Non-commutative localisation and finite domination over strongly Z-graded rings

Hüttemann, T. (2022). Non-commutative localisation and finite domination over strongly Z-graded rings. *Homology, Homotopy and Applications*, *24*(1), 373-398. https://doi.org/10.4310/HHA.2022.v24.n1.a18

NON-COMMUTATIVE LOCALISATION AND FINITE DOMINATION OVER STRONGLY $\mathbb{Z}$-GRADED RINGS

THOMAS HÜTTEMANN

ABSTRACT. Let $R = \bigoplus_{k=-\infty}^{\infty} R_k$ be a strongly $\mathbb{Z}$-graded ring, and let C^+ be a chain complex of modules over the positive subring $P = \bigoplus_{k=0}^{\infty} R_k$. The complex $C^+ \otimes_P R_0$ is contractible (*resp.*, C^+ is R_0-finitely dominated) if and only if $C^+ \otimes_P L$ is contractible, where L is a suitable non-commutative localisation of P. We exhibit universal properties of these localisations, and show by example that an R_0-finitely dominated complex need not be P-homotopy finite.

CONTENTS

Date: March 16, 2021.
2010 *Mathematics Subject Classification.* 18G35, 55U15, 16W50, 16E99.
Key words and phrases. Non-commutative localisation, finite domination, type FP, strongly graded ring, NOVIKOV homology, algebraic mapping torus, MATHER trick.

Introduction

Finite domination. Let R_0 be a unital ring, possibly non-commutative. A chain complex C of R_0-modules is called R_0-*finitely dominated* if it is a retract up to homotopy of a bounded complex of finitely generated free R_0-modules. When C is bounded and consists of projective R_0-modules, C is R_0-finitely dominated if and only if C is homotopy equivalent to a bounded complex of finitely generated projective R_0-modules [Ran85, Proposition 3.2 (ii)]; this is sometimes expressed by saying that C is "of type FP".

Non-commutative localisation. A *K-ring* is a unit-preserving homomorphism $K \longrightarrow S$ of unital rings with domain K. Let Σ be a set of homomorphisms of finitely generated projective (right) K-modules. The K-ring $f : K \longrightarrow S$ is called Σ-*inverting* if all the induced maps

$$\sigma \otimes S : P \otimes_K S \longrightarrow Q \otimes_K S\ , \qquad \big(\sigma : P \longrightarrow Q\big) \in \Sigma$$

are isomorphisms of S-modules. The *non-commutative localisation of K with respect to* Σ is the K-ring $\lambda_\Sigma : K \longrightarrow \Sigma^{-1}K$ which is initial in the category of Σ-inverting K-rings; it exists for all Σ [Sch85, Theorem 4.1].

Detecting contractibility and finite domination using non-commutative localisation. Let C^+ be a bounded chain complex consisting of finitely generated free modules over the polynomial ring $R_0[t]$, where t is a (central) indeterminate commuting with all elements of R_0. Our starting point is the following pair of results obtained by RANICKI:

Theorem. *There are sets $\tilde{\Omega}_+$ and Ω_+ of square matrices with entries in $R_0[t]$, considered as maps between finitely generated free $R_0[t]$-modules, such that*

(A) *The induced complex $C^+ \otimes_{R_0[t]} R_0$ is contractible (tensor product via the map $R_0[t] \longrightarrow R_0$, $t \mapsto 0$) if and only if the induced chain complex $C^+ \otimes_{R_0[t]} \tilde{\Omega}_+^{-1} R_0[t]$ is contractible* [Ran98, Proposition 10.13]*;*

(B) *C^+ is R_0-finitely dominated if and only if the induced chain complex $C^+ \otimes_{R_0[t]} \Omega_+^{-1} R_0[t]$ is contractible* [Ran98, Proposition 10.11].

Content of the paper. RANICKI's results are extended to a larger class of rings containing polynomial rings as special examples. Let $R = \bigoplus_{k\in\mathbb{Z}} R_k$ be a $\mathbb{Z}$-graded ring. The polynomial ring $R[t]$ has a subring, denoted $R_*[t]$, consisting of those polynomials $\sum_k r_k t^k$ with $r_k \in R_k$; up to the ring isomorphism symbolised by $t \mapsto 1$, this is the $\mathbb{N}$-graded ring $\bigoplus_{k\geq 0} R_k$. We will show that *the results above remain valid mutatis mutandis if the polynomial ring $R_0[t]$ is replaced by the $\mathbb{N}$-graded ring $R_*[t]$ throughout, where in* (B) *we additionally demand the $\mathbb{Z}$-graded ring R to be* strongly *graded*. This last conditions means that the multiplication map $R_k \otimes_{R_0} R_{-k} \longrightarrow R_0$ is surjective for all $k \in \mathbb{Z}$. It is surprising that the results rest exclusively on the (strongly) graded structure of the underlying rings, and not on the specific form of polynomial rings in one indeterminate.

Motivation. Finiteness conditions for chain complexes are studied in algebraic topology [Ran85, Ran98] and other subjects (*e.g.*, Σ-invariants in geometric group theory). The present paper develops aspects of the theory from a purely algebraic point of view, shifting the focus from (LAURENT) polynomial rings to the larger class of (strongly) $\mathbb{Z}$-graded rings instead.

Strong gradings were introduced by DADE [Dad80] to capture the quintessential properties of group rings. The extent to which strongly $\mathbb{Z}$-graded rings behave like LAURENT polynomial rings is in fact astonishing; examples include the splitting of the algebraic K-theory of the projective line (HÜTTEMANN and MONTGOMERY [HM20]), the relation between finite domination and NOVIKOV homology (HÜTTEMANN and STEERS [HS17]), and the fundamental theorem in algebraic K-theory for strongly $\mathbb{Z}$-graded rings (HÜTTEMANN [Hüt20]). The present paper adds further entries to the list of results that transfer to the strongly graded setting. Lest the reader gains the impression that this is a straightforward transcription we remark that, unlike the *statements* of the results, the *proofs* do not carry over mechanically. We also highlight in §7 a subtle example of a finiteness property that does *not* carry over as expected.

Organisation of the paper. The paper is divided into three parts, discussing $\mathbb{Z}$-graded rings and non-commutative localisation, contractible complexes, and finite domination respectively. Independently, the material is divided into numbered sections.

Conventions. All rings are unital, ring homomorphisms preserve unity, and modules are unital and right, unless stated otherwise.

Part 1. Algebraic background

1. Constructing new rings from a $\mathbb{Z}$-graded ring

For a (unital) ring R we can construct various polynomial and power series rings using a central indeterminate t; the rings $R[t]$, $R[t^{-1}]$, $R[t,t^{-1}]$, $R[[t]]$, $R[[t^{-1}]]$, $R((t)) = R[[t]][1/t]$ and $R((t^{-1})) = R[[t^{-1}]][1/t^{-1}]$ will be of relevance. Elements of these rings can be written as formal sums $\sum_k r_k t^k$, with suitable restrictions on the number and sign of indices of non-zero coefficients r_k.

Suppose now that $R = \bigoplus_{k\in\mathbb{Z}} R_k$ is equipped with the structure of a $\mathbb{Z}$-graded ring. We can then define subrings of the rings above by requiring that for all $k \in \mathbb{Z}$ the coefficient r_k of t^k lies in R_k. The resulting rings will be denoted by the symbols $R_*[t]$, $R_*[t^{-1}]$, $R_*[t,t^{-1}]$, $R_*[[t]]$, $R_*[[t^{-1}]]$, $R_*((t))$ and $R_*((t^{-1}))$, respectively. For example,

$$R_*((t)) = \bigcup_{p\geq 0} \Big\{ \sum_{k=-p}^{\infty} r_k t^k \mid \forall k\colon r_k \in R_k \Big\} \ .$$

As a graded ring, $R_*[t,t^{-1}] = R$ *via* the map symbolically described as $t \mapsto 1$. Similarly $R_*[t] = \bigoplus_{k\geq 0} R_k$ and $R_*[t^{-1}] = \bigoplus_{k\leq 0} R_k$. We write

$$t^n R_*[t] = \bigoplus_{k\geq n} R_k \quad \text{and} \quad t^n R_*[t^{-1}] = \bigoplus_{k\leq n} R_k \ , \tag{1.1}$$

which are (left and right) modules over $R_*[t]$ and $R_*[t^{-1}]$, respectively; the symbol $t^n R_*[[t^{-1}]]$ denotes the $R_*[[t^{-1}]]$-module of formal power series involving powers of t not exceeding n.

For later use we introduce notation for truncation of formal power series. For $-\infty \leq \ell < u \leq \infty$ we define

$$\mathrm{tr}_\ell^u \colon \sum_{k\in\mathbb{Z}} r_k t^k \mapsto \sum_{k=\ell}^{u} r_k t^k \ ,$$

and abbreviations in the special cases $\ell = -\infty$ and $u = \infty$,

$$\mathrm{tr}^u = \mathrm{tr}^u_{-\infty} \qquad \text{and} \qquad \mathrm{tr}_\ell = \mathrm{tr}_\ell^\infty \ . \tag{1.2}$$

For example, the map

$$\mathrm{tr}^0 \colon R_*[t] \longrightarrow R_0 \ , \quad \sum_{k=0}^{d} r_k t^k \mapsto r_0 \tag{1.3}$$

is the "constant-coefficient" ring homomorphism which is given symbolically by $t \mapsto 0$.

2. Strongly graded rings

Strongly graded rings and partitions of unity. Let $R = R_*[t, t^{-1}]$ be a $\mathbb{Z}$-graded ring. A finite sum expression $1 = \sum_j \alpha_j^{(n)} \beta_j^{(-n)}$ with $\alpha_j^{(n)} \in R_n$ and $\beta_j^{(-n)} \in R_{-n}$ is called a *partition of unity of type* $(n, -n)$. The ring $R_*[t, t^{-1}]$ is called *strongly graded* (Dade [Dad80, §1]) if there exists a partition of unity of type $(n, -n)$ for every $n \in \mathbb{Z}$; equivalently, if the multiplication map

$$\pi_n \colon R_n \otimes_{R_0} R_{-n} \longrightarrow R_0 \ , \quad x \otimes y \mapsto xy \tag{2.1}$$

is surjective for every $n \in \mathbb{Z}$.

