Chaotic Banach algebras

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Abstract

We construct an infinite dimensional non-unital Banach algebra A and $a \in A$ such that the sets $\{za^n : z \in \mathbb{C},\ n \in \mathbb{N}\}$ and $\{(\mathbf{1}+a)^n a : n \in \mathbb{N}\}$ are both dense in A, where $\mathbf{1}$ is the unity in the unitalization $A^{\#} = A \oplus \mathtt{span}\{\mathbf{1}\}$ of A. As a byproduct, we get a hypercyclic operator T on a Banach space such that $T \oplus T$ is non-cyclic and $\sigma(T) = \{1\}$.

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1 Introduction

All vector spaces in this article are over the field $\mathbb{C}$ of complex numbers. As usual, $\mathbb{R}$ is the field of real numbers, $\mathbb{T} = \{x \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leqslant 1\}$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{N}$ is the set of positive integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. If X and Y are topological vector spaces, $L(X,Y)$ stands for the space of continuous linear operators from X to Y. We write $L(X)$ instead of $L(X,X)$ and X^* instead of $L(X,\mathbb{C})$. For $T \in L(X,Y)$, the dual operator $T^* \in L(Y^*, X^*)$ is defined as usual: $T^*f = f \circ T$. Recall that $T \in L(X)$ is called *hypercyclic* (respectively, *supercyclic*) if there is $x \in X$ such that the *orbit* $O(T,x) = \{T^n x : n \in \mathbb{Z}_+\}$ (respectively, the *projective orbit* $\{zT^n x : z \in \mathbb{C},\ n \in \mathbb{Z}_+\}$) is dense in X. Such an x is called a *hypercyclic vector* (respectively, a *supercyclic vector*) for T. We refer to [1] and references therein for additional information on hypercyclicity and supercyclicity. Recall that a function $\pi : A \to \mathbb{R}_+$ defined on a complex algebra A is called *submultiplicative* if $\pi(ab) \leqslant \pi(a)\pi(b)$ for any $a, b \in A$. A *Banach algebra* is a complex (maybe non-unital) algebra A with a complete submultiplicative norm (if A is unital, it is usually also assumed that $\|\mathbf{1}\| = 1$, where $\mathbf{1}$ is the unity in A). We say that A is *non-trivial* if $A \neq \{0\}$.

Definition 1.1. Let A be a Banach algebra. We say that A is *supercyclic* if there is $a \in A$ for which $\{za^n : z \in \mathbb{C},\ n \in \mathbb{N}\}$ is dense in A. Such an a is called a *supercyclic element* of A. We say that A is *almost hypercyclic* if there is $a \in A$ for which $\{(\mathbf{1}+a)^n a : n \in \mathbb{N}\}$ is dense in A. Such an a is called an *almost hypercyclic element* of A. Finally, we say that a Banach algebra A is *chaotic* if there is $a \in A$ which is a supercyclic and an almost hypercyclic element of A. In other words, both $\{za^n : z \in \mathbb{C},\ n \in \mathbb{N}\}$ and $\{(\mathbf{1}+a)^n a : n \in \mathbb{N}\}$ are dense in A. Such an a is called a *chaotic element* of A.

In the above definition, $\mathbf{1}$ is the unit element in the unitalization $A^{\#} = A \oplus \mathtt{span}\{\mathbf{1}\}$ of A. Note that a is a supercyclic element of A if and only if a is a supercyclic vector for the multiplication operator

$$M_a \in L(A), \quad M_a b = ab \tag{1.1}$$

and a is an almost hypercyclic element of A if and only if a is a hypercyclic vector for $I + M_a$. There is no point to consider 'hypercyclic Banach algebras' in the obvious sense. Indeed, in [10] it is observed that a multiplication operator on a commutative Banach algebra is never hypercyclic. Obviously, supercyclic as well as almost hypercyclic Banach algebras are commutative and separable.

Theorem 1.2. *There exists a chaotic infinite dimensional Banach algebra A.*

In order to emphasize the value of Theorem 1.2, we would like to mention few related facts. A Banach algebra is called *radical* if it coincides with its Jackobson radical [4]. If A is a Banach algebra and X is a Banach A-bimodule [4], then $D \in L(A,X)$ is called a *derivation* if $D(ab) = (Da)b + a(Db)$ for each $a, b \in A$.

A Banach algebra A is called *weakly amenable* if every derivation $D : A \to A^*$ (with the natural bimodule structure on A^*) has the shape $Da = ax - xa$ for some $x \in A^*$. It is well-known [4] that a commutative Banach algebra A is weakly amenable if and only if there is no non-zero derivations $D : A \to X$ taking values in a commutative Banach A-bimodule X.

Theorem 1.3. *Let A be a supercyclic Banach algebra of dimension > 1. Then A is infinite dimensional, radical and weakly amenable.*

According to Theorem 1.3, Theorem 1.2 provides an infinite dimensional radical weakly amenable Banach algebra. We would like to mention the work [7] by Loy, Read, Runde, and Willis, who constructed a non-unital Banach algebra, generated by one element x and which has a bounded approximate identity of the shape $x^{n_k}/\|x^{n_k}\|$, where $\{n_k\}_{k\in\mathbb{N}}$ is a strictly increasing sequence of positive integers. Such an algebra is automatically radical and weakly amenable. Theorem 1.3 shows that the same properties are forced by supercyclicity. It is also worth mentioning that Read [8] constructed a commutative amenable radical Banach algebra, but this algebra is not generated by one element.

Proposition 1.4. *Let A be a non-trivial commutative Banach algebra and $M = cI + M_a \in L(A)$, where $a \in A$ and $c \in \mathbb{C}$. Then $M \oplus M$ is non-cyclic.*

Proof. Let $(x, y) \in A^2$. If $M_x = M_y = 0$, then $(M \oplus M)^n(x, y) = c^n(x, y)$ for every $n \in \mathbb{Z}_+$ and therefore (x, y) is not a cyclic vector for $M_a \oplus M_a$. Otherwise, the operator $T \in L(A^2, A)$, $T(u, v) = yu - xv$ is non-zero. Moreover, $T((M \oplus M)^n(x, y)) = T((c\mathbf{1} + a)^n x, (c\mathbf{1} + a)^n y) = y(c\mathbf{1} + a)^n x - x(c\mathbf{1} + a)^n y = 0$ since A is commutative. Thus $(M \oplus M)^n(x, y) \in \mathtt{ker}\, T$ for each $n \in \mathbb{Z}_+$. Since $\mathtt{ker}\, T$ is a proper closed linear subspace of A^2, (x, y) again is not a cyclic vector for $M \oplus M$. □

By Proposition 1.4, Theorem 1.2 provides hypercyclic operators T with non-cyclic $T \oplus T$. The existence of such operators used to be an open problem until De La Rosa and Read [5] (see also [2] and [1]) constructed such operators. One can observe that the spectra of the operators in [5, 2] contain a disk centered at 0 of radius > 1. On the other hand [1], any separable infinite dimensional complex Banach space supports hypercyclic operators with the spectrum being the singleton $\{1\}$. It remained unclear whether a hypercyclic operator T with non-cyclic $T \oplus T$ can have small spectrum. Theorem 1.2 provides such an operator. Indeed, by Theorem 1.2, there are an infinite dimensional Banach algebra A and $a \in A$ such that $T = I + M_a$ is hypercyclic. By Theorem 1.3, A is radical and therefore M_a is quasinilpotent. Hence the spectrum $\sigma(T)$ of T is $\{1\}$. Thus we arrive to the following corollary.

Corollary 1.5. *There exists a hypercyclic continuous linear operator T on an infinite dimensional Banach space such that $T \oplus T$ is non-cyclic and $\sigma(T) = \{1\}$.*

It seems to be of independent interest that supercyclic operators T with non-cyclic $T \oplus T$ can be found among multiplication operators on commutative Banach algebras, while hypercyclic operators T with non-cyclic $T \oplus T$ can be of the shape identity plus a multiplication operator.

2 Proof of Theorem 1.3

Since a Banach space of finite dimension > 1 supports no supercyclic operators (see [12]), a supercyclic Banach algebra of dimension > 1 must be infinite dimensional. According to [10, Proposition 3.4], an infinite dimensional commutative Banach algebra B is radical if there is $b \in B$ for which the multiplication operator M_b is supercyclic. Since a supercyclic Banach algebra of dimension > 1 is infinite dimensional, commutative and has a supercyclic multiplication operator, A is radical.

