



**QUEEN'S
UNIVERSITY
BELFAST**

On a partition with a lower expected L2-discrepancy than classical jittered sampling

Pausinger, F., & Kiderlen, M. (2022). On a partition with a lower expected L2-discrepancy than classical jittered sampling. *Journal of Complexity*, 70, Article 101616. <https://doi.org/10.1016/j.jco.2021.101616>

Published in:
Journal of Complexity

Document Version:
Peer reviewed version

Queen's University Belfast - Research Portal:
[Link to publication record in Queen's University Belfast Research Portal](#)

Publisher rights

Copyright 2021 Elsevier.

This manuscript is distributed under a Creative Commons Attribution-NonCommercial-NoDerivs License

(<https://creativecommons.org/licenses/by-nc-nd/4.0/>), which permits distribution and reproduction for non-commercial purposes, provided the author and source are cited.

General rights

Copyright for the publications made accessible via the Queen's University Belfast Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The Research Portal is Queen's institutional repository that provides access to Queen's research output. Every effort has been made to ensure that content in the Research Portal does not infringe any person's rights, or applicable UK laws. If you discover content in the Research Portal that you believe breaches copyright or violates any law, please contact openaccess@qub.ac.uk.

Open Access

This research has been made openly available by Queen's academics and its Open Research team. We would love to hear how access to this research benefits you. – Share your feedback with us: <http://go.qub.ac.uk/oa-feedback>

ON A PARTITION WITH A LOWER EXPECTED \mathcal{L}_2 -DISCREPANCY THAN CLASSICAL JITTERED SAMPLING

MARKUS KIDERLEN AND FLORIAN PAUSINGER

ABSTRACT. We prove that classical jittered sampling of the d -dimensional unit cube does *not* yield the smallest expected \mathcal{L}_2 -discrepancy among all stratified samples with $N = m^d$ points. Our counterexample can be given explicitly and consists of convex partitioning sets of equal volume.

1. INTRODUCTION

Classical *jittered sampling* with $N = m^d$ points combines the simplicity of grids with uniform random sampling by partitioning $[0, 1]^d$ into m^d axis-aligned congruent cubes and placing a random point inside each of them; see Fig. 1 (left). Jittered sampling is sometimes referred to as ‘stratified sampling’ in the literature, but we will use the term ‘stratified sampling’ in a more broad sense. Let $\boldsymbol{\Omega} = (\Omega_1, \dots, \Omega_N)$ denote a general partition of $[0, 1]^d$ into N subsets $\Omega_1, \dots, \Omega_N$ of positive volume. A *stratified sample* $\mathcal{P} = \mathcal{P}_{\boldsymbol{\Omega}}$, based on this partition, is a set of N random points, where the i th point in \mathcal{P} is chosen uniformly in the i th set of the partition (and stochastically independent of the other points), $i = 1, \dots, N$. If $N = m^d$ and the partition consists of the above mentioned axis-aligned congruent cubes, we obtain *jittered sampling* as a special case.

To analyse the irregularities of such points the concept of \mathcal{L}_p -discrepancy is commonly used. Given a set $\mathcal{P} \subset [0, 1]^d$ of N points and a vector $\mathbf{x} = (x_1, \dots, x_d)$ in the unit cube, the discrepancy function

$$(1) \quad d_{\mathcal{P}}(\mathbf{x}) = \frac{\#(\mathcal{P} \cap [0, \mathbf{x}])}{N} - |[0, \mathbf{x}]|$$

evaluates to the difference between the relative number of points in an axis-aligned box of the form $[0, \mathbf{x}] := \prod_{k=1}^d [0, x_k]$ and the volume of this box. Here, $|\cdot|$ denotes the Lebesgue measure and $\#(\mathcal{P} \cap [0, \mathbf{x}])$ counts the number of points of \mathcal{P} that lie in $[0, \mathbf{x}]$. For $1 \leq p < \infty$, the \mathcal{L}_p -discrepancy

$$\mathcal{L}_p(\mathcal{P}) := \|d_{\mathcal{P}}\|_p$$

of \mathcal{P} is defined as the L_p -norm $\|\cdot\|_p$ of the discrepancy function. We will only work with $p = 2$.

As a side remark, the well-known *star discrepancy* can be defined as the L_{∞} -norm of the discrepancy function and is generally much harder to study.

1.1. Optimal \mathcal{L}_p -discrepancy bounds. The \mathcal{L}_p -discrepancy, and in particular the special case $p = 2$, is a well studied and understood measure for the irregularities of point sets. We refer to the book [6] and the survey [7] for further details. In particular, and in contrast to other measures such as the star-discrepancy, it is known how to construct deterministic point sets with the optimal order of magnitude of the \mathcal{L}_p -discrepancy; see [2, 7, 8]. In fact, for $d \geq 2$ there are constructions of point sets $\mathcal{P} \subset [0, 1]^d$ of N points such that

$$\mathcal{L}_p(\mathcal{P}) = \Theta \left(\frac{(\log N)^{\frac{d-1}{2}}}{N} \right).$$

The optimality of these constructions follows from a seminal result of Roth [17] who derived a general lower bound for the \mathcal{L}_2 -discrepancy of arbitrary sets of N points in $[0, 1]^2$ which was

2010 *Mathematics Subject Classification.* 11K38, 60C05 (primary), and 05A18, 60D99 (secondary).

