Non-commutative localisation and finite domination over strongly $\mathbb{Z}$-graded rings


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NON-COMMUTATIVE LOCALISATION AND FINITE DOMINATION
OVER STRONGLY \( \mathbb{Z} \)-GRADED RINGS

THOMAS HÜTTEMANN

Abstract. Let \( R = \bigoplus_{k=-\infty}^{\infty} R_k \) be a strongly \( \mathbb{Z} \)-graded ring, and let \( C^+ \) be a chain complex of modules over the positive subring \( P = \bigoplus_{k=0}^{\infty} R_k \). The complex \( C^+ \otimes_P R_0 \) is contractible (resp., \( C^+ \) is \( R_0 \)-finitely dominated) if and only if \( C^+ \otimes_P L \) is contractible, where \( L \) is a suitable non-commutative localisation of \( P \). We exhibit universal properties of these localisations, and show by example that an \( R_0 \)-finitely dominated complex need not be \( P \)-homotopy finite.

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5. Complexes contractible over \( R_0 \)
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   - Algebraic half-tori and the Mather trick
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References

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**Introduction**

**Finite domination.** Let \( R_0 \) be a unital ring, possibly non-commutative.
A chain complex \( C \) of \( R_0 \)-modules is called \( R_0 \)-finitely dominated if it is a retract up to homotopy of a bounded complex of finitely generated free \( R_0 \)-modules. When \( C \) is bounded and consists of projective \( R_0 \)-modules, \( C \) is \( R_0 \)-finitely dominated if and only if \( C \) is homotopy equivalent to a bounded complex of finitely generated projective \( R_0 \)-modules [Ran85, Proposition 3.2 (ii)]; this is sometimes expressed by saying that \( C \) is “of type FP”.

**Non-commutative localisation.** A K-ring is a unit-preserving homomorphism \( K \rightarrow S \) of unital rings with domain \( K \). Let \( \Sigma \) be a set of homomorphisms of finitely generated projective (right) \( K \)-modules. The \( K \)-ring \( f : K \rightarrow S \) is called \( \Sigma \)-inverting if all the induced maps

\[
\sigma \otimes S : P \otimes_K S \longrightarrow Q \otimes_K S , \quad (\sigma : P \longrightarrow Q) \in \Sigma
\]

are isomorphisms of \( S \)-modules. The non-commutative localisation of \( K \) with respect to \( \Sigma \) is the \( K \)-ring \( \lambda_\Sigma : K \rightarrow \Sigma^{-1}K \) which is initial in the category of \( \Sigma \)-inverting \( K \)-rings; it exists for all \( \Sigma \) [Sch85, Theorem 4.1].

**Detecting contractibility and finite domination using non-commutative localisation.** Let \( C^+ \) be a bounded chain complex consisting of finitely generated free modules over the polynomial ring \( R_0[t] \), where \( t \) is a (central) indeterminate commuting with all elements of \( R_0 \). Our starting point is the following pair of results obtained by Ranicki:
**Theorem.** There are sets $\tilde{\Omega}_+$ and $\Omega_+$ of square matrices with entries in $R_0[\tau]$, considered as maps between finitely generated free $R_0[\tau]$-modules, such that

(A) The induced complex $C^+ \otimes_{R_0[\tau]} R_0$ is contractible (tensor product via the map $R_0[\tau] \to R_0$, $\tau \mapsto 0$) if and only if the induced chain complex $C^+ \otimes_{R_0[\tau]} \tilde{\Omega}_+^{-1} R_0[\tau]$ is contractible [Ran98, Proposition 10.13];

(B) $C^+$ is $R_0$-finitely dominated if and only if the induced chain complex $C^+ \otimes_{R_0[\tau]} \Omega_+^{-1} R_0[\tau]$ is contractible [Ran98, Proposition 10.11].

**Content of the paper.** Ranicki’s results are extended to a larger class of rings containing polynomial rings as special examples. Let $R = \bigoplus_{k \in \mathbb{Z}} R_k$ be a $\mathbb{Z}$-graded ring. The polynomial ring $R[\tau]$ has a subring, denoted $R_0[\tau]$, consisting of those polynomials $\sum_k r_k t^k$ with $r_k \in R_k$; up to the ring isomorphism symbolised by $\tau \mapsto 1$, this is the $\mathbb{N}$-graded ring $\bigoplus_{k \geq 0} R_k$. We will show that the results above remain valid mutatis mutandis if the polynomial ring $R_0[\tau]$ is replaced by the $\mathbb{N}$-graded ring $R_0[\tau]$ throughout, where in (B) we additionally demand the $\mathbb{Z}$-graded ring $R$ to be strongly graded. This last conditions means that the multiplication map $R_k \otimes_{R_0} R_{-k} \to R_0$ is surjective for all $k \in \mathbb{Z}$. It is surprising that the results rest exclusively on the (strongly) graded structure of the underlying rings, and not on the specific form of polynomial rings in one indeterminate.

**Motivation.** Finiteness conditions for chain complexes are studied in algebraic topology [Ran85, Ran98] and other subjects (e.g., $\Sigma$-invariants in geometric group theory). The present paper develops aspects of the theory from a purely algebraic point of view, shifting the focus from (Laurent) polynomial rings to the larger class of (strongly) $\mathbb{Z}$-graded rings instead.

Strong gradings were introduced by Dade [Dad80] to capture the quintessential properties of group rings. The extent to which strongly $\mathbb{Z}$-graded rings behave like Laurent polynomial rings is in fact astonishing; examples include the splitting of the algebraic $K$-theory of the projective line (Hüttemann and Montgomery [HM20]), the relation between finite domination and Novikov homology (Hüttemann and Steers [HS17]), and the fundamental theorem in algebraic $K$-theory for strongly $\mathbb{Z}$-graded rings (Hüttemann [Hüt20]). The present paper adds further entries to the list of results that transfer to the strongly graded setting. Lest the reader gains the impression that this is a straightforward transcription we remark that, unlike the statements of the results, the proofs do not carry over mechanically. We also highlight in §7 a subtle example of a finiteness property that does not carry over as expected.
Organisation of the paper. The paper is divided into three parts, discussing $\mathbb{Z}$-graded rings and non-commutative localisation, contractible complexes, and finite domination respectively. Independently, the material is divided into numbered sections.

Conventions. All rings are unital, ring homomorphisms preserve unity, and modules are unital and right, unless stated otherwise.

Part 1. Algebraic background

1. Constructing new rings from a $\mathbb{Z}$-graded ring

For a (unital) ring $R$ we can construct various polynomial and power series rings using a central indeterminate $t$; the rings $R[[t]], R[[t^{-1}]], R(t), R(t^{-1})$, $R[(t)], R((t)) = R[[t]][1/t]$ and $R((t^{-1})) = R[[t^{-1}]][1/t^{-1}]$ will be of relevance. Elements of these rings can be written as formal sums $\sum_k r_k t^k$, with suitable restrictions on the number and sign of indices of non-zero coefficients $r_k$.

Suppose now that $R = \bigoplus_{k \in \mathbb{Z}} R_k$ is equipped with the structure of a $\mathbb{Z}$-graded ring. We can then define subrings of the rings above by requiring that for all $k \in \mathbb{Z}$ the coefficient $r_k$ of $t^k$ lies in $R_k$. The resulting rings will be denoted by the symbols $R_\ast[[t]], R_\ast[[t^{-1}]], R_\ast[[t, t^{-1}]], R_\ast[[t]], R_\ast[[t^{-1}]], R_\ast((t))$ and $R_\ast((t^{-1}))$, respectively. For example,

$$R_\ast((t)) = \bigcup_{p \geq 0} \left\{ \sum_{k = -p}^\infty r_k t^k \mid \forall k: r_k \in R_k \right\}.$$ 

As a graded ring, $R_\ast[t, t^{-1}] = R$ via the map symbolically described as $t \mapsto 1$. Similarly $R_\ast[t] = \bigoplus_{k \geq 0} R_k$ and $R_\ast[t^{-1}] = \bigoplus_{k \leq 0} R_k$. We write

$$t^n R_\ast[t] = \bigoplus_{k \geq n} R_k \quad \text{and} \quad t^n R_\ast[t^{-1}] = \bigoplus_{k \leq n} R_k,$$

which are (left and right) modules over $R_\ast[t]$ and $R_\ast[t^{-1}]$, respectively; the symbol $t^n R_\ast[[t^{-1}]]$ denotes the $R_\ast[[t^{-1}]]$-module of formal power series involving powers of $t$ not exceeding $n$.

For later use we introduce notation for truncation of formal power series. For $-\infty \leq \ell < u \leq \infty$ we define

$$\text{tr}^u_\ell : \sum_{k \in \mathbb{Z}} r_k t^k \mapsto \sum_{k = \ell}^u r_k t^k,$$

and abbreviations in the special cases $\ell = -\infty$ and $u = \infty$,

$$\text{tr}^u = \text{tr}^u_{-\infty} \quad \text{and} \quad \text{tr}_\ell = \text{tr}^\infty_{\ell}.$$ (1.2)
For example, the map
\[ \text{tr}^0: R_\ast [t] \longrightarrow R_0, \quad \sum_{k=0}^d r_k t^k \mapsto r_0 \] (1.3)
is the “constant-coefficient” ring homomorphism which is given symbolically by \( t \mapsto 0 \).

2. Strongly graded rings

**Strongly graded rings and partitions of unity.** Let \( R = R_\ast [t, t^{-1}] \) be a \( \mathbb{Z} \)-graded ring. A finite sum expression \( 1 = \sum_i \alpha_j^{(n)} \beta_j^{(-n)} \) with \( \alpha_j^{(n)} \in R_n \) and \( \beta_j^{(-n)} \in R_{-n} \) is called a *partition of unity of type* \( (n, -n) \). The ring \( R_\ast [t, t^{-1}] \) is called *strongly graded* (DADE [Dad80, §1]) if there exists a partition of unity of type \( (n, -n) \) for every \( n \in \mathbb{Z} \); equivalently, if the multiplication map
\[ \pi_n: R_n \otimes_{R_0} R_{-n} \longrightarrow R_0, \quad x \otimes y \mapsto xy \] (2.1)
is surjective for every \( n \in \mathbb{Z} \).

