \tag{3.1}$$

Note that in particular, when $c_r = |C_r^\pi| = 1$, then we obtain the standard marginal contribution of the individual player $i$ from the coalition $\cup_{j=1}^{j=r} C_j^\pi$ given in Eq.(2.1) i.e.,

$$m_i^v(\cup_{j=1}^{j=r} C_j^\pi) = A_i^v(\cup_{j=1}^{j=r} C_j^\pi).$$

Let $\mathcal{B}(n, k)$ be the set of all finite ordered sequences $c = (c_1, c_2, ...)$ of positive integers with $\sum_{i=1} c_i = n$ and $1 \leq c_i \leq k$. Formally,

$$\mathcal{B}(n, k) = \cup_{r=1}^{n} \big\{(c_1, c_2, \cdots, c_r) \mid c_1 + c_2 + \cdots + c_r = n, \; 1 \leq c_i \leq k, \; 1 \leq i \leq r\big\} \tag{3.2}$$

Then a particular pair $(\pi, c) \in \Pi(N) \times \mathcal{B}(n, k)$ determines a unique way of entrance of the players in groups. The players following a particular way of entrance $(\pi, c)$ make an ordered sequence of pairwise disjoint groups $C^{\pi,c} = \{C_1^{\pi,c}, C_2^{\pi,c}, ..., C_m^{\pi,c}\}$ of $N$ such that $N = \cup_{r=1}^{m} C_r^{\pi,c}$ with $\max_{r=1}^{m} c_r \leq k$ where $|C_r^{\pi,c}| = c_r$. Call $C^{\pi,c} = \{C_1^{\pi,c}, C_2^{\pi,c}, ..., C_m^{\pi,c}\}$ a partition of $N$ prescribed by $(\pi, c)$ where the size of the partition $|C^{\pi,c}| = m$ (say). Define by $In(C^{\pi,c}) = \max_{j=1}^{m} c_j$ the index of a partition $C^{\pi,c}$ prescribed by the pair $(\pi, c) \in \Pi(N) \times \mathcal{B}(n, k)$.

Let $\Pi(N, k) = \Big\{C^{\pi,c} = \{C_1^{\pi,c}, C_2^{\pi,c}, ..., C_m^{\pi,c}\} \mid \pi \in \Pi(N), c \in \mathcal{B}(n, k) : In(C^{\pi,c}) \leq k\Big\}$.

It follows that for each ordered sequence of groups $C = \{C_1, C_2, ..., C_m\} \in \Pi(N,k)$, there exists a unique ordered sequence $c = (c_1, c_2, ..., c_m)$ of positive integers containing at most $n$ terms such that $\sum_{p=1}^{m} c_p = n$. Thus the members of the sequence $c$ represent the cardinalities of the groups of players within $N$. Conversely for a permutation $\pi$ on $N$ and an ordered sequence $(c_1, c_2, ..., c_m)$ of positive integers which sums upto $n$ determines a unique partition $C^{\pi,c} = \{C_1^{\pi,c}, C_2^{\pi,c}, ..., C_m^{\pi,c}\}$ of ordered groups on $N$ such that $C_i^{\pi,c} = \{\pi^{-1}(\sum_{q=1}^{i-1} c_q + 1), ..., \pi^{-1}(\sum_{q=1}^{i-1} c_q + c_i)\}$ for $1 \leq i \leq m$.

Clearly there is a bijection $\Pi(N,k) \leftrightarrow \Pi(N) \times \mathcal{B}(n,k)$ such that $C^{\pi,c} \leftrightarrow (\pi, c)$. Let $\alpha(n,k) = |\mathcal{B}(n,k)|$. Thus $\alpha(n,k)$ denotes the number of partitions with $In(C) \leq k$ that can form with $n$ players prescribed by a particular order $\pi$. This idea can be easily extended to any arbitrary coalition $S \subseteq N$ and we can define $\alpha(s,k)$ exactly in the same manner. Now, observe that $|\Pi(N,k)| = |\Pi(N)|\alpha(n,k) = n!\alpha(n,k)$. For each $C^{\pi,c} \in \Pi(N,k)$ and $i \in N$, there exists a unique $p$ such that $i \in C_p^{\pi,c}$. Define the following set.

$$P(C^{\pi,c}, i) = \{j \in N : \pi(j) < \min_{i,r \in C_p^{\pi,c}} \pi(r)\}.$$

Following Eq.(3.1), the equal share of player $i$ from her group contribution in $P(C^{\pi,c}, i) \cup C_p^{\pi,c}$ when she is in $C_p^{\pi,c} \in C^{\pi,c} \in \Pi(N,k)$ is given by,

$$A_i^v(P(C^{\pi,c}, i) \cup C_p^{\pi,c}) = \frac{1}{c_p}\left\{v(P(C^{\pi,c}, i) \cup C_p^{\pi,c}) - v(P(C^{\pi,c}, i))\right\} \tag{3.3}$$

We call $A_i^v(P(C^{\pi,c}, i) \cup C_p^{\pi,c})$ the group contribution of $i$ from $C_p^{\pi,c}$ with respect to $C^{\pi,c}$ to make it short. Now we define the $k$-lateral Shapley value as follows.

**Definition 1.** The $k$-lateral Shapley value $\Phi^k : \mathcal{G}(N) \to \mathbb{R}^n$ is a value that assigns to every player $i \in N$ her average group contributions from each member $C_p^{\pi,c}$ with respect to all the partitions $C^{\pi,c} \in \Pi(N,k)$, following all possible ways $(\pi, c)$ of entrance with the assumption that occurrence of each order of entrance has equal probability $\dfrac{1}{|\Pi(N,k)|}$. Formally we have,

$$\Phi_i^k(v) = \frac{1}{n!\alpha(n,k)} \sum_{\substack{C^{\pi,c} \in \Pi(N,k) \\ (\pi,c) \in \Pi(N) \times \mathcal{B}(n,k) \\ C_p^{\pi,c} \in C^{\pi,c} : i \in C_p^{\pi,c}}} A_i^v(P(C^{\pi,c}, i) \cup C_p^{\pi,c}) \tag{3.4}$$

**Remark 1.** Note that the contributions of the groups within a coalition described in Eq.(3.4) include the individual contributions of the player given by Eq.(2.1). This addresses the marginal prospects of $\Phi^k(v)$. Adding equal shares from the group contributions to the final payoff of a player prescribed by $\Phi^k$ gives an egalitarian flavour to the solution. Thus $\Phi^k$ brings a kind of solidarity into the model.

For our convenience, we take $\alpha(0,k) = 1$. Following standard derivations of $\alpha(s,k)$ for different combinations of the parameters $s$ and $k$ are important for the rest of the paper. The proofs of these results have been relegated to the appendix.

**Proposition 2.** For $S \subseteq N$, the quantity $\alpha(s,k)$ satisfies the following.

(a) For $s \geq k \geq 1$,

$$\alpha(s,k) = \sum_{r=1}^{s} \left\{ \tbinom{s-1}{r-1} + \sum_{i=1}^{\left[\frac{s-r}{k}\right]} (-1)^i \tbinom{r}{i} \tbinom{s-ik-1}{r-1} \right\} \tag{3.5}$$

where $[x]$ = the greatest integer less than or equal to $x$.

(b) For $k = 1$ and all $s \geq 1$, $\alpha(s,k) = 1$.

(c) For $s \leq k$, $\alpha(s,k) = 2^{s-1}$.

(d) For $s > k$, we have

$$\sum_{t=1}^{k} \alpha(s-t,k) = \alpha(s,k) \tag{3.6}$$

(e) For $s \leq k$, we have

$$\sum_{t=1}^{s} \alpha(s-t,k) = \alpha(s,k) \tag{3.7}$$

**Corollary 1.** From Eq. 3.6 and Eq. 3.7, we have

$$\sum_{t=1}^{\min\{s,k\}} \alpha(s-t,k) = \alpha(s,k) \tag{3.8}$$

**Remark 2.** For $k = 2$, from Proposition 2, we have $\alpha(0,2) = 1, \alpha(1,2) = 1, \alpha(i,2) + \alpha(i+1,2) = \alpha(i+2,2)$ for $i = 0,1,2,\cdots$. Therefore, each $\{\alpha(i,2)\}$ generates a Fibonacci sequence for $i = 0,1,2,\cdots$. This result will be helpful in computing the $k$-lateral Shapley value as can be seen at a later stage.

**Example 2** (Computational Procedure)**.** Let us take an example to illustrate the computational procedure of the $k$-lateral Shapley value described above. Take $N = \{1,2,3,4\}$ and $k = 2$. In view of Proposition 2, we have $\alpha(4,2) = 5$. Therefore there will be 5 different ordered sequences of positive integers 1 and 2 (since $k = 2$ here) for each order. They are : $c^1 = \{1,1,1,1\}$, $c^2 = \{1,2,1\}$, $c^3 = \{1,1,2\}$, $c^4 = \{2,1,1\}$ and $c^5 = \{2,2\}$. There will be $n! = 4! = 24$ orders in which the players enter the room and form groups within coalitions. Consider in particular, the order given by $\pi_1 = \{1,2,3,4\}$. Then the pair ($\pi_1$, $c^1$) determines the partition $C^{\pi_1,c^1} = \{\{1\},\{2\},\{3\},\{4\}\}$. Similarly we get the other partitions of ordered groups with respect to the pairs ($\pi_1$,$c^2$), ($\pi_1$,$c^3$), ($\pi_1$,$c^4$) and ($\pi_1$,$c^5$) as $C^{\pi_1,c^2} = \{\{1\},\{2,3\},\{4\}\}$, $C^{\pi_1,c^3} = \{\{1\},\{2\},\{3,4\}\}$, $C^{\pi_1,c^4} = \{\{1,2\},\{3\},\{4\}\}$ and $C^{\pi_1,c^5} = \{\{1,2\},\{3,4\}\}$ respectively. □

In Table 1, we identify the worths of the coalitions required for computing the group contributions with regard to each of the four partitions prescribed by order $\pi_1$.