Lemma 2.2. *If π_n is onto, then π_n is an isomorphism of R_0-R_0-bimodules.*

Proof. The map π_n is clearly left and right R_0-linear. If π_n is onto we can choose a partition of unity $1 = \sum_j \alpha_j^{(n)} \beta_j^{(-n)}$ and define the right R_0-linear map

$$\kappa_n \colon R_0 \longrightarrow R_n \otimes_{R_0} R_{-n} \ , \quad x \mapsto \sum_j \alpha_j^{(n)} \otimes \beta_j^{(-n)} x \ .$$

Then we calculate

$$\kappa_n \pi_n (x \otimes y) = \sum_j \alpha_j^{(n)} \otimes \beta_j^{(-n)} xy = \sum_j \alpha_j^{(n)} \beta_j^{(-n)} x \otimes y = x \otimes y$$

(using $\beta_j^{(-n)} x \in R_0$) so that π_n is injective. □

Lemma 2.3. *Let $R = R_*[t, t^{-1}]$ be a $\mathbb{Z}$-graded ring, and let $1 = \sum_i \alpha_i^{(m)} \beta_i^{(-m)}$ and $1 = \sum_j \bar{\alpha}_j^{(n)} \bar{\beta}_j^{(-n)}$ be two partitions of unity of types $(m, -m)$ and $(n, -n)$, respectively. Then*

$$1 = \sum_{i,j} \big(\alpha_i^{(m)} \alpha_j^{(n)}\big) \cdot \big(\beta_j^{(-n)} \beta_i^{(-m)}\big)$$

is a partition of unity of type $(m + n, -m - n)$. □

Corollary 2.4. *Partitions of unity of types $(1, -1)$ and $(-1, 1)$ exist within the $\mathbb{Z}$-graded ring $R_*[t, t^{-1}]$ if and only if it is strongly $\mathbb{Z}$-graded.* □

By direct calculation, similar to the proof of Lemma 2.2 above, one verifies:

Lemma 2.5. *Suppose that* $R = R_*[t, t^{-1}]$ *is a strongly* $\mathbb{Z}$*-graded ring, and let* $m \in \mathbb{Z}$*. The multiplication map*

$$t^{-m}R_*[t] \otimes_{R_*[t]} R_*[t, t^{-1}] \longrightarrow R_*[t, t^{-1}] \,, \quad x \otimes y \mapsto xy$$

is an isomorphism of $R_*[t]$*-*$R_*[t, t^{-1}]$*-bimodules, with inverse given by*

$$z \mapsto \sum_j \alpha_j^{(-m)} \otimes \beta_j^{(m)} z$$

for a partition of unity $1 = \sum_j \alpha_j^{(-m)} \beta_j^{(m)}$ *of type* $(-m, m)$*.* □

Note that the inverse is independent from the choice of partition of unity (since the multiplication map is). — For later use, we record an important categorical property of strongly $\mathbb{Z}$-graded rings:

Lemma 2.6. *Let* $R = R_*[t, t^{-1}]$ *be a strongly* $\mathbb{Z}$*-graded ring. The inclusion* $\beta \colon R_*[t] \xrightarrow{\subset} R_*[t, t^{-1}]$ *is an epimorphism in the category of (unital) rings.*

Proof. Let $f, g \colon R_*[t, t^{-1}] \longrightarrow S$ be ring homomorphisms satisfying the equality $f\beta = g\beta$. We need to show $f = g$. For this, let $x \in R_k$ be homogeneous of degree $k \in \mathbb{Z}$. If $k \geq 0$ we have $f(x) = f\beta(x) = g\beta(x) = g(x)$. Otherwise, choose a partition of unity $1 = \sum_j \alpha_j^{(k)} \beta_j^{(-k)}$ of type $(k, -k)$. Then $\beta_j^{(-k)}$ and $\beta_j^{(-k)} x$ lie in $R_*[t]$. Thus $f(\beta_j^{(-k)} x) = g(\beta_j^{(-k)} x)$, and we calculate

$$\begin{aligned}
f(x) &= g(1) \cdot f(x) \\
&= \sum_j g(\alpha_j^{(k)} \beta_j^{(-k)}) \cdot f(x) = \sum_j g(\alpha_j^{(k)}) \cdot g(\beta_j^{(-k)}) \cdot f(x) \\
&= \sum_j g(\alpha_j^{(k)}) \cdot f(\beta_j^{(-k)}) \cdot f(x) = \sum_j g(\alpha_j^{(k)}) \cdot f(\beta_j^{(-k)} x) \\
&= \sum_j g(\alpha_j^{(k)}) \cdot g(\beta_j^{(-k)} x) = \sum_j g(\alpha_j^{(k)} \beta_j^{(-k)} x) = g(x) \,.
\end{aligned}$$

□

Finiteness properties of strongly graded rings. The homogeneous components of strongly graded rings are finitely generated projective modules over the degree-0 subring.

Lemma 2.7. *Suppose that $R = R_*[t, t^{-1}]$ is a $\mathbb{Z}$-graded ring that admits a partition of unity of type $(1, -1)$. Then for all $n \geq 1$,*

- *R_n is finitely generated projective as a* right *R_0-module;*
- *R_{-n} is finitely generated projective as a* left *R_0-module.*

Similarly, if $R = R_*[t, t^{-1}]$ admits a partition of unity of type $(1, -1)$, then for all $n \geq 1$,

- *R_n is finitely generated projective as a* left *R_0-module;*
- *R_{-n} is finitely generated projective as a* right *R_0-module.*

Proof. Let $n \geq 1$, and let $1 = \sum_j \alpha_j^{(n)} \beta_j^{(-n)}$ be a partition of unity of type $(n, -n)$ (existence is guaranteed by Lemma 2.3). Define

$$f_j : R_n \longrightarrow R_0 \, , \quad x \mapsto \beta_j^{(-n)} x \, .$$

The maps f_j are right R_0-linear, and for all $x \in R_n$ we calculate

$$\sum_j \alpha_j^{(n)} \cdot f_j(x) = \sum_j \alpha_j^{(n)} \beta_j^{(-n)} x = x$$

so that $\big(\alpha_j^{(n)}, f_j\big)$ is a dual basis for R_n. It follows that R_n is a finitely generated projective right R_0-module by the dual basis lemma. — All the remaining claims are proved in a similar manner. $\square$

Corollary 2.8. *Suppose that $R = R_*[t, t^{-1}]$ is a strongly $\mathbb{Z}$-graded ring.*

1. *For all $n \in \mathbb{Z}$, the homogeneous component R_n of $R_*[t, t^{-1}]$ is a finitely generated projective left R_0-module and a finitely generated projective right R_0-module; in fact, R_n is an invertible R_0-bimodule.*
2. *If M is a projective (left or right) $R_*[t, t^{-1}]$-module, then M is a projective (left or right) R_0-module (with module structure given by restriction of scalars). Similarly, any projective left or right module over $R_*[t]$ or $R_*[t^{-1}]$ is a projective R_0-module.*
3. *There exists an isomorphism $R_{-m} \otimes_{R_0} R_*[t] \cong t^{-m} R_*[t]$ of finitely generated projective right $R_*[t]$-modules, for every $m \in \mathbb{Z}$. Similarly, there exists an isomorphism $R_m \otimes_{R_0} R_*[t^{-1}] \cong t^m R_*[t^{-1}]$ of finitely generated projective right $R_*[t^{-1}]$-modules.*
4. *For all $m \in \mathbb{Z}$, the module $t^{-m} R_*[t]$ is an invertible $R_*[t]$-bimodule, and hence is finitely generated projective as a left and right $R_*[t]$-module. Similarly, $t^m R_*[t^{-1}]$ is an invertible $R_*[t^{-1}]$-bimodule, and hence is finitely generated projective as a left and right $R_*[t^{-1}]$-module.*

Proof. Statements (1) and (2) follow from Lemma 2.2, Corollary 2.4 and Lemma 2.7. To prove (3) it is enough, in view of (1), to establish the isomorphism. Let $1 = \sum_j \alpha_j^{(-m)} \beta_j^{(m)}$ be a partition of unity of type $(-m, m)$. Then the multiplication map $\pi\colon R_{-m} \otimes_{R_0} R_*[t] \xrightarrow{\cong} t^{-m}R_*[t]$, sending $x \otimes y$ to xy, has inverse given by $\rho\colon z \mapsto \sum_j \alpha_j^{(-m)} \otimes \beta_j^{(m)} z$. Indeed, by straightforward calculation, $\pi\rho(z) = \sum_j \alpha_j^{(-m)} \beta_j^{(m)} z = z$ and

$$\rho\pi(x \otimes y) = \sum_j \alpha_j^{(-m)} \otimes \beta_j^{(m)} xy = \sum_j \alpha_j^{(-m)} \beta_j^{(m)} x \otimes y = x \otimes y$$

since $\beta_j^{(m)} x \in R_0$. — The proof of (4) is similar, using partitions of unity to show that $t^m R_*[t]$ is the inverse $R_*[t]$-bimodule of $t^{-m} R_*[t]$. □

3. Proto-null homotopies and proto-contractions

Let C and C' be chain complexes of right modules over the unital ring K, with differentials $d = d_k\colon C_k \longrightarrow C_{k-1}$ and $d' = d'_k$. A *proto-contraction* of C consists of module homomorphisms $s = s_k\colon C_k \longrightarrow C_{k+1}$ such that $ds + sd\colon C_k \longrightarrow C_k$ is an automorphism of C_k for all $k \in \mathbb{Z}$. Somewhat more generally, a (C, C')-*proto-null homotopy* consists of module homomorphisms $t = t_k\colon C_k \longrightarrow C'_{k+1}$ such that $g_k = d't + td\colon C_k \longrightarrow C'_k$ is an isomorphism for all $k \in \mathbb{Z}$. In fact, the maps g_k define a chain isomorphism $g\colon C \xrightarrow{\cong} C'$, and the maps t_k define a null homotopy of g.

Lemma 3.1. *A chain complex C admits a proto-contraction if and only if it is contractible. The chain complexes C and C' admit a (C, C')-proto-null homotopy if and only if $C \cong C'$ and C is contractible.*

Proof. A proto-contraction is, by definition, the same as a (C, C)-proto-null homotopy, so it suffices to prove the second statement. If there exists a chain isomorphism $g\colon C \longrightarrow C'$ with C contractible, we can choose a null homotopy t of g which constitutes a (C, C')-proto-null homotopy. Conversely, any (C, C')-proto-null homotopy t determines a null homotopic chain isomorphism $g = d't + td$, as explain above. Then $\mathrm{id}_C = g^{-1}g$ is null homotopic as well so that C is contractible. □

Given a ring homomorphism $f : K \longrightarrow S$, the family of maps s_k is called an f-*proto-contraction* if the maps $s_k \otimes \mathrm{id}$ form a proto-contraction of the induced complex $f_*(C) = C \otimes_K S$. Similarly, the family of maps t_k is called a (C, C')-f-*proto-null homotopy* if the maps $t_k \otimes \mathrm{id}$ form a $(C \otimes_K S, C' \otimes_K S)$-proto-null homotopy.

We are interested in proto-contractions for the following reason. Suppose we are given C and f as before, and another ring homomorphism

$g\colon S \longrightarrow T$. If $f_*(C) = C \otimes_K S$ is contractible then $(gf)_*(C) = C \otimes_K T \cong C \otimes_K S \otimes_S T$ is contractible as well, since taking tensor product preserves homotopies. If, however, $(gf)_*(C)$ is contractible it is not guaranteed that $f_*(C)$ is contractible. In favourable circumstances, a contraction of $(gf)_*(C)$ gives rise to a sequence of maps $s_k\colon C_k \longrightarrow C_{k+1}$ which can be shown, thanks to special properties of the maps f and g, to be an f-proto-contraction.

4. Remarks on non-commutative localisation

Let K denote an arbitrary unital, possibly non-commutative ring. For the reader's convenience we collect some standard facts about non-commutative localisation

Proposition 4.1. *Let Σ be a set of homomorphisms of finitely generated projective K-modules, and let $f : K \longrightarrow S$ be a K-ring. Write $\lambda_\Sigma\colon K \longrightarrow \Sigma^{-1}K$ for the non-commutative localisation of K with respect to Σ.*

1. *If f is Σ-inverting and injective, then λ_Σ is injective.*
2. *The non-commutative localisation $\lambda_\Sigma\colon K \longrightarrow \Sigma^{-1}K$ is an epimorphism in the category of unital rings.*
3. *Suppose that Σ is the set of all those square matrices M with entries in K such that $f(M)$ is invertible over S; we consider a square matrix of size k as a map of finitely generated free modules $\mu\colon K^k \longrightarrow K^k$ so that $f(M)$ represents the induced map $\mu \otimes S\colon S^k \longrightarrow S^k$. Let A be a square matrix with entries in K. Then $A \in \Sigma$ if and only if $\lambda_\Sigma(A)$ is invertible over the ring $\Sigma^{-1}K$.*

Proof. (1) As f is Σ-inverting, it factors as $K \xrightarrow{\lambda_\Sigma} \Sigma^{-1}K \longrightarrow S$. This forces λ_Σ to be injective if f is.

(2) Suppose we have two ring homomorphisms $\alpha, \beta\colon \Sigma^{-1}K \longrightarrow T$ with $\alpha\lambda_\Sigma = \beta\lambda_\Sigma$. This common composition is certainly Σ-inverting, so factorises *uniquely* through λ_Σ. This means precisely that $\alpha = \beta$, as required.