Key words and phrases. Jittered sampling; Stratified sampling; L_p -discrepancy.

later generalised to $d \geq 2$ and $1 < p < \infty$; see [6, Section 3.2]. While deterministic point sets with small discrepancy are widely used in the context of numerical integration, simulations of different real world phenomena may require an element of randomness. The expected (star) discrepancy of a set \mathcal{P}_N of N i.i.d. uniform random points in $[0, 1]^d$ is of order $\Theta(\sqrt{1/N})$; see [10] for the first upper bound, [1] for the first upper bound with explicit constant and [4] for the first lower bound as well as [9] for the current state of the art results in this context. This can be compared to a recent result by Doerr [5] on the precise asymptotic order of the expected star-discrepancy of a point set obtained from jittered sampling:

$$\mathbb{E}\mathcal{L}_\infty(\mathcal{P}_N) = \Theta\left(\frac{\sqrt{d}\sqrt{1 + \log(N/d)}}{N^{\frac{1}{2} + \frac{1}{2d}}}\right).$$

That jittered sampling is (asymptotically) better than equally many i.i.d. uniform random points is not only true in terms of the star-discrepancy. In fact, for $1 < p < \infty$, a stratified set derived from a partition into $N > 2$ equivolume sets *always* has a smaller expected \mathcal{L}_p -discrepancy than a set consisting of N i.i.d. random points. This *strong partition principle* was proven in [14, Theorem 1] (see also [19] for a weaker form) and raises the question which partition yields the stratified sample with the smallest mean \mathcal{L}_p -discrepancy – if such a partition exists. For simplicity we will only consider $p = 2$ here.

1.2. Generalized \mathcal{L}_p -discrepancy. Of course, any such analysis should keep the observation of Matoušek [16] in mind that the \mathcal{L}_2 -discrepancy can be misleading if the dimension d is high and the number of points is relatively small, i.e. smaller than 2^d . He observed that point sets with few points that cluster close to $(1, \dots, 1)$ can have nearly optimal \mathcal{L}_2 -discrepancy [16, Section 2]. This, and the fact that the \mathcal{L}_p -discrepancy does not yield a satisfying version of the classical Koksma-Hlawka Theorem, motivated Hickernell [12] to extend the notion. Hickernell's generalization incorporates not only the original \mathcal{L}_p -discrepancy but also the discrepancies of all projections of the point set to lower dimensional faces of the unit cube. We will briefly return to this more general discrepancy in Section 4.

1.3. Stratified sampling. Before detailing our result, we want to emphasize that our notion of *stratified sampling* is a special case of the stratification tool usually employed in statistics and simulation; see, for instance [18, Section 4.3.4] (and the corresponding notion for finite populations in e.g. [21]). There, the underlying cube $[0, 1]^d$ is partitioned into $k \leq N$ sets $\Omega_1, \dots, \Omega_k$, and $N_i \geq 1$ points are sampled i.i.d. uniformly in Ω_i independently of the other points. The resulting stratified sample $\tilde{\mathcal{P}}$ consisting of $N = N_1 + \dots + N_k$ points is only covered by our definition when $k = N$, in which case exactly one point is allocated to each of the strata, i.e. $N_1 = \dots = N_k = 1$. Often, an allocation proportional to stratum size is used, that is, one requires that $N_i = N|\Omega_i|$ for all $i = 1, \dots, k$. For given N , the question if there is a $k \leq N$, and a partition $\Omega_1, \dots, \Omega_k$ such that the sample $\tilde{\mathcal{P}}$ with allocation proportional to stratum size minimizes the expected \mathcal{L}_2 -discrepancy $\mathbb{E}\mathcal{L}_2^2(\tilde{\mathcal{P}})$, is only seemingly more general than asking for a minimizing equivolume partition with N strata, where only one point is sampled in each. This is a consequence of a version of the strong partition principle – see Remark 2, below – and is the reason why we restrict attention to one sampling point per stratum. The existence of such minimizing equivolume partitions is not trivial, but has been shown in [14] under additional regularity assumptions on the sets Ω_i . For instance, a minimizer exists among all *convex* equivolume partitions; see [14, Corollary 1].

1.4. Our result. When $N = m^d$ for some $m \in \mathbb{N}$, it is natural to ask whether classical jittered sampling yields such a minimiser among all equivolume stratified samples. Our main result gives a *negative* answer to this question, even if we require that all strata are convex.

In particular, we construct a convex, equivolume partition for each N and show that this partition improves the expected \mathcal{L}_2 -discrepancy of jittered sampling with the same number of points. It should be noted that we do not aim to maximise this improvement but rather to give

an elementary and intuitive exposition. This new partition can itself be improved as we will outline in Section 3.

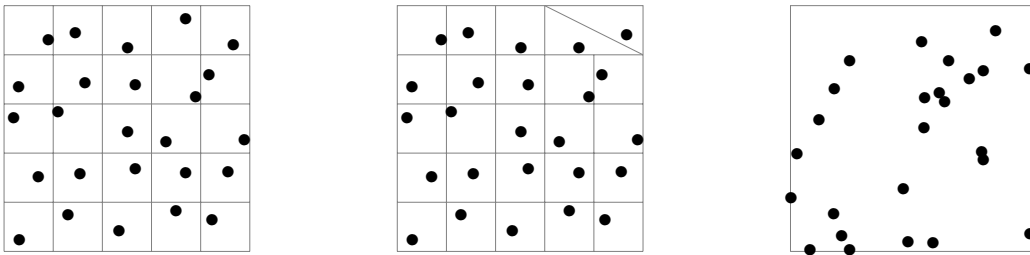


FIGURE 1. *Left:* Jittered sampling for $d = 2$ and $m = 5$. *Middle:* Improvement of jittered sampling. *Right:* $N = 25$ i.i.d. uniform random points.