Table 1: Coalitional worths required for the group contributions according to $\pi_1$

| | | $\emptyset$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{4\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{1,4\}$ | $\{2,3\}$ | $\{2,4\}$ | $\{3,4\}$ | $\{1,2,3\}$ | $\{1,2,4\}$ | $\{1,3,4\}$ | $\{2,3,4\}$ | $N$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\pi_1$ | $C^{\pi_1,c^1}$ | $v(\emptyset)$ | $v(1)$ | $\times$ | $\times$ | $\times$ | $v(1,2)$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $v(1,2,3)$ | $\times$ | $\times$ | $\times$ | $v(N)$ |
| | $C^{\pi_1,c^2}$ | $v(\emptyset)$ | $v(1)$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $v(1,2,3)$ | $\times$ | $\times$ | $\times$ | $v(N)$ |
| | $C^{\pi_1,c^3}$ | $v(\emptyset)$ | $v(1)$ | $\times$ | $\times$ | $\times$ | $v(1,2)$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $v(N)$ |
| | $C^{\pi_1,c^4}$ | $v(\emptyset)$ | $\times$ | $\times$ | $\times$ | $\times$ | $v(1,2)$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $v(1,2,3)$ | $\times$ | $\times$ | $\times$ | $v(N)$ |
| | $C^{\pi_1,c^5}$ | $v(\emptyset)$ | $\times$ | $\times$ | $\times$ | $\times$ | $v(1,2)$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $v(N)$ |

Table 2: Share of group contributions from $C^i_{\pi_1}$, $i \in N$.

| | | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\pi_1$ | $C^{\pi_1,c^1}$ | $v(1) - v(\emptyset)$ | $v(1,2) - v(1)$ | $v(1,2,3) - v(1,2)$ | $v(1,2,3,4) - v(1,2,3)$ |
| | $C^{\pi_1,c^2}$ | $v(1) - v(\emptyset)$ | $\frac{1}{2}[v(1,2,3) - v(1)]$ | $\frac{1}{2}[v(1,2,3) - v(1)]$ | $v(1,2,3,4) - v(1,2,3)$ |
| | $C^{\pi_1,c^3}$ | $v(1) - v(\emptyset)$ | $v(1,2) - v(1)$ | $\frac{1}{2}[v(1,2,3,4) - v(1,2)]$ | $\frac{1}{2}[v(1,2,3,4) - v(1,2)]$ |
| | $C^{\pi_1,c^4}$ | $\frac{1}{2}[v(1,2) - v(\emptyset)]$ | $\frac{1}{2}[v(1,2) - v(\emptyset)]$ | $v(1,2,3) - v(1,2)$ | $v(1,2,3,4) - v(1,2,3)$ |
| | $C^{\pi_1,c^5}$ | $\frac{1}{2}[v(1,2,3,4) - v(3,4)]$ | $\frac{1}{2}[v(1,2,3,4) - v(3,4)]$ | $\frac{1}{2}[v(1,2,3,4) - v(1,2)]$ | $\frac{1}{2}[v(1,2,3,4) - v(1,2)]$ |

Table 2 refers to the shares from each of the group contributions made by the players prescribed by the partition $\pi_1$. Shares due to other orders can be obtained in a similar way. Recall that in the computation of the Shapley value, each order $\pi$ gives one set of marginal contributions of the players when they form the grand coalition according to $\pi$. Here we have 5 $(= \alpha(n,k))$ sets of alternative group contributions. Note that this is sufficient to illustrate the counting procedure as the calculations for any other order $\pi$ simply follow from relabelling the players. Finally, we calculate the average shares of the players obtained from all their group contributions. In this example, these are the 2-lateral Shapley values.

**Remark 3.** Following Proposition 2 along with some standard rules of combinatorics, an equivalent expression of Eq.(3.4) is obtained as follows:

$$\Phi_i^k(v) = \sum_{S \subseteq N : i \in S} \sum_{\substack{T \subseteq S \,:\, i \in T \\ 1 \le t \le k}} \frac{(t-1)!(n-s)!(s-t)!\alpha(s-t,k)\alpha(n-s,k)}{n!\alpha(n,k)} \{v(S) - v(S \setminus T)\}. \tag{3.9}$$

**Remark 4.** Proposition 2, along with the counting procedure described above give us the following.

Given $T \subseteq N$ such that $t \le k$, the probability of forming a coalition $S$ such that $T \subseteq S$ is given by $\dfrac{(n-s)!(s-t)!t!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)}$. The average group contribution of player $i$ from $T$ is therefore given by $\frac{v(S) - v(S \setminus T)}{t}$. Now the expectation $E_i(v)$ of the average group contributions of $i \in N$ over all the coalitions $S$ and all $T \subseteq S$ such that $i \in T$, $1 \le t \le k$ is given by

$$E_i(v) = \sum_{S\subseteq N: i\in S} \sum_{\substack{T\subseteq S\,:\, i\in T\\ 1\le t\le k}} \frac{(n-s)!(s-t)!t!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)} \left\{\frac{v(S)-v(S\setminus T)}{t}\right\}$$
$$= \sum_{S\subseteq N: i\in S} \sum_{\substack{T\subseteq S\,:\, i\in T\\ 1\le t\le k}} \frac{(n-s)!(s-t)!(t-1)!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)} \left\{v(S)-v(S\setminus T)\right\},$$

It follows from Eq.(3.9) that $E_i(v) = \Phi_i^k(v)$. Thus $\Phi^k$ is the expectation of the average group contributions of player $i$ due to game $v$.

## 4 Characterization

In this section, we follow the standard Shapley procedure to characterize our $k$-lateral Shapley values. First we show that the value satisfies *Eff*, *Sym* and *Lin*. Recall from [18] that the difference between the Shapley value, the equal division rule, the solidarity value etc., can be pinpointed to an axiom that specifies a type of null players who get zero payoff under these values. Here we define two types of null players and accordingly define two alternative axioms on these null players, namely ($kNP_1$ and $kNP_2$). They replace the standard null player axiom *NP* of the Shapley value. We show that our value satisfies both these two null player axioms. For the converse part, i.e., to show that a value that satisfies *Eff*, *Sym*, *Lin* and $kNP_1$ or $kNP_2$ must be the $k$-lateral Shapley value, we adopt the following procedure. Due to *Lin* it is sufficient to define a basis for the class of games. Due to Symmetry, the $k$-lateral Shapley value gives equal shares to the members of the coalition on which the basis is defined and all the other players outside this coalition get zero payoffs following either of $kNP_1$ or $kNP_2$. It is then not hard to show that the $k$-lateral Shapley value is the unique value satisfying the aforementioned axioms.

**Proposition 3.** *The $k$-lateral Shapley value $\Phi^k$ with $k \geq 1$ satisfies Eff, Sym and Lin.*

*Proof.* The axioms *Sym* and *Lin* can be proved following the same procedure adopted in [24] for the Shapley value; therefore here we prove *Eff* only. We have from Eq.(3.9) the following.

$$\Phi_i^k(v) = \sum_{S\subseteq N: i\in S} \sum_{\substack{T\subseteq S\,:\, i\in T\\ 1\le t\le k}} \frac{(t-1)!(n-s)!(s-t)!\alpha(s-t,k)\alpha(n-s,k)}{n!\alpha(n,k)} \left\{v(S)-v(S\setminus T)\right\} \tag{4.1}$$

Rewrite Eq. (4.1) as follows.

$$\Phi_i^k(v) = \sum_{S\subseteq N: i\in N\setminus S} \sum_{\substack{T\subseteq N\setminus S\,:\, i\in T\\ 1\le t\le k}} \frac{(t-1)!(n-s-t)!s!\alpha(s,k)\alpha(n-s-t,k)}{n!\alpha(n,k)} \Big\{ v(S\cup T) - v(S) \Big\} \tag{4.2}$$

From Eq. (4.1), we have

$$\begin{aligned}\sum_{i=1}^{n} \Phi_i^k(v) &= \sum_{i=1}^{n} \sum_{S\subseteq N: i\in S} \sum_{\substack{T\subseteq S\,:\, i\in T\\ 1\le t\le k}} \frac{(t-1)!(n-s)!(s-t)!\alpha(s-t,k)\alpha(n-s,k)}{n!\alpha(n,k)} \Big\{ v(S) - v(S\setminus T) \Big\} \\ &= \sum_{S\subseteq N} \sum_{i\in S} \sum_{\substack{T\subseteq S\,:\, i\in T\\ 1\le t\le k}} \frac{(t-1)!(n-s)!(s-t)!\alpha(s-t,k)\alpha(n-s,k)}{n!\alpha(n,k)} \Big\{ v(S) - v(S\setminus T) \Big\}\end{aligned} \tag{4.3}$$

Now we will prove that the right side of Eq. (4.3) have only term $v(N)$. The term containing $v(N)$ in Eq. (4.3) is

$$\begin{aligned}\sum_{i\in N} \sum_{1\le t\le k} \frac{(t-1)!(n-n)!(n-t)!\alpha(n-t,k)}{n!\alpha(n,k)} \binom{n-1}{t-1} v(N) &= \sum_{i\in N} \frac{1}{n!\alpha(n,k)} \sum_{1\le t\le k} \alpha(n-t,k) v(N) \\ &= \sum_{i\in N} \frac{1}{n!\alpha(n,k)} \alpha(n,k) v(N) \\ &= v(N)\end{aligned}$$

From Eq. (4.2), we have

$$\begin{aligned}\sum_{i=1}^{n} \Phi_i^k(v) &= \sum_{i=1}^{n} \sum_{S\subsetneq N: i\in N\setminus S} \sum_{\substack{T\subseteq N\setminus S\,:\, i\in T\\ 1\le t\le k}} \frac{(t-1)!)(n-s-t)!s!\alpha(s,k)\alpha(n-s-t,k)}{n!\alpha(n,k)} \{ v(S\cup T) - v(S) \} \\ &= \sum_{S\subseteq N} \sum_{i\in N\setminus S} \sum_{\substack{T\subseteq N\setminus S\,:\, i\in T\\ 1\le t\le k}} \frac{(t-1)!)(n-s-t)!s!\alpha(s,k)\alpha(n-s-t,k)}{n!\alpha(n,k)} \{ v(S\cup T) - v(S) \}\end{aligned} \tag{4.4}$$