(3) Since the map f is Σ-invertible, it factors as $K \xrightarrow{\lambda_\Sigma} \Sigma^{-1}K \xrightarrow{\bar f} S$. If A is a square matrix in Σ then $\lambda_\Sigma(A)$ is invertible in $\Sigma^{-1}K$, by definition of non-commutative localisation. If the square matrix A with entries in K is such that $\lambda_\Sigma(A)$ is invertible, then $\bar f \lambda_\Sigma(A) = f(A)$ is invertible over S so that $A \in \Sigma$ by the specific choice of Σ. □

We will have occasion to use the following construction of pushout squares:

Proposition 4.2. *Let Σ be a set of homomorphisms of finitely generated projective K-modules, and let $f : K \longrightarrow S$ be a ring homomorphism. The* *square in Fig. 1 is a pushout in the category of unital rings, where* $f_*(\Sigma)$ *de-*

$$\begin{tikzcd} K \arrow[r, "\lambda_\Sigma"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner\cdot"] & \Sigma^{-1}K \arrow[d, "\bar f"'] \\ S \arrow[r, "\lambda_{f_*(\Sigma)}"] & f_*(\Sigma)^{-1}S \end{tikzcd}$$

FIGURE 1. A pushout square in the category of unital rings

notes the set of induced maps $\sigma \otimes S \colon P \otimes_K S \longrightarrow Q \otimes_K S$ *with* $\sigma \colon P \longrightarrow Q$ *an element of* Σ. *The ring homomorphism* $\bar f$ *is obtained from the universal property of* λ_Σ *as the composition* $\lambda_{f_*(\Sigma)} \circ f$ *is* Σ*-inverting. — In other words, given ring homomorphisms* $\beta \colon S \longrightarrow T$ *and* $\alpha \colon \Sigma^{-1}K \longrightarrow T$ *such that* $\alpha \circ \lambda_\Sigma = \beta \circ f$ *there exists a uniquely determined ring homomorphism* $\upsilon \colon f(\Sigma)^{-1}S \longrightarrow T$ *with* $\beta = \upsilon \circ \lambda_{f(\Sigma)}$ *and* $\alpha = \upsilon \circ \bar f$, *cf. Fig. 2.*

$$\begin{tikzcd} K \arrow[r, "\lambda_\Sigma"] \arrow[d, "f"'] & \Sigma^{-1}K \arrow[d, "\bar f"'] \arrow[ddr, "\alpha"] & \\ S \arrow[r, "\lambda_{f_*(\Sigma)}"] \arrow[drr, "\beta"'] & f_*(\Sigma)^{-1}S \arrow[dr, dashed, "\exists!\upsilon"] & \\ & & T \end{tikzcd}$$

FIGURE 2. Universal property of pushout square

Proof. As for notation, given any ring homomorphism $h \colon A \longrightarrow B$ we let h_* stand for the functor $- \otimes_A B$. — To prove the Proposition we verify that the square has the universal property of a pushout, see Fig. 2. Let $\alpha \colon \Sigma^{-1}K \longrightarrow T$ and $\beta \colon S \longrightarrow T$ be ring homomorphisms such that $\alpha\lambda_\Sigma = \beta f$. Given a map $\sigma \colon P \longrightarrow Q$ in Σ we know that

$$\beta_* f_*(\sigma) = (\beta f)_*(\sigma) = (\alpha\lambda_\Sigma)_*(\sigma) = \alpha_*(\lambda_\Sigma)_*(\sigma) ;$$

as $(\lambda_\Sigma)_*(\sigma)$ is invertible so is $\beta_* f_*(\sigma)$. Hence the map β is $f_*(\Sigma)$-inverting, and consequently factorises uniquely as $\beta = \upsilon\lambda_{f_*(\Sigma)}$, for some ring homomorphism $\upsilon \colon f_*(\Sigma)^{-1}S \longrightarrow T$. From the chain of equalities

$$\upsilon \bar f \lambda_\Sigma = \upsilon \lambda_{f_*(\Sigma)} f = \beta f = \alpha\lambda_\Sigma$$

we conclude that $\alpha = \upsilon \bar f$ since λ_Σ is an epimorphism by Proposition 4.1 (2). □

The following purely category-theoretic lemma will be applied, in the proof of Proposition 10.8, in the context of strongly graded rings and non-commutative localisation.

Lemma 4.3. *Suppose that we are given a commutative pushout square*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\ulcorner\cdot"] & B \arrow[d, "\delta"] \\ C \arrow[r, "\gamma"] & D \end{tikzcd}$$

(in any category) with β an epimorphism. Suppose further that there exists $\iota\colon C \longrightarrow B$ with $\iota\beta = \alpha$. Then $\delta\iota = \gamma$, and δ is an isomorphism.

Proof. First, since $\delta\iota\beta = \delta\alpha = \gamma\beta$, and since β is an epimorphism, we have $\delta\iota = \gamma$. Next, by the universal property of pushouts there exists a (uniquely determined) morphism $\varphi\colon D \longrightarrow B$ with $\varphi\delta = \mathrm{id}_B$ and $\varphi\gamma = \iota$. The commutative diagram of Fig. 3 can be completed along the dotted arrow by both id_D and $\delta\varphi$; by uniqueness, this means $\delta\varphi = \mathrm{id}_D$. □

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\ulcorner\cdot"] & B \arrow[d, "\delta"] \arrow[ddr, "\delta"] & \\ C \arrow[r, "\gamma"] \arrow[drr, "\delta\iota=\gamma"'] & D \arrow[dr, dashed] & \\ & & D \end{tikzcd}$$

FIGURE 3. Pushout diagram used in proof of Lemma 4.3

Part 2. $\mathbb{N}$-graded rings and complexes contractible over R_0

For this part we assume that $R = R_*[t, t^{-1}]$ is an arbitrary $\mathbb{Z}$-graded ring; in fact, we are only interested in the subring $R_*[t] = \bigoplus_{k=0}^{\infty} R_k$ which is, in effect, an arbitrary $\mathbb{N}$-graded ring.

5. COMPLEXES CONTRACTIBLE OVER R_0

We characterise complexes C of $R_*[t]$-modules such that $C \otimes_{R_*[t]} R_0$ is contractible, where the tensor product is taken *via* the "constant coefficient" ring homomorphism $\mathrm{tr}^0\colon t \mapsto 0$ of (1.3).

The map ζ. Let M be an $R_*[t]$-module. Using the notation from (1.1), we write $\zeta_M = \zeta$ for the obvious map of $R_*[t]$-modules

$$\zeta_M = \zeta \colon M \otimes_{R_*[t]} t^1 R_*[t] \longrightarrow M \otimes_{R_*[t]} t^0 R_*[t] = M \ , \quad m \otimes x \mapsto mx \tag{5.1}$$

induced by the inclusion map $t^1 R_*[t] \longrightarrow t^0 R_*[t]$. The map ζ is to be thought of as a substitute for the action of the indeterminate t. More precisely, if $R_*[t] = K[t]$ is a polynomial ring, then $t^1 R_*[t] = tK[t]$ and the composition

$$M = M \otimes_{K[t]} K[t] \xrightarrow[\tau]{\cong} M \otimes_{K[t]} \big(tK[t]\big) \xrightarrow{\zeta} M \otimes_{K[t]} K[t] = M \ ,$$

where $\tau(m \otimes r) = m \otimes tr$, is given by $m \mapsto mt$; that is, up to the isomorphism τ the map ζ coincides with the action of the indeterminate.

Invertible matrices over $R_*[[t]]$. We write an element $z \in R_*[[t]]$ as a formal power series: $z = \sum_{p \geq 0} z_p t^p$. The usual proof shows that *z is a unit in $R_*[[t]]$ if and only if $z_0 = \mathrm{tr}^0(z)$ is a unit in R_0*, cf. (1.2).

A square matrix M with entries in $R_*[[t]]$ can be written as a formal power series $M = \sum_{p \geq 0} M_p t^p$ with matrices M_p having entries in R_p; again, the usual proof shows that *the matrix M is invertible over $R_*[[t]]$ if and only if $M_0 = \mathrm{tr}^0(M)$ is invertible over R_0.*

Notation 5.2. We let $\tilde{\Omega}_+$ denote the set of all square matrices M with entries in $R_*[t]$ such that $\mathrm{tr}^0(M)$ is an invertible matrix over R_0, that is, such that M is invertible over $R_*[[t]]$.

We apply Proposition 4.1 (3) to the $R_*[t]$-ring $f \colon R_*[t] \xrightarrow{\subset} R_*[[t]]$:

Lemma 5.3. *A square matrix M with entries in the $\mathbb{N}$-graded ring $R_*[t]$ becomes invertible in $\tilde{\Omega}_+^{-1} R_*[t]$ if and only if $\mathrm{tr}^0(M)$ is invertible over R_0.* □

The localisation $\tilde{\Omega}_+^{-1} R_*[t]$. We consider an element $A^+ \in \tilde{\Omega}_+$ of size k as an endomorphism $A^+ \colon R_*[t]^k \longrightarrow R_*[t]^k$ of the finitely generated free $R_*[t]$-module $R_*[t]^k$. The non-commutative localisation

$$\lambda = \lambda_{\tilde{\Omega}_+} \colon R_*[t] \longrightarrow \tilde{\Omega}_+^{-1} R_*[t]$$

can be used to characterise the R_0-contractible complexes C^+ as follows, generalising known results for polynomial rings (Ranicki [Ran98, Proposition 10.13]):

Theorem 5.4. *Let $R_*[t] = \bigoplus_{k=0}^{\infty} R_k$ be an arbitrary $\mathbb{N}$-graded ring, and let C^+ be a bounded complex of finitely generated free $R_*[t]$-modules. The following statements are equivalent:*

(1) *The complex* $C^+ \otimes_{R_*[t]} R_0$ *is contractible, the tensor product being taken with respect to the ring map* $\mathrm{tr}^0 \colon R_*[t] \longrightarrow R_0$, $t \mapsto 0$.

(2) *The induced complex* $C^+ \otimes_{R_*[t]} R_*[[t]]$ *is contractible.*

(3) *The induced complex* $C^+ \otimes_{R_*[t]} \tilde{\Omega}_+^{-1} R_*[t]$ *is contractible.*

(4) *The map* $\zeta \colon C \otimes_{R_*[t]} t^1 R_*[t] \longrightarrow C \otimes_{R_*[t]} t^0 R_*[t]$ *from* (5.1) *is a quasi-isomorphism.*

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1): This follows from the factorisation

$$R_*[t] \xrightarrow{\subseteq} \tilde{\Omega}_+^{-1} R_*[t] \longrightarrow R_*[[t]] \xrightarrow{\mathrm{tr}^0 \colon t \mapsto 0} R_0$$

of the ring homomorphism $\mathrm{tr}^0 \colon R_*[t] \longrightarrow R_0$.

(1) $\Rightarrow$ (3): We equip the finitely generated free modules C_n^+ with arbitrary finite bases; denote the number of elements of the basis for C_n^+ by r_n so that C_n^+ is identified with $R_*[t]^{r_n}$. The differentials $d_n^+ \colon C_n^+ \longrightarrow C_{n-1}^+$ are thus represented by matrices D_n^+ of size $r_{n-1} \times r_n$ with entries in $R_*[t]$. The differentials $\mathrm{tr}^0(d_n^+)$ in the induced complex $C^+ \otimes_{R_*[t]} R_0$ are then represented by the matrices $\mathrm{tr}^0(D_n^+)$, identifying $C_n^+ \otimes_{R_*[t]} R_0$ with $R_0^{r_n}$. By hypothesis there exists a contracting homotopy consisting of a family of R_0-linear maps

$$\sigma_n^+ \colon C_n^+ \otimes_{R_*[t]} R_0 \longrightarrow C_{n+1}^+ \otimes_{R_*[t]} R_0$$

such that

$$\mathrm{tr}^0(d_{n+1}^+) \circ \sigma_n^+ + \sigma_{n-1}^+ \circ \mathrm{tr}^0(d_n^+) = \mathrm{id} \ .$$

The map σ_n^+ is represented by a matrix S_n^+ of size $r_{n+1} \times r_n$ with entries in R_0. The matrices satisfy the relation

$$\mathrm{tr}^0\Big(D_{n+1}^+ \circ S_n^+ + S_{n-1}^+ \circ D_n^+\Big) = \mathrm{tr}^0\big(D_{n+1}^+\big) \circ S_n^+ + S_{n-1}^+ \circ \mathrm{tr}^0\big(D_n^+\big) = I_{r_n} \ ,$$

a unit matrix of size r_n. This implies, by Lemma 5.3, that the matrix

$$D_{n+1}^+ \circ S_n^+ + S_{n-1}^+ \circ D_n^+$$

becomes invertible over $\tilde{\Omega}_+^{-1} R_*[t]$. Thus the S_n^+ define a $\lambda_{\tilde{\Omega}_+}$-proto-contraction of C^+, cf. §3. With Lemma 3.1 we conclude that $C^+ \otimes_{R_*[t]} \tilde{\Omega}_+^{-1} R_*[t]$ is contractible as advertised.