**Lemma 2.2.** If \( \pi_n \) is onto, then \( \pi_n \) is an isomorphism of \( R_0 \)-\( R_0 \)-bimodules.

**Proof.** The map \( \pi_n \) is clearly left and right \( R_0 \)-linear. If \( \pi_n \) is onto we can choose a partition of unity \( 1 = \sum_j \alpha_j^{(n)} \beta_j^{(-n)} \) and define the right \( R_0 \)-linear map
\[ \kappa_n: R_0 \longrightarrow R_n \otimes_{R_0} R_{-n}, \quad x \mapsto \sum_j \alpha_j^{(n)} \otimes \beta_j^{(-n)} x. \]
Then we calculate
\[ \kappa_n \pi_n (x \otimes y) = \sum_j \alpha_j^{(n)} \otimes \beta_j^{(-n)} x y = \sum_j \alpha_j^{(n)} \beta_j^{(-n)} x \otimes y = x \otimes y \]
(using \( \beta_j^{(-n)} x \in R_0 \)) so that \( \pi_n \) is injective. \( \square \)

**Lemma 2.3.** Let \( R = R_\ast [t, t^{-1}] \) be a \( \mathbb{Z} \)-graded ring, and let \( 1 = \sum_i \alpha_i^{(m)} \beta_i^{(-m)} \) and \( 1 = \sum_j \alpha_j^{(n)} \beta_j^{(-n)} \) be two partitions of unity of types \( (m, -m) \) and \( (n, -n) \), respectively. Then
\[ 1 = \sum_{i,j} (\alpha_i^{(m)} \alpha_j^{(n)}) \cdot (\beta_j^{(-n)} \beta_i^{(-m)}) \]
is a partition of unity of type \( (m + n, -m - n) \). \( \square \)

**Corollary 2.4.** Partitions of unity of types \( (1, -1) \) and \( (-1, 1) \) exist within the \( \mathbb{Z} \)-graded ring \( R_\ast [t, t^{-1}] \) if and only if it is strongly \( \mathbb{Z} \)-graded. \( \square \)
By direct calculation, similar to the proof of Lemma 2.2 above, one verifies:

**Lemma 2.5.** Suppose that \( R = R_n[t, t^{-1}] \) is a strongly \( \mathbb{Z} \)-graded ring, and let \( m \in \mathbb{Z} \). The multiplication map

\[
t^{-m}R_n[t] \otimes_{R_n[t]} R_n[t, t^{-1}] \longrightarrow R_n[t, t^{-1}], \quad x \otimes y \mapsto xy
\]

is an isomorphism of \( R_n[t] \)-\( R_n[t, t^{-1}] \)-bimodules, with inverse given by

\[
z \mapsto \sum_j \alpha_j^{(-m)} \otimes \beta_j^{(m)} z
\]

for a partition of unity \( 1 = \sum_j \alpha_j^{(-m)} \beta_j^{(m)} \) of type \((-m, m)\). \( \square \)

Note that the inverse is independent from the choice of partition of unity (since the multiplication map is). — For later use, we record an important categorical property of strongly \( \mathbb{Z} \)-graded rings:

**Lemma 2.6.** Let \( R = R_n[t, t^{-1}] \) be a strongly \( \mathbb{Z} \)-graded ring. The inclusion \( \beta : R_n[t] \longrightarrow R_n[t, t^{-1}] \) is an epimorphism in the category of (unital) rings.

**Proof.** Let \( f, g : R_n[t, t^{-1}] \longrightarrow S \) be ring homomorphisms satisfying the equality \( f \beta = g \beta \). We need to show \( f = g \). For this, let \( x \in R_n \) be homogeneous of degree \( k \in \mathbb{Z} \). If \( k \geq 0 \) we have \( f(x) = f \beta(x) = g \beta(x) = g(x) \). Otherwise, choose a partition of unity \( 1 = \sum_j \alpha_j^{(k)} \beta_j^{(-k)} \) of type \((k, -k)\). Then \( \beta_j^{(-k)} \) and \( \beta_j^{(-k)} x \) lie in \( R_n[t] \). Thus \( f(\beta_j^{(-k)} x) = g(\beta_j^{(-k)} x) \), and we calculate

\[
f(x) = g(1) \cdot f(x)
\]

\[
= \sum_j g(\alpha_j^{(k)} \beta_j^{(-k)}) \cdot f(x) = \sum_j g(\alpha_j^{(k)}) \cdot g(\beta_j^{(-k)}) \cdot f(x)
\]

\[
= \sum_j g(\alpha_j^{(k)}) \cdot f(\beta_j^{(-k)}) \cdot f(x) = \sum_j g(\alpha_j^{(k)}) \cdot f(\beta_j^{(-k)} x)
\]

\[
= \sum_j g(\alpha_j^{(k)}) \cdot g(\beta_j^{(-k)} x) = \sum_j g(\alpha_j^{(k)} \beta_j^{(-k)} x) = g(x).
\]

\( \square \)

**Finiteness properties of strongly graded rings.** The homogeneous components of strongly graded rings are finitely generated projective modules over the degree-0 subring.
Lemma 2.7. Suppose that $R = R_n[t, t^{-1}]$ is a $\mathbb{Z}$-graded ring that admits a partition of unity of type $(1, -1)$. Then for all $n \geq 1$,

- $R_n$ is finitely generated projective as a right $R_0$-module;
- $R_{-n}$ is finitely generated projective as a left $R_0$-module.

Similarly, if $R = R_n[t, t^{-1}]$ admits a partition of unity of type $(1, -1)$, then for all $n \geq 1$,

- $R_n$ is finitely generated projective as a left $R_0$-module;
- $R_{-n}$ is finitely generated projective as a right $R_0$-module.

Proof. Let $n \geq 1$, and let $1 = \sum_j \alpha_j^{(n)} \beta_j^{(-n)}$ be a partition of unity of type $(n, -n)$ (existence is guaranteed by Lemma 2.3). Define $f_j : R_n \to R_0$, $x \mapsto \beta_j^{(-n)} x$.

The maps $f_j$ are right $R_0$-linear, and for all $x \in R_n$ we calculate

$$\sum_j \alpha_j^{(n)} \cdot f_j(x) = \sum_j \alpha_j^{(n)} \beta_j^{(-n)} x = x$$

so that $(\alpha_j^{(n)}, f_j)$ is a dual basis for $R_n$. It follows that $R_n$ is a finitely generated projective right $R_0$-module by the dual basis lemma. — All the remaining claims are proved in a similar manner. □

Corollary 2.8. Suppose that $R = R_n[t, t^{-1}]$ is a strongly $\mathbb{Z}$-graded ring.

1. For all $n \in \mathbb{Z}$, the homogeneous component $R_n$ of $R_n[t, t^{-1}]$ is a finitely generated projective left $R_0$-module and a finitely generated projective right $R_0$-module; in fact, $R_n$ is an invertible $R_0$-bimodule.

2. If $M$ is a projective (left or right) $R_n[t, t^{-1}]$-module, then $M$ is a projective (left or right) $R_0$-module (with module structure given by restriction of scalars). Similarly, any projective left or right module over $R_n[t]$ or $R_n[t^{-1}]$ is a projective $R_0$-module.

3. There exists an isomorphism $R_{-m} \otimes_{R_0} R_n[t] \cong t^{-m} R_n[t]$ of finitely generated projective right $R_n[t]$-modules, for every $m \in \mathbb{Z}$. Similarly, there exists an isomorphism $R_m \otimes_{R_0} R_n[t^{-1}] \cong t^m R_n[t^{-1}]$ of finitely generated projective right $R_n[t^{-1}]$-modules.

4. For all $m \in \mathbb{Z}$, the module $t^{-m} R_n[t]$ is an invertible $R_n[t]$-bimodule, and hence is finitely generated projective as a left and right $R_n[t]$-module. Similarly, $t^m R_n[t^{-1}]$ is an invertible $R_n[t^{-1}]$-bimodule, and hence is finitely generated projective as a left and right $R_n[t^{-1}]$-module.

Proof. Statements (1) and (2) follow from Lemma 2.2, Corollary 2.4 and Lemma 2.7. To prove (3) it is enough, in view of (1), to establish the
isomorphism. Let $1 = \sum_j \alpha_j^{(-m)} \beta_j^{(m)}$ be a partition of unity of type $(-m,m)$. Then the multiplication map $\pi: R_m \otimes_R R_s[t] \xrightarrow{\cong} t^{-m}R_s[t]$, sending $x \otimes y$ to $xy$, has inverse given by $\rho: z \mapsto \sum_j \alpha_j^{(-m)} \otimes \beta_j^{(m)} z$. Indeed, by straightforward calculation, $\pi \rho (z) = \sum_j \alpha_j^{(-m)} \beta_j^{(m)} z = z$ and

$$\rho \pi (x \otimes y) = \sum_j \alpha_j^{(-m)} \otimes \beta_j^{(m)} xy = \sum_j \alpha_j^{(-m)} \beta_j^{(m)} x \otimes y = x \otimes y$$

since $\beta_j^{(m)} x \in R_0$. — The proof of (4) is similar, using partitions of unity to show that $t^mR_s[t]$ is the inverse $R_s[t]$-bimodule of $t^mR_s[t]$. \hfill \Box

3. Proto-null homotopies and proto-contractions

Let $C$ and $C'$ be chain complexes of right modules over the unital ring $K$, with differentials $d = d_k: C_k \longrightarrow C_{k-1}$ and $d' = d'_k$. A proto-contraction of $C$ consists of module homomorphisms $s = s_k: C_k \longrightarrow C_{k+1}$ such that $ds + sd: C_k \longrightarrow C_k$ is an automorphism of $C_k$ for all $k \in \mathbb{Z}$. Somewhat more generally, a $(C, C')$-proto-null homotopy consists of module homomorphisms $t = t_k: C_k \longrightarrow C'_k$ such that $g_k = d't + td: C_k \longrightarrow C'_k$ is an isomorphism for all $k \in \mathbb{Z}$. In fact, the maps $g_k$ define a chain isomorphism $g: C \xrightarrow{\cong} C'$, and the maps $t_k$ define a null homotopy of $g$.

**Lemma 3.1.** A chain complex $C$ admits a proto-contraction if and only if it is contractible. The chain complexes $C$ and $C'$ admit a $(C, C')$-proto-null homotopy if and only if $C \cong C'$ and $C$ is contractible.

**Proof.** A proto-contraction is, by definition, the same as a $(C, C)$-proto-null homotopy, so it suffices to prove the second statement. If there exists a chain isomorphism $g: C \longrightarrow C'$ with $C$ contractible, we can choose a null homotopy $t$ of $g$ which constitutes a $(C, C')$-proto-null homotopy. Conversely, any $(C, C')$-proto-null homotopy $t$ determines a null homotopic chain isomorphism $g = d't + td$, as explain above. Then $\text{id}_C = g^{-1}g$ is null homotopic as well so that $C$ is contractible. \hfill \Box

Given a ring homomorphism $f : K \longrightarrow S$, the family of maps $s_k$ is called an $f$-proto-contraction if the maps $s_k \otimes \text{id}$ form a proto-contraction of the induced complex $f_*(C) = C \otimes_K S$. Similarly, the family of maps $t_k$ is called a $(C, C')$-$f$-proto-null homotopy if the maps $t_k \otimes \text{id}$ form a $(C \otimes_K S, C' \otimes_K S)$-proto-null homotopy.

We are interested in proto-contractions for the following reason. Suppose we are given $C$ and $f$ as before, and another ring homomorphism
If \( f_*(C) = C \otimes_K S \) is contractible then \( (gf)_*(C) = C \otimes_K T \cong C \otimes_K S \otimes_S T \) is contractible as well, since taking tensor product preserves homotopies. If, however, \( (gf)_*(C) \) is contractible it is not guaranteed that \( f_*(C) \) is contractible. In favourable circumstances, a contraction of \( (gf)_*(C) \) gives rise to a sequence of maps \( s_k : C_k \rightarrow C_{k+1} \) which can be shown, thanks to special properties of the maps \( f \) and \( g \), to be an \( f \)-proto-contraction.

4. REMARKS ON NON-COMMUTATIVE LOCALISATION

Let \( K \) denote an arbitrary unital, possibly non-commutative ring. For the reader’s convenience we collect some standard facts about non-commutative localisation

**Proposition 4.1.** Let \( \Sigma \) be a set of homomorphisms of finitely generated projective \( K \)-modules, and let \( f : K \rightarrow S \) be a \( K \)-ring. Write \( \lambda_\Sigma : K \rightarrow \Sigma^{-1}K \) for the non-commutative localisation of \( K \) with respect to \( \Sigma \).

1. If \( f \) is \( \Sigma \)-inverting and injective, then \( \lambda_\Sigma \) is injective.
2. The non-commutative localisation \( \lambda_\Sigma : K \rightarrow \Sigma^{-1}K \) is an epimorphism in the category of unital rings.
3. Suppose that \( \Sigma \) is the set of all those square matrices \( M \) with entries in \( K \) such that \( f(M) \) is invertible over \( S \); we consider a square matrix of size \( k \) as a map of finitely generated free modules \( \mu : K^k \rightarrow K^k \) so that \( f(M) \) represents the induced map \( \mu \otimes S : S^k \rightarrow S^k \). Let \( A \) be a square matrix with entries in \( K \). Then \( A \in \Sigma \) if and only if \( \lambda_\Sigma(A) \) is invertible over the ring \( \Sigma^{-1}K \).