Theorem 1. *Let $m \in \mathbb{N}$, $m \geq 2$ be given. The set of m^d random points in $[0, 1]^d$ obtained from jittered sampling does not have the minimal expected \mathcal{L}_2 -discrepancy among all stratified samples from convex equivolume partitions with the same number of points.*

1.5. Outline. In the next section we modify the partition of jittered sampling to obtain another partition, depicted for $d = 2$ in Figure 1 (middle), and show that this new partition gives rise to a better stratified point set. We further explore this construction in Section 3 and discuss the maximal gain that can be achieved with our method for $d = 2$. Various open questions and directions for future research are discussed in Section 4.

2. AN ELEMENTARY PROOF OF THEOREM 1

To begin with, we summarize a number of known results for stratified sampling. As we will need them for underlying sets other than the unit cube, we consider in the following stratified samples in a fixed compact convex set $K \subset \mathbb{R}^d$ with $|K| > 0$. Let Ω be a *partition* of K into $N \in \mathbb{N}$ sets $\Omega_1, \dots, \Omega_N$ of positive volume. This means that $\Omega_1 \cup \dots \cup \Omega_N = K$ and that two different partitioning sets have no interior points in common. Let \mathcal{P}_Ω be the corresponding stratified sample. We will call a partition *convex* if all the sets $\Omega_1, \dots, \Omega_N$ are convex.

Generalizing (1) the discrepancy function $d_{\mathcal{P}}(\mathbf{x})$ of a finite set of points $\mathcal{P} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subset K$ is given by

$$(2) \quad d_{\mathcal{P}}(\mathbf{x}) = \frac{\#(\mathcal{P} \cap (-\infty, \mathbf{x}])}{N} - \frac{|K \cap (-\infty, \mathbf{x}])|}{|K|},$$

$\mathbf{x} \in K$ and $(-\infty, \mathbf{x}] := \prod_{k=1}^d (-\infty, x_k]$. If \mathcal{P}_Ω is a stratified sample based on the partition Ω of K , the discrepancy function at \mathbf{x} is a random variable. It was shown in [14, Proposition 1] for $K = [0, 1]^d$ that $d_{\mathcal{P}_\Omega}(\mathbf{x})$ has mean 0 for all $\mathbf{x} \in K$ if and only if the partition is *equivolume*, that is, if $|\Omega_1| = \dots = |\Omega_N|$. The proof extends literally to all compact convex K with positive volume. From now on, all the partitions Ω we consider in this paper will be equivolume.

For $p \geq 1$, the *mean \mathcal{L}_p -discrepancy* is usually defined as $\mathbb{E}\mathcal{L}_p^p(\mathcal{P}) = \mathbb{E}\|d_{\mathcal{P}}\|_p^p$, where $\|\cdot\|_p$ is now the L_p -norm of functions on K with respect to the uniform distribution on K . It should correctly be called ‘mean p -th power \mathcal{L}_p -discrepancy’. Due to Tonelli’s theorem we see that

$$\mathbb{E}\mathcal{L}_p^p(\mathcal{P}_\Omega) = \frac{1}{|K|} \int_K \mathbb{E}[d_{\mathcal{P}_\Omega}(\mathbf{x})]^p d\mathbf{x}.$$

As Ω is equivolume, $d_{\mathcal{P}_\Omega}(\mathbf{x})$ is centered, so

$$(3) \quad \mathbb{E}\mathcal{L}_p^p(\mathcal{P}_\Omega) = \frac{1}{|K|} \int_K M_p \left(\frac{\#(\mathcal{P}_\Omega \cap (-\infty, \mathbf{x}])}{N} \right) d\mathbf{x},$$

where

$$M_p(Y) = \mathbb{E}|Y - \mathbb{E}Y|^p$$

is the p th central moment of a random variable Y . The variable $\#(\mathcal{P} \cap (-\infty, \mathbf{x}])$ is the sum of N independent (but not identically distributed) Bernoulli variables with success probabilities $q_1(\mathbf{x}), \dots, q_N(\mathbf{x})$, where

$$(4) \quad q_i(\mathbf{x}) = \frac{|\Omega_i \cap (-\infty, \mathbf{x}]|}{|\Omega_i|} = \frac{N}{|K|} |\Omega_i \cap (-\infty, \mathbf{x}]|.$$

The distribution of $\#(\mathcal{P} \cap (-\infty, \mathbf{x}])$ is usually called *Poisson-binomial distribution* with N trials and parameter vector $\mathbf{q}(\mathbf{x}) = (q_1(\mathbf{x}), \dots, q_N(\mathbf{x}))$. Its mean is

$$(5) \quad \sum_{i=1}^N q_i(\mathbf{x}) = \frac{N}{|K|} |K \cap (-\infty, \mathbf{x}]|.$$

Remark 2. *Similar arguments also apply to the generalized stratified sample $\tilde{\mathcal{P}}$ based on a partition $\Omega_1, \dots, \Omega_k$ of $[0, 1]^d$ with allocation proportional to size ($N_i = N|\Omega_i|$ for $i = 1, \dots, k$), as outlined in the introduction. The equivalent of (3) is now*

$$\mathbb{E}\mathcal{L}_p^p(\tilde{\mathcal{P}}) = \int_{[0,1]^d} M_p \left(\frac{\#(\tilde{\mathcal{P}} \cap [0, \mathbf{x}])}{N} \right) d\mathbf{x},$$

and specializes for $p = 2$ to

$$(6) \quad \mathbb{E}\mathcal{L}_2^2(\tilde{\mathcal{P}}) = \frac{1}{N^2} \sum_{i=1}^k \int_{[0,1]^d} \text{Var}(\#(\tilde{\mathcal{P}} \cap \Omega_i \cap [0, \mathbf{x}])) d\mathbf{x},$$

where the independence of sampling points in different strata was used. Equation (6) shows that the mean \mathcal{L}_2 -discrepancy is a sum of integrated variances originating from the k strata. In stratum Ω_i the N_i points are i.i.d. uniform, and [18, Proposition 4.3.1] implies that sub-stratifying this set into N_i equivolume subsets, and choosing one point uniformly in each of these sub-strata does not increase the variance. Applying this to all strata Ω_i with $N_i > 1$ yields a new stratified sample based on N equivolume strata with a mean \mathcal{L}_2 -discrepancy that is not larger than (6).