It remains to show that that A is weakly amenable. Assume the contrary. Then there is a commutative Banach A-bimodule X and a non-zero derivation $D \in L(A, X)$. Since A is supercyclic, there is $a \in A$ such that $\{za^n : z \in \mathbb{C},\ n \in \mathbb{N}\}$ is dense in A. Since $\dim A > 1$, $\Omega_m = \{za^n : z \in \mathbb{C},\ n \geqslant m\}$ is dense in A for each $m \in \mathbb{N}$. Consider the operator $M \in L(A, X)$, $Mb = bDa$. Since X is commutative and D is a derivation, we have $D(a^n) = na^{n-1}Da$ for $n \geqslant 2$. If $M = 0$, then $D(a^n) = na^{n-1}Da = nM(a^{n-1}) = 0$ for $n \geqslant 2$. Hence D vanishes on the dense set Ω_2. Since D is continuous, $D = 0$, which is a contradiction.

Hence $M \neq 0$ and therefore $M^* \neq 0$. Thus there is $f \in X^*$ such that $g = M^* f^*$ is a non-zero element of A^*. Then for each $n \in \mathbb{N}$, we have $g(a^n) = M^* f(a^n) = f(a^n Da) = \frac{f(D(a^{n+1}))}{n+1}$. Hence

$$|g(a^n)| = \tfrac{|f(D(a^{n+1}))|}{n+1} \leqslant \tfrac{C\|a^n\|}{n+1}, \quad \text{where } C = \|D\|\,\|f\|\,\|a\|.$$

Now let $m \in \mathbb{N}$ be such that $\frac{C}{m+1} < \frac{\|g\|}{2}$ and $W = \bigl\{u \in A : |g(u)| > \frac{\|g\|\|u\|}{2}\bigr\}$. Clearly W is non-empty and open. By the last display, $\Omega_m \cap W = \varnothing$, which contradicts the density of Ω_m in A. This contradiction completes the proof of Theorem 1.3.

3 Proof of Theorem 1.2

From now on, $\mathbb{P}$ is the algebra $\mathbb{C}[z]$ of polynomials with complex coefficients in one variable z. Clearly, $\mathbb{P}_0 = \{p \in \mathbb{P} : p(0) = 0\}$ is an ideal in $\mathbb{P}$ of codimension 1. There is a sequence $\{p_n\}_{n\in\mathbb{N}}$ in $\mathbb{P}_0$ such that

$$\{p_n : n \in \mathbb{N}\} \text{ is dense in } \mathbb{P}_0 \text{ with respect to any seminorm on } \mathbb{P}_0. \tag{3.1}$$

Indeed, (3.1) is satisfied if, for instance, $\{p_n : n \in \mathbb{N}\}$ is the set of all polynomials in $\mathbb{P}_0$ with coefficients from a fixed dense countable subset of $\mathbb{C}$, containing 0.

Lemma 3.1. *Let π be a non-zero submultiplicative seminorm on $\mathbb{P}_0$ and $\{p_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathbb{P}_0$ satisfying* (3.1). *Assume also that there exist sequences $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ of positive integers and a sequence $\{c_k\}_{k\in\mathbb{N}}$ of complex numbers such that $\pi(c_k z^{n_k} - p_k) \to 0$ and $\pi(z(1+z)^{m_k} - p_k) \to 0$. Then π is a norm and the completion A of $(\mathbb{P}_0, \pi)$ is an infinite dimensional chaotic Banach algebra with z as a chaotic element.*

Proof. Let $I = \{q \in \mathbb{P}_0 : \pi(q) = 0\}$. Since π is submultiplicative, I is an ideal in $\mathbb{P}_0$ and therefore in $\mathbb{P}$. Since π is non-zero, $I \neq \mathbb{P}_0$. Thus $\mathbb{P}_0/I$ with the norm $\|q+I\| = \pi(q)$ is a non-trivial complex algebra with a submultiplicative norm. Since $\pi(z(1+z)^{m_k} - p_k) \to 0$, (3.1) implies that the operator $p+I \mapsto (1+z)p+I$ on $\mathbb{P}_0/I$ is hypercyclic with the hypercyclic vector $z+I$. Since there is no hypercyclic operator on a non-trivial finite dimensional normed space [12], $\mathbb{P}_0/I$ is infinite dimensional and therefore I has infinite codimension in $\mathbb{P}$. Since the only ideal in $\mathbb{P}$ of infinite codimension is $\{0\}$, $I = \{0\}$ and therefore π is a norm.

Thus the completion A of $(\mathbb{P}_0, \pi)$ is an infinite dimensional Banach algebra. Conditions $\pi(c_k z^{n_k} - p_k) \to 0$ and $\pi(z(1+z)^{m_k} - p_k) \to 0$ together with (3.1) imply that A is chaotic with z as a chaotic element. $\square$

It remains to construct a seminorm on $\mathbb{P}_0$, which will allow us to apply Lemma 3.1.

3.1 Ideals in $\mathbb{A}^{[k]}$ and submultiplicative norms on $\mathbb{P}$

For $k \in \mathbb{N}$, we consider the commutative Banach algebra $\mathbb{A}^{[k]}$ of the power series

$$a = \sum_{n\in\mathbb{Z}_+^k} a_n u_1^{n_1} \dots u_k^{n_k}, \quad \text{with } \|a\|_{[k]} = \sum_{n\in\mathbb{Z}_+^k} |a_n| < \infty$$

with the natural multiplication. We will treat the elements of $\mathbb{A}^{[k]}$ both as power series and as continuous functions $u \mapsto a(u_1, \dots, u_k)$ on $\overline{\mathbb{D}}^k$, holomorphic on $\mathbb{D}^k$. Note that as a Banach space $\mathbb{A}^{[k]}$ is $\ell_1(\mathbb{Z}_+^k)$. In particular, the underlying Banach space of $\mathbb{A}^{[k]}$ can be treated as the dual space of $c_0(\mathbb{Z}_+^k)$, which allows us to speak about the weak-$*$ topology on $\mathbb{A}^{[k]}$.

For a non-empty open subset U of $\mathbb{C}$ we also consider the complex algebra $\mathcal{H}_U$ of holomorphic functions $f : U \to \mathbb{C}$ endowed with the Fréchet space topology of uniform convergence on compact subsets of U. For $\gamma > 0$, we write $\mathcal{H}_\gamma$ instead of $\mathcal{H}_{\gamma\mathbb{D}}$.

If $\xi = (\xi_1, \dots, \xi_k) \in \mathbb{P}_0^k$ and $a \in \mathbb{A}^{[k]}$, we can consider $a(\xi_1, \dots, \xi_k)$ as a power series

$$a(\xi_1, \dots, \xi_k)(z) = a(\xi_1(z), \dots, \xi_k(z)) = \sum_{m=1}^{\infty} \alpha_m(a, \xi) z^m, \tag{3.2}$$

which converges uniformly on the compact subsets of the disk $\gamma(\xi)\mathbb{D}$, where

$$\gamma(\xi) = \sup\{c > 0 : \xi_j(c\mathbb{D}) \subseteq \mathbb{D} \ \text{ for } \ 1 \leqslant j \leqslant k\} > 0.$$

By the Hadamard formula, $\overline{\lim\limits_{m\to\infty}}\, |\alpha_m(a,\xi)|^{1/m} \leqslant \frac{1}{\gamma(\xi)}$ for each $a \in \mathbb{A}^{[k]}$. By the uniform boundedness principle, $\overline{\lim\limits_{m\to\infty}}\, \|\alpha_m(\cdot,\xi)\|^{1/m} \leqslant \frac{1}{\gamma(\xi)}$, where the norm is taken in $\left(\mathbb{A}^{[k]}\right)^*$. Hence the map

$$\Phi_\xi : \mathbb{A}^{[k]} \to \mathcal{H}_{\gamma(\xi)}, \quad \Phi_\xi(a) = a(\xi_1, \dots, \xi_k)$$

is a continuous algebra homomorphism from the Banach algebra $\mathbb{A}^{[k]}$ to the Fréchet algebra $\mathcal{H}_{\gamma(\xi)}$ of holomorphic complex valued functions on the disk $\gamma(\xi)\mathbb{D}$.

Remark 3.2. Note that if U is a connected non-empty open subset of $\mathbb{C}$ and all zeros of a polynomial $p \in \mathbb{P}$ of degree $n \in \mathbb{N}$ are in U, then the ideal J_p, generated by p in the algebra $\mathcal{H}_U$ is closed and has codimension n. It consists of all $f \in \mathcal{H}_U$ such that every zero of p of order $k \in \mathbb{N}$ is also a zero of f of order $\geqslant k$. We write $p\big|f$ to denote the inclusion $f \in J_p$. Note that $\mathcal{H}_U = J_p \oplus \mathtt{span}\,\{1, z, \dots, z^{n-1}\}$.