(1) $\Leftrightarrow$ (4): From the short exact sequence

$$0 \longrightarrow C^+ \otimes_{R_*[t]} t^1 R_*[t] \xrightarrow{\zeta} C^+ \otimes_{R_*[t]} t^0 R_*[t] \longrightarrow C^+ \otimes_{R_*[t]} R_0 \longrightarrow 0$$

we infer that the canonical map $\mathrm{cone}(\zeta) \longrightarrow C^+ \otimes_{R_*[t]} R_0$ is a quasi-isomorphism. Thus ζ is a quasi-isomorphism if and only if $\mathrm{cone}(\zeta)$ is acyclic if and only if $C^+ \otimes_{R_*[t]} R_0$ is acyclic; as the latter complex consists of projective R_0-modules, this is equivalent with $C^+ \otimes_{R_*[t]} R_0$ being contractible. □

Theorem 5.5 (Universal property of $\tilde{\Omega}_+^{-1} R_*[t]$). *Let $R_*[t]$ be an arbitrary $\mathbb{N}$-graded ring. The localisation $\lambda\colon R_*[t] \longrightarrow \tilde{\Omega}_+^{-1} R_*[t]$ is the universal $R_*[t]$-ring making R_0-contractible chain complexes contractible. That is, suppose that $f : R_*[t] \longrightarrow S$ is an $R_*[t]$-ring such that for every bounded complex of finitely generated free $R_*[t]$-modules C^+, contractibility of $C^+ \otimes_{R_*[t]} R_0$ implies contractibility of $C^+ \otimes_{R_*[t]} S$. Then there is a factorisation*

$$R_*[t] \xrightarrow{\lambda} \tilde{\Omega}_+^{-1} R_*[t] \xrightarrow{\eta} S$$

of f, with a uniquely determined ring homomorphism η.

Proof. It was shown in Theorem 5.4 that the $R_*[t]$-ring $\tilde{\Omega}_+^{-1} R_*[t]$ makes R_0-contractible chain complexes contractible. Thus it is enough to verify that f is $\tilde{\Omega}_+$-inverting; the universal property of non-commutative localisation then yields the desired factorisation and its uniqueness. Consider the element $A^+ \in \tilde{\Omega}_+$ as a chain complex

$$C^+ = \left(R_*[t]^k \xrightarrow{A^+} R_*[t]^k\right) .$$

As A^+ becomes invertible over $\tilde{\Omega}_+^{-1} R_*[t]$, the complex $C^+ \otimes_{R_*[t]} \tilde{\Omega}_+^{-1} R_*[t]$ is contractible, hence so is $C^+ \otimes_{R_*[t]} R_0$ by Theorem 5.4. This makes $C^+ \otimes_{R_*[t]} S$ contractible, by hypothesis on f, whence A^+ becomes invertible in S as required. □

Part 3. Strongly graded rings and finite domination

We now turn to the theory of R_0-finite domination of $R_*[t]$-module complexes. We characterise finite domination *via* NOVIKOV homology (Theorem 8.1) and *via* a non-commutative localisation of $R_*[t]$ (Theorem 10.6). *We assume throughout that $R = R_*[t, t^{-1}]$ is a strongly $\mathbb{Z}$-graded ring.*

6. ALGEBRAIC HALF-TORI AND THE MATHER TRICK

Algebraic half-tori and the MATHER trick. Let $1 = \sum_j \alpha_j^{(1)} \beta_j^{(-1)}$ be a partition of unity of type $(1, -1)$ in $R = R_*[t, t^{-1}]$. Given an arbitrary

$R_*[t]$-module M, let $\mu = \mu_M$ denote the map

$$\mu\colon M \otimes_{R_0} t^1 R_*[t] \longrightarrow M \otimes_{R_0} t^0 R_*[t]\,, \quad m \otimes x \mapsto \sum_j m\alpha_j^{(1)} \otimes \beta_j^{(-1)} x\;. \tag{6.1}$$

The map μ is R_0-balanced (hence well-defined) and independent of the choice of partition of unity since it can be written as the composition

$$\begin{aligned} M \otimes_{R_0} t^1 R_*[t] &\cong M \otimes_{R_0} R_0 \otimes_{R_0} t^1 R_*[t] \\ &\cong M \otimes_{R_0} R_1 \otimes_{R_0} R_{-1} \otimes_{R_0} t^1 R_*[t] \longrightarrow M \otimes_{R_0} t^0 R_*[t] \end{aligned}$$

where the second isomorphism is induced by $\pi_1^{-1}\colon R_0 \xrightarrow{\cong} R_1 \otimes_{R_0} R_{-1}$, cf. Lemma 2.2, and the last arrow is induced by the multiplication maps $M \otimes_{R_0} R_1 \longrightarrow M$ and $R_{-1} \otimes_{R_0} t^1 R_*[t] \longrightarrow t^0 R_*[t]$.

As a matter of notation, we also introduce the inclusion map

$$\iota\colon M \otimes_{R_0} t^1 R_*[t] \longrightarrow M \otimes_{R_0} t^0 R_*[t]\,, \quad m \otimes x \mapsto m \otimes x\;.$$

Moreover, it is convenient at this point to choose, once and for all, additional partitions of unity

$$1 = \sum_{j_n} \alpha_{j_n}^{(n)} \beta_{j_n}^{(-n)}$$

of type $(n, -n)$, for all $n \geq 0$ $(n \neq 1)$. These exist in view of our standing assumption for this part, that the ring $R = R_*[t, t^{-1}]$ is strongly graded.

Lemma 6.2 (Canonical resolution)**.** *Suppose that $R = R_*[t, t^{-1}]$ is a strongly $\mathbb{Z}$-graded ring. Let M be a right $R_*[t]$-module. There is a short exact sequence of $R_*[t]$-modules*

$$0 \longrightarrow M \otimes_{R_0} t^1 R_*[t] \xrightarrow{\iota-\mu} M \otimes_{R_0} t^0 R_*[t] \xrightarrow{\pi} M \longrightarrow 0\,, \tag{6.3}$$

where μ is as in (6.1)*, $\iota(m \otimes x) = m \otimes x$ and $\pi(m \otimes x) = mx$.*

Proof. This is similar to the proof of Proposition 3.2 in [HS17]. Since $\sum_j \alpha_j^{(1)} \beta_j^{(-1)} = 1$ we have $\pi\iota = \pi\mu$ and hence $\pi(\iota - \mu) = 0$. It is thus enough to show that the sequence is split exact when considered as a sequence of R_0-modules.

To begin with, the map $\sigma(m) = m \otimes 1$ is certainly an R_0-linear section of π. Next, we define the R_0-linear map

$$\rho\colon M \otimes_{R_0} t^0 R_*[t] \longrightarrow M \otimes_{R_0} t^1 R_*[t]$$

on elements of the form $m \otimes x_d$, with $x_d \in R_d$, by the formula

$$\rho(m \otimes x_d) = \sum_{k=0}^{d-1} \sum_{j_k} m\alpha_{j_k}^{(k)} \otimes \beta_{j_k}^{(-k)} x_d \ .$$

We note the particular cases

$$\begin{aligned}
&\rho(m \otimes x_0) = 0 \ , \\
&\rho(m \otimes x_1) = m \otimes x_1 \ , \\
&\rho(m \otimes x_2) = m \otimes x_2 + \sum_j m\alpha_j^{(1)} \otimes \beta_j^{(-1)} x_2 \ .
\end{aligned}$$

The summands $s_k = \sum_{j_k} m\alpha_{j_k}^{(k)} \otimes \beta_{j_k}^{(-k)} x_d$, and hence the map ρ, do not depend on the particular choice of partition of unity. This is because s_k is the image of $m \otimes x_d$ under the the composition

$$\begin{aligned}
&M \otimes_{R_0} R_d \cong M \otimes_{R_0} R_0 \otimes_{R_0} R_d \\
&\qquad \xrightarrow[\cong]{\pi_k^{-1}} M \otimes_{R_0} R_k \otimes_{R_0} R_{-k} \otimes_{R_0} R_d \xrightarrow{\sigma} M \otimes_{R_0} R_{-k+d} \ ,
\end{aligned}$$

where $\sigma(m \otimes a \otimes b \otimes x) = ma \otimes bx$, and π_k^{-1} does not depend on choices by Lemma 2.2.

We have $\rho \circ (\iota - \mu) = \mathrm{id}$ since, for an element $x_d \in R_d$, we calculate

$$\begin{aligned}
\rho \circ (\iota - \mu)(m \otimes x_d) &= \rho(m \otimes x_d) - \sum_j \rho(m\alpha_j^{(1)} \otimes \beta_j^{(-1)} x_d) \\
&= \sum_{k=0}^{d-1} \sum_{j_k} m\alpha_{j_k}^{(k)} \otimes \beta_{j_k}^{(-k)} x_d - \sum_{k=0}^{d-2} \sum_{j_k} \sum_j m\alpha_j^{(1)} \alpha_{j_k}^{(k)} \otimes \beta_{j_k}^{(-k)} \beta_j^{(-1)} x_d \\
&\underset{(\diamond)}{=} \sum_{k=0}^{d-1} \sum_{j_k} m\alpha_{j_k}^{(k)} \otimes \beta_{j_k}^{(-k)} x_d - \sum_{k=0}^{d-2} \sum_{j_{k+1}} m\alpha_{j_{k+1}}^{(k+1)} \otimes \beta_{j_{k+1}}^{(-k-1)} x_d \\
&= \sum_{j_0} m\alpha_{j_0}^{(0)} \otimes \beta_{j_0}^{(0)} x_d = \sum_{j_0} m\alpha_{j_0}^{(0)} \beta_{j_0}^{(0)} \otimes x_d = m \otimes x_d \ ;
\end{aligned}$$

the equality labelled $(\diamond)$ makes use of Lemma 2.3, and of the fact that summands of the form s_{k+1} do not depend on choice of the partition of unity involved so that

$$\sum_{j_k} \sum_j m\alpha_j^{(1)} \alpha_{j_k}^{(k)} \otimes \beta_{j_k}^{(-k)} \beta_j^{(-1)} x_d = s_{k+1} = \sum_{j_{k+1}} m\alpha_{j_{k+1}}^{(k+1)} \otimes \beta_{j_{k+1}}^{(-k-1)} x_d \ .$$