As stratified points are independent and M_2 is additive for independent variables, (3) allows us to state the following explicit formula when $p = 2$ which was stated in [14, Proposition 2] for $K = [0, 1]^d$.

Proposition 3. *If Ω is an equivolume partition of a compact convex set $K \subset \mathbb{R}^d$ with $|K| > 0$ then*

$$\mathbb{E}\mathcal{L}_2^2(\mathcal{P}_\Omega) = \frac{1}{N^2|K|} \sum_{i=1}^N \int_K q_i(\mathbf{x})(1 - q_i(\mathbf{x})) d\mathbf{x},$$

where $q_i(\mathbf{x})$ is given in (4).

The relation in Proposition 3 actually shows that only the integrals of q_i^2 are required, as (5) gives

$$(7) \quad \mathbb{E}\mathcal{L}_2^2(\mathcal{P}_\Omega) = \frac{1}{N|K|^2} \int_K |K \cap (-\infty, \mathbf{x}]| d\mathbf{x} - \frac{1}{N^2|K|} \sum_{i=1}^N \int_K q_i^2(\mathbf{x}) d\mathbf{x}.$$

The proof of Theorem 1 can be reduced to a comparison of two partitions in the plane depicted in Figure 2.

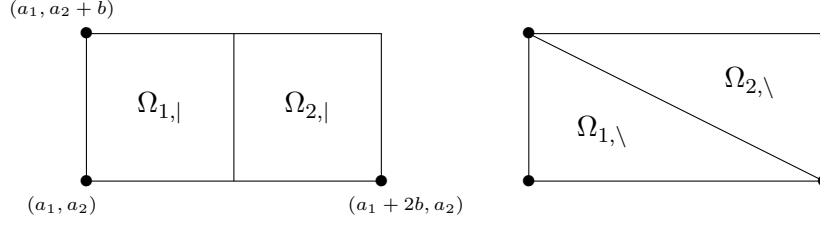


FIGURE 2. The two partitions of the rectangle studied in Lemma 4.

Lemma 4. Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ and $b > 0$ be given. Consider the rectangle

$$I = [a_1, a_1 + 2b] \times [a_2, a_2 + b]$$

and its two equivolume partitions $\Omega_{|} = (\Omega_{1,|}, \Omega_{2,|})$ into two closed squares and $\Omega_{\setminus} = (\Omega_{1,\setminus}, \Omega_{2,\setminus})$ into two closed triangles with

$$\Omega_{1,|} = [a_1, a_1 + b] \times [a_2, a_2 + b], \quad \Omega_{1,\setminus} = \text{conv}\{\mathbf{a}, \mathbf{a} + (2b, 0), \mathbf{a} + (0, b)\},$$

where conv denotes the convex hull. Then

$$\mathbb{E}\mathcal{L}_2^2(\mathcal{P}_{\Omega_{\setminus}}) < \mathbb{E}\mathcal{L}_2^2(\mathcal{P}_{\Omega_{|}}).$$

Proof. By Proposition 3, we have for any equivolume partition $\Omega = (\Omega_1, \Omega_2)$ of I that

$$(8) \quad \mathbb{E}\mathcal{L}_2^2(\mathcal{P}_{\Omega}) = \frac{1}{4} \sum_{i=1}^2 \int_I q_i(\mathbf{x})(1 - q_i(\mathbf{x})) \frac{d\mathbf{x}}{|I|},$$

where $|I| = 2b^2$ and

$$q_i(\mathbf{x}) = \frac{|\Omega_i \cap [0, \mathbf{x}]|}{|\Omega_i|} = b^{-2} |\Omega_i \cap [0, \mathbf{x}]|.$$

The integral in (8) is independent of \mathbf{a} , so we may assume $\mathbf{a} = \mathbf{0}$ from now on. Furthermore, the substitution $\mathbf{y} = b\mathbf{x}$ also shows independence of b , so we may assume $b = 1$. We have

$$(9) \quad q_{1,|}(x_1, x_2) = \min\{x_1, 1\}x_2, \quad q_{2,|}(x_1, x_2) = \max\{x_1 - 1, 0\}x_2,$$

(see Figure 3) and thus

$$B_{1,|} = \int_I q_{1,|}^2(\mathbf{x}) d\mathbf{x} = \int_{\Omega_{1,|}} x_1^2 x_2^2 d\mathbf{x} + \int_{\Omega_{2,|}} x_2^2 d\mathbf{x} = \frac{1}{9} + \frac{1}{3} = \frac{4}{9},$$

$$B_{2,|} = \int_I q_{2,|}^2(\mathbf{x}) d\mathbf{x} = \int_{\Omega_{2,|}} (x_1 - 1)^2 x_2^2 d\mathbf{x} = \frac{1}{9}.$$

As $|I| = 2$ we get from (7) that

$$8\mathbb{E}\mathcal{L}_2^2(\mathcal{P}_{\Omega_{|}}) = 1 - (B_{1,|} + B_{2,|}) = \frac{4}{9}.$$