We use the following notation. If $\xi \in \mathbb{P}_0^k$ and $q \in \mathbb{P}$ has all its zeros in the disk $\gamma(\xi)\mathbb{D}$, then

$$I_{\xi,q} = \{a \in \mathbb{A}^{[k]} : q\big|\Phi_\xi(a)\} \tag{3.3}$$

with $\Phi_\xi(a)$ considered as an element of $\mathcal{H}_{\gamma(\xi)}$. In the case $q = z^n$ with $n \in \mathbb{N}$, we have

$$I_{\xi,z^n} = \{a \in \mathbb{A}^{[k]} : \alpha_j(a,\xi) = 0 \ \text{ for } \ 0 \leqslant j < n\}, \tag{3.4}$$

where $\alpha_j(a,\xi)$ are defined in (3.2). Finally,

$$I_\xi = \mathtt{ker}\,\Phi_\xi = \bigcap_{n=1}^{\infty} I_{\xi,z^n}. \tag{3.5}$$

Lemma 3.3. *Let $\xi = (\xi_1, \dots, \xi_k) \in \mathbb{P}_0^k$ be such that $\xi_1 = z$. Then I_ξ is a closed ideal in $\mathbb{A}^{[k]}$ and for each $q \in \mathbb{P}$, whose zeros are in the disk $\gamma(\xi)\mathbb{D}$, $I_{\xi,q}$ is closed ideal in $\mathbb{A}^{[k]}$ of codimension* $\deg q$. *Moreover, $I_\xi \subset I_{\xi,q}$ and*

$$\|a + I_{\xi,z^n}\|_{A^{[k]}/I_{\xi,z^n}} \to \|a + I_\xi\|_{A^{[k]}/I_\xi} \quad \textit{as } n \to \infty \textit{ for each } a \in \mathbb{A}^{[k]}. \tag{3.6}$$

Furthermore, if $q_n \in \mathbb{P}$ for $n \in \mathbb{N} \cup \{\infty\}$ are polynomials of degree $m \in \mathbb{N}$, whose zeros are in $\gamma(\xi)\mathbb{D}$ and the sequence $\{q_n\}_{n\in\mathbb{N}}$ converges to q_∞ as $n \to \infty$ (in the usual sense in the finite dimensional space of polynomials of degree $\leqslant m$), then

$$\|a + I_{\xi,q_n}\|_{A^{[k]}/I_{\xi,q_n}} \to \|a + I_{\xi,q_\infty}\|_{A^{[k]}/I_{\xi,q_\infty}} \quad \textit{as } n \to \infty \textit{ for each } a \in \mathbb{A}^{[k]}. \tag{3.7}$$

Proof. For brevity, we denote $\gamma = \gamma(\xi)$. It is straightforward to verify that $\Phi_\xi : \mathbb{A}^{[k]} \to \mathcal{H}_\gamma$ is not just continuous but also weak-$*$ continuous. That is Φ_ξ is continuous when $\mathbb{A}^{[k]}$ is equipped with the weak-$*$ topology and the Fréchet space $\mathcal{H}_\gamma$ carries its weak (=weak-$*$) topology.

Since $I_{\xi,q} = \Phi_\xi^{-1}(J_q)$ for every polynomial q whose zeros are all in $\gamma\mathbb{D}$ and $I_\xi = \Phi_\xi^{-1}(0)$, we see that the ideals $I_{\xi,q}$ and I_ξ are weak-$*$ closed and therefore closed in $\mathbb{A}^{[k]}$. Using the equality $\xi_1 = z$, one can readily see that $I_{\xi,q} \oplus \mathtt{span}\,\{1, u_1, \dots, u_1^r\} = \mathbb{A}^{[k]}$, where $r = \deg q - 1$. Thus $I_{\xi,q}$ has codimension $\deg q$ in $\mathbb{A}^{[k]}$.

Obviously, $I_\xi \subseteq I_{\xi,z^{n+1}} \subseteq I_{\xi,z^n}$ for every $n \in \mathbb{N}$. Therefore, the sequence $\|a + I_{\xi,z^n}\|_{A^{[k]}/I_{\xi,z^n}}$ is increasing and bounded above by $\|a + I_\xi\|_{A^{[k]}/I_\xi}$ for every $a \in \mathbb{A}^{[k]}$. Hence

$$c = \lim_{n\to\infty} \|a + I_{\xi,z^n}\|_{A^{[k]}/I_{\xi,z^n}} \leqslant \|a + I_\xi\|_{A^{[k]}/I_\xi} = c_1.$$

The proof of (3.6) will be complete if we show that $c_1 \leqslant c$.

By definition of the quotient norms, we can find $b_n \in I_{\xi,z^n}$ such that $\|a+b_n\|_{[k]} \to c$. By the Banach–Alaoglu theorem, the bounded sequence $\{b_n\}$ has a weak-$*$ accumulation point b in $\mathbb{A}^{[k]}$. Since $b_m \in T_{\xi,z^n}$ for $m \geqslant n$ and each I_{ξ,z^n} is weak-$*$ closed, b belongs to every I_{ξ,z^n} and therefore to their intersection I_ξ: $b \in I_\xi$. Since the norm is weak-$*$ upper semicontinuous (a straightforward consequence of the Hahn–Banach theorem) and $\|a+b_n\|_{[k]} \to c$, we have $\|a+b\|_{[k]} \leqslant c$. Since $b \in I_\xi$, $c_1 = \|a+I_\xi\|_{A^{[k]}/I_\xi} \leqslant \|a+b\| \leqslant c$, which completes the proof of (3.6).

It remains to prove (3.7). Let $m \in \mathbb{N}$ and $\mathcal{P}_m$ be the $(m+1)$-dimensional space of polynomials of degree $\leqslant m$. Let also $q_n \in \mathcal{P}_m$ for $n \in \mathbb{N} \cup \{\infty\}$ be polynomials of degree exactly m, whose zeros are all in $\gamma\mathbb{D}$ and the sequence $\{q_n\}_{n\in\mathbb{N}}$ converges to q_∞ as $n \to \infty$ in the finite dimensional space $\mathcal{P}_m$. Since $\xi_1 = z$, $\mathbb{P} \subset \Phi_\xi(\mathbb{A}^{[k]})$. Indeed, $\Phi_\xi(a) = p$ if $p \in \mathbb{P}$ and $a(u_1,\dots,u_k) = p(u_1)$. Furthermore, $\Phi_\xi^{-1}(\mathbb{P})$ contains the unital subalgebra generated by $u_1,\dots,u_k$ and therefore is dense in $\mathbb{A}^{[k]}$. It is an easy exercise that in every topological vector space X the intersection $L \cap M$ of a dense in X linear subspace L and a finite codimensional closed subspace M is dense in M. It follows that

$$\Phi_\xi^{-1}(\mathbb{P} \cap J_q) = \Phi_\xi^{-1}(\mathbb{P}) \cap I_{\xi,q} \text{ is dense in } I_{\xi,q}$$

for every polynomial q whose zeros are all in $\gamma\mathbb{D}$.

Now take $a \in \Phi_\xi^{-1}(\mathbb{P})$ and denote $c_n = \|a + I_{\xi,q_n}\|_{A^{[k]}/I_{\xi,q_n}}$ for $n \in \mathbb{N} \cup \{\infty\}$. By the above display, for each $\varepsilon > 0$, we can pick $b \in \Phi_\xi^{-1}(\mathbb{P} \cap J_{q_\infty})$ such that $\|a+b\|_{[k]} \leqslant c_\infty + \varepsilon$. Since $b \in \Phi_\xi^{-1}(\mathbb{P} \cap J_{q_\infty})$, $\Phi_\xi(b) = pq_\infty$ for some $p \in \mathbb{P}$. Since $p\mathcal{P}_m$ is an $(m+1)$-dimensional subspace of $\mathbb{P} \subset \Phi_\xi(\mathbb{A}^{[k]})$, we can find an $(m+1)$-dimensional subspace L of $\mathbb{A}^{[k]}$ such that $R = \Phi_\xi\big|_L : L \to p\mathcal{P}_m$ is a linear isomorphism. Set $b_n = b + R^{-1}(p(q_n - q_\infty))$. Since every linear operator on a finite dimensional topological vector space is continuous and $pq_n \to pq_\infty$, we have $\|b_n - b\|_{[k]} \to 0$. On the other hand, the construction of b_n yields $\Phi_\xi(b_n) = pq_n \in J_{q_n}$. Hence $b_n \in I_{\xi,q_n}$ and therefore