It remains to verify the equality $\sigma \circ \pi + (\iota - \mu) \circ \rho = \mathrm{id}$. For this, let $x \in R_d$ and $m \in M$, and calculate:

$$(\iota-\mu)\circ\rho(m\otimes x_d)=(\iota-\mu)\Big(\sum_{k=0}^{d-1}\sum_{j_k} m\alpha_{j_k}^{(k)}\otimes\beta_{j_k}^{(-k)}x_d\Big)$$
$$=\sum_{k=0}^{d-1}\sum_{j_k} m\alpha_{j_k}^{(k)}\otimes\beta_{j_k}^{(-k)}x_d-\sum_{j}\sum_{k=0}^{d-1}\sum_{j_k} m\alpha_{j_k}^{(k)}\alpha_j^{(1)}\otimes\beta_j^{(-1)}\beta_{j_k}^{(-k)}x_d$$
$$\underset{(\diamond)}{=}\sum_{k=0}^{d-1}\sum_{j_k} m\alpha_{j_k}^{(k)}\otimes\beta_{j_k}^{(-k)}x_d-\sum_{k=0}^{d-1}\sum_{j_{k+1}} m\alpha_{j_{k+1}}^{(k+1)}\otimes\beta_{j_{k+1}}^{(-k-1)}x_d$$
$$=\sum_{j_0} m\alpha_{j_0}^{(0)}\otimes\beta_{j_0}^{(-0)}x_d-\sum_{j_d} m\alpha_{j_d}^{(d)}\otimes\beta_{j_d}^{(-d)}x_d$$
$$=\sum_{j_0} m\alpha_{j_0}^{(0)}\beta_{j_0}^{(-0)}\otimes x_d-\sum_{j_d} m\alpha_{j_d}^{(d)}\beta_{j_d}^{(-d)}x_d\otimes 1$$
$$=m\otimes x_d-mx_d\otimes 1=(\mathrm{id}-\sigma\circ\pi)(m\otimes x_d)\ .$$

(As before, the equality marked $(\diamond)$ holds because summands of the form s_{k+1} do not depend on choice of the partition of unity involved.) This finishes the proof. □

Definition 6.4. Let C^+ be a complex of $R_*[t]$-modules. The mapping cone $\mathfrak{H}^+(C^+)$ of the map $\iota-\mu$,

$$\mathfrak{H}^+(C^+)=\mathrm{cone}\Big(C^+\otimes_{R_0}t^1R_*[t]\xrightarrow{\iota-\mu}C^+\otimes_{R_0}t^0R_*[t]\Big)\ ,$$

is called the *algebraic half-torus of C^+*.

Corollary 6.5. *Let C^+ be a complex of $R_*[t]$-modules. The canonical map*

$$\mathfrak{H}^+(C^+)=\mathrm{cone}\big(C^+\otimes_{R_0}t^1R_*[t]\xrightarrow{\iota-\mu}C^+\otimes_{R_0}t^0R_*[t]\big)\longrightarrow C^+$$

induced by the short exact sequence (6.3) *is a quasi-isomorphism. If C^+ is bounded below and consists of projective $R_*[t]$-modules, the map is a homotopy equivalence of $R_*[t]$-module complexes.*

Proof. This is a direct consequence of standard homological algebra and Lemma 6.2 above. □

The following result, though technical, is central to the theory of finite domination. By the previous Corollary we can replace any complex C^+ of $R_*[t]$-modules by an algebraic half-torus, up to quasi-isomorphism; the MATHER trick is the observation that we can further replace the complex C^+ within the mapping cone of the half-torus construction by an R_0-module complex homotopy equivalent to C^+.

Proposition 6.6 (The algebraic MATHER trick for algebraic half-tori)**.** *Let $R=R_*[t,t^{-1}]$ be a strongly $\mathbb{Z}$-graded ring, let C^+ be a complex of*

$R_*[t]$-modules, and let D be a complex of R_0-modules. Let $\alpha\colon C^+ \longrightarrow D$ and $\beta\colon D \longrightarrow C^+$ be mutually inverse chain homotopy equivalences of R_0-module complexes with $H\colon \mathrm{id} \simeq \alpha\beta$ a specified homotopy. Write ψ for the $R_*[t]$-module complex map

$$\psi = (\alpha\otimes\mathrm{id})\circ(\iota-\mu)\circ(\beta\otimes\mathrm{id})\colon \quad D\otimes_{R_0} t^1R_*[t] \longrightarrow D\otimes_{R_0} t^0R_*[t] \ .$$

Then the square diagram (6.7) in Fig. 4 commutes up to a preferred homotopy J induced by H, given by the formula

$$J = (\alpha\otimes\mathrm{id})\circ(\iota-\mu)\circ(H\otimes\mathrm{id})\colon (\alpha\otimes\mathrm{id})\circ(\iota-\mu) \simeq \psi\circ(\alpha\otimes\mathrm{id}) \ .$$

The homotopy J induces a preferred chain map

$$\Xi\colon \mathfrak{H}^+(C^+) = \mathrm{cone}(\iota-\mu) \longrightarrow \mathrm{cone}(\psi)$$

which is a quasi-isomorphism. If both C^+ and D are bounded below complexes of projective R_0-modules, the map $\mathfrak{H}^+(C^+) \longrightarrow \mathrm{cone}(\psi)$ is a homotopy equivalence of $R_*[t]$-module complexes.

$$\begin{tikzcd}
C^+\otimes_{R_0} t^1R_*[t] \arrow[r, "\iota-\mu"] \arrow[d, "\alpha\otimes\mathrm{id}"'] & C^+\otimes_{R_0} t^0R_*[t] \arrow[d, "\alpha\otimes\mathrm{id}"'] \\
D\otimes_{R_0} t^1R_*[t] \arrow[r, "\psi"] & D\otimes_{R_0} t^0R_*[t]
\end{tikzcd} \tag{6.7}$$

FIGURE 4. The MATHER trick square

Proof. By construction, J is a homotopy from $(\alpha\otimes\mathrm{id})\circ(\iota-\mu)$ to $\psi\circ(\alpha\otimes\mathrm{id})$. Hence we obtain a chain map of the mapping cones of the horizontal maps in the diagram,

$$\alpha_* = \begin{pmatrix}\alpha\otimes\mathrm{id} & 0 \\ J & \alpha\otimes\mathrm{id}\end{pmatrix}\colon \mathfrak{H}^+(C^+) = \mathrm{cone}(\iota-\mu) \longrightarrow \mathrm{cone}(\psi) \ ;$$

this map is a quasi-isomorphism since α is a homotopy equivalence (so the induced map on homology will be represented by a lower triangular matrix with isomorphisms on the main diagonal). □

Corollary 6.8. *If C^+ is an R_0-finitely dominated bounded below chain complex of projective $R_*[t]$-modules, then C^+ is $R_*[t]$-finitely dominated, that is, C^+ is homotopy equivalent to a bounded complex of finitely generated projective $R_*[t]$-modules.*

Proof. As C^+ is R_0-finitely dominated we can choose an R_0-linear chain homotopy equivalence $\alpha\colon C^+ \longrightarrow D$ from C^+ to a bounded complex D

of finitely generated projective R_0-modules. By Corollary 6.5 and Proposition 6.6 there are quasi-isomorphisms

$$C^+ \xleftarrow{\simeq} \mathfrak{H}^+(C^+) \xrightarrow{\simeq} \mathrm{cone}(\psi) \tag{6.9}$$

with $\psi\colon D \otimes_{R_0} t^1 R_*[t] \longrightarrow D \otimes_{R_0} t^0 R_*[t]$ as defined in Proposition 6.6 a map of bounded complexes of finitely generated projective $R_*[t]$-modules. It follows that C^+ is quasi-isomorphic, hence homotopy equivalent, to a bounded complex of finitely generated projective $R_*[t]$-modules as claimed. □

7. Finite domination and homotopy finiteness

It is an interesting question whether in the situation of Corollary 6.8 the complex C^+ is actually $R_*[t]$-*homotopy finite*, that is, homotopy equivalent to a bounded complex of finitely generated *free* $R_*[t]$-modules. In general this turns out to be false; however, when working with $R_*[t,t^{-1}]$ instead of $R_*[t]$ the analogous question has a positive answer. — As before, let $R = R_*[t,t^{-1}]$ be a strongly $\mathbb{Z}$-graded ring.

Proposition 7.1. *Suppose that C is a bounded complex of finitely generated projective $R_*[t,t^{-1}]$-modules. If C is R_0-finitely dominated, then C is $R_*[t,t^{-1}]$-homotopy finite, i.e., C is homotopy equivalent to a bounded complex of finitely generated free $R_*[t,t^{-1}]$-modules.*

Proof. Let D be a bounded chain complex of finitely generated projective R_0-modules chain homotopy equivalent to C. Then by the Mather trick for algebraic tori [HS17, Lemma 3.7], C is homotopy equivalent, as an $R_*[t,t^{-1}]$-module complex, to the mapping cone of a certain self map of the induced complex $D \otimes_{R_0} R_*[t,t^{-1}]$. Hence the finiteness obstruction of C in $\tilde{K}_0\big(R_*[t,t^{-1}]\big)$ vanishes whence C is $R_*[t,t^{-1}]$-homotopy finite. □

The analogous statement holds over a *polynomial* ring $R_0[t]$ with a central indeterminate t: *An R_0-finitely dominated, bounded $R_0[t]$-module complex C^+ of finitely generated projective modules is $R_0[t]$-homotopy finite.* For $C^+ \simeq \mathrm{cone}(\psi)$ as in (6.9) and (6.7), and the finiteness obstruction of the mapping cone vanishes since $tR_0[t] \cong R_0[t]$. — In general, however, this line of reasoning fails when working over $R_*[t]$.

Example 7.2. *There exist a strongly $\mathbb{Z}$-graded ring $R = R_*[t,t^{-1}]$ together with a bounded complex C^+ of finitely generated projective $R_*[t]$-modules such that C^+ is R_0-finitely dominated but not homotopy equivalent to a* *bounded complex of finitely generated free $R_*[t]$-modules.* Specifically[1], let K be a field and let $R = R_*[t, t^{-1}]$ be the LEAVITT K-algebra of type $(1, 1)$, that is, the (non-commutative) K-algebra on generators A, B, C, D subject to the relations

$$AB + CD = 1 \ , \quad BA = DC = 1 \ , \quad BC = DA = 0 \ ;$$

we declare that A and C have degree -1, while B and D are given degree 1. This is a $\mathbb{Z}$-graded ring since all relations are homogeneous of degree 0. It is strongly graded by Corollary 2.4 as the relations $AB + CD = 1$ and $BA = 1$ provide partitions of unity of types $(-1, 1)$ and $(1, -1)$, respectively. It is known that R_0 can be identified with an increasing union $\bigcup_{n \geq 0} \mathbf{Mat}_{2^n}(K)$ of matrix algebras, using the block-diagonal embeddings $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$. It follows that R_0 has IBN, and since the projection map $R_*[t] = \bigoplus_{k \geq 0} R_k \longrightarrow R_0$ is a ring homomorphism, so does $R_*[t]$. — The $R_*[t]$-module $Q = t^1 R_*[t]$ is finitely generated projective by Corollary 2.8 (3), and the map

$$R_*[t] \xrightarrow{\cong} Q \oplus Q \ , \quad r \mapsto (Br, Dr) \tag{7.3}$$

is an isomorphism of $R_*[t]$-modules with inverse $(x, y) \mapsto Ax + Cy$. In addition, Q is not stably free: if $Q \oplus R_*[t]^m \cong R_*[t]^n$, then by (7.3) also

$$R_*[t]^{2n} \cong \big(Q \oplus R_*[t]^m\big) \oplus \big(Q \oplus R_*[t]^m\big) \cong R_*[t]^{2m+1} \ ;$$

as $R_*[t]$ has IBN, the inequality $2n \neq 2m + 1$ renders this impossible. The class of Q in $\tilde{K}_0\big(R_*[t]\big)$ is thus non-zero, and has in fact order 2 in view of the isomorphism (7.3) . Thus the inclusion map $Q \longrightarrow R_*[t]$, considered as a chain complex C^+, is an example of a bounded complex of finitely generated projective $R_*[t]$-modules not homotopy equivalent to a bounded complex of finitely generated free $R_*[t]$-modules. On the other hand, the complex C^+ is certainly R_0-finitely dominated since the cokernel of the inclusion map $Q = t^1 R_*[t] \longrightarrow R_*[t]$ is isomorphic to R_0.