Furthermore, we have (see Figure 3)

$$q_{1,\setminus}(\mathbf{x}) = \begin{cases} x_1 x_2, & \text{if } \mathbf{x} \in \Omega_{1,\setminus}, \\ x_1 x_2 - \frac{1}{4}(x_1 + 2x_2 - 2)^2, & \text{if } \mathbf{x} \in \Omega_{2,\setminus}, \end{cases}$$

and

$$q_{2,\setminus}(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} \in \Omega_{1,\setminus}, \\ \frac{1}{4}(x_1 + 2x_2 - 2)^2, & \text{if } \mathbf{x} \in \Omega_{2,\setminus}. \end{cases}$$

For simplicity we set $\Delta(\mathbf{x}) := \frac{1}{4}(x_1 + 2x_2 - 2)^2$. Using the fact that $\Omega_{1,\setminus} = \{(x_1, x_2) \in I : 0 \leq x_2 \leq 1 - x_1/2\}$, we obtain

$$B_{1,\setminus} = \int_I q_{1,\setminus}^2(\mathbf{x}) d\mathbf{x} = \int_{\Omega_{1,\setminus}} x_1^2 x_2^2 d\mathbf{x} + \int_{\Omega_{2,\setminus}} (x_1 x_2 - \Delta(\mathbf{x}))^2 d\mathbf{x} = \frac{2}{45} + \frac{22}{45} = \frac{24}{45},$$

$$B_{2,\setminus} = \int_I q_{2,\setminus}^2(\mathbf{x}) \, d\mathbf{x} = \int_{\Omega_{2,\setminus}} \Delta^2(\mathbf{x}) \, d\mathbf{x} = \frac{1}{15}.$$

Therefore,

$$8\mathbb{E}\mathcal{L}_2^2(\mathcal{P}_{\Omega_\setminus}) = 1 - (B_{1,\setminus} + B_{2,\setminus}) = \frac{18}{45} = \frac{36}{90} < \frac{4}{9}. \quad \square$$

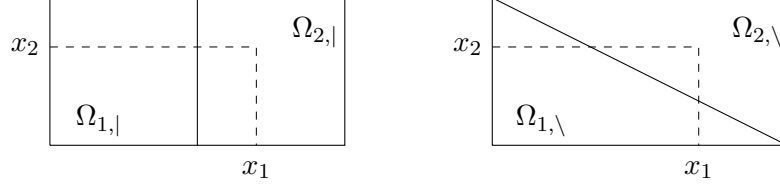


FIGURE 3. Illustration of the two different partitions of the rectangle $I = [0, 2] \times [0, 1]$.

This idea can be extended in a straightforward manner to dimensions $d \geq 2$ as the following lemma shows.

Lemma 5. *Let $\mathbf{a} \in \mathbb{R}^d$ and $b > 0$ be given. Consider the rectangle*

$$(10) \quad I = [a_1, a_1 + 2b] \times \prod_{i=2}^d [a_i, a_i + b]$$

and its two equivolume partitions $\Omega_{\setminus} = (\Omega_{1,\setminus}, \Omega_{2,\setminus})$ into two closed hypercubes and $\Omega_{|} = (\Omega_{1,|}, \Omega_{2,|})$ into two closed, regular triangular hyperprisms with

$$\Omega_{1,|} = \prod_{i=1}^d [a_i, a_i + b],$$

$$\Omega_{1,\setminus} = \text{conv}\{(a_1, a_2), (a_1, a_1 + 2b), (a_2, a_2 + b)\} \times \prod_{i=3}^d [a_i, a_i + b].$$

Then

$$(11) \quad \mathbb{E}\mathcal{L}_2^2(\mathcal{P}_{\Omega_{|}}) - \mathbb{E}\mathcal{L}_2^2(\mathcal{P}_{\Omega_{\setminus}}) = \frac{1}{20}3^{-d} > 0,$$

for all $d \geq 2$.

Proof. We start with the vertical case. For $i = 1, 2$ we have

$$q_{i,|}(\mathbf{x}) = q_{i,|}(x_1, x_2) \prod_{j=3}^d x_j,$$

where $q_{i,|}(x_1, x_2)$ denotes the two-dimensional function defined in (9). Hence,

$$\int_I q_{i,|}^2(\mathbf{x}) \, d\mathbf{x} = B_{i,|} \int_{[0,1]^{d-2}} \prod_{j=3}^d x_j^2 \, d(x_3, \dots, x_d) = \frac{1}{3^{d-2}} B_{i,|}$$

where $B_{i,|}$ was calculated in the proof of Lemma 4. As

$$\int_I |[0, \mathbf{x}]| \, d\mathbf{x} = \int_{[0,1]^{d-2}} \prod_{j=3}^d x_j \, d(x_3, \dots, x_d) = \frac{1}{2^{d-2}},$$

Equation (7) shows

$$8 \mathbb{E}\mathcal{L}_2^2(\mathcal{P}_{\Omega_{|}}) = \frac{1}{2^{d-2}} - \frac{1}{3^{d-2}}(B_{1,|} + B_{2,|}) = \frac{4}{2^d} - \frac{5}{3^d}.$$

Exactly the same line of arguments applies to the partition with hyperprisms, so

$$8 \mathbb{E} \mathcal{L}_2^2(\mathcal{P}_{\Omega_\setminus}) = \frac{1}{2^{d-2}} - \frac{1}{3^{d-2}}(B_{1,\setminus} + B_{2,\setminus}) = \frac{4}{2^d} - \frac{3}{5} \frac{1}{3^{d-2}}.$$

We see that

$$8 \mathbb{E} \mathcal{L}_2^2(\mathcal{P}_{\Omega_1}) - 8 \mathbb{E} \mathcal{L}_2^2(\mathcal{P}_{\Omega_\setminus}) = \frac{2}{5 \cdot 3^d}$$

for all $d \geq 2$ showing the assertion. \square

This lemma combined with Proposition 3 is the key ingredient in the proof of our main result.