**Proof.**

1. As \( f \) is \( \Sigma \)-inverting, it factors as \( K \rightarrow \frac{\lambda_\Sigma}{\lambda_\Sigma} \rightarrow \Sigma^{-1}K \rightarrow S \). This forces \( \lambda_\Sigma \) to be injective if \( f \) is.

2. Suppose we have two ring homomorphisms \( \alpha, \beta : \Sigma^{-1}K \rightarrow T \) with \( \alpha \lambda_\Sigma = \beta \lambda_\Sigma \). This common composition is certainly \( \Sigma \)-inverting, so factorises uniquely through \( \lambda_\Sigma \). This means precisely that \( \alpha = \beta \), as required.

3. Since the map \( f \) is \( \Sigma \)-invertible, it factors as \( K \rightarrow \frac{\lambda_\Sigma}{\lambda_\Sigma} \rightarrow \Sigma^{-1}K \rightarrow S \). If \( A \) is a square matrix in \( \Sigma \) then \( \lambda_\Sigma(A) \) is invertible in \( \Sigma^{-1}K \), by definition of non-commutative localisation. If the square matrix \( A \) with entries in \( K \) is such that \( \lambda_\Sigma(A) \) is invertible, then \( f \lambda_\Sigma(A) = f(A) \) is invertible over \( S \) so that \( A \in \Sigma \) by the specific choice of \( \Sigma \).

We will have occasion to use the following construction of pushout squares:

**Proposition 4.2.** Let \( \Sigma \) be a set of homomorphisms of finitely generated projective \( K \)-modules, and let \( f : K \rightarrow S \) be a ring homomorphism. The
square in Fig. 1 is a pushout in the category of unital rings, where \( f_{\ast}(\Sigma) \) de-

\[
\begin{array}{ccc}
K & \xrightarrow{\lambda_{\Sigma}} & \Sigma^{-1}K \\
f & \downarrow \sigma & \downarrow \tilde{f} \\
S & \xrightarrow{\lambda_{f_{\ast}(\Sigma)}} & f_{\ast}(\Sigma)^{-1}S
\end{array}
\]

\textbf{Figure 1.} A pushout square in the category of unital rings

notes the set of induced maps \( \sigma \otimes S : P \otimes K S \longrightarrow Q \otimes K S \) with \( \sigma : P \longrightarrow Q \) an element of \( \Sigma \). The ring homomorphism \( \tilde{f} \) is obtained from the universal property of \( \lambda_{\Sigma} \) as the composition \( \lambda_{f_{\ast}(\Sigma)} \circ f \) is \( \Sigma \)-inverting. — In other words, given ring homomorphisms \( \beta : S \longrightarrow T \) and \( \alpha : \Sigma^{-1}K \longrightarrow T \) such that \( \alpha \circ \lambda_{\Sigma} = \beta \circ f \) there exists a uniquely determined ring homomorphism \( \upsilon : f_{\ast}(\Sigma)^{-1}S \longrightarrow T \) with \( \beta = \upsilon \circ \lambda_{f_{\ast}(\Sigma)} \) and \( \alpha = \upsilon \circ \tilde{f} \), cf. Fig. 2.

\[
\begin{array}{ccc}
K & \xrightarrow{\lambda_{\Sigma}} & \Sigma^{-1}K \\
f & \downarrow \lambda_{f_{\ast}(\Sigma)} & \downarrow \tilde{f} \\
S & \xrightarrow{f_{\ast}(\Sigma)^{-1}S} & T
\end{array}
\]

\textbf{Figure 2.} Universal property of pushout square

\textbf{Proof.} As for notation, given any ring homomorphism \( h : A \longrightarrow B \) we let \( h_{\ast} \) stand for the functor \(- \otimes_A B\). — To prove the Proposition we verify that the square has the universal property of a pushout, see Fig. 2. Let \( \alpha : \Sigma^{-1}K \longrightarrow T \) and \( \beta : S \longrightarrow T \) be ring homomorphisms such that \( \alpha \lambda_{\Sigma} = \beta f \). Given a map \( \sigma : P \longrightarrow Q \) in \( \Sigma \) we know that

\[
\beta_{\ast} f_{\ast}(\sigma) = (\beta f)_{\ast}(\sigma) = (\alpha \lambda_{\Sigma})_{\ast}(\sigma) = \alpha_{\ast}(\lambda_{\Sigma})_{\ast}(\sigma);
\]

as \( (\lambda_{\Sigma})_{\ast}(\sigma) \) is invertible so is \( \beta_{\ast} f_{\ast}(\sigma) \). Hence the map \( \beta \) is \( f_{\ast}(\Sigma) \)-inverting, and consequently factorises uniquely as \( \beta = \upsilon \lambda_{f_{\ast}(\Sigma)} \), for some ring homomorphism \( \upsilon : f_{\ast}(\Sigma)^{-1}S \longrightarrow T \). From the chain of equalities

\[
u \tilde{f} \lambda_{\Sigma} = \upsilon \lambda_{f_{\ast}(\Sigma)} f = \beta f = \alpha \lambda_{\Sigma}\]

we conclude that \( \alpha = \upsilon \tilde{f} \) since \( \lambda_{\Sigma} \) is an epimorphism by Proposition 4.1 (2). \( \square \)
The following purely category-theoretic lemma will be applied, in the proof of Proposition 10.8, in the context of strongly graded rings and non-commutative localisation.

**Lemma 4.3.** Suppose that we are given a commutative pushout square

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\beta \downarrow & \nearrow & \delta \\
C & \xrightarrow{\gamma} & D
\end{array}
\]

(in any category) with \(\beta\) an epimorphism. Suppose further that there exists \(\iota : C \rightarrow B\) with \(\iota \beta = \alpha\). Then \(\delta \iota = \gamma\), and \(\delta\) is an isomorphism.

**Proof.** First, since \(\delta \iota \beta = \delta \alpha = \gamma \beta\), and since \(\beta\) is an epimorphism, we have \(\delta \iota = \gamma\). Next, by the universal property of pushouts there exists a (uniquely determined) morphism \(\varphi : D \rightarrow B\) with \(\varphi \delta = \text{id}_B\) and \(\varphi \gamma = \iota\). The commutative diagram of Fig. 3 can be completed along the dotted arrow by both \(\text{id}_D\) and \(\delta \varphi\); by uniqueness, this means \(\delta \varphi = \text{id}_D\). □

![Figure 3. Pushout diagram used in proof of Lemma 4.3](image)

**Part 2.** \(\mathbb{N}\)-graded rings and complexes contractible over \(R_0\)

For this part we assume that \(R = R_\ast[t, t^{-1}]\) is an arbitrary \(\mathbb{Z}\)-graded ring; in fact, we are only interested in the subring \(R_\ast[t] = \bigoplus_{k=0}^{\infty} R_k\) which is, in effect, an arbitrary \(\mathbb{N}\)-graded ring.

**5. Complexes contractible over \(R_0\)**

We characterise complexes \(C\) of \(R_\ast[t]\)-modules such that \(C \otimes_{R_\ast[t]} R_0\) is contractible, where the tensor product is taken via the “constant coefficient” ring homomorphism \(\text{tr}^0 : t \mapsto 0\) of (1.3).
The map $\zeta$. Let $M$ be an $R_\ast[[t]]$-module. Using the notation from (1.1), we write $\zeta_M = \zeta$ for the obvious map of $R_\ast[[t]]$-modules

$$\zeta_M : M \otimes_{R_\ast[[t]]} t^1 R_\ast[[t]] \rightarrow M \otimes_{R_\ast[[t]]} t^0 R_\ast[[t]] = M, \quad m \otimes x \mapsto mx$$

induced by the inclusion map $t^1 R_\ast[[t]] \rightarrow t^0 R_\ast[[t]]$. The map $\zeta$ is to be thought of as a substitute for the action of the indeterminate $t$. More precisely, if $R_\ast[[t]] = K[[t]]$ is a polynomial ring, then $t^1 R_\ast[[t]] = tK[[t]]$ and the composition

$$M = M \otimes_{K[[t]]} K[[t]] \xrightarrow{\cong} M \otimes_{K[[t]]} (tK[[t]]) \xrightarrow{\zeta} M \otimes_{K[[t]]} K[[t]] = M,$$

where $\tau(m \otimes r) = m \otimes tr$, is given by $m \mapsto mt$; that is, up to the isomorphism $\tau$ the map $\zeta$ coincides with the action of the indeterminate.

Invertible matrices over $R_\ast[[t]]$. We write an element $z \in R_\ast[[t]]$ as a formal power series: $z = \sum_{p \geq 0} z_p t^p$. The usual proof shows that $z$ is a unit in $R_\ast[[t]]$ if and only if $z_0 = t^0(z)$ is a unit in $R_0$, cf. (1.2).

A square matrix $M$ with entries in $R_\ast[[t]]$ can be written as a formal power series $M = \sum_{p \geq 0} M_p t^p$ with matrices $M_p$ having entries in $R_p$; again, the usual proof shows that the matrix $M$ is invertible over $R_\ast[[t]]$ if and only if $M_0 = t^0(M)$ is invertible over $R_0$.

Notation 5.2. We let $\tilde{\Omega}_+$ denote the set of all square matrices $M$ with entries in $R_\ast[[t]]$ such that $t^0(M)$ is an invertible matrix over $R_0$, that is, such that $M$ is invertible over $R_\ast[[t]]$.

We apply Proposition 4.1 (3) to the $R_\ast[[t]]$-ring $f : R_\ast[[t]] \rightarrow R_\ast[[t]]$:

**Lemma 5.3.** A square matrix $M$ with entries in the $\mathbb{N}$-graded ring $R_\ast[[t]]$ becomes invertible in $\tilde{\Omega}_+^{-1} R_\ast[[t]]$ if and only if $t^0(M)$ is invertible over $R_0$. \( \square \)

The localisation $\tilde{\Omega}_+^{-1} R_\ast[[t]]$. We consider an element $A^+ \in \tilde{\Omega}_+$ of size $k$ as an endomorphism $A^+ : R_\ast[[t]]^k \rightarrow R_\ast[[t]]^k$ of the finitely generated free $R_\ast[[t]]$-module $R_\ast[[t]]^k$. The non-commutative localisation

$$\lambda = \lambda_{\tilde{\Omega}_+} : R_\ast[[t]] \rightarrow \tilde{\Omega}_+^{-1} R_\ast[[t]]$$

can be used to characterise the $R_0$-contractible complexes $C^+$ as follows, generalising known results for polynomial rings (Ranicki [Ran98, Proposition 10.13]):

**Theorem 5.4.** Let $R_\ast[[t]] = \bigoplus_{k=0}^{\infty} R_k$ be an arbitrary $\mathbb{N}$-graded ring, and let $C^+$ be a bounded complex of finitely generated free $R_\ast[[t]]$-modules. The following statements are equivalent:
(1) The complex $C^+ \otimes_{R_\ast[t]} R_0$ is contractible, the tensor product being taken with respect to the ring map $t^0\colon R_\ast[t] \longrightarrow R_0$, $t \mapsto 0$.

(2) The induced complex $C^+ \otimes_{R_\ast[t]} R_\ast[[t]]$ is contractible.

(3) The induced complex $C^+ \otimes_{R_\ast[t]} \tilde{\Omega}_+^{-1} R_\ast[t]$ is contractible.

(4) The map $\zeta\colon C \otimes_{R_\ast[t]} t^1 R_\ast[t] \longrightarrow C \otimes_{R_\ast[t]} t^0 R_\ast[t]$ from (5.1) is a quasi-isomorphism.

