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Hüttemann, T., & Zhang, Z. (2020). An elementary description of $K_1(R)$ without elementary matrices. *Algebra and Discrete Mathematics*, 30(1), 79. <https://doi.org/10.12958/adm1568>

Published in:
Algebra and Discrete Mathematics

Document Version:
Peer reviewed version

Queen's University Belfast - Research Portal:
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AN ELEMENTARY DESCRIPTION OF $K_1(R)$ WITHOUT ELEMENTARY MATRICES

THOMAS HÜTTEMANN AND ZUHONG ZHANG

ABSTRACT. Let R be a ring with unit. Passing to the colimit with respect to the standard inclusions $\mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n+1, R)$ (which add a unit vector as new last row and column) yields, by definition, the stable linear group $\mathrm{GL}(R)$; the same result is obtained, up to isomorphism, when using the “opposite” inclusions (which add a unit vector as new first row and column). In this note it is shown that passing to the colimit along both these families of inclusions simultaneously recovers the algebraic K -group $K_1(R) = \mathrm{GL}(R)/E(R)$ of R , giving an elementary description that does not involve elementary matrices explicitly.

Let R be an associative ring with unit element 1, and let $\mathrm{GL}(n, R)$ denote the group of invertible $n \times n$ -matrices with entries in R . The usual stabilisation maps

$$i_{n+1}^n: \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n+1, R), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

are used to define the stable general linear group $\mathrm{GL}(R) = \bigcup_{n \geq 3} \mathrm{GL}(n, R)$, or, phrased in categorical language,

$$\mathrm{GL}(R) = \mathrm{colim} \left(\mathrm{GL}(3, R) \xrightarrow{i_4^3} \mathrm{GL}(4, R) \xrightarrow{i_5^4} \mathrm{GL}(5, R) \xrightarrow{i_6^5} \dots \right). \quad (1)$$

The canonical group homomorphisms $\iota_n: \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(R)$ are injective and satisfy the relation

$$\iota_{n+1} \circ i_{n+1}^n = \iota_n. \quad (2)$$

There are other “block-diagonal” embedding $i_j^n: \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n+1, R)$, for $1 \leq j \leq n+1$, characterised by saying that the j th row and j th column of $i_j^n(A)$ are j th unit vectors, and that deleting these from $i_j^n(A)$ recovers the matrix A . We will determine the result of stabilising over first and last embeddings simultaneously, that is, we identify the categorical colimit M of the following group-valued infinite diagram:

$$\mathrm{GL}(3, R) \begin{array}{c} \xrightarrow{i_1^3} \\ \xrightarrow{i_4^3} \end{array} \mathrm{GL}(4, R) \begin{array}{c} \xrightarrow{i_1^4} \\ \xrightarrow{i_5^4} \end{array} \mathrm{GL}(5, R) \begin{array}{c} \xrightarrow{i_1^5} \\ \xrightarrow{i_6^5} \end{array} \dots \begin{array}{c} \xrightarrow{i_1^{n-1}} \\ \xrightarrow{i_n^{n-1}} \end{array} \mathrm{GL}(n, R) \begin{array}{c} \xrightarrow{i_1^n} \\ \xrightarrow{i_{n+1}^n} \end{array} \mathrm{GL}(n+1, R) \begin{array}{c} \xrightarrow{i_1^{n+1}} \\ \xrightarrow{i_{n+2}^{n+1}} \end{array} \dots \quad (3)$$

By the general theory of colimits, the group M comes equipped with canonical group homomorphisms $\alpha_n: \mathrm{GL}(n, R) \longrightarrow M$ satisfying the relations

$$\alpha_{n+1} \circ i_j^n = \alpha_n \quad (j = 1, n+1). \quad (4)$$

Theorem. *The group M is canonically isomorphic to $K_1(R)$.*

Proof. First we observe that in M we have the commutation relation

$$\alpha_n(X)\alpha_n(Y) = \alpha_n(Y)\alpha_n(X) \quad \text{for all } X, Y \in \mathrm{GL}(n, R). \quad (5)$$

Indeed, by (4) we can re-write

$$\alpha_n(X) = \alpha_{2n}(i_{2n}^{2n-1} i_{2n-1}^{2n-2} \dots i_{n+1}^n(X)) \quad \text{and} \quad \alpha_n(Y) = \alpha_{2n}(i_1^{2n-1} i_1^{2n-2} \dots i_1^n(Y)),$$

and the arguments of α_{2n} are block-diagonal matrices of the form

$$\begin{pmatrix} X & 0 \\ 0 & I_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_n & 0 \\ 0 & Y \end{pmatrix}$$

which commute in $\mathrm{GL}(2n, R)$; hence their images under α_{2n} must commute as well.

Date: October 10, 2019.

2010 *Mathematics Subject Classification.* Primary 19B99; secondary 16E20.

Key words and phrases. K -theory; invertible matrix; elementary matrix.