Proof of Theorem 1. Let Ω be the partition of $[0, 1]^d$ into m^d congruent closed cubes, where we may assume that the cube containing the vector $(1, \dots, 1)$ is Ω_2 and the neighboring cube containing $(1 - m^{-1}, 1, 1, \dots, 1)$ is Ω_1 .

We compare Ω with the partition $\tilde{\Omega}$, where $\tilde{\Omega}_i = \Omega_i$ for $i = 3, \dots, m^d$. The remaining two partition sets must cover the interval I in (10) with $b = 1/m$, $a_1 = 1 - 2b$, $a_2 = \dots = a_d = 1 - b$, and we put $\tilde{\Omega}_1 = \Omega_{1,\setminus}$, $\tilde{\Omega}_2 = \Omega_{2,\setminus}$ with the notation of Lemma 5. Both partitions are equivolume and consist of convex sets.

From Proposition 3, we have

$$(12) \quad m^{2d} \mathbb{E} \mathcal{L}_2^2(\mathcal{P}_\Omega) = \sum_{i=1}^{m^d} \int_{[0,1]^d} q_i(\mathbf{x})(1 - q_i(\mathbf{x})) d\mathbf{x},$$

and a corresponding formula for $\tilde{\Omega}$, where $q_i(\mathbf{x})$ is replaced by $\tilde{q}_i(\mathbf{x})$. For all $\mathbf{x} \notin I$, the two integrands coincide. Furthermore, when $\mathbf{x} \in I$ only the contributions of the first two partitioning sets can differ, so

$$(13) \quad \begin{aligned} & m^{2d} [\mathbb{E} \mathcal{L}_2^2(\mathcal{P}_\Omega) - \mathbb{E} \mathcal{L}_2^2(\mathcal{P}_{\tilde{\Omega}})] \\ &= \sum_{i=1}^2 \int_I q_i(\mathbf{x})(1 - q_i(\mathbf{x})) d\mathbf{x} - \sum_{i=1}^2 \int_I \tilde{q}_i(\mathbf{x})(1 - \tilde{q}_i(\mathbf{x})) d\mathbf{x} \\ &= 4|I| \left[\mathbb{E} \mathcal{L}_2^2(\mathcal{P}_{\Omega_1}) - \mathbb{E} \mathcal{L}_2^2(\mathcal{P}_{\Omega_\setminus}) \right] > 0, \end{aligned}$$

where Proposition 3 and Lemma 5 were used in the last line. \square

Remark 6. Since $|I| = \frac{2}{m^d}$, relations (11) and (13) imply $\mathbb{E} \mathcal{L}_2^2(\mathcal{P}_\Omega) - \mathbb{E} \mathcal{L}_2^2(\mathcal{P}_{\tilde{\Omega}}) \in \mathcal{O}(m^{-3d})$, which corresponds to a gain of order $\mathcal{O}(N^{-3})$.

3. EXPLOITING THE LOCAL IMPROVEMENT

In this section, we discuss the potential and limits of the above idea to replace two neighboring partition squares in a jittered sample by a double-triangular partition, restricting considerations to the two-dimensional case ($d = 2$). We will show that this modification only improves the mean \mathcal{L}_2 -discrepancy if the squares involved are sufficiently close to the upper boundary of $[0, 1]^2$. The main goal of the present section is to show that the local ameliorations – if applied at multiple locations – improve jittered sampling with N points by a term of order $N^{-3/2}$ in contrast to the term of order N^{-3} as discussed in Remark 6.

Although explicit calculations are possible (and will be stated later omitting details), we prefer to give qualitative arguments that reveal the underlying structure of the problem more clearly. The key ingredient of this discussion is concerned with projections of random point sets onto the two axis directions. In the resulting one-dimensional setting, the following two observations describe the worst and the best random point pair distributions explicitly. The condition (14) corresponds to the ‘equivolume condition’ when the variables Y_1 and Y_2 stem from a stratification of $[0, 1]$ into two sets.

Lemma 7. Let Z_1 and Z_2 be i.i.d. uniform random variables in $[0, 1]$. Furthermore, let Y_1 and Y_2 be independent random variables in $[0, 1]$. If the cumulative distribution functions obey

$$(14) \quad \frac{F_{Y_1}(x) + F_{Y_2}(x)}{2} = x, \quad x \in [0, 1],$$

then

$$\mathbb{E}\mathcal{L}_2^2(\{Y_1, Y_2\}) \leq \mathbb{E}\mathcal{L}_2^2(\{Z_1, Z_2\}),$$

with equality if and only if (Y_1, Y_2) has the same distribution as (Z_1, Z_2) .

Proof. If (Y_1, Y_2) are random points derived from an equivolume partition of $[0, 1]$ into two sets, this statement is a consequence of the strong partition principle [14, Theorem 1]. In the more general situation considered here, the proof is literally the same if $q_i(x)$ in that proof is replaced by $F_{Y_i}(x)$. \square

Lemma 8. Let X_1 and X_2 be independent random variables, uniform in $[0, 1/2]$ and $[1/2, 1]$, respectively. Furthermore, let Y_1 and Y_2 be independent random variables in $[0, 1]$. If the cumulative distribution functions obey (14), then

$$\mathbb{E}\mathcal{L}_2^2(\{Y_1, Y_2\}) \geq \mathbb{E}\mathcal{L}_2^2(\{X_1, X_2\}),$$

with equality if and only if (Y_1, Y_2) has the same distribution as (X_1, X_2) or (X_2, X_1) .