Adding Eq. (4.3) and Eq. (4.4), we have

$$\begin{aligned}2\sum_{i=1}^{n} \Phi_i^k(v) = \sum_{S\subseteq N} \Bigg\{ &\sum_{i\in S} \sum_{\substack{T\subseteq S\,:\, i\in T\\ 1\le t\le k}} \frac{(t-1)!(n-s)!(s-t)!\alpha(s-t,k)\alpha(n-s,k)}{n!\alpha(n,k)} \{ v(S) - v(S\setminus T) \} \\ &+ \sum_{i\in N\setminus S} \sum_{\substack{T\subseteq N\setminus S\,:\, i\in T\\ 1\le t\le k}} \frac{(t-1)!)(n-s-t)!s!\alpha(s,k)\alpha(n-s-t,k)}{n!\alpha(n,k)} \{ v(S\cup T) - v(S) \} \Bigg\}\end{aligned} \tag{4.5}$$

Suppose that $S \subsetneq N$. Then the term containing $v(S)$ in Eq.(4.5) is given by,

$$\sum_{i \in S} \sum_{\substack{T \subseteq S : i \in T \\ 1 \le t \le k}} \frac{(t-1)!(n-s)!(s-t)!\alpha(s-t,k)\alpha(n-s,k)}{n!\alpha(n,k)} v(S)$$
$$- \sum_{i \in N \setminus S} \sum_{\substack{T \subseteq N \setminus S : i \in T \\ 1 \le t \le k}} \frac{(t-1)!(n-s-t)!s!\alpha(s,k)\alpha(n-s-t,k)}{n!\alpha(n,k)} v(S)$$
$$= \sum_{i \in S} \sum_{1 \le t \le \min\{k,s\}} \frac{(t-1)!(n-s)!(s-t)!\alpha(s-t,k)\alpha(n-s,k)}{n!\alpha(n,k)} \binom{s-1}{t-1} v(S)$$
$$- \sum_{i \in N \setminus S} \sum_{1 \le t \le \min\{k,n-s\}} \frac{(t-1)!(n-s-t)!s!\alpha(s,k)\alpha(n-s-t,k)}{n!\alpha(n,k)} \binom{n-s-1}{t-1} v(S)$$
$$= \sum_{i \in S} \sum_{1 \le t \le \min\{k,s\}} \frac{(n-s)!(s-1)!\alpha(s-t,k)\alpha(n-s,k)}{n!\alpha(n,k)} v(S)$$
$$- \sum_{i \in N \setminus S} \sum_{1 \le t \le \min\{k,n-s\}} \frac{(n-s-1)!s!\alpha(s,k)\alpha(n-s-t,k)}{n!\alpha(n,k)} v(S)$$
$$= \sum_{i \in S} \frac{(n-s)!(s-1)!\alpha(n-s,k)}{n!\alpha(n,k)} v(S) \sum_{1 \le t \le \min\{k,s\}} \alpha(s-t,k)$$
$$- \sum_{i \in N \setminus S} \frac{(n-s-1)!s!\alpha(s,k)}{n!\alpha(n,k)} v(S) \sum_{1 \le t \le \min\{k,n-s\}} \alpha(n-s-t,k)$$

By proposition 2(f), $\sum_{t=1}^{\min\{k,s\}} \alpha(s-t,k) = \alpha(s,k)$. Therefore the term containing $v(S)$ in Eq. (4.5) becomes

$$\sum_{i \in S} \frac{(n-s)!(s-1)!\alpha(n-s,k)}{n!\alpha(n,k)} \alpha(s,k) v(S) - \sum_{i \in N \setminus S} \frac{(n-s-1)!s!\alpha(s,k)}{n!\alpha(n,k)} \alpha(n-s,k) v(S)$$
$$= s \frac{(n-s)!(s-1)!\alpha(n-s,k)}{n!\alpha(n,k)} \alpha(s,k) v(S) - (n-s) \frac{(n-s-1)!s!\alpha(s,k)}{n!\alpha(n,k)} \alpha(n-s,k) v(S)$$
$$= \frac{(n-s)!s!\alpha(n-s,k)}{n!\alpha(n,k)} \alpha(s,k) v(S) - (n-s) \frac{(n-s)!s!\alpha(s,k)}{n!\alpha(n,k)} \alpha(n-s,k) v(S)$$
$$= 0$$

It follows that $\sum_{i=1}^{n} \Phi_i^k(v) = v(N)$. □

In view of Proposition 3, $\Phi^k$ is an ESL value. Therefore by Proposition 1, there exists a unique collection of real constants $B = \{b_s : s \in \{0, 1, 2, 3, ..., n\}\}$ with $b_0 = 0$ and $b_n = 1$ such that for every $v \in \mathcal{G}(N)$,

$$\Phi_i^k(v) = \sum_{S \subset N \setminus i} \frac{s!(n-s-1)!}{n!} \Big\{ b_{s+1} v(S \cup i) - b_s v(S) \Big\} \tag{4.6}$$

**Proposition 4.** *The $k$-lateral Shapley value $\Phi^k$ with $k \geq 1$ is in the form Eq.(4.6) with the sequence of non negative real numbers $B = \{b_s : s \in 0, 1, 2, ..., n\}$ where $b_s = \dfrac{\alpha(n-s,k)\alpha(s,k)}{\alpha(n,k)}$ for $s \geq 1$ and $b_0 = 0$.*

*Proof.* Rearranging the terms in Eq.(3.9) we obtain

$$\begin{aligned}
\Phi_i^k(v) &= \sum_{S \subseteq N : i \in S} \sum_{\substack{T \subseteq S \,:\, i \in T \\ 1 \leq t \leq k}} \frac{(n-s)!(s-t)!(t-1)!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)} \Big\{ v(S) - v(S \setminus T) \Big\} \\
&= \sum_{S \subseteq N : i \in S} \sum_{1 \leq t \leq \min\{k,s\}} \binom{s-1}{t-1} \frac{(n-s)!(s-t)!(t-1)!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)} v(S) \\
&\quad - \sum_{S \subseteq N : i \in S} \sum_{\substack{T \subseteq S \,:\, i \in T \\ 1 \leq t \leq k}} \frac{(n-s)!(s-t)!(t-1)!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)} v(S \setminus T) \\
&= \sum_{S \subseteq N : i \in S} \sum_{1 \leq t \leq \min\{k,s\}} \binom{s-1}{t-1} \frac{(n-s)!(s-t)!(t-1)!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)} v(S) \\
&\quad - \sum_{R \subseteq N : i \in R} \sum_{\substack{T \subseteq R \,:\, i \in T \\ 1 \leq t \leq k}} \frac{(n-r)!(r-t)!(t-1)!\alpha(n-r,k)\alpha(r-t,k)}{n!\alpha(n,k)} v(R \setminus T) \qquad (4.7)
\end{aligned}$$

Let $P = T \setminus i$, $Q = (R \setminus T) \cup i$. Then $i \in Q$, $Q \setminus i = R \setminus T$. Therefore $T \subset (N \setminus Q) \cup i \implies T \setminus i \subset N \setminus Q \implies P \subset N \setminus Q$. Again

$$\begin{aligned}
T &= P \cup i \\
R &= Q \cup T \cup i = Q \cup T = Q \cup P \cup i = Q \cup P \\
Q \cap P &= \{(R \setminus T) \cup i\} \cap (T \setminus i) = \emptyset \\
N \setminus R &= N \setminus (Q \cup P)
\end{aligned}$$

Therefore $p = t-1, r = q+p, n-r = n-q-p, r-t = q-1$. We have from Eq.(4.7) and Proposition 2(f),

$$\begin{aligned}
\Phi_i^k(v) &= \sum_{S \subseteq N : i \in S} \sum_{1 \leq t \leq \min\{k,s\}} \frac{(n-s)!(s-1)!\alpha(n-s,k)\alpha(s-t,k)}{n!\alpha(n,k)} v(S) \\
&\quad - \sum_{Q \subseteq N : i \in Q} \sum_{\substack{P \subseteq N \setminus Q \\ 0 \leq p \leq \min\{k-1, n-s\}}} \frac{(n-q-p)!(q-1)!p!\alpha(n-q-p,k)\alpha(q-1,k)}{n!\alpha(n,k)} v(Q \setminus i)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{S \subseteq N : i \in S} \frac{(n-s)!(s-1)!\alpha(n-s,k)}{n!\alpha(n,k)} \sum_{1 \leq t \leq \min\{k,s\}} \alpha(s-t,k) v(S) \\
&\quad - \sum_{Q \subseteq N : i \in Q} \sum_{0 \leq p \leq \min\{k-1, n-q\}} \binom{n-q}{p} \frac{(n-q-p)!(q-1)!p!\alpha(n-q-p,k)\alpha(q-1,k)}{n!\alpha(n,k)} v(Q \setminus i)
\end{aligned}$$

$$= \sum_{S\subseteq N: i\in S} \frac{(n-s)!(s-1)!\alpha(n-s,k)\alpha(s,k)}{n!\alpha(n,k)} v(S)$$
$$- \sum_{Q\subseteq N: i\in Q} \sum_{0\leq p\leq \min\{k-1,n-q\}} \frac{(n-q)!(q-1)!\alpha(n-q-p,k)\alpha(q-1,k)}{n!\alpha(n,k)} v(Q\setminus i)$$

$$= \sum_{S\subseteq N: i\in S} \frac{(n-s)!(s-1)!\alpha(n-s,k)\alpha(s,k)}{n!\alpha(n,k)} v(S)$$
$$- \sum_{Q\subseteq N: i\in Q} \frac{(n-q)!(q-1)!\alpha(q-1,k)}{n!\alpha(n,k)} \sum_{1\leq p+1\leq \min\{k,n-q+1\}} \alpha(n-q-p,k) v(Q\setminus i)$$

By proposition 2(f), $\sum_{1\leq p+1\leq \min\{k,n-q+1\}} \alpha(n-q-p,k) = \alpha(n-q+1,k)$. Therefore

$$\Phi_i^k(v) = \sum_{S\subseteq N: i\in S} \frac{(n-s)!(s-1)!\alpha(n-s,k)\alpha(s,k)}{n!\alpha(n,k)} v(S)$$
$$- \sum_{Q\subseteq N: i\in Q} \frac{(n-q)!(q-1)!\alpha(q-1,k)\alpha(n-q+1,k)}{n!\alpha(n,k)} v(Q\setminus i)$$
$$= \sum_{S\subseteq N: i\in S} \frac{(n-s)!(s-1)!\alpha(n-s,k)\alpha(s,k)}{n!\alpha(n,k)} v(S)$$
$$- \sum_{S\subseteq N: i\in S} \frac{(n-s)!(s-1)!\alpha(s-1,k)\alpha(n-s+1,k)}{n!\alpha(n,k)} v(S\setminus i)$$
$$= \sum_{S\subseteq N: i\in S} \frac{(n-s)!(s-1)!}{n!} \left\{ \frac{\alpha(n-s,k)\alpha(s,k)}{\alpha(n,k)} v(S) - \frac{\alpha(n-s+1,k)\alpha(s-1,k)}{\alpha(n,k)} v(S\setminus i) \right\} \tag{4.8}$$