$$c_n = \|a + I_{\xi,q_n}\|_{A^{[k]}/I_{\xi,q_n}} \leqslant \|a+b_n\|_{[k]} \to \|a+b\|_{[k]} \leqslant c_\infty + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\overline{\lim\limits_{n\to\infty}}\, c_n \leqslant c_\infty$. In order to verify (3.7) for a it now suffices to show that $\underline{\lim\limits_{n\to\infty}}\, c_n \geqslant c_\infty$. Pick $d_n \in I_{\xi,q_n}$ such that $\|a+d_n\|_{[k]} - c_n \to 0$. Clearly, $\{d_n\}$ is bounded. By the Banach–Alaoglu theorem, every bounded sequence in $\mathbb{A}^{[k]}$ has a weak-$*$ convergent subsequence. Thus, if $\underline{\lim\limits_{n\to\infty}}\, c_n \geqslant c_\infty$ fails, passing to a subsequence, if necessary, we can assume that $c_n \to r < c_\infty$ and $\{d_n\}$ is weak-$*$ convergent to $d \in \mathbb{A}^{[k]}$. Since $d_n \in I_{\xi,q_n}$, $\Phi_\xi(d_n) = q_n f_n$ with $f_n \in \mathcal{H}_\gamma$ for every $n \in \mathbb{N}$. Since $\{d_n\}$ is weak-$*$ convergent to d and Φ_ξ is weak-$*$ continuous, $\{q_n f_n\}$ converges weakly in $\mathcal{H}_\gamma$ to $\Phi_\xi(d) \in \mathcal{H}_\gamma$. Since $\mathcal{H}_\gamma$ is a nuclear (and therefore Montel) Fréchet space, weak and strong convergence of sequences in $\mathcal{H}_\gamma$ coincide. Hence $q_n f_n \to \Phi_\xi(d)$ in the Fréchet space $\mathcal{H}_\gamma$. That is the sequence $\{q_n f_n\}$ of holomorphic on $\gamma\mathbb{D}$ functions converges to $\Phi_\xi(d)$ uniformly on compact subsets of $\gamma\mathbb{D}$. Since $q_n \to q_\infty$ and all zeros of q_∞ are in $\gamma\mathbb{D}$, there exists $\gamma' < \gamma$ and $\delta > 0$ such that $|q_n(z)| \geqslant \delta$ whenever $\gamma' < |z| < \gamma$. It follows that $\{f_n\}$ converges uniformly on compact sets of the last annulus. Since each f_n is holomorphic on $\gamma\mathbb{D}$, $\{f_n\}$ converges in $\mathcal{H}_\gamma$ to some $f_\infty \in \mathcal{H}_\gamma$. Since $q_n \to q_\infty$, we have $q_n f_n \to q_\infty f_\infty$ in $\mathcal{H}_\gamma$. Thus $\Phi_\xi(d) = q_\infty f_\infty \in J_{q_\infty}$ and therefore $d \in I_{\xi,q_\infty}$. Using the weak-$*$ upper semicontinuity of the norm, we obtain

$$c_\infty \leqslant \|a+d\|_{[k]} \leqslant \underline{\lim_{n\to\infty}}\, \|a+d_n\| = \lim_{n\to\infty} c_n = r < c_\infty.$$

This contradiction completes the proof of (3.7) for a form the dense subset $\Phi_\xi^{-1}(\mathbb{P})$ of $\mathbb{A}^{[k]}$. Note also that if a locally uniformly continuous and locally uniformly bounded sequence of maps between two complete metric spaces converges to a continuous map pointwise on a dense set, then it converges everywhere. It remains to notice that each seminorm $a \mapsto \|a + I_{\xi,q_n}\|_{A^{[k]}/I_{\xi,q_n}}$ is Lipschitz with constant 1 and is bounded above by the norm $\|\cdot\|_{[k]}$. Thus the previous remark allows to extend the validity of (3.7) for $a \in \Phi_\xi^{-1}(\mathbb{P})$ to its validity for each $a \in \mathbb{A}^{[k]}$. This completes the proof of (3.7) and of the lemma. □

As we have already mentioned, $\mathbb{P} \subseteq \Phi_\xi(\mathbb{A}^{[k]})$ if $\xi \in \mathbb{P}_0^k$ and $\xi_1 = z$. Hence we can use the above ideals to define seminorms on $\mathbb{P}$. Since $I_\xi = \ker \Phi_\xi$ and $\Phi_\xi(\mathbb{A}^{[k]}) \supseteq \mathbb{P}$, we can write

$$\pi_\xi : \mathbb{P} \to \mathbb{R}_+, \quad \pi_\xi(p) = \|\Phi_\xi^{-1}(p)\|_{\mathbb{A}^{[k]}/I_\xi} = \inf\{\|a\|_{[k]} : a \in \mathbb{A}^{[k]},\ \Phi_\xi(a) = p\}. \tag{3.8}$$

By Lemma 3.3, I_ξ is a closed ideal in $\mathbb{A}^{[k]}$ and therefore π_ξ is a submultiplicative norm on $\mathbb{P}$.

If additionally $q \in \mathbb{P}$ has all its zeros in the disk $\gamma(\xi)\mathbb{D}$, then using the closeness of the ideal $I_{\xi,q}$ in $\mathbb{A}^{[k]}$ and the inclusion $I_\xi \subset I_{\xi,q}$, we can define

$$\pi_{\xi,q} : \mathbb{P} \to \mathbb{R}_+, \quad \pi_{\xi,q}(p) = \|\Phi_\xi^{-1}(p) + I_{\xi,q}\|_{\mathbb{A}^{[k]}/I_{\xi,q}} = \inf\{\|a\|_{[k]} : a \in \mathbb{A}^{[k]},\ q\big|(p - \Phi_\xi(a))\}. \tag{3.9}$$

The function $\pi_{\xi,q}$ is a submultiplicative seminorm on $\mathbb{P}$.

Lemma 3.4. *Let $k \in \mathbb{N}$, $\xi' = (\xi_1, \dots, \xi_{k+1}) \in \mathbb{P}_0^{k+1}$ with $\xi_1 = z$ and $\xi = (\xi_1, \dots, \xi_k)$. Then $\pi_{\xi'}(p) \leqslant \pi_\xi(p)$ for all $p \in \mathbb{P}$. Moreover, if U is a connected open subset of $\gamma(\xi)\mathbb{D}$, $0 \in U$, $\xi_{k+1}(U) \subseteq \mathbb{D}$ and $q \in \mathbb{P} \setminus \{0\}$ is a divisor of ξ_{k+1} and has all its zeros in U, then $\pi_{\xi,q}(p) \leqslant \pi_{\xi'}(p)$ for every $p \in \mathbb{P}$.*

Proof. For any $p \in \mathbb{P}$ and $a \in \mathbb{A}^{[k]}$ satisfying $\Phi_\xi(a) = p$, we have $\Phi_{\xi'}(b) = p$ and $\|a\|_{[k]} = \|b\|_{[k+1]}$ with $b(u_1, \dots, u_{k+1}) = a(u_1, \dots, u_k)$. By (3.8), $\pi_{\xi'}(p) \leqslant \pi_\xi(p)$ for each $p \in \mathbb{P}$. Now assume that U is a connected open subset of $\gamma(\xi)\mathbb{D}$, $0 \in U$, $\xi_{k+1}(U) \subseteq \mathbb{D}$ and $q \in \mathbb{P} \setminus \{0\}$ is a divisor of ξ_{k+1} and has all its zeros in U. Let $p \in \mathbb{P}$ and $a \in \mathbb{A}^{[k+1]}$ be such that $\Phi_{\xi'}(a) = p$. By definition of $\mathbb{A}^{[k+1]}$,

$$a = b_0 + \sum_{n=1}^{\infty} b_n u_{k+1}^n, \quad \text{where } b_j \in \mathbb{A}^{[k]} \text{ and } \ \|a\|_{[k+1]} = \sum_{j=0}^{\infty} \|b_j\|_{[k]}. \tag{3.10}$$

By the definitions of Φ_ξ and $\Phi_{\xi'}$, we get

$$p = \Phi_{\xi'}(a) = \sum_{n=0}^{\infty} \Phi_\xi(b_n)\xi_{k+1}^n \quad \text{in } \mathcal{H}_{\gamma(\xi')}. \tag{3.11}$$

By (3.10), the series $\sum b_n$ converges absolutely in the Banach space $\mathbb{A}^{[k]}$. Since $\Phi_\xi : \mathbb{A}^{[k]} \to \mathcal{H}_\gamma$ is a continuous linear operator, the series $\sum \Phi_\xi(b_n)$ converges absolutely in the Fréchet space $\mathcal{H}_{\gamma(\xi)}$ and therefore in the Fréchet space $\mathcal{H}_U$. Since $\xi_{k+1}(U) \subseteq \mathbb{D}$, the series in (3.11) converges in $\mathcal{H}_U$. Since U is open, connected and contains 0, the sum of the series in (3.11) and p coincide as functions on U by the uniqueness theorem: they are both holomorphic on U and have the same Taylor series at 0. Since $q\big|\xi_{k+1}$, (3.11) implies that $q\big|(p - \Phi_\xi(b_0))$ in $\mathcal{H}_U$. Since all zeros of q are in U, $q\big|(p - \Phi_\xi(b_0))$ in $\mathcal{H}_{\gamma(\xi)}$. By (3.9) and (3.10), $\pi_{\xi,q}(p) \leqslant \|b_0\|_{[k]} \leqslant \|a\|_{[k+1]}$. Since a is an arbitrary element of $\mathbb{A}^{[k+1]}$ satisfying $\Phi_{\xi'}(a) = p$, (3.8) implies that $\pi_{\xi,q}(p) \leqslant \pi_{\xi'}(p)$. □

Lemma 3.5. *Let $q \in \mathbb{P}_0$, $n \in \mathbb{N}$ and $k > 0$ be such that $\deg q < n$. For every $c > 0$, let $\delta(c) = (2kc)^{-1/n}$ and $q_c = k(cz^n - q) \in \mathbb{P}_0$. Then for every sufficiently large $c > 0$, $q_c(\delta(c)\mathbb{D}) \subseteq \mathbb{D}$ and all zeros of q_c belong to $\delta(c)\mathbb{D}$.*

Proof. Obviously, $\lim\limits_{c\to\infty} \delta(c) = 0$. Since $q(0) = 0$, there is $\alpha > 0$ such that $|q(z)| \leqslant \alpha|z|$ for all $z \in \mathbb{D}$. Clearly, it suffices to show that $q_c(\delta(c)\mathbb{D}) \subseteq \mathbb{D}$ and all zeros of q_c belong to $\delta(c)\mathbb{D}$ whenever $\delta(c) < \min\{1, \frac{1}{2k\alpha}\}$.