8. R_0-FINITE DOMINATION OF $R_*[t]$-MODULE COMPLEXES

We now develop a homological criterion to detect if a chain complex C^+ of $R_*[t]$-modules is R_0-finitely dominated when considered as a complex of R_0-modules *via* restriction of scalars. This happens if and only if C^+ has trivial NOVIKOV homology in the sense that the induced chain complex $C^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!)$ is acyclic.

[1] I am indebted to R. Hazrat for bringing this example to my attention.

Theorem 8.1. *Let $R = R_*[t, t^{-1}]$ be a strongly $\mathbb{Z}$-graded ring, and let C^+ be a bounded chain complex of finitely generated projective $R_*[t]$-modules. The following statements are equivalent:*

(1) *The complex C^+ is R_0-finitely dominated.*
(2) *The complex $C^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!)$ is contractible (i.e., C^+ has trivial* Novikov *homology).*

Proof. (1) $\Rightarrow$ (2): As C^+ is R_0-finitely dominated, we find a bounded complex D of finitely generated projective R_0-modules, and mutually inverse R_0-linear chain homotopy equivalences $\alpha\colon C^+ \longrightarrow D$ and $\beta : D \longrightarrow C^+$. Let ψ be as in Proposition 6.6; together with Corollary 6.5, the Mather trick asserts that the $R_*[t]$-module complexes C^+ and cone(ψ) are quasi-isomorphic and thus are chain homotopy equivalent (as both are bounded below and consist of projective $R_*[t]$-modules). Thus $C^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!)$ is homotopy equivalent to cone$(\psi) \otimes_{R_*[t]} R_*(\!(t^{-1})\!)$. The latter complex in turn is isomorphic to the mapping cone of the chain map

$$D \otimes_{R_0} R_*(\!(t^{-1})\!) \longrightarrow D \otimes_{R_0} R_*(\!(t^{-1})\!)$$

sending the element $x \otimes \sum_{i \leq k} r_i t^i$ to

$$\alpha\beta(x) \otimes \sum_{j \leq k} r_j t^j - \sum_j \alpha\big(\beta(x)\alpha_j^{(1)}\big) \otimes \sum_{j \leq k} \beta_j^{(-1)} r_j t^{j-1} ,$$

where we write elements of $R_*(\!(t^{-1})\!)$ as formal Laurent series in t^{-1}; note that in the target of the map, $\beta_j^{(-1)} r_j$ is the coefficient of t^{j-1} as $\beta_j^{(-1)}$ has degree -1.

As D consists of finitely presented R_0-modules, we can identify both the tensor products

$$D \otimes_{R_0} R_*[t] \otimes_{R_*[t]} R_*(\!(t^{-1})\!) = D \otimes_{R_0} R_*(\!(t^{-1})\!)$$

and

$$D \otimes_{R_0} t^1 R_*[t] \otimes_{R_*[t]} R_*(\!(t^{-1})\!) = D \otimes_{R_0} R_*(\!(t^{-1})\!)$$

with the twisted right-truncated power of D [HS17, Proposition 3.13], that is,

$$D \otimes_{R_0} R_*(\!(t^{-1})\!) \cong \prod_{n \leq 0} \big(D \otimes_{R_0} R_n\big) \oplus \bigoplus_{n > 0} \big(D \otimes_{R_0} R_n\big) .$$

Thus we rewrite cone$(\psi) \otimes_{R_*[t]} R_*(\!(t^{-1})\!)$ as the right-truncated totalisation [Hüt11, Definition 1.1] of a double complex

$$Z_{p,q} = (D_{p+q+1} \otimes_{R_0} R_p) \oplus (D_{p+q} \otimes_{R_0} R_p)$$

with vertical differential $d^v\colon Z_{p,q} \longrightarrow Z_{p,q-1}$ and horizontal differential $d^h\colon Z_{p,q} \longrightarrow Z_{p-1,q}$ given by the formulæ

$$d^v(x\otimes a, y\otimes b) = \big(-d(x)\otimes a, \alpha\beta(x)\otimes a + d(y)\otimes b\big)\,,$$
$$d^h(x\otimes a, y\otimes b) = \Big(-\sum \alpha\big(\beta(y)\alpha_j^{(1)}\big)\otimes \beta_j^{(-1)}b, 0\Big)\,.$$

The symbol "d", without any decorations, refers to the differential of the chain complex D. The columns are acyclic since $Z_{p,*}$ is a shift suspension of $\mathrm{cone}(\alpha\beta)\otimes_{R_0} R_p$ and the chain map $\alpha\beta$ is homotopic to an identity map. It follows that $C^+\otimes_{R_*[t]} R_*(\!(t^{-1})\!) \simeq \mathrm{cone}(\psi)\otimes_{R_*[t]} R_*(\!(t^{-1})\!)$ is acyclic [Hüt11, Proposition 1.2], and hence contractible.

(2) ⇒ (1): As C^+ consists of finitely generated projective $R_*[t]$-modules, there exists another bounded complex B^+ with zero differentials, consisting of finitely generated projective $R_*[t]$-modules, such that $A^+ = B^+\oplus C^+$ is a bounded complex of finitely generated free $R_*[t]$-modules. We equip A^+_k with a basis with r_k elements, and identify A^+_k with $R_*[t]^{r_k}$ henceforth. The differential $d_k\colon A^+_k \longrightarrow A^+_{k-1}$ is thus represented by a matrix D_k with entries in $R_*[t]$.

Suppose, for ease of notation, that C^+ is concentrated in chain levels between 0 and m. We can choose integers

$$d_m \le d_{m-1} \le \ldots \le d_0 = -1$$

so that D_k defines a map $d_k^-\colon t^{d_k}R_*[[t^{-1}]]^{r_k} \longrightarrow t^{d_{k-1}}R_*[[t^{-1}]]^{r_{k-1}}$; we only need to ensure that no entry of D_k has a non-zero homogeneous component of degree exceeding $d_{k-1}-d_k$. We let S denote the chain complex thus defined, with $S_k = t^{d_k}R_*[[t^{-1}]]^{r_k}$ and differentials D_k. Similarly, we let N denote the chain complex with $N_k = R_*(\!(t^{-1})\!)^{r_k}$ and differentials D_k. Note that S is a subcomplex of N.

For any $d\le -1$ there is a short exact sequence of R_0-modules

$$0 \longrightarrow t^dR_*[[t^{-1}]]\oplus R_*[t] \xrightarrow{(-1\ 1)} R_*(\!(t^{-1})\!) \longrightarrow \bigoplus_{j=d+1}^{-1} R_j \longrightarrow 0 \quad (8.2)$$

with last term a finitely generated projective R_0-module by Corollary 2.8, as $R_*[t,t^{-1}]$ is strongly graded. It follows that there is a short exact sequence of R_0-module complexes

$$0 \longrightarrow S\oplus A^+ \xrightarrow[\beta]{(-1\ 1)} N \longrightarrow P \longrightarrow 0 \qquad (8.3)$$

with P a bounded complex of finitely generated projective R_0-modules. In chain degree k this sequence is actually just the r_k-fold direct sum of (8.2) with itself, for $d=d_k$.

From the sequence (8.3) we infer that the map from the mapping cone of β to P is a quasi-isomorphism. Now recall $A^+ = B^+ \oplus C^+$ and observe the consequent splitting

$$N = A^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!) = B^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!) \ \oplus \ C^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!) \ . \quad (8.4)$$

By hypothesis $C^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!)$ is contractible; thus N is quasi-isomorphic, *via* the projection map, to $B^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!)$. As taking mapping cones is homotopy invariant, we can replace N by the latter complex and conclude that P is quasi-isomorphic to the mapping cone of the map

$$\gamma \colon S \oplus A^+ = S \oplus B^+ \oplus C^+ \xrightarrow{(-1\ 1\ 0)} B^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!) \ .$$

As γ is the zero map on the C^+-summand, the mapping cone of γ contains C^+ as a direct summand. Hence in the derived category of the ring R_0, the complex C^+ is a retract of P. Since both complexes are bounded and consist of projective R_0-modules, we conclude that C^+ is a retract up to homotopy of P whence C^+ is R_0-finitely dominated as claimed. □

9. $R_*[t]$-Fredholm matrices

Let $R = R_*[t, t^{-1}]$ be a $\mathbb{Z}$-graded ring, and let A^+ be a non-zero square matrix of size k with entries in $R_*[t, t^{-1}]$. For suitable $m \in \mathbb{Z}$, multiplication by A^+ defines an $R_*[t]$-module homomorphism

$$A^+ = \mu(A^+, m) \colon R_*[t]^k \longrightarrow \big(t^{-m} R_*[t]\big)^k \ , \quad x \mapsto A^+ \cdot x \ ;$$

"suitable" means, in fact, that $-m$ is not larger than the minimal degree of non-zero homogeneous components of entries of A^+. Suppose now that in addition to such m we fix an integer $n > m$ so that the map $\mu(A^+, n)$ is defined as well.

Lemma 9.1. *There is an isomorphism of R_0-modules*

$$\operatorname{coker} \mu(A^+, n) \cong \operatorname{coker} \mu(A^+, m) \oplus \bigoplus_{k=-n}^{-m-1} R_k \ .$$

Proof. The direct sum of the exact sequence of R_0-modules

$$R_*[t]^k \xrightarrow{\mu(A^+, m)} \big(t^{-m} R_*[t]\big)^k \longrightarrow \operatorname{coker} \mu(A^+, m) \longrightarrow 0$$

with the exact sequence

$$0 \longrightarrow \bigoplus_{k=-n}^{-m-1} R_k \xrightarrow{=} \bigoplus_{k=-n}^{-m-1} R_k \longrightarrow 0$$

yields a new exact sequence, which is precisely the sequence

$$R_*[t]^k \xrightarrow{\mu(A^+,n)} \big(t^{-n}R_*[t]\big)^k \longrightarrow \operatorname{coker}\mu(A^+,m) \oplus \bigoplus_{k=-n}^{-m-1} R_k \longrightarrow 0 \ .$$

Hence $\operatorname{coker}\mu(A^+,n) \cong \operatorname{coker}\mu(A^+,m) \oplus \bigoplus_{k=-n}^{-m-1} R_k$ as R_0-modules. □

Corollary 9.2. *Suppose that $R = R_*[t,t^{-1}]$ is* strongly *$\mathbb{Z}$-graded. In the situation of Lemma 9.1, the module* $\operatorname{coker}\mu(A^+,n)$ *is a finitely generated projective R_0-module if and only if* $\operatorname{coker}\mu(A^+,m)$ *is.*

Proof. This is a consequence of Corollary 2.8 (1) and Lemma 9.1. □

Proposition 9.3. *Suppose that $R = R_*[t,t^{-1}]$ is a strongly $\mathbb{Z}$-graded ring. Let A^+ be a $k\times k$-matrix with entries in $R_*[t,t^{-1}]$, and let $m \in \mathbb{Z}$ be "suitable" in the sense that multiplication by A^+ yields a map of finitely generated projective $R_*[t]$-modules $A^+ = \mu(A^+,m)\colon R_*[t]^k \longrightarrow t^{-m}R_*[t]^k$, $x \mapsto A^+\cdot x$ (see discussion above) which we may consider as a chain complex concentrated in chain degrees 1 and* 0*. The following statements are equivalent:*

(1) *The chain complex A^+ is R_0-finitely dominated.*
(2) *The induced chain complex $A^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!)$ is contractible.*
(3) *The map A^+ is invertible over $R_*(\!(t^{-1})\!)$, that is, the map*

$$R_*(\!(t^{-1})\!)^k \longrightarrow R_*(\!(t^{-1})\!)^k \ , \quad x \mapsto A^+\cdot x$$

is an isomorphism.
(4) *The matrix A^+ is invertible in the ring of all square matrices of size k with entries in $R_*(\!(t^{-1})\!)$.*
(5) *The map $\mu(A^+,m)$ is injective, and* $\operatorname{coker}\mu(A^+,m)$ *is a finitely generated projective R_0-module.*

Moreover, the validity of these statements does not depend on the specific choice of a suitable $m \in \mathbb{Z}$.