Let $b_s = \frac{\alpha(n-s,k)\alpha(s,k)}{\alpha(n,k)}$ for $s \geq 1$ and $b_0 = 0$. Then $B = \{b_s : s \in 0,1,2,...,n\}$ is a real sequence of non negative real numbers with $b_0 = 0$, $b_n = 1$ and the result follows. □

**Remark 5.** Eq.(4.8) provides an alternative representation of $\Phi^k$. For $k = n$, $\alpha(s,n) = 2^{s-1}$ where $1 \leq s \leq n$ and $\alpha(0,n) = 1$. Let $b_s = \dfrac{\alpha(n-s,k)\alpha(s,k)}{\alpha(n,k)}$ for $1 \leq s \leq n$. Therefore $b_s = \dfrac{1}{2}$ for $k = n$, $1 \leq s < n$ and $b_n = 1$. Take $b_0 = 0$. Then the $n$-lateral Shapley value becomes,

$$\Phi_i^n(v) = \sum_{S\subseteq N: i\in S} \frac{(n-s)!(s-1)!}{n!} \left\{ b_s v(S) - b_{s-1} v(S\setminus i) \right\}$$
$$= \sum_{S\subsetneq N: i\in S} \frac{(n-s)!(s-1)!}{n!} \left\{ \frac{1}{2} v(S) - \frac{1}{2} v(S\setminus i) \right\} + \left\{ \frac{v(N)}{n} - \frac{v(N\setminus i)}{2n} \right\}$$
$$= \sum_{S\subsetneq N: i\in S} \frac{(n-s)!(s-1)!}{n!} \left\{ \frac{1}{2} v(S) - \frac{1}{2} v(S\setminus i) \right\} + \left\{ \frac{v(N)}{2n} - \frac{v(N\setminus i)}{2n} \right\} + \frac{v(N)}{2n}$$

$$= \frac{1}{2} \sum_{S \subseteq N: i \in S} \frac{(n-s)!(s-1)!}{n!} \Big\{ v(S) - v(S \setminus i) \Big\} + \frac{v(N)}{2n}$$
$$= \frac{1}{2} \Big\{ \Phi_i^{Sh}(v) + \Phi_i^{ED}(v) \Big\}$$

Therefore $\Phi^n = \frac{1}{2}\{\Phi^{Sh} + \Phi^{ED}\}$. It follows that when $k = n$, the value $\Phi^n$ divides the half of the worth of the grand coalition $v(N)$ equally among the players and the other half is divided among the players as par the Shapley value. This is indeed the $\alpha$-egalitarian Shapley value first proposed by Joosten [16] and later discussed in details by van den Brink et al., [28], with $\alpha = \frac{1}{2}$.

**Remark 6.** In view of Remark 5, we explore now, for any $k \in \{1, 2, ..., n\}$ if there exists a constant $\alpha_k \in [0, 1]$ such that $\alpha_k \Phi^{Sh} + (1 - \alpha_k)\Phi^{ED} = \Phi^k$. Note that for $S \subsetneq N$, $\Phi^{ED}(\tilde{u}_S) = 0$. Therefore $\alpha_k \Phi_i^{Sh}(\tilde{u}_S) = \Phi_i^k(\tilde{u}_S)$ for all $i \in N$ and $S \subsetneq N$. For any non empty subset $S \subsetneq N$ we have,

$$\Phi_i^{Sh}(\tilde{u}_S) = \begin{cases} \dfrac{(s-1)!(n-s)!}{n!} & \text{if } i \in S \\ -\dfrac{s!(n-s-1)!}{n!} & \text{otherwise} \end{cases}$$

and

$$\Phi_i^k(\tilde{u}_S) = \begin{cases} \dfrac{(s-1)!(n-s)!\alpha(s,k)\alpha(n-s,k)}{n!\alpha(n,k)} & \text{if } i \in S \\ -\dfrac{s!(n-s-1)!\alpha(s,k)\alpha(n-s)}{n!\alpha(n,k)} & \text{otherwise} \end{cases}$$

It follows that $\alpha_k = \dfrac{\alpha(s,k)\alpha(n-s)}{\alpha(n,k)}$ which depends on the size $s$ of coalition $S$ for $n > 3$. Therefore $\alpha_k$ cannot be a constant for $n > 3$. It follows that $\Phi^k$ is not a convex combination of the Shapley value and the Equal division rule for $1 < k < n$ and $n > 3$.

## 4.1 Two types of null-players

Recall that in our counting process, we allowed the players to enter one by one following a particular order but waited till a group of size no more than $k$ had formed. We computed the contribution of this group which is then divided equally among the players in that group. In this way we allowed the players to finally form the grand coalition. Based on the counting of the groups formed henceforth, we define two types of null players: call them coalitional null players of type I and II, followed by their respective null player axioms.

### The $k$-Coalitional null player of Type-I

Let $S$ be a coalition with $i \in S$. Consider the partition $C = \{C_1, ..., C_{p-1}, C_p, C_{p+1}, ..., C_m\}$ of ordered groups of $N$ with $\text{In}(C) \leq k$ such that $S = \cup_{j=1}^{p} C_j$. It follows that there is an order $\pi_C$ of $N$ such that the players in $C$ enter according to $\pi_C$. Consider the subset

$C_S = \{C_1, C_2, ..., C_{p-1}, C_p\}$ of $C$. $C_S$ is a partition of $S$ prescribed by $\pi_C$. The number of such partitions of $S$ prescribed by any order is $\alpha(s,k)$. The total number of partitions of $N$ prescribed by any order in which $S$ is the union of first $p$ members, $1 \leq p \leq s$ is therefore given by $\alpha(s,k)\alpha(n-s,k)$. Then following Eq.(3.3), the total group contribution of all the players in the group $C_p$ when player $i$ enters $S = \cup_{j=1}^{p} C_j$ in the last i.e., $\pi_C(s) = i$ with respect to $C$ is given by $c_p A_i^v(P(C,i) \cup C_p)$. Now define the $k$-lateral group contribution $M_i^{(S,k)}(v)$ of player $i$ from the coalition $S$, by the average of the total group contributions of all the players in the group $C_p$ when player $i$ enters $S = \cup_{j=1}^{p} C_j$ in the last i.e., $\pi_C(s) = i$ with respect to all $C \in \Pi(N,k)$.
Formally we have,

$$M_i^{(S,k)}(v) = \frac{1}{(s-1)!\alpha(s,k)\alpha(n-s,k)} \sum_{\substack{C \in \Pi(N,k): \\ S = \cup_{q=1}^{p} C_q, \pi_C(s)=i}} c_p A_i^v(P(C,i) \cup C_p)$$

$$= \sum_{\substack{T \subset S: i \in T \\ 1 \leq t \leq k}} \frac{(s-t)!\alpha(s-t,k)(t-1)!}{(s-1)!\alpha(s,k)} \Big\{ v(S) - v(S \setminus T) \Big\}$$

**Definition 2.** Given $v \in \mathcal{G}(N)$, a player $i \in N$ is called a $k$-coalitional null player of type I or a $k^1$-coalitional null player in short, if her $k$-lateral group contributions $M_i^{(S,k)}(v) = 0$ for all coalitions $S$ such that $i \in S$.

Observe that when $k = 1$, the $1^1$-coalitional null player is the standard null player characterizing the Shapley value. Further we note that the $k^1$-coalitional null player and her group members with respect to each partition, on an average makes no contribution to the corresponding coalition. Thus the $k^1$-coalitional null player not only contributes nothing of her own on an average but also she forces her group members to keep their average contributions zero. Therefore, it is justified to award her zero payoff under the $k$-lateral Shapley value. In the following, we have the $k^1$-coalitional null player axiom or the $kNP_1$ in short.

**Axiom** 5. $k^1$-Coalitional Null Player ($kNP_1$): For $v \in \mathcal{G}(N)$ and for any $k^1$-coalitional null player $i \in N$ of $v$, $\Phi_i^k(v) = 0$.

**Proposition 5.** *The $k$-lateral Shapley value $\Phi^k$, $k \geq 1$ satisfies $kNP_1$.*

*Proof.* The proof is immediate from Eq.(3.9). □

**The $k$-Coalitional null player of Type-II**

Recall that every sequence of positive integers $c = (c_1, ..., c_p)$ with $1 \leq p \leq s$ and $0 < c_p \leq k$ and the order $\pi$ determine a partition $C_S^\pi = \{C_1^\pi, ..., C_p^\pi\}$ of ordered groups so that $S =$

$\cup_{j=1}^{p} C_j^{\pi}$. Then $\alpha(s,k)$ is the number of such partitions of $S$ according to the particular order $\pi$. The total number of partitions of $N$ prescribed by $\pi$ in which $S$ is the union of first $p$ members, $1 \leq p \leq s$ is therefore given by $\alpha(s,k)\alpha(n-s,k)$. Thus the probability that $S$ is chosen from $N$ with this property prescribed by a particular order $\pi$ is given by $\frac{\alpha(s,k)\alpha(n-s,k)}{\alpha(n,k)}$. Let a random variable take the value $v(S) > 0$ when $S$ is formed such that for some $p$ with $1 \leq p \leq s$, $S$ is the union of first $p$ members of the partitions of index $\leq k$, prescribed by $\pi$ and $v(S) = 0$ otherwise. Then the expectation that the random variable takes $v(S)$ when $S$ is chosen from $N$ according to the above rule is given by $\frac{\alpha(s,k)\alpha(n-s,k)}{\alpha(n,k)}v(S)$. Let us call it the expected $k$-lateral worth of $S$. Now fix an $i$ from $N$. Find those $S$'s of $N$ in which $\pi(s) = i$, i.e., $i$ is the last member to enter in $S$. Then the expected $k$-lateral worth of $S \setminus i$ is found to be $\frac{\alpha(s-1,k)\alpha(n-s+1,k)}{\alpha(n,k)}v(S \setminus i)$. Based on this formulation, we now define the $k$-coalitional null player of type II or in short the $k^2$-coalitional null player and the corresponding $k^2$-coalitional null player axiom or $kNP_2$ in short.