Let $c > 0$ be such that $\delta(c) < \min\{1, \frac{1}{2k\alpha}\}$. If $z \in \delta(c)\mathbb{D}$, then $|kcz^n| < kc\delta(c)^n = \frac{kc}{2kc} = \frac{1}{2}$ and $|kq(z)| \leqslant k\alpha\delta(c) < \frac{k\alpha}{2k\alpha} = \frac{1}{2}$. Hence $|q_c(z)| \leqslant |kcz^n| + |kq(z)| < \frac{1}{2} + \frac{1}{2} = 1$. Thus $q_c(\delta(c)\mathbb{D}) \subseteq \mathbb{D}$.

Now if $|z| = \delta(c)$, then $|kcz^n| = kc\delta(c)^n = \frac{kc}{2kc} = \frac{1}{2}$, but $|kq(z)| \leqslant k\alpha\delta(c) < \frac{k\alpha}{2k\alpha} = \frac{1}{2}$. By the Rouché theorem [6], $q_c = kcz^n - kq$ has the same number of zeros (counting with multiplicity) in $\delta(c)\mathbb{D}$ as kcz^n. The latter has $n = \deg q_c$ zeros in $\delta(c)\mathbb{D}$. Hence all the zeros of q_c are in $\delta(c)\mathbb{D}$. □

The proof of the next lemma is postponed until Section 4.

Lemma 3.6. *Let $k, \delta > 0$, $p \in \mathbb{P} \setminus \{0\}$ and $m \in \mathbb{N}$. Then for every sufficiently large $n \in \mathbb{N}$, there exists a connected open set $W_n \subset \mathbb{C}$ such that $0 \in W_n \subseteq \delta\mathbb{D}$ and the polynomial $q_n = kz((1+z)^n - p)$ has at least m zeros (counting with multiplicity) in W_n and satisfies $q_n(W_n) \subseteq \mathbb{D}$.*

Corollary 3.7. *Let $k > 0$, $p \in \mathbb{P} \setminus \{0\}$ and $m \in \mathbb{N}$. Then there is $n_0 \in \mathbb{N}$ and sequences $\{W_n\}_{n \geqslant n_0}$ of connected non-empty open subsets of $\mathbb{C}$ containing 0 and $\{r_n\}_{n \geqslant n_0}$ of degree m polynomials such that $r_n \to z^m$, $\lim\limits_{n\to\infty} \sup\limits_{z \in W_n} |z| = 0$, each r_n is a divisor of $q_n = kz((1+z)^n - p)$, $q_n(W_n) \subseteq \mathbb{D}$ and all zeros of r_n are in W_n for each $n \geqslant n_0$.*

Proof. Applying Lemma 3.6 with $\delta = 2^{-k}$ for $k \in \mathbb{Z}_+$, we find a strictly increasing sequence $\{n_k\}_{k \in \mathbb{Z}_+}$ of positive integers such that for every $k \in \mathbb{Z}_+$ and every $n \geqslant n_k$, there is a connected open subset $W_{k,n}$ of $\mathbb{C}$ for which

$$\begin{array}{l} 0 \in W_{k,n} \subseteq 2^{-k}\mathbb{D},\ q_n(W_{k,n}) \subseteq \mathbb{D} \text{ and} \\ q_n \text{ has at least } m \text{ zeros in } W_{k,n} \text{ for every } k \in \mathbb{Z}_+ \text{ and } n \geqslant n_k. \end{array} \tag{3.12}$$

The latter means that we can pick $\lambda_{k,n,1}, \dots, \lambda_{k,n,m} \in W_{k,n}$ such that $r_{k,n} = \prod\limits_{j=1}^{m} (z - \lambda_{k,n,j})$ is a divisor of q_n. Now for every $n \geqslant n_0$, we define $r_n = r_{k,n}$ and $W_n = W_{k,n}$ whenever $n_k \leqslant n < n_{k+1}$. According to (3.12), each r_n is a divisor of q_n, each r_n has all its zeros in W_n, $q_n(W_n) \subseteq \mathbb{D}$ and $W_n \subseteq 2^{-k}\mathbb{D}$ provided $n_k \leqslant n < n_{k+1}$. Hence $\lim\limits_{n\to\infty} \sup\limits_{z \in W_n} |z| = 0$ and $r_n \to z^m$. ☐

3.2 Proof of Theorem 1.2 modulo Lemma 3.6

Now we take Lemma 3.6 as granted and prove Theorem 1.2. Fix a sequence $\{p_n\}_{n \in \mathbb{N}}$ in $\mathbb{P}_0 \setminus \{0\}$ satisfying (3.1). We describe an inductive procedure of constructing sequences $\{\xi_k\}_{k \in \mathbb{N}}$ in $\mathbb{P}_0$, $\{n_k\}_{k \in \mathbb{N}}$ of natural numbers and $\{c_{2k}\}_{k \in \mathbb{N}}$ of positive numbers such that

(A0) $\xi_1 = z$ and $n_1 = 1$;
(A1) $\pi_{\xi_{[k]}}(z) > \frac{1}{2}$ for each $k \in \mathbb{N}$, where $\xi_{[k]} = (\xi_1, \dots, \xi_k) \in \mathbb{P}_0^k$;
(A2) $n_k > n_{k-1}$ for $k \geqslant 2$;
(A3) $\xi_k = k(c_k z^{n_k} - p_{k/2})$ for even $k \geqslant 2$ and $\xi_k = k(z(1+z)^{n_k} - p_{(k-1)/2})$ for odd $k \geqslant 3$.

First, we take $n_1 = 1$, $\xi_1 = z$ and observe that $\pi_{\xi_{[1]}}(a_0 + a_1 z + \dots + a_m z^m) = |a_0| + \dots + |a_m|$. In particular, $\pi_{\xi_{[1]}}(z) = 1 > \frac{1}{2}$. Thus (A0–A3) for $k = 1$ are satisfied and we have got the basis of induction. It remains to describe the induction step. Let $k \geqslant 2$ and ξ_j, n_j for $j < k$ and c_j for $j < k$ satisfying (A0–A3) are already constructed. We shall construct ξ_k, n_k and c_k (if k is even), satisfying (A1–A3).

Denote $\gamma = \gamma(\xi_{[k-1]})$. By Lemma 3.3, $\pi_{\xi_{[k-1]}, z^n}(z) \to \pi_{\xi_{[k-1]}}(z)$ as $n \to \infty$. By (A1) for $k-1$, $\pi_{\xi_{[k-1]}}(z) > \frac{1}{2}$. Hence we can pick $m \in \mathbb{N}$ such that

$$\pi_{\xi_{[k-1]}, z^n}(z) > \tfrac{1}{2} \text{ for every } n \geqslant m. \tag{3.13}$$

Case 1: k is even. By (3.13), there is $n_k \in \mathbb{N}$ such that $n_k > \max\{n_{k-1}, \deg p_{k/2}\}$ and $\pi_{\xi_{[k-1]}, z^{n_k}}(z) > \frac{1}{2}$. For $c > 0$, we consider the degree n_k polynomial $q_c = k(cz^{n_k} - p_{k/2}) \in \mathbb{P}_0$ and denote $\delta(c) = (2kc)^{-1/n_k}$. Clearly, $\delta(c) \to 0$ as $c \to \infty$. By Lemma 3.5,

$$\delta(c) < \gamma,\ q_c(\delta(c)\mathbb{D}) \subseteq \mathbb{D} \text{ and all zeros of } q_c \text{ are in } \delta(c)\mathbb{D} \text{ for all sufficiently large } c > 0. \tag{3.14}$$