1 Let $E(n, R)$ denote the subgroup of $\mathrm{GL}(n, R)$ generated by the elementary matrices [Bas68, §V.1].
 2 Since $E(n, R) = [E(n, R), E(n, R)]$ for all $n \geq 3$, cf. [Bas68, Corollary V.1.5], the commutation rela-
 3 tion (5) implies

$$4 \quad E(n, R) \subseteq \ker(\alpha_n). \quad (6)$$

5 Since the diagram (1) defining $\mathrm{GL}(R)$ is contained in the diagram (3) defining M there is a canonical
 6 group homomorphism $\alpha: \mathrm{GL}(R) \longrightarrow M$ described completely by $\alpha \circ \iota_n = \alpha_n$, that is, $\alpha|_{\mathrm{GL}(n, R)} = \alpha_n$.

7 Let $E(R) = \bigcup_{n \geq 3} E(n, R) \subseteq \mathrm{GL}(R)$ be the stabilisation *via* the embeddings i_{n+1}^n . In view of (6) above
 8 we have the inclusion

$$9 \quad E(R) = \bigcup_{n \geq 3} E(n, R) \subseteq \ker \alpha. \quad (7)$$

10 The group $E(R)$ is normal in $\mathrm{GL}(R)$, cf. [Bas68, Theorem V.2.1], and $K_1(R) = \mathrm{GL}(R)/E(R)$ is an
 11 ABELian group [Bas68, p. 229]. We write $\pi: \mathrm{GL}(R) \longrightarrow K_1(R)$ for the canonical projection. Let
 12 $\pi_n = \pi \circ \iota_n$ denote the restriction of π to $\mathrm{GL}(n, R)$, and write $[X] = \pi_n(X)$ for the class of $X \in \mathrm{GL}(n, R)$
 13 in $K_1(R)$. By (7) we obtain a factorisation $\lambda: K_1(R) \longrightarrow M$ of α with $\lambda \circ \pi = \alpha$. Explicitly, λ is described
 14 by the formula

$$15 \quad \lambda: [X] = \pi_n(X) \mapsto \alpha \circ \iota_n(X) = \alpha_n(X), \quad \text{for } X \in \mathrm{GL}(n, R). \quad (8)$$

16 We observe the relation $\pi_{n+1} \circ i_j^n = \pi_n$. Indeed, for $X \in \mathrm{GL}(n, R)$ the matrices $i_j^n(X)$ and $i_{n+1}^n(X)$
 17 are related by the expression

$$18 \quad i_j^n(X) = P^{-1} i_{n+1}^n(X) P$$

19 for a permutation matrix $P \in \mathrm{GL}(n+1, R)$. It follows that said two matrices have the same image
 20 under π_{n+1} in the ABELian group $K_1(R)$ whence, using (2),

$$21 \quad \pi_{n+1} \circ i_j^n(X) = \pi_{n+1} \circ i_{n+1}^n(X) = \pi \circ \iota_{n+1} \circ i_{n+1}^n(X) = \pi \circ \iota_n(X) = \pi_n(X).$$

22 The various maps π_n thus form a ‘‘cone’’ on the diagram (3) and induce a map $\varrho: M \longrightarrow K_1(R)$ such
 23 that

$$24 \quad \pi_n = \varrho \circ \alpha_n. \quad (9)$$

25 We verify the equality $\lambda \circ \varrho = \mathrm{id}_M$. By the universal property of colimits, it is enough to show that
 26 $\lambda \circ \varrho \circ \alpha_n = \alpha_n$ for all n . But for $X \in \mathrm{GL}(n, R)$ we calculate

$$27 \quad \lambda \circ \varrho \circ \alpha_n(X) \stackrel{(9)}{=} \lambda \circ \pi_n(X) \stackrel{(8)}{=} \alpha_n(X),$$

28 using relation (9) and the explicit description (8) of λ above.

29 Using the same relations again, in opposite order, we finally verify that $\varrho \circ \lambda = \mathrm{id}_{K_1(R)}$. Let $X \in$
 30 $\mathrm{GL}(n, R)$ represent the element $[X] \in K_1(R)$ as before; then $\varrho \circ \lambda([X]) \stackrel{(8)}{=} \varrho \circ \alpha_n(X) \stackrel{(9)}{=} \pi_n(X) = [X]$. \square

31 With minor changes the argument also shows that the the colimit of the group-valued diagram

$$32 \quad \begin{array}{ccccccc} & \xrightarrow{i_1^3} & \xrightarrow{i_1^4} & \xrightarrow{i_1^5} & \xrightarrow{i_1^{n-1}} & \xrightarrow{i_1^n} & \xrightarrow{i_1^{n+1}} \\ \mathrm{GL}(3, R) & \xrightarrow{\quad} & \mathrm{GL}(4, R) & \xrightarrow{\quad} & \mathrm{GL}(5, R) & \xrightarrow{\quad} & \cdots \\ & \xrightarrow{i_4^3} & \xrightarrow{i_5^4} & \xrightarrow{i_6^5} & \xrightarrow{i_n^{n-1}} & \xrightarrow{i_{n+1}^n} & \xrightarrow{i_{n+2}^{n+1}} \end{array}$$

33 is canonically isomorphic to $K_1(R)$.

34 REFERENCES

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