Proof. If (Y_1, Y_2) are random points derived from an equivolume partition of $[0, 1]$ into two sets, this statement is coinciding with [14, Corollary 2]. In the more general situation considered here, the proof is literally the same if $q_i(x)$ in that proof is replaced by $F_{Y_i}(x)$. \square

These observations are now used to determine the improvement obtained by inserting a double-triangular partition that is not necessarily positioned at the upper right. Let Ω be the jittered partition of the unit square with $N = m^2$ sets, $m \geq 2$. We fix a vector $\mathbf{z} = (z_1, z_2) \in [0, 1]^2$ with $z_1 \in \{0, \dots, \frac{m-2}{m}\}$, $z_2 \in \{0, \dots, \frac{m-1}{m}\}$, and compare Ω with the partition $\tilde{\Omega}$, which only deviates from Ω in that two neighboring partitioning cubes at ‘position’ \mathbf{z} of Ω are replaced by triangles. Choosing an appropriate enumeration, we may put $\tilde{\Omega}_i = \Omega_i$ for $i = 3, \dots, m^d$. The remaining two partition sets must cover the interval I in (10) with $b = 1/m$, $a_1 = 1 - z_1 - 2b$, $a_2 = 1 - z_2 - b$, and we put $\tilde{\Omega}_1 = \Omega_{1, \setminus}$, $\tilde{\Omega}_2 = \Omega_{2, \setminus}$ with the notation of Lemma 5. This is illustrated in Figure 4 (Left). Both partitions are equivolume and consist of convex sets. For $z_1 = z_2 = 0$ the triangular partition is placed as in Figure 1 (Middle). Equation (12) in the proof of Theorem 1 for Ω and the corresponding relation for $\tilde{\Omega}$ are still valid with $d = 2$.

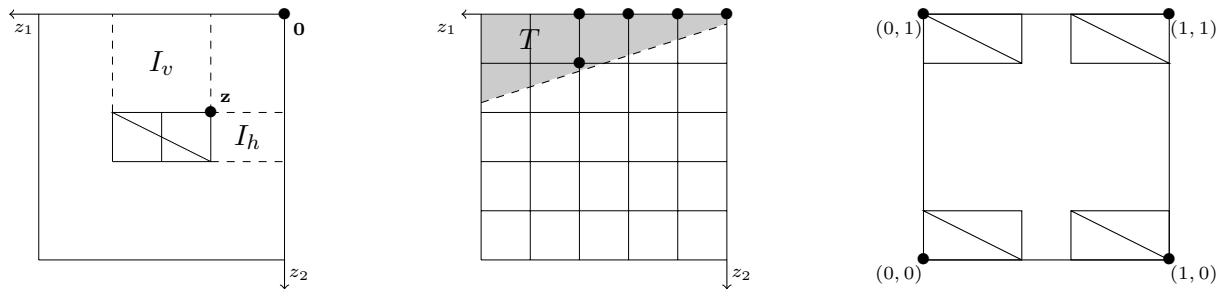


FIGURE 4. *Left:* Coordinates of a rectangle where the jittered partition is modified together with the sets I_h and I_v . *Middle:* Illustration of (18). The dots represent points $\mathbf{z} \in T$ at which a rectangle can be placed in order to improve the expected discrepancy. *Right:* Illustration of the different point sets in Table 2. Each partition contains exactly one of the four modified rectangles and is otherwise identical with the original jittered partition.

For all $i \neq 1, 2$ the corresponding summands associated to Ω and $\tilde{\Omega}$ coincide. For $i \in \{1, 2\}$, they coincide for $\mathbf{x} \notin I \cup I_h \cup I_v$ with the ‘horizontal’ and ‘vertical’ rectangular sets

$$I_h = [1 - z_1, 1] \times [1 - z_2 - \frac{1}{m}, 1 - z_2], \quad I_v = [1 - z_1 - \frac{2}{m}, 1 - z_1] \times [1 - z_2, 1],$$

as in this case, $[0, \mathbf{x}]$ either is disjoint with I or contains it completely. Hence,

$$(15) \quad m^4 [\mathbb{E}\mathcal{L}_2^2(\mathcal{P}_\Omega) - \mathbb{E}\mathcal{L}_2^2(\mathcal{P}_{\tilde{\Omega}})] = \sum_{J \in \{I, I_h, I_v\}} S(J),$$

with

$$S(J) = \sum_{i=1}^2 \int_J q_i(\mathbf{x})(1 - q_i(\mathbf{x}))d\mathbf{x} - \sum_{i=1}^2 \int_J \tilde{q}_i(\mathbf{x})(1 - \tilde{q}_i(\mathbf{x}))d\mathbf{x}.$$

This also explains why we worked with a modification of jittered sampling *in the upper right corner* in Section 2, as then only the positive term $S(I)$ contributes. As before, Proposition 3 and Lemma 5 imply

$$S(I) = 4|I| [\mathbb{E}\mathcal{L}_2^2(\mathcal{P}_{\Omega_I}) - \mathbb{E}\mathcal{L}_2^2(\mathcal{P}_{\tilde{\Omega}_I})] = c_0 m^{-2},$$

with $c_0 = \frac{2}{45}$. Putting $J = I_h$ and observing that $q_i(\mathbf{x})$ and $\tilde{q}_i(\mathbf{x})$ do not depend on the first entry of $\mathbf{x} \in I_h$, we obtain

$$(16) \quad S(I_h) = z_1 \left[\sum_{i=1}^2 \int_{1-z_2-\frac{1}{m}}^{1-z_2} q_i(1-z_1, x)(1 - q_i(1-z_1, x))dx - \sum_{i=1}^2 \int_{1-z_2-\frac{1}{m}}^{1-z_2} \tilde{q}_i(1-z_1, x)(1 - \tilde{q}_i(1-z_1, x))dx \right].$$