Since $\frac{1}{kc} q_c = z^{n_k} - \frac{1}{c} p_{k-1} \to z^{n_k}$ as $c \to \infty$, Lemma 3.3 implies that

$$\pi_{\xi_{[k-1]}, q_c}(p) = \pi_{\xi_{[k-1]}, \frac{1}{kc} q_c}(p) \to \pi_{\xi_{[k-1]}, z^{n_k}}(p) \ \text{ as } c \to \infty \text{ for every } p \in \mathbb{P}. \tag{3.15}$$

Using (3.15), (3.14) and the inequality $\pi_{\xi_{[k-1]}}(z) > \frac{1}{2}$, we can choose $c_k > 0$ large enough in such a way that $\delta = \delta(c_k) < \gamma$, all zeros of $\xi_k = q_{c_k} = k(c_k z^{n_k} - p_{k/2})$ are in $\delta\mathbb{D}$, $\xi_k(\delta\mathbb{D}) \subseteq \mathbb{D}$ and $\pi_{\xi_{[k-1]}, \xi_k}(z) > \frac{1}{2}$. By

Lemma 3.4, $\pi_{\xi_{[k]}}(p) \geqslant \pi_{\xi_{[k-1]},\xi_k}(p)$ for every $p \in \mathbb{P}$. In particular, $\pi_{\xi_{[k]}}(z) \geqslant \pi_{\xi_{[k-1]},\xi_k}(z) > \frac{1}{2}$. It remains to notice that (A1–A3) are satisfied.

Case 2: k is odd. By (3.13), $\pi_{\xi_{[k-1]},z^m}(z) > \frac{1}{2}$. By Corollary 3.7, there is $l \in \mathbb{N}$ and sequences $\{W_n\}_{n\geqslant l}$ of connected non-empty open subsets of $\mathbb{C}$ containing 0 and $\{r_n\}_{n\geqslant l}$ of degree m polynomials such that $r_n \to z^m$, $\lim\limits_{n\to\infty} \sup\limits_{z\in W_n} |z| = 0$, each r_n is a divisor of $q_n = k(z(1+z)^n - p_{(k-1)/2})$, $q_n(W_n) \subseteq \mathbb{D}$ and all zeros of r_n are in W_n for each $n \in \mathbb{N}$. By Lemma 3.3, $\pi_{\xi_{[k-1]},r_n}(z) \to \pi_{\xi_{[k-1]},z^m}(z) > \frac{1}{2}$ as $n \to \infty$ and therefore we can pick $n_k > \max\{l, n_{k-1}\}$ such that $\pi_{\xi_{[k-1]},r_{n_k}}(z) > \frac{1}{2}$ and $W_{n_k} \subseteq \gamma\mathbb{D}$. Put $\xi_k = q_{n_k} = k(z(1+z)^{n_k} - p_{(k-1)/2})$. By Lemma 3.4, $\pi_{\xi_{[k]}}(z) \geqslant \pi_{\xi_{[k-1]},r_{n_k}}(z) > \frac{1}{2}$. It remains to notice that (A1–A3) are again satisfied.

This concludes the inductive construction of the sequences $\{\xi_k\}_{k\in\mathbb{N}}$, $\{n_k\}_{k\in\mathbb{N}}$ and $\{c_{2k}\}_{k\in\mathbb{N}}$ satisfying (A0–A3). By Lemma 3.4, $\pi_{\xi_{[k+1]}}(p) \leqslant \pi_{\xi_{[k]}}(p)$ for every $p \in \mathbb{P}$. Thus, $\{\pi_{\xi_{[k]}}\}_{k\in\mathbb{N}}$ is a pointwise decreasing sequence of submultiplicative norms on $\mathbb{P}$. Hence the formula $\pi(p) = \lim\limits_{k\to\infty} \pi_{\xi_{[k]}}(p)$ defines a submultiplicative seminorm on $\mathbb{P}$. By (A1), $\pi_{\xi_{[k]}}(z) > \frac{1}{2}$ for each $k \in \mathbb{N}$ and therefore $\pi(z) \geqslant \frac{1}{2} > 0$. Hence π is non-zero. From (3.8) it immediately follows that $\pi_{\xi_{[k]}}(\xi_k) \leqslant 1$ for every $k \in \mathbb{N}$. Indeed, $\|u_k\|_{[k]} = 1$ and $\Phi_{\xi_{[k]}}(u_k) = \xi_k$. Hence $\pi(\xi_k) \leqslant \pi_{\xi_{[k]}}(\xi_k) \leqslant 1$. By (A3), $\xi_{2k} = 2k(c_{2k}z^{n_{2k}} - p_k)$ for $k \in \mathbb{N}$. Hence $\pi(c_{2k}z^{n_{2k}} - p_k) \leqslant \frac{1}{2k}$ for every $k \in \mathbb{N}$ and therefore $\pi(c_{2k}z^{n_{2k}} - p_k) \to 0$. By (A3), $\xi_{2k+1} = (2k+1)(z(1+z)^{n_{2k+1}} - p_k)$ for $k \in \mathbb{N}$. Hence $\pi(z(1+z)^{n_{2k+1}} - p_k) \leqslant \frac{1}{2k+1}$ for every $k \in \mathbb{N}$ and therefore $\pi(z(1+z)^{n_{2k+1}} - p_k) \to 0$. Thus all conditions of Lemma 3.1 are satisfied. By Lemma 3.1, the restriction of π to $\mathbb{P}_0$ is a submultiplicative norm on $\mathbb{P}_0$ and the completion of the normed algebra $(\mathbb{P}_0, \pi)$ is an infinite dimensional chaotic Banach algebra with z being a chaotic element. The proof of Theorem 1.2 modulo Lemma 3.6 is complete.

4 Proof of Lemma 3.6

Our main instrument is the argument principle [6]. We recall the related basic concepts. An *oriented path* Γ in $\mathbb{C}$ with the *source* $s(\Gamma)$ and the *end* $e(\Gamma)$ is a set of the shape $\Gamma = \varphi([a,b])$, where $\varphi : [a,b] \to \mathbb{C}$ is continuous, $\varphi(a) = s(\Gamma)$, $\varphi(b) = e(\Gamma)$ and $\varphi\big|_{(a,b)}$ is injective. Such a map φ is a *parametrization* of the path Γ. The oriented path Γ is *closed* if $s(\Gamma) = e(\Gamma)$. If Γ is an oriented path in $\mathbb{C}$ and $f : \Gamma \to \mathbb{C} \setminus \{0\}$ is continuous, we can find continuous $\varphi : [a,b] \to \Gamma$ and $\psi : [a,b] \to \mathbb{R}$ such that $\varphi(a) = s(\Gamma)$, $\varphi(b) = e(\Gamma)$ and $\frac{f(\varphi(t))}{|f(\varphi(t))|} = e^{i\psi(t)}$ for every $t \in [a,b]$. The number $\frac{\psi(b)-\psi(a)}{2\pi}$ does not depend on the choice of φ and ψ and is called the *winding number of f along the path* Γ and denoted $w(f,\Gamma)$. Alternatively, $2\pi w(f,\Gamma)$ is the *variation of the argument of f along* Γ.

We need few well-known properties of the winding numbers. If Γ and Γ' are two non-closed oriented paths with $e(\Gamma) = s(\Gamma')$ and $(\Gamma \setminus \{e(\Gamma), s(\Gamma)\}) \cap (\Gamma' \setminus \{e(\Gamma'), s(\Gamma')\}) = \varnothing$, then $\Gamma \cup \Gamma'$ can be naturally considered as an oriented path with the source $s(\Gamma)$ and the end $e(\Gamma')$. Then

$$w(f, \Gamma\cup\Gamma') = w(f,\Gamma) + w(f,\Gamma') \quad \text{for each continuous } f : \Gamma\cup\Gamma' \to \mathbb{C}\setminus\{0\}. \tag{4.1}$$

Variants of the following elementary property exist in the literature under different names, one of which is the *dog on a leash lemma*. If Γ is an oriented path in $\mathbb{C}$ and $f, g : \Gamma \to \mathbb{C}$ are continuous, then

$$|w(f+g,\Gamma) - w(f,\Gamma)| < 1/2 \quad \text{if } |g(z)| < |f(z)| \text{ for each } z \in \Gamma. \tag{4.2}$$

It is easy to see that if Γ is an oriented path, $f : \Gamma \to \mathbb{C} \setminus \{0\}$ is continuous and $|w(f,\Gamma)| \geqslant n/2$ with $n \in \mathbb{N}$, then f crosses every line in $\mathbb{C}$ passing through 0 at least n times. In other words, if $c \in \mathbb{T}$, then

$$|w(f,\Gamma)| < \tfrac{n+1}{2} \quad \text{if } \{z \in \Gamma : f(z) \in c\mathbb{R}\} \text{ consists of at most } n \text{ points.} \tag{4.3}$$

We use the above property to prove the following lemma.