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1): This follows from the factorisation

$$R_\ast[t] \xrightarrow{\zeta} \tilde{\Omega}_+^{-1} R_\ast[t] \longrightarrow R_\ast[[t]] \xrightarrow{t^0\colon t \mapsto 0} R_0$$

of the ring homomorphism $t^0\colon R_\ast[t] \longrightarrow R_0$.

(1) $\Rightarrow$ (3): We equip the finitely generated free modules $C_n^+$ with arbitrary finite bases; denote the number of elements of the basis for $C_n^+$ by $r_n$ so that $C_n^+$ is identified with $R_\ast[t]_{r_n}$. The differentials $d_n^+\colon C_n^+ \longrightarrow C_{n-1}^+$ are thus represented by matrices $D_n^+$ of size $r_{n-1} \times r_n$ with entries in $R_\ast[t]$. The differentials $t^0(d_n^+)$ in the induced complex $C^+ \otimes_{R_\ast[t]} R_0$ are then represented by the matrices $t^0(D_n^+)$, identifying $C_n^+ \otimes_{R_\ast[t]} R_0$ with $R_0^{r_n}$. By hypothesis there exists a contracting homotopy consisting of a family of $R_0$-linear maps

$$\sigma_n^+\colon C_n^+ \otimes_{R_\ast[t]} R_0 \longrightarrow C_{n+1}^+ \otimes_{R_\ast[t]} R_0$$

such that

$$t^0(D_n^+) \cdot \sigma_n^+ + \sigma_{n-1}^+ \cdot t^0(D_n^+) = \text{id}.$$ 

The map $\sigma_n^+$ is represented by a matrix $S_n^+$ of size $r_{n+1} \times r_n$ with entries in $R_0$. The matrices satisfy the relation

$$t^0(D_{n+1}^+) \cdot S_n^+ + S_{n-1}^+ \cdot D_n^+ = t^0(D_{n+1}^+) \cdot S_n^+ + S_{n-1}^+ \cdot t^0(D_n^+) = I_{r_n},$$

a unit matrix of size $r_n$. This implies, by Lemma 5.3, that the matrix

$$D_{n+1}^+ \cdot S_n^+ + S_{n-1}^+ \cdot D_n^+$$

becomes invertible over $\tilde{\Omega}_+^{-1} R_\ast[t]$. Thus the $S_n^+$ define a $\lambda_{\tilde{\Omega}_+}$-proto-contraction of $C^+$, cf. §3. With Lemma 3.1 we conclude that $C^+ \otimes_{R_\ast[t]} \tilde{\Omega}_+^{-1} R_\ast[t]$ is contractible as advertised.

(1) $\iff$ (4): From the short exact sequence

$$0 \longrightarrow C^+ \otimes_{R_\ast[t]} t^1 R_\ast[t] \xrightarrow{\zeta} C^+ \otimes_{R_\ast[t]} t^0 R_\ast[t] \longrightarrow C^+ \otimes_{R_\ast[t]} R_0 \longrightarrow 0$$

we infer that the canonical map $\text{cone}(\zeta) \longrightarrow C^+ \otimes_{R_\ast[t]} R_0$ is a quasi-isomorphism. Thus $\zeta$ is a quasi-isomorphism if and only if $\text{cone}(\zeta)$ is
acyclic if and only if \( C^+ \otimes_{R_0} R_0 \) is acyclic; as the latter complex consists of projective \( R_0 \)-modules, this is equivalent with \( C^+ \otimes_{R_0} R_0 \) being contractible.

\[ \square \]

**Theorem 5.5** (Universal property of \( \tilde{\Omega}_+^{1}R_s[t] \)). Let \( R_s[t] \) be an arbitrary \( \mathbb{N} \)-graded ring. The localisation \( \lambda : R_s[t] \longrightarrow \tilde{\Omega}_+^{1}R_s[t] \) is the universal \( R_s[t] \)-ring making \( R_0 \)-contractible chain complexes contractible. That is, suppose that \( f : R_s[t] \longrightarrow S \) is an \( R_s[t] \)-ring such that for every bounded complex of finitely generated free \( R_s[t] \)-modules \( C^+ \), contractibility of \( C^+ \otimes_{R_s[t]} R_0 \) implies contractibility of \( C^+ \otimes_{R_s[t]} S \). Then there is a factorisation

\[ R_s[t] \xrightarrow{\lambda} \tilde{\Omega}_+^{1}R_s[t] \xrightarrow{\eta} S \]

of \( f \), with a uniquely determined ring homomorphism \( \eta \).

**Proof.** It was shown in Theorem 5.4 that the \( R_s[t] \)-ring \( \tilde{\Omega}_+^{1}R_s[t] \) makes \( R_0 \)-contractible chain complexes contractible. Thus it is enough to verify that \( f \) is \( \tilde{\Omega}_+^{1} \)-inverting; the universal property of non-commutative localisation then yields the desired factorisation and its uniqueness. Consider the element \( A^+ \in \tilde{\Omega}_+ \) as a chain complex

\[ C^+ = \left( R_s[t]^k \xrightarrow{A^+} R_s[t]^k \right) \]

As \( A^+ \) becomes invertible over \( \tilde{\Omega}_+^{1}R_s[t] \), the complex \( C^+ \otimes_{R_s[t]} \tilde{\Omega}_+^{1}R_s[t] \) is contractible, hence so is \( C^+ \otimes_{R_s[t]} R_0 \) by Theorem 5.4. This makes \( C^+ \otimes_{R_s[t]} S \) contractible, by hypothesis on \( f \), whence \( A^+ \) becomes invertible in \( S \) as required. \( \square \)

**Part 3. Strongly graded rings and finite domination**

We now turn to the theory of \( R_0 \)-finite domination of \( R_s[t] \)-module complexes. We characterise finite domination via **Novikov homology** (Theorem 8.1) and via a non-commutative localisation of \( R_s[t] \) (Theorem 10.6). We assume throughout that \( R = R_s[t, t^{-1}] \) is a strongly \( \mathbb{Z} \)-graded ring.

### 6. Algebraic half-tori and the Mather trick

**Algebraic half-tori and the Mather trick.** Let \( 1 = \sum_j a_j^{(1)} \beta_j^{(-1)} \) be a partition of unity of type \((1, -1)\) in \( R = R_s[t, t^{-1}] \). Given an arbitrary
Let $M$ be a right $R_s[t]$-module, let $\mu = \mu_M$ denote the map

$$\mu: M \otimes_{R_0} t^1 R_s[t] \longrightarrow M \otimes_{R_0} t^0 R_s[t], \quad m \otimes x \mapsto \sum_j m \alpha_j^{(1)} \otimes \beta_j^{(-1)} x.$$  \hfill (6.1)

The map $\mu$ is $R_0$-balanced (hence well-defined) and independent of the choice of partition of unity since it can be written as the composition

$$M \otimes_{R_0} t^1 R_s[t] \cong M \otimes_{R_0} R_0 \otimes_{R_0} t^1 R_s[t] \cong M \otimes_{R_0} t^0 R_s[t],$$

where the second isomorphism is induced by $\pi^{-1}_1: R_0 \longrightarrow R_1 \otimes_{R_0} R_{-1}$, cf. Lemma 2.2, and the last arrow is induced by the multiplication maps $M \otimes_{R_0} R_1 \longrightarrow M$ and $R_{-1} \otimes_{R_0} t^1 R_s[t] \longrightarrow t^0 R_s[t]$.

As a matter of notation, we also introduce the inclusion map

$$\iota: M \otimes_{R_0} t^1 R_s[t] \longrightarrow M \otimes_{R_0} t^0 R_s[t], \quad m \otimes x \mapsto m \otimes x.$$  

Moreover, it is convenient at this point to choose, once and for all, additional partitions of unity

$$1 = \sum_{j_n} \alpha_{j_n}^{(n)} \beta_{j_n}^{(-n)}$$

of type $(n, -n)$, for all $n \geq 0$ ($n \neq 1$). These exist in view of our standing assumption for this part, that the ring $R = R_s[t, t^{-1}]$ is strongly graded.

**Lemma 6.2 (Canonical resolution).** Suppose that $R = R_s[t, t^{-1}]$ is a strongly $\mathbb{Z}$-graded ring. Let $M$ be a right $R_s[t]$-module. There is a short exact sequence of $R_s[t]$-modules

$$0 \longrightarrow M \otimes_{R_0} t^1 R_s[t] \overset{\iota-\mu}{\longrightarrow} M \otimes_{R_0} t^0 R_s[t] \overset{\pi}{\longrightarrow} M \longrightarrow 0,$$  \hfill (6.3)

where $\mu$ is as in (6.1), $\iota(m \otimes x) = m \otimes x$ and $\pi(m \otimes x) = mx$.

**Proof.** This is similar to the proof of Proposition 3.2 in [HS17]. Since $\sum_j \alpha_j^{(1)} \beta_j^{(-1)} = 1$ we have $\pi \iota = \pi \mu$ and hence $\pi(\iota - \mu) = 0$. It is thus enough to show that the sequence is split exact when considered as a sequence of $R_0$-modules.

To begin with, the map $\sigma(m) = m \otimes 1$ is certainly an $R_0$-linear section of $\pi$. Next, we define the $R_0$-linear map

$$\rho: M \otimes_{R_0} t^0 R_s[t] \longrightarrow M \otimes_{R_0} t^1 R_s[t]$$
We note the particular cases 
\[ \rho(m \otimes x_d) = \begin{cases} 
0 , & m \in \mathbb{R} \\
0 & m \otimes x_1, \\
= m \otimes x_2 + \sum_j ma_j^{(1)} \otimes \beta_j^{-1} x_2. 
\end{cases} \]

The summands \( s_k = \sum_{j_k} m \alpha_{j_k}^{(k)} \otimes \beta_{j_k}^{-k} x_d, \) and hence the map \( \rho, \) do not depend on the particular choice of partition of unity. This is because \( s_k \) is the image of \( m \otimes x_d \) under the composition \( M \otimes_{R_0} R_d \cong M \otimes_{R_0} R_0 \otimes_{R_0} R_d \) 
\[ \xrightarrow{\pi_k^{-1}} M \otimes_{R_0} R_k \otimes_{R_0} R_{-k} \otimes_{R_0} R_d \xrightarrow{\sigma} M \otimes_{R_0} R_{-k+d}, \]
where \( \sigma(m \otimes a \otimes b \otimes x) = ma \otimes bx, \) and \( \pi_k^{-1} \) does not depend on choices by Lemma 2.2.