**Definition 3.** Given $v \in \mathcal{G}(N)$, a player $i \in N$ is said to be a $k$-coalitional null player of type II or a $k^2$-coalitional null player in short of $v$ if for all coalitions $S$ such that $i \in S$, the expected $k$-lateral worths of $S$ and $S \setminus i$ are identical. Thus formally, $i \in N$ is a $k^2$-coalitional null player if for each $S \subseteq N$ such that $i \in S$,

$$\frac{\alpha(s,k)\alpha(n-s,k)}{\alpha(n,k)}v(S) = \frac{\alpha(s-1,k)\alpha(n-s+1,k)}{\alpha(n,k)}v(S \setminus i) \tag{4.9}$$

It follows from Eq.(4.9) that for $k = 1$, the $1^2$-null player becomes the null player. When $k > 1$, the $k$-null player contributes nothing to the coalitions on an average when both her individual (groups of size 1) and group contributions are measured. Therefore, it is justified to award the $k^2$-coalitional null player zero payoff under the $k$-lateral Shapley value. We have the following $k^2$-coalitional null player axiom or the $kNP_2$ in short.

**Axiom** 6. $k^2$-Coalitional Null Player ($kNP_2$): For $v \in \mathcal{G}(N)$ and for any $k^2$-coalitional null player $i \in N$ of $v$, $\Phi_i^k(v) = 0$.

**Proposition 6.** *The $k$-lateral Shapley value $\Phi^k$, $k \geq 1$ satisfies $kNP_2$.*

*Proof.* Immediately follows from the definition. □

**Remark 7.** Define a new game $\bar{v}$ on $N$ as follows: $\bar{v} = \frac{\alpha(s,k)\alpha(n-s,k)}{\alpha(n,k)}v(S)$ for all $S \subseteq N$. Call it the associate game of $v$ with respect to group contributions. Then we have $\Phi_i^k(v) = \Phi_i^{Sh}(\bar{v})$ for all $i \in N$. Therefore the $k$-lateral Shapley value over the game $v$ is the Shapley value over its associate game $\bar{v}$. Furthermore, player $i$ is a $k^2$-coalitional null player of game $v$ if and only if $i$ is a null player of $\bar{v}$.

**Remark 8.** It is interesting to note that the two types of null players defined above are neither equivalent nor they imply each other. Take for example, a game $v$ on $N = \{1, 2, 3\}$ as follows.

$v(\{1\}) = 0$, $v(\{2\}) = v(\{3\}) = 2$, $v(\{1, 2\}) = v(\{1, 3\}) = 1$, $v(\{2, 3\}) = 2$ and $v(\{1, 2, 3\}) = 2$. Suppose $k = 2$. Then player 1 is a $k^1$-null player but not a $k^2$-null player of $v$. Consider another game $w$ on $N = \{1, 2, 3\}$ as follows. $w(1) = 0$, $w(2) = w(3) = w(\{1, 2\}) = w(\{1, 3\}) = 1$, $w(\{2, 3\}) = 3$ and $w(\{1, 2, 3\}) = 4$. Here, for the same $k = 2$, player 1 is a $k^2$-null player but not a $k^1$-null player of $w$.

**The Characterization Theorem**

In this section, we prove a couple of characterization theorems for the $k$-lateral Shapley value using the axioms *Eff, Lin, Sym* and either of $kNP_1$ or $kNP_2$. In view of propositions 3 and 5, the characterization only requires to show that if a value satisfies *Eff, Lin, Sym* and either of $kNP_1$ or $kNP_2$ it must be given by Eq.(3.4) or equivalently Eq.(3.9). Both the proofs are constructive and we start with the introduction of a couple of new bases for the class $\mathcal{G}(N)$ of games. Every $v \in \mathcal{G}(N)$ is then expressed as a linear combination of these bases. Therefore, following *Lin* it will suffice to obtain the expression of the $k$-lateral Shapley value for these bases. Let us start with the axioms *Eff, Lin, Sym* or ESL in short and the axiom $kNP_1$. To complete the proofs we need the following propositions. Their proofs are kept in the Appendix.

**Proposition 7.** *For a non empty coalition $T \subseteq N$, define the function $D_T : 2^N \to \mathbb{R}$ by*

$$D_T(S) = \begin{cases} f(s,t), & \text{if } T \subsetneq S \\ 1, & \text{if } T = S \\ 0, & \text{otherwise} \end{cases} \tag{4.10}$$

*where the value $f(s,t)$ of the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is recursively given by,*

$$f(s,t) = \sum_{1 \le m \le min\{k, s-t\}} \frac{\alpha(s-m,k)}{\alpha(s,k)} \frac{\binom{s-t-1}{m-1} f(s-m,t)}{\binom{s-1}{m-1}} \tag{4.11}$$

*with $f(s,s) = 1$ for all $1 \le s \le n$. Then $D_T$ is a TU game for each $T \subseteq N$. Moreover, the set of games $\{D_T : T \subseteq N, T \neq \emptyset\}$ is a basis for $\mathcal{G}(N)$.*

**Proposition 8.** *For an ESL value $\Phi$ having $kNP_1$ and $T \subset N$, $\Phi(D_T)$ is uniquely determined by*

$$\Phi_i(D_T) = \begin{cases} \dfrac{f(n,t)}{t}, & \text{if } i \in T \\ 0, & \text{otherwise} \end{cases} \tag{4.12}$$

*where the value $f(n,t)$ is given by Eq.(4.11).*

Note that the expression for $f(s,t)$ given by Eq.(4.11) is obtained in the following manner. The underlying assumption for getting this expression is that each $i \notin T$ is a $k^1$-coalitional null player, i.e., $M_i^{(S,k)}(D_T) = 0$ for all $S \subseteq N$, $i \in S$ and for each non empty coalition $i \notin T \subseteq N$. It follows that,

$$\sum_{\substack{M \subseteq S\,:\, i \in M \\ 1 \le m \le k}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} \{D_T(S) - D_T(S \setminus M)\} = 0$$

$$\Rightarrow \sum_{\substack{M \subseteq S\,:\, i \in M \\ 1 \le m \le k}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} \{D_T(S)\}$$

$$= \sum_{\substack{M \subseteq S\,:\, i \in M \\ 1 \le m \le k}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} \{D_T(S \setminus M)\}.$$

Therefore,

$$\sum_{1 \le m \le k} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} \binom{s-1}{m-1} D_T(S)$$

$$= \sum_{\substack{M \subseteq S\,:\, i \in M \\ 1 \le m \le k}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} D_T(S \setminus M)$$

Thus we have,

$$D_T(S) = \sum_{\substack{M \subseteq S\,:\, i \in M \\ 1 \le m \le k}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} D_T(S \setminus M)$$

and

$$f(s,t) = \sum_{\substack{M \subseteq S \setminus T\,:\, i \in M \\ 1 \le m \le k}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} D_T(S \setminus M)$$

$$= \sum_{1 \le m \le \min\{k, s-t\}} \frac{(m-1)!(s-m)!\alpha(s-m,k)}{(s-1)!\alpha(s,k)} \binom{s-t-1}{m-1} f(s-m,t)$$

$$= \sum_{1 \le m \le \min\{k, s-t\}} \frac{\alpha(s-m,k)}{\alpha(s,k)} \frac{\binom{s-t-1}{m-1} f(s-m,t)}{\binom{s-1}{m-1}}, \text{ with } f(s,s) = 1 \text{ such that } 1 \le s \le n.$$

In this way, we get a recursive method of obtaining all the values of $f(s,t)$. Next, we have a characterization theorem for the $k$-lateral Shapley value $\Phi^k$.

**Theorem 1.** *The $k$-lateral Shapley value $\Phi^k$ is the unique value that satisfies Eff, Lin, Sym and $kNP_1$.*

*Proof.* Since $\{D_T : T \subset N, T \neq \emptyset\}$ is a basis for $\mathcal{G}(N)$ therefore any game $v \in \mathcal{G}(N)$ can be expressed uniquely as $v = \sum_{T \subset N: T \neq \emptyset} \gamma_T^v D_T$ where $\gamma_T^v \in \mathbb{R} : T \subseteq N$. Since $\Phi^k$ is linear therefore $\Phi_i^k(v) = \sum_{T \subset N: T \neq \emptyset} \gamma_T^v \Phi_i^k(D_T)$. By Proposition (8), $\Phi_i^k(D_T)$ is uniquely determined by Eq.(7.3). This completes the proof. □

For the second characterization, we define the $k$-unanimity game denoted by $W_T : 2^N \to R$ for each non empty coalition $T \subseteq N$ in explicit form as follows.

$$W_T(S) = \begin{cases} \dfrac{\alpha(n-t,k)\alpha(t,k)}{\alpha(n-s,k)\alpha(s,k)}, & \text{if } T \subseteq S \\ 0, & \text{otherwise} \end{cases} \tag{4.13}$$

Note that $W_T$ is identical with the unanimity game for $k = 1$. For $T = S$, $W_T(T) = 1$.

**Remark 9.** For $T \neq \emptyset$, the game $W_T$ possesses the following properties.

(a) $W_T(T) = 1$ for $T \subset N$.

(b) $W_T(S) = 0$ for $T \not\subset S$.

(c) $W_T(S) = \dfrac{\alpha(n-s+1,k)\alpha(s-1,k)}{\alpha(n-s,k)\alpha(s,k)} W_T(S \setminus i)$ for $T \subset S \setminus i$.

**Proposition 9.** *For an ESL value $\Phi$ having $kNP_2$ and $T \subset N$, $\Phi(W_T)$ is uniquely determined by*

$$\Phi_i(W_T) = \begin{cases} \dfrac{\alpha(n-t,k)\alpha(t,k)}{t\alpha(n,k)}, & \text{if } i \in T \\ 0, & \text{otherwise} \end{cases} \tag{4.14}$$

The proof of Proposition 9 is kept in the Appendix.