Definition 9.4. A square matrix matrix with entries in $R_*[t,t^{-1}]$ satisfying one (and hence all) of the conditions listed in Proposition 9.3 is called an $R_*[t]$-FREDHOLM *matrix*. The set of all $R_*[t]$-FREDHOLM matrices (of arbitrary finite size) is denoted by the symbol Ω_+.

Proof of Proposition 9.3. Condition (5) is insensitive to the precise value of the suitable integer m, in view of Corollary 9.2.

The equivalence of conditions (1) and (2) is Theorem 8.1 above. Statements (3) and (4) are trivially equivalent.

By Lemma 2.5, the multiplication map

$$t^{-m}R_*[t] \otimes_{R_*[t]} R_*[t,t^{-1}] \longrightarrow R_*[t,t^{-1}] \ , \quad x\otimes y \mapsto xy$$

is an isomorphism of $R_*[t,t^{-1}]$-modules. It follows that there is a chain of isomorphisms

$$\begin{aligned} t^{-m}R_*[t]\otimes_{R_*[t]}R_*(\!(t^{-1})\!) &\cong t^{-m}R_*[t]\otimes_{R_*[t]}R_*[t,t^{-1}]\otimes_{R_*[t,t^{-1}]}R_*(\!(t^{-1})\!) \\ &\cong R_*[t,t^{-1}]\otimes_{R_*[t,t^{-1}]}R_*(\!(t^{-1})\!)\cong R_*(\!(t^{-1})\!) \end{aligned}$$

with composition the multiplication map. In view of this, statements (2) and (3) are equivalent.

If (5) holds then the chain complex A^+ is R_0-homotopy equivalent to the module $\operatorname{coker}\mu(A^+,m)$, considered as a chain complex concentrated in degree 0, which shows that (1) is satisfied in this case.

Suppose finally that (3) holds; we will show that (5) is valid as well. We infer from the commutative square

$$\begin{tikzcd} R_*[t]^k \arrow[r, "\mu(A^+,m)"] \arrow[d, "\subset"'] & t^{-m}R_*[t]^k \arrow[d, "\subset"'] \\ R_*(\!(t^{-1})\!)^k \arrow[r, "A^+", "\cong"'] & R_*(\!(t^{-1})\!)^k \end{tikzcd}$$

that the map $\mu(A^+,m)$ must be injective. Thus it remains to verify that $\operatorname{coker}\mu(A^+,m)$ is a finitely generated projective R_0-module. Assuming $m\geq 1$, as we may in view of Corollary 9.2, we can embed $\mu(A^+,m)$ into a commutative diagram of R_0-modules

$$\begin{tikzcd} t^{-1}R[[t^{-1}]]^k \arrow[r, "\subset"] \arrow[d, "A^+"'] & R(\!(t^{-1})\!)^k \arrow[d, "A^+"'] & R_*[t]^k \arrow[l, "\supset"'] \arrow[d, "A^+"', "\mu(A^+,m)"] \\ t^qR[[t^{-1}]]^k \arrow[r, "\subset"] & R(\!(t^{-1})\!)^k & t^{-m}R_*[t]^k \arrow[l, "\supset"'] \end{tikzcd} \tag{9.5}$$

where $q\geq 0$ is sufficiently large; it is sufficient that q exceeds the maximal degree of any non-zero homogeneous component of the entries of A^+. Now in any ABELian category, a diagram $\mathscr{D}=\big(X\xrightarrow{\xi}Y\xleftarrow{\zeta}Z\big)$ gives rise to an exact sequence, natural in $\mathscr{D}$, of the form

$$0\longrightarrow \ker(\xi-\zeta)\longrightarrow X\oplus Z\xrightarrow{\xi-\zeta}Y\longrightarrow \operatorname{coker}(\xi-\zeta)\longrightarrow 0\ .$$

We apply this to the rows of diagram (9.5) above, noting the the coker term is trivial in both cases (since $q,m\geq 0$). The kernel, on the other hand, is trivial in case of the top row, and is the finitely generated projective R_0-module $P=\bigoplus_{-m}^{q}R_j$ for the bottom row. We arrive at the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & t^{-1}R(\!(t^{-1})\!)^k \oplus R_*[t]^k \arrow[r] \arrow[d, "A^+ \oplus A^+"'] & R(\!(t^{-1})\!) \arrow[r] \arrow[d, "A^+"'] & 0 \\
0 \arrow[r] & P \arrow[r] & t^q R(\!(t^{-1})\!)^k \oplus t^{-m}R_*[t]^k \arrow[r] & R(\!(t^{-1})\!) \arrow[r] & 0
\end{tikzcd}$$

As the right-hand vertical map is an isomorphism by hypothesis (3), the SNAKE lemma yields an isomorphism of P with the cokernel of the middle vertical map, which contains $\operatorname{coker}\mu(A^+, m) \colon R_*[t]^k \longrightarrow t^{-m}R_*[t]^k$ as a direct summand. This shows that $\operatorname{coker}\mu(A^+, m)$ is a finitely generated projective R_0-module as desired. □

10. THE FREDHOLM LOCALISATIONS $\Omega_+^{-1}R_*[t]$ AND $\Omega_+^{-1}R_*[t, t^{-1}]$

We now turn our attention to the non-commutative localisations

$$\alpha \colon R_*[t] \longrightarrow \Omega_+^{-1}R_*[t] \quad \text{and} \quad \gamma \colon R_*[t, t^{-1}] \longrightarrow \Omega_+^{-1}R_*[t, t^{-1}]$$

where Ω_+ denotes the set of $R_*[t]$-FREDHOLM matrices as in Definition 9.4. To be precise, we define $\alpha = \lambda_{\Omega_+} \colon R_*[t] \longrightarrow \Omega_+^{-1}R_*[t]$ as the the non-commutative localisation inverting all the maps

$$\mu(A^+, m) \colon R_*[t]^k \longrightarrow t^{-m}R_*[t]^k \tag{10.1}$$

of finitely generated projective $R_*[t]$-modules, where $k \geq 1$ is arbitrary, $A^+ \in \Omega_+$ has size k, and $m \in \mathbb{Z}$ is suitable in the sense of §9. As A^+ satisfies property (4) of Proposition 9.3, the universal property of non-commutative localisation yields a factorisation

$$R_*[t] \xrightarrow{\alpha} \Omega_+^{-1}R_*[t] \longrightarrow R(\!(t^{-1})\!) \tag{10.2}$$

of the inclusion map; in particular, α is injective. — Similarly, we define $\gamma = \lambda_{\Omega_+} \colon R_*[t, t^{-1}] \longrightarrow \Omega_+^{-1}R_*[t, t^{-1}]$ as the the non-commutative localisation inverting all the maps

$$A^+ \colon R_*[t, t^{-1}]^k \longrightarrow R_*[t, t^{-1}]^k \tag{10.3}$$

of finitely generated free $R_*[t, t^{-1}]$-modules, where $k \geq 1$ is arbitrary and $A^+ \in \Omega_+$ has size k. As A^+ satisfies property (4) of Proposition 9.3, the universal property of non-commutative localisation yields a factorisation

$$R_*[t, t^{-1}] \xrightarrow{\gamma} \Omega_+^{-1}R_*[t, t^{-1}] \longrightarrow R(\!(t^{-1})\!) \tag{10.4}$$

of the inclusion map; in particular, γ is injective.

Applying the functor - $\otimes_{R_*[t]} R_*[t, t^{-1}]$ to a map as in (10.1) yields a map as in (10.3), by Lemma 2.5. Thus $\gamma|_{R_*[t]} \colon R_*[t] \longrightarrow \Omega_+^{-1} R_*[t, t^{-1}]$ inverts all the maps (10.1) and factorises through a ring homomorphism $\delta \colon \Omega_+^{-1} R_*[t] \longrightarrow \Omega_+^{-1} R_*[t, t^{-1}]$. That is, the maps α and γ fit into the commutative square diagram of Fig. 5 which, by Proposition 4.2, is a pushout square in the category of unital rings.

$$\begin{tikzcd} R_*[t] \arrow[r, "\alpha"] \arrow[d, "\subset"', "\beta"] \arrow[dr, phantom, "\ulcorner\cdot"] & \Omega_+^{-1} R_*[t] \arrow[d, "\delta"'] \\ R_*[t, t^{-1}] \arrow[r, "\gamma"] & \Omega_+^{-1} R_*[t, t^{-1}] \end{tikzcd} \tag{10.5}$$

FIGURE 5. Pushout square of FREDHOLM localisations

Theorem 10.6. *Let $R = R_*[t, t^{-1}]$ be a strongly $\mathbb{Z}$-graded ring, and let C^+ be a bounded chain complex of finitely generated projective $R_*[t]$-modules. The following statements are equivalent:*

(1) *The chain complex C^+ is R_0-finitely dominated.*
(2) *The induced chain complex $C^+ \otimes_{R_*[t]} \Omega_+^{-1} R_*[t]$ is contractible.*
(3) *The induced chain complex $C^+ \otimes_{R_*[t]} \Omega_+^{-1} R_*[t, t^{-1}]$ is contractible.*

Proof. (1) ⇒ (2): Suppose that C^+ is R_0-finitely dominated. For ease of notation we assume $C_n^+ = 0$ for $n < 1$. By taking direct sum with contractible one-step complexes of the form $P \xrightarrow{=} P$, with P suitable finitely generated projective $R_*[t]$-modules, we obtain a new bounded chain complex A^+ concentrated in non-negative chain levels, which is homotopy equivalent to C^+ such that all chain modules A_n^+ are finitely generated free over $R_*[t]$, with the possible exception of A_0^+ which is finitely generated projective over $R_*[t]$. Explicitly, let N be maximal with $C_N^+ \neq 0$. We can choose finitely generated projective $R_*[t]$-module $Q_N^+, Q_{N-1}^+, \cdots, Q_1^+$, in this order, such that $C_n^+ \oplus Q_n^+ \oplus Q_{n+1}^+$ is finitely generated free ($1 \leq n \leq N$, with $Q_k^+ = 0$ for $k > N$); the bounded complex

A^+ can then take the form

$$\begin{array}{ccccccccc}
Q_1^+ & \xleftarrow{=} & Q_1^+ & & Q_3^+ & \xleftarrow{=} & Q_3^+ & & Q_5^+ \xleftarrow{=} \cdots \\
 & & \oplus & & \oplus & & \oplus & & \oplus \\
 & & C_1^+ & \longleftarrow & C_2^+ & \longleftarrow & C_3^+ & \longleftarrow & C_4^+ \longleftarrow \cdots \\
 & & \oplus & & \oplus & & \oplus & & \oplus \\
 & & Q_2^+ & \xleftarrow{=} & Q_2^+ & & Q_4^+ & \xleftarrow{=} & Q_4^+ \qquad \cdots
\end{array}$$

so that C^+ is a direct summand of A^+, and both the inclusion $C^+ \longrightarrow A^+$ and the projection $A^+ \longrightarrow C^+$ are homotopy equivalences.

For $n > 0$ the module $A_n^+ = C_n^+ \oplus Q_n^+ \oplus Q_{n+1}^+$ is finitely generated free. We choose a basis consisting of r_n elements, thereby identifying A_n^+ with the finite direct sum $\bigoplus_{r_n} R_*[t] = \big(R_*[t]\big)^{r_n}$. The nth chain module of the induced complex $A^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!)$ is identified with $R_*(\!(t^{-1})\!)^{r_n}$.

For $n > 1$ the differential $A_n^+ \longrightarrow A_{n-1}^+$ of A^+ can be thought of as a matrix D_n^+ with entries in $R_*[t]$. The differential D_1^+ is the homomorphism given by projection onto $A_0^+ = Q_1^+$.