We have \( \rho \circ (\iota - \mu) = \text{id} \) since, for an element \( x_d \in R_d, \) we calculate 
\[ \rho \circ (\iota - \mu)(m \otimes x_d) = \rho(m \otimes x_d) - \sum_j \rho(ma_j^{(1)} \otimes \beta_j^{-1} x_d) \]
\[ = \sum_{k=0}^{d-1} \sum_{j_k} m \alpha_{j_k}^{(k)} \otimes \beta_{j_k}^{-k} x_d - \sum_{k=0}^{d-2} \sum_{j_k} \sum_j m \alpha_j^{(1)} \alpha_{j_k}^{(k)} \otimes \beta_{j_k}^{-k} \beta_j^{-1} x_d \]
\[ = \sum_{j_0}^{d-1} \sum_{j_k} m \alpha_{j_0}^{(0)} \otimes \beta_{j_0}^{(0)} x_d = \sum_{j_0} m \alpha_{j_0}^{(0)} \otimes \beta_{j_0}^{(0)} x_d = m \otimes x_d; \]
the equality labelled \( \circ \) makes use of Lemma 2.3, and of the fact that summands of the form \( s_{k+1} \) do not depend on choice of the partition of unity involved so that 
\[ \sum_{j_k} \sum_j m \alpha_j^{(1)} \alpha_{j_k}^{(k)} \otimes \beta_{j_k}^{-k} \beta_j^{-1} x_d = s_{k+1} = \sum_{j_{k+1}} m \alpha_{j_{k+1}}^{(k+1)} \otimes \beta_{j_{k+1}}^{-k-1} x_d. \]

It remains to verify the equality \( \sigma \circ \pi + (\iota - \mu) \circ \rho = \text{id}. \) For this, let \( x \in R_d \) and \( m \in M, \) and calculate:
\((\tau - \mu) \circ \rho (m \otimes x_d) = (\tau - \mu) \left( \sum_{k=0}^{d-1} \sum_{j \in J_k} ma_{j_k}^{(k)} \otimes \beta_{j_k}^{(-k)} x_d \right)\)

\[
= \sum_{k=0}^{d-1} \sum_{j \in J_k} ma_{j_k}^{(k)} \otimes \beta_{j_k}^{(-k)} x_d - \sum_{k=0}^{d-1} \sum_{k+1} ma_{j_{k+1}}^{(k+1)} \otimes \beta_{j_{k+1}}^{(-k-1)} x_d
\]

\[
= \sum_{j_0} ma_{j_0}^{(0)} \otimes \beta_{j_0}^{(-0)} x_d - \sum_{j_d} ma_{j_d}^{(d)} \otimes \beta_{j_d}^{(-d)} x_d
\]

\[
= m \otimes x_d - mx_d \otimes 1 = (id - \sigma \circ \pi)(m \otimes x_d).
\]

(As before, the equality marked \((*)\) holds because summands of the form \(s_{k+1}\) do not depend on choice of the partition of unity involved.) This finishes the proof. \(\square\)

**Definition 6.4.** Let \(C^+\) be a complex of \(R_+[t]\)-modules. The mapping cone \(\mathcal{H}^+(C^+)\) of the map \(\tau - \mu\),

\[
\mathcal{H}^+(C^+) = \text{cone} \left( C^+ \otimes_{R_0} t^1 R_+[t] \xrightarrow{\tau - \mu} C^+ \otimes_{R_0} t^0 R_+[t] \right),
\]

is called the **algebraic half-torus of** \(C^+\).

**Corollary 6.5.** Let \(C^+\) be a complex of \(R_+[t]\)-modules. The canonical map

\[
\mathcal{H}^+(C^+) = \text{cone} \left( C^+ \otimes_{R_0} t^1 R_+[t] \xrightarrow{\tau - \mu} C^+ \otimes_{R_0} t^0 R_+[t] \right) \longrightarrow C^+
\]

induced by the short exact sequence (6.3) is a quasi-isomorphism. If \(C^+\) is bounded below and consists of projective \(R_+[t]\)-modules, the map is a homotopy equivalence of \(R_+[t]\)-module complexes.

**Proof.** This is a direct consequence of standard homological algebra and Lemma 6.2 above. \(\square\)

The following result, though technical, is central to the theory of finite domination. By the previous Corollary we can replace any complex \(C^+\) of \(R_+[t]\)-modules by an algebraic half-torus, up to quasi-isomorphism; the **MATHER trick** is the observation that we can further replace the complex \(C^+\) within the mapping cone of the half-torus construction by an \(R_0\)-module complex homotopy equivalent to \(C^+\).

**Proposition 6.6** (The algebraic **MATHER trick** for algebraic half-tori). Let \(R = R_+[t, t^{-1}]\) be a strongly \(\mathbb{Z}\)-graded ring, let \(C^+\) be a complex of
$R_\ast[t]$-modules, and let $D$ be a complex of $R_0$-modules. Let $\alpha: C^+ \rightarrow D$ and $\beta: D \rightarrow C^+$ be mutually inverse chain homotopy equivalences of $R_0$-module complexes with $H: \text{id} \simeq \alpha \beta$ a specified homotopy. Write $\psi$ for the $R_\ast[t]$-module complex map

$$\psi = (\alpha \otimes \text{id}) \circ (\iota - \mu) \circ (\beta \otimes \text{id}): D \otimes_{R_0} t^1R_\ast[t] \rightarrow D \otimes_{R_0} t^0R_\ast[t].$$

Then the square diagram (6.7) in Fig. 4 commutes up to a preferred homotopy $J$ induced by $H$, given by the formula

$$J = (\alpha \otimes \text{id}) \circ (\iota - \mu) \circ (H \otimes \text{id}) : (\alpha \otimes \text{id}) \circ (\iota - \mu) \simeq \psi \circ (\alpha \otimes \text{id}).$$

The homotopy $J$ induces a preferred chain map

$$\Xi: \mathcal{S}^+(C^+) = \text{cone}(\iota - \mu) \rightarrow \text{cone}(\psi)$$

which is a quasi-isomorphism. If both $C^+$ and $D$ are bounded below complexes of projective $R_0$-modules, the map $\mathcal{S}^+(C^+) \rightarrow \text{cone}(\psi)$ is a homotopy equivalence of $R_\ast[t]$-module complexes.

![Figure 4. The Mather trick square](image)

**Proof.** By construction, $J$ is a homotopy from $(\alpha \otimes \text{id}) \circ (\iota - \mu)$ to $\psi \circ (\alpha \otimes \text{id})$. Hence we obtain a chain map of the mapping cones of the horizontal maps in the diagram,

$$\alpha \otimes \text{id} : \mathcal{S}^+(C^+) = \text{cone}(\iota - \mu) \rightarrow \text{cone}(\psi);$$

this map is a quasi-isomorphism since $\alpha$ is a homotopy equivalence (so the induced map on homology will be represented by a lower triangular matrix with isomorphisms on the main diagonal).

**Corollary 6.8.** If $C^+$ is an $R_0$-finitely dominated bounded below chain complex of projective $R_\ast[t]$-modules, then $C^+$ is $R_\ast[t]$-finitely dominated, that is, $C^+$ is homotopy equivalent to a bounded complex of finitely generated projective $R_\ast[t]$-modules.

**Proof.** As $C^+$ is $R_0$-finitely dominated we can choose an $R_0$-linear chain homotopy equivalence $\alpha: C^+ \rightarrow D$ from $C^+$ to a bounded complex $D$.
of finitely generated projective $R_0$-modules. By Corollary 6.5 and Proposition 6.6 there are quasi-isomorphisms
\[ C^+ \xrightarrow{\cong} \operatorname{ cone}(\psi) \] (6.9)
with $\psi : D \otimes_{R_0} t^1 R_0[t] \longrightarrow D \otimes_{R_0} t^0 R_0[t]$ as defined in Proposition 6.6 a map of bounded complexes of finitely generated projective $R_0[t]$-modules. It follows that $C^+$ is quasi-isomorphic, hence homotopy equivalent, to a bounded complex of finitely generated projective $R_0[t]$-modules as claimed.

7. Finite domination and homotopy finiteness

It is an interesting question whether in the situation of Corollary 6.8 the complex $C^+$ is actually $R_0[t]$-homotopy finite, that is, homotopy equivalent to a bounded complex of finitely generated free $R_0[t]$-modules. In general this turns out to be false; however, when working with $R_0[t, t^{-1}]$ instead of $R_0[t]$ the analogous question has a positive answer. — As before, let $R = R_0[t, t^{-1}]$ be a strongly $\mathbb{Z}$-graded ring.

**Proposition 7.1.** Suppose that $C$ is a bounded complex of finitely generated projective $R_0[t, t^{-1}]$-modules. If $C$ is $R_0$-finitely dominated, then $C$ is $R_0[t, t^{-1}]$-homotopy finite, i.e., $C$ is homotopy equivalent to a bounded complex of finitely generated free $R_0[t, t^{-1}]$-modules.

**Proof.** Let $D$ be a bounded chain complex of finitely generated projective $R_0$-modules chain homotopy equivalent to $C$. Then by the Mather trick for algebraic tori [HS17, Lemma 3.7], $C$ is homotopy equivalent, as an $R_0[t, t^{-1}]$-module complex, to the mapping cone of a certain self map of the induced complex $D \otimes_{R_0} R_0[t, t^{-1}]$. Hence the finiteness obstruction of $C$ in $\tilde{K}_0(R_0[t, t^{-1}])$ vanishes whence $C$ is $R_0[t, t^{-1}]$-homotopy finite.

The analogous statement holds over a polynomial ring $R_0[t]$ with a central indeterminate $t$: An $R_0$-finitely dominated, bounded $R_0[t]$-module complex $C^+$ of finitely generated projective modules is $R_0[t]$-homotopy finite. For $C^+ \cong \operatorname{ cone}(\psi)$ as in (6.9) and (6.7), and the finiteness obstruction of the mapping cone vanishes since $tR_0[t] \cong R_0[t]$. — In general, however, this line of reasoning fails when working over $R_0[t]$.

**Example 7.2.** There exist a strongly $\mathbb{Z}$-graded ring $R = R_0[t, t^{-1}]$ together with a bounded complex $C^+$ of finitely generated projective $R_0[t]$-modules such that $C^+$ is $R_0$-finitely dominated but not homotopy equivalent to a
bounded complex of finitely generated free $R_\ast[t]$-modules. Specifically\footnote{I am indebted to R. Hazrat for bringing this example to my attention.}, let\linebreak $K$ be a field and let $R = R_\ast[t, t^{-1}]$ be the LEAVITT $K$-algebra of type $(1, 1)$,\linebreak that is, the (non-commutative) $K$-algebra on generators $A, B, C, D$ subject\linebreak to the relations

$$AB + CD = 1, \quad BA = DC = 1, \quad BC = DA = 0;$$

we declare that $A$ and $C$ have degree $-1$, while $B$ and $D$ are given degree 1.\linebreak This is a $\mathbb{Z}$-graded ring since all relations are homogeneous of degree 0.\linebreak It is strongly graded by Corollary 2.4 as the relations $AB + CD = 1$ and\linebreak $BA = 1$ provide partitions of unity of types $(1, 1)$ and $(1, -1)$, respectively.\linebreak It is known that $R_0$ can be identified with an increasing union\linebreak $\bigcup_{n \geq 0} \text{Mat}_{2n}(K)$ of matrix algebras, using the block-diagonal embeddings\linebreak $x \mapsto \left( \begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix} \right)$. It follows that $R_0$ has IBN, and since the projection map\linebreak $R_\ast[t] = \bigoplus_{k \geq 0} R_k \longrightarrow R_0$ is a ring homomorphism, so does $R_\ast[t]$. —\linebreak The $R_\ast[t]$-module $Q = t^1 R_\ast[t]$ is finitely generated projective by Corol-\linebreak lary 2.8 (3), and the map

$$R_\ast[t] \xrightarrow{\cong} Q \oplus Q, \quad r \mapsto (Br, Dr) \quad (7.3)$$

is an isomorphism of $R_\ast[t]$-modules with inverse $(x, y) \mapsto Ax + Cy$. In\linebreak addition, $Q$ is not stably free: if $Q \oplus R_\ast[t]^m \cong R_\ast[t]^n$, then by (7.3) also\linebreak $R_\ast[t]^{2n} \cong (Q \oplus R_\ast[t]^m) \oplus (Q \oplus R_\ast[t]^m) \cong R_\ast[t]^{2m+1}$;

as $R_\ast[t]$ has IBN, the inequality $2n \neq 2m + 1$ renders this impossible.\linebreak The class of $Q$ in $\bar{K}_0(R_\ast[t])$ is thus non-zero, and has in fact order 2 in\linebreak view of the isomorphism (7.3). Thus the inclusion map $Q \longrightarrow R_\ast[t]$,\linebreak considered as a chain complex $C^+$, is an example of a bounded complex of\linebreak finitely generated projective $R_\ast[t]$-modules not homotopy equivalent to a\linebreak bounded complex of finitely generated free $R_\ast[t]$-modules. On the\linebreak other hand, the complex $C^+$ is certainly $R_0$-finitely dominated since the\linebreak cokernel of the inclusion map $Q = t^1 R_\ast[t] \longrightarrow R_\ast[t]$ is isomorphic to $R_0$.