**Theorem 2.** *The $k$-lateral Shapley value $\Phi^k$ is the unique value that satisfies Eff, Lin, Sym and $kNP_2$.*

*Proof.* Consider the set $\{W_T | T \subseteq N, T \neq \emptyset\}$. By a similar procedure as in Prop. 7, we can show that the set $\{W_T | T \subseteq N, T \neq \emptyset\}$ forms a basis of $\mathcal{G}(N)$. Any game $v \in \mathcal{G}(N)$ can be expressed uniquely as $v = \sum_{T \subset N : T \neq \emptyset} \gamma_T^v W_T$ where $\gamma_T^v = \sum_{S \subset T : S \neq \emptyset} (-1)^{t-s} \dfrac{\alpha(n-s,k)\alpha(s,k)}{\alpha(n-t,k)\alpha(t,k)} v(S)$. Using the expression of $\gamma_T^v$ we derive the following.

$$\begin{aligned}
\sum_{T \subset N : T \neq \emptyset} \gamma_T^v W_T(S) &= \sum_{T \subset S : T \neq \emptyset} \gamma_T^v \frac{\alpha(n-t,k)\alpha(t,k)}{\alpha(n-s,k)\alpha(s,k)} \\
&= \sum_{T \subset S : T \neq \emptyset} \frac{\alpha(n-t,k)\alpha(t,k)}{\alpha(n-s,k)\alpha(s,k)} \sum_{R \subset T : R \neq \emptyset} (-1)^{t-r} \frac{\alpha(n-r,k)\alpha(r,k)}{\alpha(n-t,k)\alpha(t,k)} v(R) \\
&= \sum_{T \subset S : T \neq \emptyset} \sum_{R \subset T : R \neq \emptyset} (-1)^{t-r} \frac{\alpha(n-r,k)\alpha(r,k)}{\alpha(n-s,k)\alpha(s,k)} v(R) \\
&= \sum_{R \subset S : R \neq \emptyset} \sum_{T \subset S : R \subset T} (-1)^{t-r} \frac{\alpha(n-r,k)\alpha(r,k)}{\alpha(n-s,k)\alpha(s,k)} v(R) \\
&= \sum_{R \subset S : R \neq \emptyset} \left\{ \sum_{T \subset S : R \subset T} (-1)^{t-r} \right\} \frac{\alpha(n-r,k)\alpha(r,k)}{\alpha(n-s,k)\alpha(s,k)} v(R) \\
&= v(S) + \sum_{R \subset S : R \neq \emptyset} \left\{ \sum_{t=r : s \neq r}^{s} (-1)^{t-r} \binom{s-r}{t-r} \right\} \frac{\alpha(n-r,k)\alpha(r,k)}{\alpha(n-s,k)\alpha(s,k)} v(R)
\end{aligned}$$

Since $\sum_{t=r:s\neq r}^{s}(-1)^{t-r}\binom{s-r}{t-r}=0$ therefore $\sum_{T\subset N:T\neq\emptyset}\gamma_T^v W_T(S)=v(S)$.

Since $\Phi^k$ is linear therefore $\Phi_i^k(v)=\sum_{T\subset N:T\neq\emptyset}\gamma_T^v\Phi_i^k(W_T)$. By Proposition (9), $\Phi_i^k(W_T)$ is uniquely determined by Eq.(4.14). Therefore $\Phi_i^k(v)$ is also uniquely determined by

$$\Phi_i^k(v)=\sum_{T\subset N:i\in T}\gamma_T^v\frac{\alpha(n-t,k)\alpha(t,k)}{t\alpha(n,k)}$$

□

**Remark 10.** Note that in the proof of Theorem 1, the basis $D_T$ is obtained recursively while in Theorem 2, the basis $W_T$ is expressed in a closed form to illustrate the procedure of obtaining the $k$-lateral Shapley value in an explicit way, however to show only the existence and uniqueness, such explicit forms are seemingly redundant.

## 4.2 Logical Independence

Logical independence of the axioms of Theorem 1 can be illustrated by the following examples.

(a) The value $\beta^k:\mathcal{G}(N)\to R^N$ given by

$$\beta_i^k(v)=\frac{1}{2^{|N|-1}}\sum_{\substack{S\subseteq N\\ :i\in S}}\sum_{\substack{T\subseteq S:\, i\in T\\ 1\leq t\leq k,}}\frac{(s-t)!(t-1)!\alpha(s-t,k)}{(s-1)!\alpha(s,k)}\Big\{v(S)-v(S\setminus T)\Big\}$$

for all $i\in N$ satisfies $kNP_1$, *Lin* and *Sym* but does not satisfy *Eff*.

(b) The value $\gamma^k:\mathcal{G}(N)\to R^N$ given by

$$\gamma_i^k(v)=\frac{1}{2^{|N|-1}}\sum_{\substack{S\subseteq N\\ :i\in S}}\left\{\frac{\alpha(n-s,k)\alpha(s,k)}{\alpha(n,k)}v(S)-\frac{\alpha(n-s+1,k)\alpha(s-1,k)}{\alpha(n,k)}v(S\setminus i)\right\}$$

for all $i\in N$ satisfies $kNP_2$, *Lin* and *Sym* but does not satisfy *Eff*.

(c) The equal division rule $\Phi^{ED}$ satisfies *Eff*, *Sym* and *Lin* but it does neither satisfy $kNP_1$ nor $kNP_2$.

(d) The value $\bar{\Phi}^k:\mathcal{G}(N)\to R^N$ given by $\bar{\Phi}_i^k(v)=\dfrac{\beta_i^k(v)}{\sum_{j\in N}\beta_j^k(v)}v(N)$ $\left(\text{or } \bar{\Phi}_i^k(v)=\dfrac{\gamma_i^k(v)}{\sum_{j\in N}\gamma_j^k(v)}v(N)\right)$ for all $i\in N$ satisfies $kNP_1$ (or $kNP_2$), *Eff* and *Sym* but does not satisfy *Lin* if $\sum_{j\in N}\beta_j^k(v)\neq 0$ $\left(\text{or } \sum_{j\in N}\gamma_j^k(v)\neq 0\right)$.

(e) Consider the basis $\{D_T : T \subseteq N, T \neq \emptyset\}$ of the games defined by the Eq. (7.3) for the class $\mathcal{G}(N)$. Suppose that $i = \min_{j \in T} j$. Let $\beta$ be a value such that $\beta_i(D_T) = D_T(N)$ and $\beta_j(D_T) = 0$ for $j \in N, j \neq i$. Extend $\beta$ linearly for all games in $\mathcal{G}(N)$. $\beta$ satisfies $kNP_1$, *Lin* and *Eff* but does not satisfy *Sym*.

(f) Consider the basis $\{W_T : T \subseteq N, T \neq \emptyset\}$ of the games defined by the Eq. (4.13) for the class $\mathcal{G}(N)$. Suppose that $i = \min_{j \in T} j$. Let $\gamma$ be a value such that $\gamma_i(W_T) = W_T(N)$ and $\gamma_j(W_T) = 0$ for $j \in N, j \neq i$. Extend $\gamma$ linearly for all games in $\mathcal{G}(N)$. $\gamma$ satisfies $kNP_2$, *Lin* and *Eff* but does not satisfy *Sym*.

## 5 Examples

In this section, we first consider the motivating example of Section 1 to highlight explicitly the $k$-lateral interactions among the players in a TU-game and how they influence the $k$-lateral Shapley value for different choices of $k$. In a subsequent example, we try to highlight where and how group contributions capture the idea of solidarity among players.

**Example 3** (Continue from example 1)**.** Recall the game $(N, v)$ in Example 1, where $N = \{1, 2, 3, 4\}$ and the coalition function $v$ was defined by: $v(S) = 0$ if $\{1, 2\} \not\subseteq S$, $v(1, 2) = 2$, $v(1, 2, 3) = 4$, $v(1, 2, 4) = 6$ and $v(1, 2, 3, 4) = 8$. The $k$-lateral Shapley value $\Phi^k(v)$ for different choices of $k$ including the Shapley value where $k = 1$ are given below.

$$\begin{aligned}
\Phi^1(v) &= (3.0, 3.0, 0.6, 1.4) = \Phi^{Sh}(v) \\
\Phi^2(v) &= (2.6, 2.6, 1.2, 1.6) \\
\Phi^3(v) &= (2.57, 2.57, 1.24, 1.62) \\
\Phi^4(v) &= (2.5, 2.5, 1.33, 1.67)
\end{aligned}$$

We have already mentioned that this is a special example where all the players are individually non-productive. Also neither player 1 nor 2 alone can have a non-zero contribution to a coalition. They are productive only when they are together in a coalition. Similarly 3 and 4 can generate worths only when they join with both 1 and 2. Thus in this stylized example, we want to see how and why the marginal productivities of player 1 and 2 can be compensated by solidarity towards 3 and 4. The Shapley value considers the individual contributions ($k = 1$) of 1 and 2, even though they are dependent on each other in generating the worth of the grand coalition. Such dependence among players in deciding to join or leave a coalition is not explicitly seen in Shapley formulations. Thus under the Shapley value their productivity is the highest. However, when we consider players' group contributions, with an increase of the size of the groups, sharing is more egalitarian. Therefore, more solidarity for player 3 and 4 is ensured as the group contributions are shared equally among all the players.