As C^+ is $R_*[t,t^{-1}]$-finitely dominated, Theorem 8.1 ensures that the induced complex $C^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!)$ is contractible; we choose homomorphisms

$$\tau_n \colon C_n^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!) \longrightarrow C_{n+1}^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!)$$

forming a chain contraction. These maps give rise to a contraction σ^+ of $A^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!)$, by defining

$$\sigma_n \colon A_n^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!) \longrightarrow A_{n+1}^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!)$$

by the formula

$$\sigma_n^+ = \begin{cases} Q_1^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!) \xrightarrow{\subseteq} A_1^+ \otimes_{R_*[t]} R_*(\!(t^{-1})\!) & \text{for } n = 0\,, \\ \big(C_n^+ \oplus Q_n^+ \oplus Q_{n+1}^+\big) \otimes_{R_*[t]} R_*(\!(t^{-1})\!) & \\ \qquad \xrightarrow[(\sigma_n, 0, \mathrm{id})]{} \big(C_{n+1}^+ \oplus Q_{n+2}^+ \oplus Q_{n+1}^+\big) \otimes_{R_*[t]} R_*(\!(t^{-1})\!) & \text{for } n > 0\,. \end{cases}$$

Note that the map σ_0^+ is defined over $R_*[t]$. For $n > 0$ we think of σ_n^+ as matrices with entries in $R_*(\!(t^{-1})\!)$ such that $D_{n+1}^+ \cdot \sigma_n^+ + \sigma_{n-1}^+ \cdot D_n^+$ is a unit matrix of size r_n. We can truncate the entries of the matrices σ_n^+ below at some suitable integer $m \ll 0$ (not depending on n) to obtain matrices $S_n^+ = \mathrm{tr}_m(\sigma_n^+)$ with entries in $R_*[t,t^{-1}]$ such that $E_n = D_{n+1}^+ \cdot S_n^+ + S_{n-1}^+ \cdot D_n^+$, for $n \geq 2$, is the sum of a unit matrix, and a matrix the non-zero entries of which have homogeneous components of strictly negative degree. Thus E_n is invertible over $R_*((t^{-1}))$ so that $E_n \in \Omega_+$. Similarly, writing S_0^+ for the homomorphism σ_0^+ we see that $E_1 = D_2^+ \cdot S_1^+ + S_0^+ \cdot D_1^+$ and $E_0 = D_1^+ \circ S_0^+$ are invertible in $R_*((t^{-1}))$ whence $E_1, E_0 \in \Omega_+$ as well. Here we make use of the fact that $\sigma_0^+ = S_0^+$ and D_1^+ are defined over $R_*[t]$; in fact $E_0 = \mathrm{id}_{Q_1}$, and the matrix representing $S_0^+ \cdot D_1^+$ has entries in $R_*[t]$ and is hence unaffected by truncation.

We now define a new $R_*[t]$-module chain complex B^+ by setting $B_n^+ = \big(t^m R_*[t]\big)^{r_n} = A_n^+ \otimes_{R_*[t]} t^m R_*[t]$, with differentials given by the matrices D_n^+ for $n > 1$, and the projection map onto the direct summand $Q_1^+ \otimes_{R_*[t]} \otimes t^m R_*[t]$ for $n = 1$. The matrices S_n^+ for $n > 0$, and the homomorphism S_0^+, define module homomorphisms $A_n^+ \longrightarrow B_{n+1}^+$ constituting an (A^+, B^+)-α-proto-null homotopy, cf. §3, since $E_n = D_{n+1}^+ \cdot S_n^+ + S_{n-1}^+ \cdot D_n^+$ is an element of Ω_+ as explained above. Here $\alpha\colon R_*[t] \longrightarrow \Omega_+ R_*[t]$ is the localisation map as in (10.5). It follows that $A^+ \otimes_{R_*[t]} \Omega_+^{-1} R_*[t]$ is contractible by Lemma 3.1, hence so is $C^+ \otimes_{R_*[t]} \Omega_+^{-1} R_*[t]$.

(2) ⇒ (3): Immediate from the factorisation

$$R_*[t] \xrightarrow{\alpha} \Omega_+^{-1} R_*[t] \longrightarrow \Omega_+^{-1} R_*[t, t^{-1}]$$

of γ, see (10.5) in Fig. 5.

(3) ⇒ (1): Immediate from the factorisation (10.4) and Theorem 8.1. □

Theorem 10.7 (Universal property of $\Omega_+^{-1} R_*[t]$). *Suppose that $R_*[t, t^{-1}]$ is a strongly $\mathbb{Z}$-graded ring. The localisation $\lambda\colon R_*[t] \longrightarrow \Omega_+^{-1} R_*[t]$ is the universal $R_*[t]$-ring making R_0-finitely dominated chain complexes contractible. That is, suppose that $f : R_*[t] \longrightarrow S$ is an $R_*[t]$-ring such that for every bounded complex of finitely generated projective $R_*[t]$-modules C^+ which is R_0-finitely dominated, the complex $C^+ \otimes_{R_*[t]} S$ is contractible. Then there is a factorisation $R_*[t] \xrightarrow{\lambda} \Omega_+^{-1} R_*[t] \xrightarrow{\eta} S$ of f, with a uniquely determined ring homomorphism η.*

Proof. It was shown in Theorem 10.6 above that $\Omega_+^{-1} R_*[t]$ makes R_0-finitely dominated chain complexes contractible. Thus it is enough to show that f inverts the maps $\mu(A^+, m)$ of (10.1) for any $A^+ \in \Omega_+$ and any suitable $m \in \mathbb{Z}$. By definition of Ω_+ the *complex* $\mu(A^+, m)$ is R_0-finitely dominated so that, by hypothesis on f, the complex $\mu(A^+, m) \otimes_{R_*[t]} S$ is contractible. This says precisely that f inverts the map $\mu(A^+, m)$. □

One can also show that *the localisation* $\lambda\colon R_*[t,t^{-1}] \longrightarrow \Omega_+^{-1}R_*[t,t^{-1}]$ *is the universal* $R_*[t,t^{-1}]$*-ring making* R_0*-finitely dominated, bounded chain complexes of finitely generated projective* $R_*[t]$*-modules complexes contractible.*

We finish with proving that $\delta\colon \Omega_+^{-1}R_*[t] \longrightarrow \Omega_+^{-1}R_*[t,t^{-1}]$ is an isomorphism if $R_*[t,t^{-1}]$ contains a homogeneous unit of non-zero degree.

Proposition 10.8. *Suppose that* $R = R_*[t,t^{-1}]$ *is a strongly* $\mathbb{Z}$*-graded ring. Suppose there exists a homogeneous unit of positive degree in* $R_*[t,t^{-1}]$*. Then there is an injective ring homomorphism* $\iota\colon R_*[t,t^{-1}] \longrightarrow \Omega_+^{-1}R_*[t]$ *with* $\iota\beta = \alpha$*, and* $\delta\colon \Omega_+^{-1}R_*[t] \longrightarrow \Omega_+^{-1}R_*[t,t^{-1}]$ *is an isomorphism satisfying* $\delta\iota = \gamma$*.*

Proof. Let $u \in R_d \cap R_*[t,t^{-1}]^\times$, with $d > 0$. Then the 1×1-matrix (u) is an $R_*[t]$-FREDHOLM matrix since the cokernel of the map

$$R_*[t] \longrightarrow R_*[t]\,, \quad r \mapsto ur$$

is the finitely generated projective R_0-module $\bigoplus_0^{d-1} R_j$. The induced map

$$\Omega_+^{-1}R_*[t] \longrightarrow \Omega_+^{-1}R_*[t]$$

is given by multiplication with $\alpha(u) \in \Omega_+^{-1}R_*[t]$. Since the induced map is an isomorphism, $\alpha(u)$ is invertible in $\Omega_+^{-1}R_*[t]$.

Given any $x \in R_*[t,t^{-1}]$ there exists $k \geq 0$ with $u^k x \in R_*[t]$ and thus $\alpha(u^k x) \in \Omega_+^{-1}R_*[t]$; we define $\iota(x) = \alpha(u)^{-k} \cdot \alpha(u^k x) \in \Omega_+^{-1}R_*[t]$. The element $\iota(x)$ does not depend on the choice of k, for if $\ell > k$ we have

$$\alpha(u)^{-\ell} \cdot \alpha(u^\ell x) \;=\; \alpha(u)^{-k}\alpha(u)^{-\ell+k} \cdot \alpha(u^{\ell-k}u^k x) \;=\; \alpha(u)^{-k} \cdot \alpha(u^k x)\ ,$$

since $u^{\ell-k} \in R_*[t]$ and since α is a ring homomorphism. Note that $\iota(x) = \alpha(x)$ for $x \in R_*[t]$, and that $\iota(u^{-1}) = \alpha(u)^{-1}$.

Suppose that $x \in R_*[t,t^{-1}]$ and $k,\ell \geq 0$ are such that $u^k x u^{-\ell} \in R_*[t]$. Then $\alpha(u^k x u^{-\ell}) = \alpha(u^k x)\alpha(u)^{-\ell}$, since both sides equal $\alpha(u^k x)$ after multiplication with $\alpha(u)^\ell$. Consequently, for $x, y \in R_*[t,t^{-1}]$ and $k,\ell \geq 0$ with $u^\ell y,\ u^k x u^{-\ell} \in R_*[t]$ we calculate

$$\begin{aligned}
\iota(xy) &= \iota(xu^{-\ell}u^\ell y)\\
&= \alpha(u)^{-k} \cdot \alpha(u^k x u^{-\ell} u^\ell y)\\
&= \alpha(u)^{-k} \cdot \alpha(u^k x u^{-\ell}) \cdot \alpha(u^\ell y)\\
&= \alpha(u)^{-k} \cdot \alpha(u^k x) \cdot \alpha(u)^{-\ell} \cdot \alpha(u^\ell y) = \iota(x) \cdot \iota(y)\ .
\end{aligned}$$

Since ι is clearly additive, the map $\iota\colon R_*[t,t^{-1}] \longrightarrow \Omega_+^{-1}R_*[t]$ is thus a ring homomorphism. Moreover, ι is injective as $\iota(x) = \alpha(u)^{-k} \cdot \alpha(u^k x)$ vanishes if and only if $\alpha(u^k x)$ vanishes. It follows from Lemmas 2.6 and 4.3 that the ring homomorphism δ is an isomorphism and satisfies $\delta\iota = \gamma$. □

References

[Dad80] Everett Clarence Dade. Group-graded rings and modules. *Math. Z.*, 174(3):241–262, 1980.

[HM20] Thomas Hüttemann and Tasha Montgomery. The algebraic K-theory of the projective line associated with a strongly $\mathbb{Z}$-graded ring. *J. Pure Appl. Algebra*, 224(12):106425, 20, 2020.

[HS17] Thomas Hüttemann and Luke Steers. Finite domination and Novikov homology over strongly $\mathbb{Z}$-graded rings. *Israel J. Math.*, 221(2):661–685, 2017.

[Hüt11] Thomas Hüttemann. Double complexes and vanishing of Novikov cohomology. *Serdica Math. J.*, 37(4):295–304 (2012), 2011.

[Hüt20] Thomas Hüttemann. The "fundamental theorem" for the algebraic K-theory of strongly $\mathbb{Z}$-graded rings, 2020. Preprint: arXiv 2003.01506.

[Ran85] Andrew Ranicki. The algebraic theory of finiteness obstruction. *Math. Scand.*, 57(1):105–126, 1985.

[Ran98] Andrew Ranicki. *High-dimensional knot theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 1998. Algebraic surgery in codimension 2, With an appendix by Elmar Winkelnkemper.

[Sch85] Aidan H. Schofield. *Representation of rings over skew fields*, volume 92 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1985.

Thomas Hüttemann, Queen's University Belfast, School of Mathematics and Physics, Mathematical Sciences Research Centre, Belfast BT7 1NN, Northern Ireland, UK
ORCiD: 0000-0002-3323-4503

E-mail address: `t.huettemann@qub.ac.uk`

URL: `https://pure.qub.ac.uk/en/persons/thomas-huettemann`

URL: `https://t-huettemann.github.io/`