Lemma 4.1. *If the oriented path Γ in $\mathbb{C}$ is an interval of a straight line, f is a polynomial of degree at most $m \in \mathbb{Z}_+$ and $g : \Gamma \to \mathbb{C}$ is a continuous map taking values in a line in $\mathbb{C}$ passing through zero such that $f(z) + g(z) \neq 0$ for every $z \in \Gamma$, then $w(f+g,\Gamma) < \frac{m+1}{2}$.*

Proof. Since Γ is an interval of a straight line we can parametrize Γ by $\varphi : [0,1] \to \mathbb{C}$, $\varphi(t) = at + b$ with $a, b \in \mathbb{C}$, $a \neq 0$. Since g takes values in a line in $\mathbb{C}$ passing through zero, there is $c \in \mathbb{T}$ such that $g(z) \in c^{-1}\mathbb{R}$ for $z \in \Gamma$. Since the function $h(t) = \mathtt{Im}\, cf(at+b)$ is a polynomial with real coefficients of degree at most m, it either vanishes identically on $[0,1]$ or has at most m zeros on $[0,1]$.

If $h \equiv 0$, then $f + g : I \to \mathbb{C}$ takes values in the line $c^{-1}\mathbb{R}$. Hence $w(f+g,\Gamma) = 0 < \frac{m+1}{2}$. If $h \not\equiv 0$, then the set $C = \{t \in [0,1] : h(t) = 0\}$ consists of at most m points. It is easy to see that the set $C' = \{z \in \Gamma : (f+g)(z) \in c^{-1}\mathbb{R}\}$ coincides with $\{at+b : t \in C\}$ and therefore C' consists of at most m points. By (4.3), $w(f+g,\Gamma) < \frac{m+1}{2}$. ☐

Finally, we remind the *argument principle.*

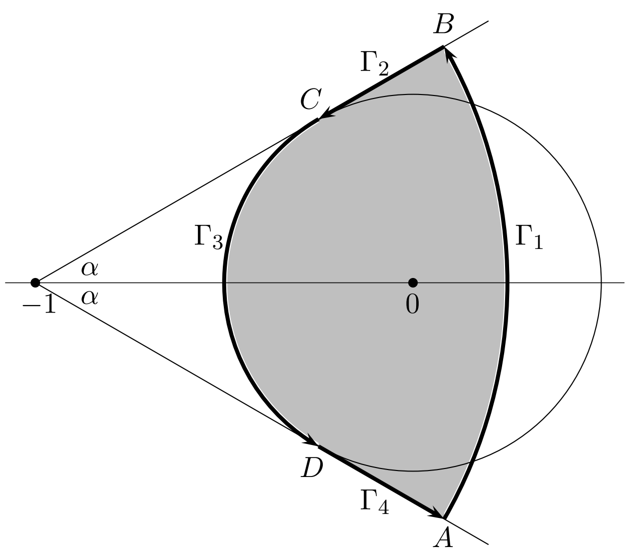
Argument Principle. *Let U be a bounded open subset of $\mathbb{C}$, whose boundary is a closed oriented path Γ, which encircles U counterclockwise. Let also $f : \overline{U} \to \mathbb{C}$ be a continuous function such that f is holomorphic on U and $0 \notin f(\Gamma)$. Then $w(f,\Gamma)$ is exactly the number of zeros of f in U counted with multiplicity.*

We are ready to prove Lemma 3.6. Let $k, \delta > 0$, $p \in \mathbb{P} \setminus \{0\}$ and $m \in \mathbb{N}$. We have to show that for every sufficiently large $n \in \mathbb{N}$, there exists a connected open set $W_n \subset \mathbb{C}$ such that $0 \in W_n \subseteq \delta\mathbb{D}$ and the polynomial $q_n = kz((1+z)^n - p)$ has at least m zeros in W_n and satisfies $q_n(W_n) \subseteq \mathbb{D}$.

Since at most one of the polynomials q_n can be zero, there is $n_0 \in \mathbb{N}$ such that $q_n \neq 0$ for $n \geqslant n_0$. Let $c > 1$ be such that $|p(z)| \leqslant c$ for every $z \in \mathbb{D}$. Pick $\alpha \in (0,1)$ such that $\alpha < \delta$, $\alpha < \frac{1}{3kc}$, the circle $(\sin\alpha)\mathbb{T}$ contains no zeros of p and the rays $\{-1 + te^{i\alpha} : t > 0\}$ and $\{-1 + te^{-i\alpha} : t > 0\}$ contain no zeros of q_n for every $n \geqslant n_0$. For every $n \in \mathbb{N}$, let $\varepsilon_n = (2c)^{1/n}$. Clearly $\{\varepsilon_n\}$ is a strictly decreasing sequence of positive numbers convergent to 1. Now for each $n \in \mathbb{N}$, we consider the open set $W_n \subset \mathbb{C}$ defined by the formula:

$$W_n = \{-1 + re^{i\beta} : -\alpha < \beta < \alpha,\ \cos\beta - \sqrt{\cos^2\beta - \cos^2\alpha} < r < \varepsilon_n\}.$$

It is easy to see that W_n is convex and therefore connected, open and contains 0. The following picture shows the set W_n.



with W_n being the gray area,

$A = -1 + \varepsilon_n e^{-i\alpha}$,
$B = -1 + \varepsilon_n e^{i\alpha}$
$C = -1 + (\cos\alpha)e^{i\alpha}$,
$D = -1 + (\cos\alpha)e^{-i\alpha}$,
$\Gamma_1 = \{-1 + \varepsilon_n e^{it} : t \in [-\alpha,\alpha]\}$,
$\Gamma_2 = \{-1 - te^{i\alpha} : t \in [-\varepsilon_n, -\cos\alpha]\}$,
$\Gamma_3 = \{(\sin\alpha)e^{it} : t \in [\frac{\pi}{2} + \alpha, \frac{3\pi}{2} - \alpha]\}$
and $\Gamma_4 = \{-1 + te^{-i\alpha} : t \in [\cos\alpha, \varepsilon_n]\}$.

The boundary ∂W_n, oriented in such a way that it encircles W_n counterclockwise, is the concatenation of 4 oriented paths $\partial W_n = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ defined above. Clearly Γ_1 is an arc of the circle $-1 + \varepsilon_n\mathbb{T}$, Γ_3 is an arc of the circle $(\sin\alpha)\mathbb{T}$, while Γ_2 and Γ_4 are intervals of the straight lines $-1 + e^{i\alpha}\mathbb{R}$ and $-1 + e^{-i\alpha}\mathbb{R}$ respectively. In each case the parametrization is chosen to agree with the right orientation. First, observe that the farthest from 0 points of ∂W_n are $B = -1 + \varepsilon_n e^{i\alpha}$ and $A = -1 + \varepsilon_n e^{-i\alpha}$. Hence W_n is contained in the disk $|-1 + \varepsilon_n e^{i\alpha}|\,\mathbb{D}$. Since $|-1 + \varepsilon_n e^{i\alpha}| \to |-1 + e^{i\alpha}| = 2\sin\frac{\alpha}{2} < \alpha$ as $n \to \infty$, we have

$$W_n \subset \alpha\mathbb{D} \subset \delta\mathbb{D} \text{ for each sufficiently large } n. \tag{4.4}$$

Since $\alpha < 1$, we also have $W_n \subset \mathbb{D}$ for n large enough. Since $|p(z)| \leqslant c$ for $z \in \mathbb{D}$, $|(1+z)^n| \leqslant 2c$ for $z \in -1 + \varepsilon_n\mathbb{D}$ and $W_n \subset -1 + \varepsilon_n\mathbb{D}$, we see that $|(1+z)^n - p(z)| \leqslant 3c$ for all $z \in W_n$ for all sufficiently large

n. Since $\alpha < \frac{1}{3kc}$ and $\sup\limits_{z\in W_n} |z| < \alpha$ for all n large enough, we have $|q_n(z)| < k\alpha|(1+z)^n - p(z)| \leqslant 3ck\alpha < 1$ for $z \in W_n$ for all sufficiently large n. Hence

$$q_n(W_n) \subseteq \mathbb{D} \text{ for each sufficiently large } n. \tag{4.5}$$