**Example 4** (Solidarity within and between groups). Consider the game $v = u_{\{1\}} + 2u_{\{2,3\}}$ with 3 players. There are two groups of players whose productivities are independent, namely, $\{1\}$ and $\{2,3\}$. Moreover, players 2 and 3 are symmetric and the per-capita productivity in both groups is the same, namely 1. Therefore, there neither is a need to express solidarity within these groups nor between these groups, i.e., players should be rewarded according to their common individual productivity, which is 1. The Shapley value and the egalitarian Shapley value award each player with payoff 1. For $k = 2$, the $k$-lateral Shapley value for this game is also $(1,1,1)$ since the group contributions of the players have nothing to add to their payoffs. On the contrary, take the game $v = 2u_{\{1,2\}} + 3u_{\{3,4,5\}}$ with 5 players. In this game, one can easily see that there are two independent groups $\{1,2\}$ and $\{3,4,5\}$. Therefore, there is no need for solidarity among these two groups. However, the players within each group being individually non-productive are dependent on each other in producing a non-zero worth. Despite all the players have equal per-capita productivities, players 1 and 2 make more non-zero group contributions than 3, 4 and 5. So, it is difficult to give equal weightage to the players of the two groups even though the per-capita productivities of all the players are same. The $k-$lateral Shapley value $\Phi^k(v)$ for different choices of $k$ $(1 \leq k \leq 5)$ are obtained as follows.

$$\Phi^1(v) = \Big(1, 1, 1, 1, 1\Big)$$
$$\Phi^2(v) = \left(\frac{163}{160}, \frac{163}{160}, \frac{158}{160}, \frac{158}{160}, \frac{158}{160}\right)$$
$$\Phi^3(v) = \left(\frac{789}{780}, \frac{789}{780}, \frac{774}{780}, \frac{774}{780}, \frac{774}{780}\right)$$
$$\Phi^4(v) = \left(\frac{15}{14}, \frac{15}{14}, \frac{20}{21}, \frac{20}{21}, \frac{20}{21}\right)$$
$$\Phi^5(v) = \Big(1, 1, 1, 1, 1\Big)$$

# 6 Conclusion

This paper proposes a new class of values for TU-games – the $k$-lateral Shapley value – that considers group contributions of players within a coalition. All the Shapley like marginalistic values implicitly assume that players individually and independently decide to join or leave a coalition of their own. However, there are instances where players within a coalition are influenced by each other on making such decisions and finally they make collective decisions. Since a marginalistic value awards payoffs to the players based on their own contributions, their reliance on the others in generating the worth should be given due consideration. This led us to define the notion of group contributions. Our value computes the average of all such individual and group contributions of the players. Thus, to summarize, our model primarily focuses on the finer contributions of players in groups and their agreement for the equal division of the worths they generate. By several examples, we highlight different levels of interactions among players in groups. The characterization of the new values is done using standard axioms of Efficiency, Symmetry, and Linearity along with two new axioms: the $k$-null player axioms of type I and type II. Under the assumption of the null player out property [10], null players can be deleted from the game without consequences for the allocation of the non-null players. Similar axioms may be formulated for the $k$-coalitional null players. Moreover, the notion of group contributions can also be used to generalize other Shapley like values. These we keep for our future studies.

**Acknowledgement**

The work is supported by UKIERI [184-15/2017(IC)]. We are greatly indebted to the AE and the three anonymous reviewers for their valuable suggestions.

## References

[1] Aumann, R., and Dreze, J (1974) Cooperative games with coalition structures. *International Journal of Game Theory*, 3, 217-237.

[2] Beal, S., Rémilla, and Solal, P (2017) Axiomatization and implementation of a class of solidarity values for TU games. *Theory and Decisions*, 83, 61-94.

[3] Borkotokey, S., Sarangi, S. and Kumar, R (2015) A solution concept for network games : The role of multilateral interactions. *European Journal of Operational Research*, 243(3) 912-920.

[4] Branzei, R., Dimitrov, D. and Tijs, S (2008) *Models in Cooperative Game Theory: Crisp, Fuzzy and Multichoice Games*. Springer, Berlin Heidelberg.

[5] Calvo, E (2008) Random marginal and random removal values. *International Journal of Game Theory*, 37(4) 533-563.

[6] Casajus, A. and Huettner, F (2014a) On a class of solidarity values. *European Journal of Operational Research*, 236, 583-591.

[7] Casajus, A. and Huettner, F. (2014b) Weakly monotonic solutions for cooperative games. *Journal of Economic Theory* 154, 162–172.

[8] Casajus A, Huettner F (2013) Null players, solidarity and the egalitarian Shapley values. *J Math Econ*, 49, 58-61.

[9] Chun, Y (1991) On the symmetric and weighted Shapley values. *International Journal of Game Theory*, 20, 183-190.

[10] Derks, JJM and Haller, HH (1999) Null players out? Linear values for games with variable supports, *International Game Theory Review*, 1(3-4) 301-314.

[11] Dubey, P (1975) On the uniqueness of the Shapley value. *International Journal of Game Theory*, 4(3) 131-139.

[12] Dubey, P (1982) The Shapley value as aircraft landing fees-revisited. *Management Science*, 28(8) 869-874.

[13] Farrell, J. and Scotchmer, S (1998) Partnerships, *The Quarterly Journal of Economics* 103 (2) 279-297.

[14] Grabisch, M (2009) An Axiomatic Approach to the Concept of Interaction among Players in Cooperative Games. *International Journal of Game Theory*, 28(04) 547-565.

[15] Hamiache, G (2012) A Matrix Approach to TU Games with Coalition and Communication Structures. *Social Choice and Welfare*, 38, 85. https://doi.org/10.1007/s00355-010-0519-9.

[16] Joosten, R (1996) *Dynamics, equilibria and values dissertation*. Maastricht University.

[17] Kamijo, Y (2009) A two-step Shapley value in a cooperative game with a coalition structure. *International Journal of Game Theory*, 11(02) 207-214.

[18] Kamijo, Y., Kongo, T (2012) Whose deletion does not affect your payoff? The difference between the Shapley value, the egalitarian value, the solidarity value, and the Banzhaf value. *European Journal of Operational Research*, 216 638–646.

[19] Malawski M (2013) Procedural values for cooperative games. *International Journal of Game Theory*, 42(1) 305-324.

[20] Nowak, A. S. and Radzik, T (1994) A solidarity value for n-person transferable utility games, *International Journal of Game Theory*, 23(1) 43-48.

[21] Owen, G (1977) Values of games with a priori unions, in *Essays in Mathematical Economics and Game Theory*, ed. by R. Henn, and O. Moeschlin, 76-88. Springer-Verlag, Berlin.

[22] Potters, J., Poos, R., Tijs, S. and Muto, S (1989) Clan games. *Games and Economic Behavior*, 1, 275-293.

[23] Radzik, T., Driessen, T (2016) Modeling values for TU-games using generalized versions of consistency, standardness, and the null player property. *Math Meth Oper Res*, 83, 179-205.

[24] Shapley, L (1953) A value for n-person games, in Kuhn, H. and Tucker, A.W. (eds.), *Contribution to the Theory of games II*, Princeton, New Jersey, Princeton University Press, 307-317.

[25] van den Brink, R (2001) An axiomatization of the Shapley value using a fairness property. *International Journal of Game Theory*, 30, 309–319.

[26] van den Brink, R (2007) Null or nullifying players: The difference between the Shapley value and equal division solutions. *Journal of Economic Theory*, 136, 767-775.

[27] van den Brink, R., Funaki, Y (2015) Implementation and axiomatization of discounted Shapley values. *Social Choice and Welfare*, 2 (45) 329–344.

[28] van den Brink, R., Funaki, Y. Ju, Y (2013) Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian Shapley values. *Social Choice and Welfare*, 40, 693-714.

[29] van den Brink, R. and Pintér, M (2015) On Axiomatizations of the Shapley Value for Assignment Games. *Journal of Mathematical Economics*, 60, 110–114.

[30] van den Brink, R., Levinsky, R. and Zeleny, M (2015) On proper Shapley values for monotone TU-games. *International Journal of Game Theory*, 44, 449–471.

[31] Weber, R (1988) Probabilistic values for games, in Roth A.E. (ed), *The Shapley value: essays in honor of Lloyd Shapley*, Cambridge University Press.

[32] Young, H (1985) Monotonic solutions of cooperative games. *International Journal of Game Theory*, 14(2) 65-72.

# 7 Appendix

**Proof of Proposition 2**.

(a) Inspired by the construction of $\mathcal{B}(n,k)$ given by 3.2, we can set $\mathcal{B}(s,k)$ for any $1 \leq s \leq n$ as follows:

$$\mathcal{B}(s,k) = \cup_{r=1}^{s}\{(x_1,x_2,...,x_r) : x_1+x_2+...+x_r = s, 1 \leq x_i \leq k, 1 \leq i \leq r\}.$$

Therefore,

$$\alpha(s,k) = \sum_{r=1}^{s} |\{(x_1,x_2,...,x_r) : x_1+x_2+...+x_r = s, 1 \leq x_i \leq k, 1 \leq i \leq r\}| \quad (7.1)$$

In general, the number of positive integer solutions of $x_1+x_2+...+x_r = s$ is $\binom{s-1}{r-1}$. Suppose that $i$ be the least number of variables with $x_j > k$, $j \in \{1,2,\cdots,r\}$.

In this case, the number of positive integer solutions of $x_1+x_2+...+x_r = s$ is $\binom{r}{i}\binom{s-ik-1}{r-1}$. Therefore, the number of integer solutions of $x_1+x_2+...+x_r = s$ with $1 \leq x_i \leq k, 1 \leq i \leq r$ is

$$\binom{s-1}{r-1} + \sum_{i=1}^{\left[\frac{s-r}{k}\right]} (-1)^i \binom{r}{i}\binom{s-ik-1}{r-1}$$

It follows that,

$$\alpha(s,k) = \sum_{r=1}^{s} \left\{ \binom{s-1}{r-1} + \sum_{i=1}^{\left[\frac{s-r}{k}\right]} (-1)^i \binom{r}{i}\binom{s-ik-1}{r-1} \right\}^{2}$$

□

(b) For $k = 1$ and $r = s$, the only solution of $x_1+x_2+...+x_r = s$ is $(1, 1, 1, ..., 1)$. For $r < s$, there is no solution in $x_1+x_2+...+x_r = s$ with $k = 1$. The result follows trivially. □

(c) By (a),

$$\alpha(s,k) = \sum_{r=1}^{s} |\{(x_1, x_2, ..., x_r) : x_1 + x_2 + ... + x_r = s, 1 \leq x_i \leq k, 1 \leq i \leq r\}|$$

Since $x_1 + x_2 + ... + x_r = s$, therefore, each $x_i \leq s$ for all $1 \leq r \leq s$.
Since $s \leq k$, therefore, for all $i \in \{1, 2, ..., s\}$ we must have $x_i \leq k$ .

[2] To explain the proof, consider a particular case say, $s = 9, k = 3, r = 3$. Then,

$$\alpha(9,3) = \sum_{r=1}^{9} |\{(x_1, x_2, ..., x_r) : x_1 + x_2 + ... + x_r = 9, 1 \leq x_i \leq 3, 1 \leq i \leq r\}|$$

The number of positive integer solutions of $x_1 + x_2 + x_3 = 9$ is $\binom{9-1}{3-1} = \binom{8}{2}$. Suppose that the least number of variables with $x_j > 3$ is $i$.