8. $R_0$-FINITE DOMINATION OF $R_\ast[t]$-MODULE COMPLEXES

We now develop a homological criterion to detect if a chain complex $C^+$\linebreak of $R_\ast[t]$-modules is $R_0$-finitely dominated when considered as a complex of\linebreak $R_0$-modules via restriction of scalars. This happens if and only if $C^+$ has\linebreak trivial NOVIKOV homology in the sense that the induced chain complex\linebreak $C^+ \otimes_{R_\ast[t]} R_\ast((t^{-1}))$ is acyclic.
Theorem 8.1. Let \( R = R_\ast [t, t^{-1}] \) be a strongly \( \mathbb{Z} \)-graded ring, and let \( C^+ \) be a bounded chain complex of finitely generated projective \( R_\ast [t] \)-modules. The following statements are equivalent:

1. The complex \( C^+ \) is \( R_0 \)-finitely dominated.
2. The complex \( C^+ \otimes_{R_\ast [t]} R_\ast ((t^{-1})) \) is contractible (i.e., \( C^+ \) has trivial Novikov homology).

Proof. (1) \( \Rightarrow \) (2): As \( C^+ \) is \( R_0 \)-finitely dominated, we find a bounded complex \( D \) of finitely generated projective \( R_0 \)-modules, and mutually inverse \( R_0 \)-linear chain homotopy equivalences \( \alpha : C^+ \longrightarrow D \) and \( \beta : D \longrightarrow C^+ \).

Let \( \psi \) be as in Proposition 6.6; together with Corollary 6.5, the Mather trick asserts that the \( R_\ast [t] \)-module complexes \( C^+ \) and \( \text{cone}(\psi) \) are quasi-isomorphic and thus are chain homotopy equivalent (as both are bounded below and consist of projective \( R_\ast [t] \)-modules). Thus \( C^+ \otimes_{R_\ast [t]} R_\ast ((t^{-1})) \) is homotopy equivalent to \( \text{cone}(\psi) \otimes_{R_\ast [t]} R_\ast ((t^{-1})) \). The latter complex in turn is isomorphic to the mapping cone of the chain map

\[
D \otimes_{R_0} R_\ast ((t^{-1})) \longrightarrow D \otimes_{R_0} R_\ast ((t^{-1}))
\]

sending the element \( x \otimes \sum_{i \leq k} r_i t^i \) to

\[
\alpha \beta(x) \otimes \sum_{j \leq k} r_j t^j - \sum_j \alpha(\beta(x) a_j^{(1)}) \otimes \sum_{j \leq k} \beta_j^{(-1)} r_j t^{j-1},
\]

where we write elements of \( R_\ast ((t^{-1})) \) as formal Laurent series in \( t^{-1} \); note that in the target of the map, \( \beta_j^{(-1)} r_j \) is the coefficient of \( t^{-1} \) as \( \beta_j^{(-1)} \) has degree \(-1\).

As \( D \) consists of finitely presented \( R_0 \)-modules, we can identify both the tensor products

\[
D \otimes_{R_0} R_\ast [t] \otimes_{R_\ast [t]} R_\ast ((t^{-1})) = D \otimes_{R_0} R_\ast ((t^{-1}))
\]

and

\[
D \otimes_{R_0} t^1 R_\ast [t] \otimes_{R_\ast [t]} R_\ast ((t^{-1})) = D \otimes_{R_0} R_\ast ((t^{-1}))
\]

with the twisted right-truncated power of \( D \) [HS17, Proposition 3.13], that is,

\[
D \otimes_{R_0} R_\ast ((t^{-1})) \cong \prod_{n \leq 0} (D \otimes_{R_0} R_n) \oplus \bigoplus_{n > 0} (D \otimes_{R_0} R_n).
\]

Thus we rewrite \( \text{cone}(\psi) \otimes_{R_\ast [t]} R_\ast ((t^{-1})) \) as the right-truncated totalisation [Hüt11, Definition 1.1] of a double complex

\[
Z_{p,q} = (D_{p+q+1} \otimes_{R_0} R_{p+1}) \oplus (D_{p+q} \otimes_{R_0} R_p)
\]

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with vertical differential \( d^v : Z_{p,q} \longrightarrow Z_{p,q-1} \) and horizontal differential \( d^h : Z_{p,q} \longrightarrow Z_{p-1,q} \) given by the formulæ

\[
d^v(x \otimes a, y \otimes b) = \left( -d(x) \otimes a, \alpha \beta(x) \otimes a + d(y) \otimes b \right),
\]

\[
d^h(x \otimes a, y \otimes b) = \left( -\sum a(\beta(y)\alpha_i^{(1)}) \otimes \beta_i^{(-1)} b, 0 \right).
\]

The symbol “\( d \)”, without any decorations, refers to the differential of the chain complex \( D \). The columns are acyclic since \( Z_{p,i} \) is a shift suspension of \( \text{cone}(\alpha \beta) \otimes R_0 R_p \) and the chain map \( \alpha \beta \) is homotopic to an identity map. It follows that \( C^+ \otimes_{R_0[t]} R_s(\langle t^{-1} \rangle) \cong \text{cone}(\psi) \otimes_{R_0[t]} R_s(\langle t^{-1} \rangle) \) is acyclic \([\text{Hüt11}, \text{Proposition 1.2}]\), and hence contractible.

(2) \( \Rightarrow \) (1): As \( C^+ \) consists of finitely generated projective \( R_s[t] \)-modules, there exists another bounded complex \( B^+ \) with zero differentials, consisting of finitely generated projective \( R_s[t] \)-modules, such that \( A^+ = B^+ \otimes C^+ \) is a bounded complex of finitely generated free \( R_s[t] \)-modules.

We equip \( A^+_k \) with a basis with \( r_k \) elements, and identify \( A^+_k \) with \( R_s[t]^{r_k} \) henceforth. The differential \( d^+_k : A^+_k \longrightarrow A^+_{k-1} \) is thus represented by a matrix \( D_k \) with entries in \( R_s[t] \).

Suppose, for ease of notation, that \( C^+ \) is concentrated in chain levels between 0 and \( m \). We can choose integers

\[
d_m \leq d_{m-1} \leq \ldots \leq d_0 = -1
\]

so that \( D_k \) defines a map \( d^-_k : t^{d_k} R_s(\langle t^{-1} \rangle)^{r_k} \longrightarrow t^{d_k-1} R_s(\langle t^{-1} \rangle)^{r_{k-1}} \); we only need to ensure that no entry of \( D_k \) has a non-zero homogeneous component of degree exceeding \( d_{k-1} - d_k \). We let \( S \) denote the chain complex thus defined, with \( S_k = t^{d_k} R_s(\langle t^{-1} \rangle)^{r_k} \) and differentials \( D_k \). Similarly, we let \( N \) denote the chain complex with \( N_k = R_s(\langle t^{-1} \rangle)^{r_k} \) and differentials \( D_k \). Note that \( S \) is a subcomplex of \( N \).

For any \( d \leq -1 \) there is a short exact sequence of \( R_0 \)-modules

\[
0 \longrightarrow t^d R_s(\langle t^{-1} \rangle) \oplus R_s[t] \xrightarrow{(-1,1)} R_s(\langle t^{-1} \rangle) \longrightarrow \bigoplus_{j=d+1}^{\infty} R_j \longrightarrow 0 \quad (8.2)
\]

with last term a finitely generated projective \( R_0 \)-module by Corollary 2.8, as \( R_s[t, t^{-1}] \) is strongly graded. It follows that there is a short exact sequence of \( R_0 \)-module complexes

\[
0 \longrightarrow S \oplus A^+ \xrightarrow{(-1,1)} N \longrightarrow P \longrightarrow 0 \quad (8.3)
\]

with \( P \) a bounded complex of finitely generated projective \( R_0 \)-modules. In chain degree \( k \) this sequence is actually just the \( r_k \)-fold direct sum of \((8.2)\) with itself, for \( d = d_k \).
From the sequence (8.3) we infer that the map from the mapping cone of \( \beta \) to \( P \) is a quasi-isomorphism. Now recall \( A^+ = B^+ \oplus C^+ \) and observe the consequent splitting

\[
N = A^+ \otimes_{R_0[t]} R_0((t^{-1})) = B^+ \otimes_{R_0[t]} R_0((t^{-1})) \oplus C^+ \otimes_{R_0[t]} R_0((t^{-1})) .
\] (8.4)

By hypothesis \( C^+ \otimes_{R_0[t]} R_0((t^{-1})) \) is contractible; thus \( N \) is quasi-isomorphic, via the projection map, to \( B^+ \otimes_{R_0[t]} R_0((t^{-1})) \). As taking mapping cones is homotopy invariant, we can replace \( N \) by the latter complex and conclude that \( P \) is quasi-isomorphic to the mapping cone of the map

\[
\gamma : S \oplus A^+ = S \oplus B^+ \oplus C^+ \xrightarrow{(-1 \ 1 \ 0)} B^+ \otimes_{R_0[t]} R_0((t^{-1})) .
\]

As \( \gamma \) is the zero map on the \( C^+ \)-summand, the mapping cone of \( \gamma \) contains \( C^+ \) as a direct summand. Hence in the derived category of the ring \( R_0 \), the complex \( C^+ \) is a retract of \( P \). Since both complexes are bounded and consist of projective \( R_0 \)-modules, we conclude that \( C^+ \) is a retract up to homotopy of \( P \) whence \( C^+ \) is \( R_0 \)-finitely dominated as claimed. \( \square \)

9. \( R_0[t] \)-Fredholm matrices

Let \( R = R_0[t, t^{-1}] \) be a \( \mathbb{Z} \)-graded ring, and let \( A^+ \) be a non-zero square matrix of size \( k \) with entries in \( R_0[t, t^{-1}] \). For suitable \( m \in \mathbb{Z} \), multiplication by \( A^+ \) defines an \( R_0[t] \)-module homomorphism

\[
A^+ = \mu(A^+, m) : R_0[t]^k \xrightarrow{k \times (t^{-m}R_0[t])^k} x \mapsto A^+ \cdot x ;
\]

“suitable” means, in fact, that \(-m\) is not larger than the minimal degree of non-zero homogeneous components of entries of \( A^+ \). Suppose now that in addition to such \( m \) we fix an integer \( n > m \) so that the map \( \mu(A^+, n) \) is defined as well.

**Lemma 9.1.** There is an isomorphism of \( R_0 \)-modules

\[
\text{coker } \mu(A^+, n) \cong \text{coker } \mu(A^+, m) \oplus \bigoplus_{k=-n}^{m-1} R_k .
\]

**Proof.** The direct sum of the exact sequence of \( R_0 \)-modules

\[
R_0[t]^k \xrightarrow{\mu(A^+, m)} (t^{-m}R_0[t])^k \xrightarrow{\text{coker } \mu(A^+, m)} 0
\]

with the exact sequence

\[
0 \xrightarrow{\bigoplus_{k=-n}^{m-1} R_k} \left( \bigoplus_{k=-n}^{m-1} R_k \right) \xrightarrow{-m-1} 0
\]
yields a new exact sequence, which is precisely the sequence

\[ R_*[t]^k \xrightarrow{\mu(A^+, n)} (t^{-n}R_*[t])^k \xrightarrow{coker \mu(A^+, m) \oplus \bigoplus_{k=-n}^{-m-1} R_k} 0. \]

Hence \( coker \mu(A^+, n) \cong coker \mu(A^+, m) \oplus \bigoplus_{k=-n}^{-m-1} R_k \) as \( R_0 \)-modules. \( \Box \)

**Corollary 9.2.** Suppose that \( R = R_*[t, t^{-1}] \) is strongly \( \mathbb{Z} \)-graded. In the situation of Lemma 9.1, the module \( coker \mu(A^+, n) \) is a finitely generated projective \( R_0 \)-module if and only if \( coker \mu(A^+, m) \) is.