According to (4.4) and (4.5), it suffices to show that $r_n = (1+z)^n - p$ has at least m zeros in W_n for each sufficiently large n. Since r_n have no zeros on the rays $\{-1 + te^{i\alpha} : t > 0\}$ and $\{-1 + te^{-i\alpha} : t > 0\}$ for every $n \geqslant n_0$, r_n have no zeros on $\Gamma_2 \cup \Gamma_4$ for all n large enough. Since $|(1+z)^n| = 2c$ for $z \in \Gamma_1$ and $|p(z)| \leqslant c$ for $z \in \Gamma_1$ ($\Gamma_1 \subset \mathbb{D}$ for n large enough), we see that $r_n(z) \neq 0$ for $z \in \Gamma_1$ for all sufficiently large n. Since $\Gamma_3 \subset (\sin\alpha)\mathbb{T}$ and p has no zeros on the circle $(\sin\alpha)\mathbb{T}$, $\min\limits_{z\in\Gamma_3} |p(z)| = c_0 > 0$. It is easy to see that Γ_3 does not depend on n and is a compact subset of the disk $-1 + \mathbb{D}$. Hence $(1+z)^n$ converges uniformly to 0 on Γ_3 as $n \to \infty$. Thus $|p(z)| > |(1+z)^n|$ and therefore $r_n(z) \neq 0$ for $z \in \Gamma_3$ for all n large enough. Summarizing, we see that

$$0 \notin r_n(\partial W_n) \text{ for each sufficiently large } n.$$

By the argument principle and (4.1), the number $\nu(n)$ of zeros of r_n in W_n satisfies

$$\nu(n) = w(r_n, \partial W_n) = \sum_{j=1}^{4} w(r_n, \Gamma_j) \ \text{ for all sufficiently large } n. \tag{4.6}$$

Since on each of Γ_2 and Γ_4, the function $(1+z)^n$ takes values in a line in $\mathbb{C}$ passing through zero and Γ_2 and Γ_4 are intervals of straight lines, Lemma 4.1 implies that

$$|w(r_n, \Gamma_2)| < \tfrac{\deg p + 1}{2} \ \text{ and } \ |w(r_n, \Gamma_4)| < \tfrac{\deg p + 1}{2} \ \text{ for every sufficiently large } n. \tag{4.7}$$

Since $|(1+z)^n| < |p(z)|$ for $z \in \Gamma_3$ for any n large enough, (4.2) implies that

$$|w(r_n, \Gamma_3)| < |w(p, \Gamma_3)| + \tfrac{1}{2} \ \text{ for every sufficiently large } n. \tag{4.8}$$

Finally, since $|p(z)| < |(1+z)^n|$ for $z \in \Gamma_1$ for any n large enough, (4.2) implies that

$$w(r_n, \Gamma_1) > w((1+z)^n, \Gamma_1) - \tfrac{1}{2} \ \text{ for every sufficiently large } n.$$

A direct computation shows that $w((1+z)^n, \Gamma_1) = 2n\alpha$. Hence by the last display,

$$w(r_n, \Gamma_1) > 2n\alpha - \tfrac{1}{2} \ \text{ for every sufficiently large } n. \tag{4.9}$$

Combining (4.6–4.9), we get

$$\nu(n) > 2n\alpha - 2 - |w(p, \Gamma_3)| - \deg p \quad \text{for every sufficiently large } n.$$

Since Γ_3 does not depend on n, $\nu(n) \to \infty$ as $n \to \infty$. Hence r_n and therefore q_n has at least m zeros in W_n for each n large enough. The proof of Lemma 3.6 and that of Theorem 1.2 is complete.

5 Remarks and open questions

1. Our construction of a chaotic Banach algebra provides little control over its Banach space structure. Thus the following interesting questions arise.

Question 5.1. *Which separable infinite dimensional Banach spaces admit a multiplication turning them into a supercyclic or into an almost hypercyclic Banach algebra? In particular, is there a multiplication on ℓ_2, turning it into a chaotic Banach algebra?*

2. The structural properties of the class of supercylic or almost hypercyclic Banach algebras remain a complete mystery.

3. Let $\mathcal{H}$ be the Hilbert space of Hilbert–Schmidt operators on ℓ_2. With respect to the composition multiplication, $\mathcal{H}$ is a non-commutative non-unital Banach algebra. Let also $S \in \mathcal{H}$ be defined by its action on the basic vectors as follows: $Se_0 = 0$, $Se_n = n^{-1}e_{n-1}$ if $n \geqslant 1$. Consider the left multiplication by S operator $\Phi \in L(\mathcal{H})$, $\Phi(T) = ST$. Using the hypercyclicity and supercyclicity criteria [1], it is easy to see that Φ is supercyclic and $I + \Phi$ is hypercyclic. Thus supercyclicity of a multiplication operator and hypercyclicity of a perturbation of the identity by a multiplication operator on a non-commutative Banach algebra is a much simpler phenomenon.

4. We would also like to raise the following question. We say that a Banach algebra A is *wildly chaotic* if it has a supercyclic element a such that for every $z \in \mathbb{T}$, the set $\{a(z + a)^n : n \in \mathbb{N}\}$ is dense in A.

Question 5.2. *Does there exist a wildly chaotic infinite dimensional Banach algebra?*

Note that our construction can be modified to make $\{a(z+a)^n : n \in \mathbb{N}\}$ dense in A for each z from a given countable subset of $\mathbb{T}$.

5. Corollary 1.5 ensures the existence of a hypercyclic operator T with $\sigma(T) = \{1\}$ and $T \oplus T$ being non-cyclic. This naturally leads to the question whether such operators exist on every separable infinite dimensional Banach space.

Question 5.3. *Let X be a separable infinite dimensional Banach space. Does there exist a $T \in L(X)$ such that T is hypercyclic, $T \oplus T$ is non-cyclic and $\sigma(T) = \{1\}$? What is the answer for $X = \ell_2$?*

The above question is related to the following question of Bayart and Matheron [2].

Question 5.4. *Does every separable infinite dimensional Banach space admit a hypercyclic operator T such that $T \oplus T$ is non-cyclic?*

6. Bayart and Matheron [1] ask whether there exists a hypercyclic strongly continuous operator semigroup $\{T_t\}_{t\geqslant 0}$ on a Banach space X such that the semigroup $\{T_t \oplus T_t\}_{t\geqslant 0}$ acting on $X \oplus X$ is non-hypercyclic. As we have already mentioned, Theorem 1.2 provides a quasinilpotent operator M_a on the Banach space A such that $I + M_a$ is hypercyclic, while $(I + M_a) \oplus (I + M_a)$ is non-hypercyclic. Since M_a is quasinilpotent,

$$S = \ln(I + M_a) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} M_a^n$$

is a well-defined (also quasinilpotent) continuous linear operator on A. Hence we can consider the operator norm continuous semigroup $\{e^{tS}\}_{t\geqslant 0}$, which contains all powers of $I + M_a$: $e^{nS} = (I + M_a)^n$ for $n \in \mathbb{N}$. It follows that $\{e^{tS}\}_{t\geqslant 0}$ is hypercyclic. On the other hand, $e^S \oplus e^S = (I+M_a)\oplus(I+M_a)$ is a non-hypercyclic member of the semigroup $\{e^{tS} \oplus e^{tS}\}_{t\geqslant 0}$. According to Conejero, Müller and Peris [3], T_t is hypercyclic for every $t > 0$ if $\{T_t\}_{t\geqslant 0}$ is a hypercyclic strongly continuous operator semigroup. Hence $\{e^{tS} \oplus e^{tS}\}_{t\geqslant 0}$ is non-hypercyclic which answers negatively the above mentioned question of Bayart and Matheron.

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References

[1] F. Bayart and E. Matheron, *Dynamics of linear operators,* Cambridge University Press, Cambridge, 2009

[2] F. Bayart and E. Matheron, *Hypercyclic operators failing the hypercyclicity criterion on classical Banach spaces,* J. Funct.Anal. **250** (2007), 426–441

[3] J. Conejero, V. Müller and A. Peris, *Hypercyclic behaviour of operators in a hypercyclic C_0-semigroup,* J. Funct. Anal. **244** (2007), 342–348

[4] H. Dales, *Banach algebras and automatic continuity,* London Mathematical Society Monographs **24**, Oxford University Press, New York, 2000

[5] M. De La Rosa and C. Read, *A hypercyclic operator whose direct sum $T \oplus T$ is not hypercyclic,* J. Operator Theory **61** (2009), 369–380

[6] T. Gamelin, *Complex analysis,* Springer, New York, 2001

[7] R. Loy, C. Read, V. Runde and G. Willis, *Amenable and weakly amenable Banach algebras with compact multiplication,* Funct. Anal. **171** (2000), 78–114

[8] C. Read, *Commutative, radical amenable Banach algebras,* Studia Math. **140** (2000), 199–212

[9] A. Robertson and W. Robertson, *Topological vector spaces,* Cambridge University Press, Cambridge, 1980

[10] S. Shkarin, *Operators commuting with the Volterra operator are not weakly supercyclic,* Integral Equations Operator Theory [to appear], Electronic: DOI:10.1007/s00020-010-1790-y

[11] B. Van der Waerden, *Algebra* **I**, Springer, New York, 1991

[12] J. Wengenroth, *Hypercyclic operators on non-locally convex spaces,* Proc. Amer. Math. Soc. **131** (2003), 1759–1761