For $i = 1$, there are three possibilities: $x_1 > 3$ or $x_2 > 3$ or $x_3 > 3$. Suppose $x_1 > 3$. Let $y_1 = x_1 - 3$. Then $y_1 > 0$ and the equation $x_1 + x_2 + x_3 = 9$ becomes $y_1 + x_2 + x_3 = 9 - 3$. The number of positive integer solutions of $y_1 + x_2 + x_3 = 9 - 3$ is $\binom{9-3-1}{3-1}$. So, the number of such solutions of $x_1 + x_2 + x_3 = 9$ with $x_1 > 3$ is $\binom{9-3-1}{3-1}$. It follows that the number of solutions with at least one variable greater than 3 is $\binom{3}{1}\binom{9-3-1}{3-1}$.

For $i = 2$, there are $\binom{3}{2}$ possibilities: $x_1$, $x_2 > 3$ or $x_1$, $x_3 > 3$ or $x_2$, $x_3 > 3$. Suppose that $x_1$, $x_2 > 3$. Let $y_1 = x_1-3$, $y_2 = x_2-3$. Then $y_1$, $y_2 > 0$ and the equation $x_1+x_2+x_3 = 9$ becomes $y_1+y_2+x_3 = 9-2\cdot3$. The number of positive solutions of $y_1+y_2+x_3 = 9-2\cdot3$ is $\binom{9-2\cdot3-1}{3-1}$ which is same as that of $x_1+x_2+x_3 = 9$ with $x_1, x_2 > 3$. Thus, the number of solutions with at least two variables greater than 3 is given by $\binom{3}{2}\binom{9-2\cdot3-1}{3-1}$. Since $\frac{s-r}{k} = \frac{6}{3} = 2$, therefore, at most two variables can be greater than 3 in the positive integer solutions of $x_1 + x_2 + x_3 = 9$. Thus, the number of positive integer solutions with at least one variable greater than 3 is $\binom{3}{1}\binom{9-3-1}{3-1} - \binom{3}{2}\binom{9-2\cdot3-1}{3-1}$. It follows that the number of positive integer solutions of $x_1 + x_2 + x_3 = 9$ with $1 \leq x_i \leq 3$ is $\binom{9-1}{3-1} - \binom{3}{1}\binom{9-3-1}{3-1} + \binom{3}{2}\binom{9-2\cdot3-1}{3-1}$. Thus, the number of such solutions of $x_1 + x_2 + ... + x_r = 9$ with $1 \leq x_i \leq 3$,$1 \leq i \leq r$ is

$$\binom{9-1}{r-1} + \sum_{i=1}^{\left[\frac{9-r}{3}\right]} (-1)^i \binom{r}{i}\binom{9-3i-1}{r-1}.$$

It follows that,

$$\begin{aligned}\alpha(s,k) &= \sum_{r=1}^{s} |\left\{(x_1,x_2,...,x_r) : x_1+x_2+...+x_r = s, 1 \le x_i \le k, 1 \le i \le r\right\}| \\ &= \sum_{r=1}^{s} |\left\{(x_1,x_2,...,x_r) : x_1+x_2+...+x_r = s, 1 \le i \le r\right\}| \\ &= \sum_{r=1}^{s} \binom{s-1}{r-1} \\ &= 2^{s-1}\end{aligned}$$

□

(d) Following Eq.(7.1) we have,

$$\alpha(s,k) = \sum_{r=1}^{s} |\{(x_1,x_2,...,x_r) : x_1+x_2+...+x_r = s, 1 \le x_i \le k, 1 \le i \le r\}|.$$

Therefore,

$$\sum_{t=1}^{k} \alpha(s-t,k) = \sum_{t=1}^{k}\sum_{r=1}^{s-t} |\{(x_1,x_2,...,x_r) : x_1+x_2+...+x_r = s-t, 1 \le x_i \le k, 1 \le i \le r\}| \tag{7.2}$$

Since $s > k$, we must have $r \geq 2$ for satisfying $x_1 + x_2 + ... + x_r = s$, $1 \le x_i \le k$, $1 \le i \le r$. It follows that,

$$\begin{aligned}\alpha(s,k) &= \sum_{r=2}^{s} |\left\{(x_1,x_2,...,x_r) : x_1+x_2+...+x_r = s, 1 \le x_i \le k, 1 \le i \le r\right\}| \\ &= \sum_{r=2}^{s} |\left\{(x_1,x_2,...,x_r) : x_1+x_2+...+x_{r-1} = s - x_r, 1 \le x_i \le k, 1 \le i \le r\right\}| \\ &= \sum_{r=2}^{s}\sum_{x_r=1}^{k} |\left\{(x_1,x_2,...,x_{r-1}) : x_1+x_2+...+x_{r-1} = s - x_r, 1 \le x_i \le k, 1 \le i \le r\right\}| \\ &= \sum_{r=1}^{s-1}\sum_{t=1}^{k} |\left\{(x_1,x_2,...,x_r) : x_1+x_2+...+x_r = s-t, 1 \le x_i \le k, 1 \le i \le r\right\}| \\ &= \sum_{t=1}^{k}\sum_{r=1}^{s-t} |\left\{(x_1,x_2,...,x_r) : x_1+x_2+...+x_r = s-t, 1 \le x_i \le k, 1 \le i \le r\right\}|\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha(s,k) &= \sum_{t=1}^{k}\sum_{r=1}^{s-t} |\left\{(x_1,x_2,...,x_r) : x_1+x_2+...+x_r = s-t, 1 \le x_i \le k, 1 \le i \le r\right\}| \\ &= \sum_{t=1}^{k} \alpha(s-t,k)\end{aligned}$$

□

(e) Observe that $\alpha(0, k) = 1$ when $k > 0$ and for $1 \leq s \leq k$, $\alpha(s, k) = 2^{s-1}$. Therefore,

$$\begin{aligned}\sum_{1\leq t\leq s} \alpha(s-t, k) &= \sum_{1\leq t\leq s-1} 2^{s-t-1} + \alpha(0, k) \\ &= \sum_{0\leq t\leq s-2} 2^t + 1 \\ &= 2^{s-1} - 1 + 1 \\ &= 2^{s-1} \\ &= \alpha(s, k)\end{aligned}$$

□

**Proof of Proposition 7**

We show that the set of games $\{D_T\}$ for all $T \subset N$, $T \neq \emptyset$ form a basis of $\mathcal{G}(N)$. Let $d = 2^n - 1$. Since the class of unanimity games $\{u_S | S \subseteq N, S \neq \emptyset\}$ makes a basis for the vector space $\mathcal{G}(N)$, therefore the dimension of $\mathcal{G}(N)$ is $d$. Let $S_1$, $S_2$, ..., $S_d$ be a fixed sequence containing all non empty subsets of $N$ such that $n = s_1 \leq s_2 \leq ... \leq s_d = 1$. Let $A = [a_{ij}]$ be the $d \times d$ matrix defined by $a_{ij} = D_{S_i}(S_j), i, j = 1, 2, 3, ..., d$. Then $a_{ii} = D_{S_i}(S_i) = 1$. For $i > j$, $s_i \geq s_j$. Then either $s_i = s_j$ or $s_i > s_j$. Suppose that $s_i = s_j$. Since $S_i \neq S_j$ therefore $S_i \nsubseteq S_j$. Let $s_i > s_j$. Then $S_i \nsubseteq S_j$. It follows that $a_{ij} = 0$ for $i > j$. Thus $A = [a_{ij}]$ is an upper triangular matrix with diagonal entries 1 meaning $det(A) = 1$. Therefore, the set $\{D_{S_i} : i = 1, 2, ..., d\}$ is comprised of $d$ independent vectors in $\mathcal{G}(N)$. Since any linearly independent set containing $d$ vectors form a basis of $\mathcal{G}(N)$ therefore $\{D_{S_i} | i = 1, 2, 3, ..., d\}$ forms a basis of $\mathcal{G}(N)$. □

**Proof of Proposition 8**

By *Eff*, we have $\sum_{i\in N} \Phi_i(D_T) = D_T(N) = f(n, t)$.
By Proposition 7, $i \in N \setminus T$ is a $k^1$-null player in the game $D_T$. Since $\Phi$ satisfies $kNP_1$ therefore $\Phi_i(D_T) = 0$ for $i \in N \setminus T$. Further, any two players $i, j \in T$ are symmetric which implies $\Phi_i(D_T) = \Phi_j(D_T)$. Thus we have,

$$\Phi_i(D_T) = \begin{cases} \dfrac{f(n,t)}{t}, & \text{if } i \in T \\ 0, & \text{otherwise} \end{cases} \tag{7.3}$$

This completes the proof. □

**Proof of Proposition 9**

By *Eff*, we have $\sum_{i\in N} \Phi_i(W_T) = W_T(N) = \dfrac{\alpha(n-t, k)\alpha(t, k)}{\alpha(n, k)}$.
For $i \in N \setminus T$, if $S \subset N \setminus i$ then $T \subset S \implies T \subset S \setminus i$. If $T \not\subset S$, $i \notin T$ then $T \not\subset S \setminus i$.

Therefore $W_T(S) = W_T(S \setminus i) = 0$ for $T \not\subset S, i \notin T$. For $T \subset S, i \notin T$, by Remark [9], we have $W_T(S) = \dfrac{\alpha(n-s+1,k)\alpha(s-1,k)}{\alpha(n-s,k)\alpha(s,k)} W_T(S \setminus i)$. Therefore $i \in N \setminus T$ is a $k$-null player of type II in the $k$-unanimity game $W_T$. Since $\Phi$ satisfies $kNP_2$ therefore $\Phi_i(W_T) = 0$ for $i \in N \setminus T$. Any two players $i, j \in T$ are symmetric therefore $\Phi_i(W_T) = \Phi_j(W_T)$ for all $i, j \in T$. Thus

$$\Phi_i(W_T) = \begin{cases} \dfrac{\alpha(n-t,k)\alpha(t,k)}{t\alpha(n,k)}, & \text{if } i \in T \\ 0, & \text{otherwise} \end{cases}$$

□