**Proof.** This is a consequence of Corollary 2.8 (1) and Lemma 9.1. \( \Box \)

**Proposition 9.3.** Suppose that \( R = R_*[t, t^{-1}] \) is a strongly \( \mathbb{Z} \)-graded ring. Let \( A^+ \) be a \( k \times k \)-matrix with entries in \( R_*[t, t^{-1}] \), and let \( m \in \mathbb{Z} \) be “suitable” in the sense that multiplication by \( A^+ \) yields a map of finitely generated projective \( R_* \)-modules \( A^+ = \mu(A^+, m) : R_*[t]^k \rightarrow t^{-m}R_*[t]^k, \ x \mapsto A^+ \cdot x \) (see discussion above) which we may consider as a chain complex concentrated in chain degrees 1 and 0. The following statements are equivalent:

1. The chain complex \( A^+ \) is \( R_0 \)-finitely dominated.
2. The induced chain complex \( A^+ \otimes_{R_*[t]} R_*(t^{-1}) \) is contractible.
3. The map \( A^+ \) is invertible over \( R_*(t^{-1}) \), that is, the map
   \[ R_*(t^{-1})^k \rightarrow R_*(t^{-1})^k, \ x \mapsto A^+ \cdot x \]
   is an isomorphism.
4. The matrix \( A^+ \) is invertible in the ring of all square matrices of size \( k \) with entries in \( R_*(t^{-1}) \).
5. The map \( \mu(A^+, m) \) is injective, and \( coker \mu(A^+, m) \) is a finitely generated projective \( R_0 \)-module.

Moreover, the validity of these statements does not depend on the specific choice of a suitable \( m \in \mathbb{Z} \).

**Definition 9.4.** A square matrix matrix with entries in \( R_*[t, t^{-1}] \) satisfying one (and hence all) of the conditions listed in Proposition 9.3 is called an \( R_*[t] \)-**FREDHOLM matrix**. The set of all \( R_*[t] \)-FREDHOLM matrices (of arbitrary finite size) is denoted by the symbol \( \Omega^+_* \).

**Proof of Proposition 9.3.** Condition (5) is insensitive to the precise value of the suitable integer \( m \), in view of Corollary 9.2.

The equivalence of conditions (1) and (2) is Theorem 8.1 above. Statements (3) and (4) are trivially equivalent.

By Lemma 2.5, the multiplication map

\[ t^{-m}R_*[t] \otimes_{R_*[t]} R_*[t, t^{-1}] \rightarrow R_*[t, t^{-1}], \quad x \otimes y \mapsto xy \]
is an isomorphism of $R_s[t, t^{-1}]$-modules. It follows that there is a chain of isomorphisms

$$t^{-m}R_s[t] \otimes_{R_s[t]} R_s((t^{-1})) \cong t^{-m}R_s[t] \otimes_{R_s[t]} R_s[t, t^{-1}] \otimes_{R_s[t, t^{-1}]} R_s((t^{-1}))$$

$$\cong R_s[t, t^{-1}] \otimes_{R_s[t, t^{-1}]} R_s((t^{-1})) \cong R_s((t^{-1}))$$

with composition the multiplication map. In view of this, statements (2) and (3) are equivalent.

If (5) holds then the chain complex $A^+$ is $R_0$-homotopy equivalent to the module $\text{coker} \mu(A^+, m)$, considered as a chain complex concentrated in degree 0, which shows that (1) is satisfied in this case.

Suppose finally that (3) holds; we will show that (5) is valid as well. We infer from the commutative square

$$R_s[t] \xrightarrow{\mu(A^+, m)} t^{-m}R_s[t]$$

that the map $\mu(A^+, m)$ must be injective. Thus it remains to verify that $\text{coker} \mu(A^+, m)$ is a finitely generated projective $R_0$-module. Assuming $m \geq 1$, as we may in view of Corollary 9.2, we can embed $\mu(A^+, m)$ into a commutative diagram of $R_0$-modules

$$0 \rightarrow \text{ker}(\xi - \zeta) \rightarrow X \oplus Z \xrightarrow{\xi - \zeta} Y \rightarrow \text{coker}(\xi - \zeta) \rightarrow 0.$$  

We apply this to the rows of diagram (9.5) above, noting the the coker term is trivial in both cases (since $q, m \geq 0$). The kernel, on the other hand, is trivial in case of the top row, and is the finitely generated projective $R_0$-module $P = \bigoplus_{-m}^{q} R_j$ for the bottom row. We arrive at the following
commutative diagram with exact rows:

\[
\begin{array}{c}
0 \longrightarrow t^{-1}R((t^{-1}))^k \oplus R_n[t] \longrightarrow R((t^{-1})) \longrightarrow 0 \\
A^+ \oplus A^+ \downarrow \quad \downarrow A^+ \\
0 \longrightarrow P \longrightarrow t^0R((t^{-1}))^k \oplus t^{-m}R_n[t]^k \longrightarrow R((t^{-1})) \longrightarrow 0
\end{array}
\]

As the right-hand vertical map is an isomorphism by hypothesis (3), the Snake lemma yields an isomorphism of \(P\) with the cokernel of the middle vertical map, which contains \(\text{coker} \mu(A^+, m): R_n[t]^k \longrightarrow t^{-m}R_n[t]^k\) as a direct summand. This shows that \(\text{coker} \mu(A^+, m)\) is a finitely generated projective \(R_0\)-module as desired. □

10. The Fredholm Localisations \(\Omega_+^{-1}R_n[t]\) and \(\Omega_+^{-1}R_n[t, t^{-1}]\)

We now turn our attention to the non-commutative localisations

\[
\alpha: R_n[t] \longrightarrow \Omega_+^{-1}R_n[t] \quad \text{and} \quad \gamma: R_n[t, t^{-1}] \longrightarrow \Omega_+^{-1}R_n[t, t^{-1}]
\]

where \(\Omega_+\) denotes the set of \(R_n[t]\)-Fredholm matrices as in Definition 9.4.

To be precise, we define \(\alpha = \lambda_{\Omega_+}: R_n[t] \longrightarrow \Omega_+^{-1}R_n[t]\) as the the non-commutative localisation inverting all the maps

\[
\mu(A^+, m): R_n[t]^k \longrightarrow t^{-m}R_n[t]^k \quad (10.1)
\]

of finitely generated projective \(R_n[t]\)-modules, where \(k \geq 1\) is arbitrary, \(A^+ \in \Omega_+\) has size \(k\), and \(m \in \mathbb{Z}\) is suitable in the sense of §9. As \(A^+\) satisfies property (4) of Proposition 9.3, the universal property of non-commutative localisation yields a factorisation

\[
R_n[t] \longrightarrow \Omega_+^{-1}R_n[t] \longrightarrow R((t^{-1})) \quad (10.2)
\]

of the inclusion map; in particular, \(\alpha\) is injective. — Similarly, we define \(\gamma = \lambda_{\Omega_+}: R_n[t, t^{-1}] \longrightarrow \Omega_+^{-1}R_n[t, t^{-1}]\) as the the non-commutative localisation inverting all the maps

\[
A^+: R_n[t, t^{-1}]^k \longrightarrow R_n[t, t^{-1}]^k \quad (10.3)
\]

of finitely generated free \(R_n[t, t^{-1}]\)-modules, where \(k \geq 1\) is arbitrary and \(A^+ \in \Omega_+\) has size \(k\). As \(A^+\) satisfies property (4) of Proposition 9.3, the universal property of non-commutative localisation yields a factorisation

\[
R_n[t, t^{-1}] \longrightarrow \Omega_+^{-1}R_n[t, t^{-1}] \longrightarrow R((t^{-1})) \quad (10.4)
\]

of the inclusion map; in particular, \(\gamma\) is injective.
Applying the functor $- \otimes_{R_0[t]} R_0[t, t^{-1}]$ to a map as in (10.1) yields a map as in (10.3), by Lemma 2.5. Thus $\gamma|_{R_0[t]}: R_0[t] \longrightarrow \Omega_+^{-1} R_0[t, t^{-1}]$ inverts all the maps (10.1) and factorises through a ring homomorphism $\delta: \Omega_+^{-1} R_0[t] \longrightarrow \Omega_+^{-1} R_0[t, t^{-1}]$. That is, the maps $\alpha$ and $\gamma$ fit into the commutative square diagram of Fig. 5 which, by Proposition 4.2, is a pushout square in the category of unital rings.

\[
\begin{array}{ccc}
R_0[t] & \xrightarrow{\alpha} & \Omega_+^{-1} R_0[t] \\
\downarrow{\beta} & \nearrow{\gamma} & \downarrow{\delta} \\
R_0[t, t^{-1}] & \xrightarrow{\gamma} & \Omega_+^{-1} R_0[t, t^{-1}]
\end{array}
\]  

(10.5)

**Figure 5.** Pushout square of Fredholm localisations

**Theorem 10.6.** Let $R = R_0[t, t^{-1}]$ be a strongly $\mathbb{Z}$-graded ring, and let $C^+$ be a bounded chain complex of finitely generated projective $R_0[t]$-modules. The following statements are equivalent:

1. The chain complex $C^+$ is $R_0$-finitely dominated.
2. The induced chain complex $C^+ \otimes_{R_0[t]} \Omega_+^{-1} R_0[t]$ is contractible.
3. The induced chain complex $C^+ \otimes_{R_0[t]} \Omega_+^{-1} R_0[t, t^{-1}]$ is contractible.

**Proof.** (1) $\Rightarrow$ (2): Suppose that $C^+$ is $R_0$-finitely dominated. For ease of notation we assume $C^+_n = 0$ for $n < 1$. By taking direct sum with contractible one-step complexes of the form $P \longrightarrow P$, with $P$ suitable finitely generated projective $R_0[t]$-modules, we obtain a new bounded chain complex $A^+$ concentrated in non-negative chain levels, which is homotopy equivalent to $C^+$ such that all chain modules $A^+_n$ are finitely generated free over $R_0[t]$, with the possible exception of $A^+_0$ which is finitely generated projective over $R_0[t]$. Explicitly, let $N$ be maximal with $C^+_N \neq 0$. We can choose finitely generated projective $R_0[t]$-module $Q^+_N, Q^+_{N-1}, \cdots, Q^+_1$, in this order, such that $C^+_n \oplus Q^+_n \oplus Q^+_{n+1}$ is finitely generated free ($1 \leq n \leq N$, with $Q^+_k = 0$ for $k > N$); the bounded complex
\( A^+ \) can then take the form
\[
\begin{array}{cccccc}
Q_1^+ & \leftarrow & Q_1^+ & Q_3^+ & \leftarrow & Q_3^+ & Q_5^+ & \leftarrow & \cdots \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
C_1^+ & \leftarrow & C_2^+ & \leftarrow & C_3^+ & \leftarrow & C_4^+ & \leftarrow & \cdots \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
Q_2^+ & \leftarrow & Q_2^+ & Q_4^+ & \leftarrow & Q_4^+ & \cdots
\end{array}
\]
so that \( C^+ \) is a direct summand of \( A^+ \), and both the inclusion \( C^+ \hookrightarrow A^+ \) and the projection \( A^+ \twoheadrightarrow C^+ \) are homotopy equivalences.

For \( n > 0 \) the module \( A^+_n = C^+_n \oplus Q^+_n \oplus Q^+_n \oplus Q^+_n \oplus Q^+_n \) is finitely generated free. We choose a basis consisting of \( r_n \) elements, thereby identifying \( A^+_n \) with the finite direct sum \( \oplus r_n R_n[t] = (R_n[t])^r_n \). The \( n \)th chain module of the induced complex \( A^+ \otimes_{R, [t]} R_n((t^{-1})) \) is identified with \( R_n((t^{-1}))^r_n \).

For \( n > 1 \) the differential \( A^+_n \twoheadrightarrow A^+_{n-1} \) of \( A^+ \) can be thought of as a matrix \( D^+_n \) with entries in \( R_n[t] \). The differential \( D^+_1 \) is the homomorphism given by projection onto \( A^+_0 = Q^+_1 \).

As \( C^+ \) is \( R_n[t, t^{-1}] \)-finitely dominated, Theorem 8.1 ensures that the induced complex \( C^+ \otimes_{R, [t]} R_n((t^{-1})) \) is contractible; we choose homomorphisms
\[
\tau_n : C^+_n \otimes_{R, [t]} R_n((t^{-1})) \twoheadrightarrow C^+_{n+1} \otimes_{R, [t]} R_n((t^{-1}))
\]
forming a chain contraction. These maps give rise to a contraction \( \sigma^+_n \) of \( A^+ \otimes_{R, [t]} R_n((t^{-1})) \), by defining
\[
\sigma_n : A^+_n \otimes_{R, [t]} R_n((t^{-1})) \twoheadrightarrow A^+_{n+1} \otimes_{R, [t]} R_n((t^{-1}))
\]
by the formula
\[
\sigma^+_n = \begin{cases} 
Q^+_1 \otimes_{R, [t]} R_n((t^{-1})) \subseteq A^+_1 \otimes_{R, [t]} R_n((t^{-1})) & \text{for } n = 0, \\
(C^+_n \oplus Q^+_n \oplus Q^+_n \oplus Q^+_n) \otimes_{R, [t]} R_n((t^{-1})) & \\
(\sigma_{n+1}, 0, \text{id}) \left( C^+_{n+1} \oplus Q^+_n \oplus Q^+_n \oplus Q^+_n \otimes_{R, [t]} R_n((t^{-1})) \right) & \text{for } n > 0.
\end{cases}
\]
Note that the map \( \sigma^+_0 \) is defined over \( R_n[t] \). For \( n > 0 \) we think of \( \sigma^+_n \) as matrices with entries in \( R_n((t^{-1})) \) such that \( D^+_{n+1} \cdot \sigma^+_n + \sigma^+_n \cdot D^+_{n+1} \) is a unit matrix of size \( r_n \). We can truncate the entries of the matrices \( \sigma^+_n \) below at some suitable integer \( m \ll 0 \) (not depending on \( n \)) to obtain matrices \( S^+_n = \text{tr}_m(\sigma^+_n) \) with entries in \( R_n[t, t^{-1}] \) such that \( E_n = D^+_{n+1} \cdot S^+_n + S^+_n \cdot D^+_{n+1} \), for \( n \geq 2 \), is the sum of a unit matrix, and a matrix the non-zero entries of
which have homogeneous components of strictly negative degree. Thus $E_n$ is invertible over $R_n((t^{-1}))$ so that $E_n \in \Omega_+$. Similarly, writing $S_0^+$ for the homomorphism $\sigma_0^+$ we see that $E_1 = D_2^+ \cdot S_1^+ + S_0^+ \cdot D_1^+$ and $E_0 = D_1^+ \circ S_0^+$ are invertible in $R_n((t^{-1}))$ whence $E_1, E_0 \in \Omega_+$ as well. Here we make use of the fact that $\sigma_0^+ = S_0^+$ and $D_1^+$ are defined over $R_n[t]$; in fact $E_0 = \text{id}_{Q_1}$, and the matrix representing $S_0^+ \cdot D_1^+$ has entries in $R_n[t]$ and is hence unaffected by truncation.

We now define a new $R_n[t]$-module chain complex $B^+$ by setting $B_n^+ = \left( t^m R_n[t] \right) \gamma_s = A_n^+ \otimes_{R_n[t]} t^m R_n[t]$, with differentials given by the matrices $D_n^+$ for $n > 1$, and the projection map onto the direct summand $Q_1^+ \otimes_{R_n[t]} t^m R_n[t]$ for $n = 1$. The matrices $S_n^+$ for $n > 0$, and the homomorphism $\sigma_0^+$, define module homomorphisms $A_n^+ \longrightarrow B_n^+$ constituting an $(A^+, B^+)$-\(\alpha\)-proto-null homotopy, cf. §3, since $E_n = D_{n+1}^+ \cdot S_n^+ + S_{n-1}^+ \cdot D_n^+$ is an element of $\Omega_+$ as explained above. Here $\alpha : R_n[t] \longrightarrow \Omega_+ R_n[t]$ is the localisation map as in (10.5). It follows that $A^+ \otimes_{R_n[t]} \Omega_+^{-1} R_n[t]$ is contractible by Lemma 3.1, hence so is $C^+ \otimes_{R_n[t]} \Omega_+^{-1} R_n[t]$.

(2) $\Rightarrow$ (3): Immediate from the factorisation $R_n[t] \overset{\alpha}{\longrightarrow} \Omega_+^{-1} R_n[t] \longrightarrow \Omega_+^{-1} R_n[t, t^{-1}]$ of $\gamma$, see (10.5) in Fig. 5.

(3) $\Rightarrow$ (1): Immediate from the factorisation (10.4) and Theorem 8.1. □

**Theorem 10.7** (Universal property of $\Omega_+^{-1} R_n[t]$). Suppose that $R_n[t, t^{-1}]$ is a strongly $\mathbb{Z}$-graded ring. The localisation $\lambda : R_n[t] \longrightarrow \Omega_+^{-1} R_n[t]$ is the universal $R_n[t]$-ring making $R_0$-finitely dominated chain complexes contractible. That is, suppose that $f : R_n[t] \longrightarrow S$ is an $R_n[t]$-ring such that for every bounded complex of finitely generated projective $R_n[t]$-modules $C$ which is $R_0$-finitely dominated, the complex $C^+ \otimes_{R_n[t]} S$ is contractible. Then there is a factorisation $R_n[t] \overset{\lambda}{\longrightarrow} \Omega_+^{-1} R_n[t] \overset{\eta}{\longrightarrow} S$ of $f$, with a uniquely determined ring homomorphism $\eta$.

**Proof.** It was shown in Theorem 10.6 above that $\Omega_+^{-1} R_n[t]$ makes $R_0$-finitely dominated chain complexes contractible. Thus it is enough to show that $f$ inverts the maps $\mu(A^+, m)$ of (10.1) for any $A^+ \in \Omega_+$ and any suitable $m \in \mathbb{Z}$. By definition of $\Omega_+$ the complex $\mu(A^+, m)$ is $R_0$-finitely dominated so that, by hypothesis on $f$, the complex $\mu(A^+, m) \otimes_{R_n[t]} S$ is contractible. This says precisely that $f$ inverts the map $\mu(A^+, m)$. □
One can also show that the localisation $\lambda : R_s[t, t^{-1}] \longrightarrow \Omega^+_s R_s[t, t^{-1}]$ is the universal $R_s[t, t^{-1}]$-ring making $R_0$ finitely dominated, bounded chain complexes of finitely generated projective $R_s[t]$-modules complexes contractible.

We finish with proving that $\delta : \Omega^+_s R_s[t] \longrightarrow \Omega^+_s R_s[t, t^{-1}]$ is an isomorphism if $R_s[t, t^{-1}]$ contains a homogeneous unit of non-zero degree.

**Proposition 10.8.** Suppose that $R = R_s[t, t^{-1}]$ is a strongly $\mathbb{Z}$-graded ring. Suppose there exists a homogeneous unit of positive degree in $R_s[t, t^{-1}]$. Then there is an injective ring homomorphism $\iota : R_s[t, t^{-1}] \longrightarrow \Omega^+_s R_s[t]$ with $\iota \beta = \alpha$, and $\delta : \Omega^+_s R_s[t] \longrightarrow \Omega^+_s R_s[t, t^{-1}]$ is an isomorphism satisfying $\delta \iota = \gamma$.

**Proof.** Let $u \in R_d \cap R_s[t, t^{-1}]$, with $d > 0$. Then the $1 \times 1$-matrix $(u)$ is an $R_s[t]$-Fredholm matrix since the cokernel of the map

$$R_s[t] \longrightarrow R_s[t], \quad r \mapsto ur$$

is the finitely generated projective $R_0$-module $\bigoplus_0^{d-1} R_j$. The induced map

$$\Omega^+_s R_s[t] \longrightarrow \Omega^+_s R_s[t]$$

is given by multiplication with $\alpha(u) \in \Omega^+_s R_s[t]$. Since the induced map is an isomorphism, $\alpha(u)$ is invertible in $\Omega^+_s R_s[t]$.

Given any $x \in R_s[t, t^{-1}]$ there exists $k \geq 0$ with $u^k x \in R_s[t]$ and thus $\alpha(u^k x) \in \Omega^+_s R_s[t]$; we define $\iota(x) = \alpha(u)^{-k} \cdot \alpha(u^k x) \in \Omega^+_s R_s[t]$. The element $\iota(x)$ does not depend on the choice of $k$, for if $\ell > k$ we have

$$\alpha(u)^{-\ell} \cdot \alpha(u^\ell x) = \alpha(u)^{-k} \cdot \alpha(u^{\ell-k} u^k x) = \alpha(u)^{-k} \cdot \alpha(u^k x),$$

since $u^{\ell-k} \in R_s[t]$ and since $\alpha$ is a ring homomorphism. Note that $\iota(x) = \alpha(x)$ for $x \in R_s[t]$, and that $\iota(u^{-1}) = \alpha(u)^{-1}$.

Suppose that $x \in R_s[t, t^{-1}]$ and $k, \ell \geq 0$ are such that $u^k xu^{-\ell} \in R_s[t]$. Then $\alpha(u^k xu^{-\ell}) = \alpha(u^k x) \alpha(u)^{-\ell}$, since both sides equal $\alpha(u^k x)$ after multiplication with $\alpha(u)^\ell$. Consequently, for $x, y \in R_s[t, t^{-1}]$ and $k, \ell \geq 0$ with $u^\ell y, u^k xu^{-\ell} \in R_s[t]$ we calculate

$$\iota(xy) = \iota(xu^{-\ell}u^\ell y)$$

$$= \alpha(u)^{-k} \cdot \alpha(u^k xu^{-\ell}u^\ell y)$$

$$= \alpha(u)^{-k} \cdot \alpha(u^k xu^{-\ell}) \cdot \alpha(u^\ell y)$$

$$= \alpha(u)^{-k} \cdot \alpha(u^k x) \cdot \alpha(u)^{-\ell} \cdot \alpha(u^\ell y) = \iota(x) \cdot \iota(y).$$
Since \( \iota \) is clearly additive, the map \( \iota : R_\ast[t, t^{-1}] \to \Omega^{-1}_\ast R_\ast[t] \) is thus a ring homomorphism. Moreover, \( \iota \) is injective as \( \iota(x) = \alpha(u)^{-k} \cdot \alpha(u^kx) \) vanishes if and only if \( \alpha(u^kx) \) vanishes. It follows from Lemmas 2.6 and 4.3 that the ring homomorphism \( \delta \) is an isomorphism and satisfies \( \delta \iota = \gamma \).

**References**