Using

$$q_1(1-z_1, x) = q_2(1-z_1, x) = m(x - (1-z_2 - \frac{1}{m}))$$

and a substitution, the first summand in the parenthesis coincides with

$$\frac{2}{m} \int_0^1 x(1-x)dx = \frac{4}{m} \int_0^1 \text{Var}\left(\frac{1}{2} \sum_{i=1}^2 1_{Z_i \leq x}\right)dx = 4\mathbb{E}\mathcal{L}_2^2(\{Z_1, Z_2\})m^{-1},$$

where Z_1, Z_2 are i.i.d. uniform in $[0, 1]$. This reflects the fact that the projections of the two jitter-points in I onto the y -axis are two independent uniform points. A similar argument shows that the second summand on the right side of (16) coincides with $4\mathbb{E}\mathcal{L}_2^2(\{Y_1, Y_2\})m^{-1}$ with two independent non-uniform variables Y_1 and Y_2 satisfying (14), so (16) becomes

$$S(I_h) = 4z_1 [\mathbb{E}\mathcal{L}_2^2(\{Z_1, Z_2\}) - \mathbb{E}\mathcal{L}_2^2(\{Y_1, Y_2\})]m^{-1} = c_1 z_1 m^{-1}$$

with $c_1 > 0$ due to Lemma 7. The third contribution $S(I_v)$ in (15) can be treated similarly, but now the projection of the two jitter-points in I onto the x -axis yields the best one-dimensional stratification and Lemma 8 implies

$$S(I_v) = -c_2 z_2 m^{-1}$$

with some $c_2 > 0$. Summarizing, (15) becomes

$$(17) \quad \mathbb{E}\mathcal{L}_2^2(\mathcal{P}_\Omega) - \mathbb{E}\mathcal{L}_2^2(\mathcal{P}_{\tilde{\Omega}}) = c_0 m^{-6} + c_1 z_1 m^{-5} - c_2 z_2 m^{-5},$$

where all occurring constants are positive.

This result shows that one can in general not improve jittered sampling by replacing two arbitrary horizontally neighboring jitter squares with triangles. More specifically, we get an improvement if and only if the numbers z_1, z_2 describing the upper right corner satisfy

$$(18) \quad z_2 < \frac{c_0}{c_2} m^{-1} + \frac{c_1}{c_2} z_1.$$

Geometrically, (18) is satisfied if and only if \mathbf{z} is an element of the set

$$T = \{\mathbf{z} \in [0, 1]^2 : z_2 < \frac{c_0}{mc_2} + \frac{c_1}{c_2} z_1\};$$

see Figure 4 (Middle) for the case $m = 5$.

To illustrate our results, it is necessary to explicitly know the involved constants. We obtain

$$c_0 = \frac{2}{45}, \quad c_1 = \frac{1}{15}, \quad c_2 = \frac{1}{5},$$

where c_0 was determined in Lemma 5, and c_1, c_2 were derived by elementary calculations. In accordance with the qualitative arguments above, c_0, c_1, c_2 and hence the slope of the bounding line in the definition of T are independent of m .

In Table 2 we present numerical results to further illustrate (18). We compare the empirical mean of the \mathcal{L}_2 -discrepancy of 1000 individual point sets for each N . In particular, we compare instances of jittered sampling \mathcal{P}_{jit} to modified point sets in which we replaced exactly two adjacent boxes with triangles. To be more precise, we moved the original modified rectangle once into each of the 4 corners of the unit square. We denote these different sets simply by \mathcal{P}_{xy} indicating which of the four vertices (x, y) of the unit square is a vertex of the modified rectangle. We see that moving the rectangle to the upper left corner increases the gain; i.e. considering $\mathbf{z} = (1 - 2/m, 0)$ instead of the original $\mathbf{z} = (0, 0)$. Furthermore, moving the rectangle to the lower right corner worsens the result; i.e. considering $\mathbf{z} = (0, 1 - 1/m)$. Of course, this is both in line with our theoretical analysis.

For example, for \mathcal{P}_{01} we expect a gain of size $c_1 m^{-5}$ according to (17). As $c_1 = 1/15$ and $N = m^2 = 10^2$ the expected gain is $\mathcal{O}(10^{-6})$ which is in correspondence with the empirical gain of $\approx 0.857 \cdot 10^{-6}$. Similarly, for \mathcal{P}_{11} , we expect a gain of size $c_0 m^{-6}$, i.e. of order $\mathcal{O}(10^{-7})$, which corresponds to the empirical difference $\approx 0.85 \cdot 10^{-7}$.

$\mathbb{E}\mathcal{L}_2(\cdot)$	\mathcal{P}_{jit}	\mathcal{P}_{11}	\mathcal{P}_{01}	\mathcal{P}_{10}	\mathcal{P}_{00}
$N = 7^2$	0.000476834	0.00047629	0.000473918	0.000486402	0.000481983
$N = 10^2$	0.00016377	0.000163685	0.000162913	0.000165225	0.000165369
$N = 14^2$	0.0000599499	0.0000599455	0.0000598861	0.0000601582	0.0000602246

TABLE 1. Comparison of expected \mathcal{L}_2 -discrepancy of classical jittered sampling with our different modifications. The values in the table give the empirical mean of the \mathcal{L}_2 -discrepancy of 1000 individual samples for each N . We calculated the discrepancy of individual samples with Warnock's formula.

Now, modifying the jittered partition successively at several positions leads to an accumulated improvement. In fact, note that the proof of (17) still works even if an arbitrary partition Ω instead of the jitter partition is used, as long as Ω coincides with the jittered partitions within I . Hence, replacing every second rectangle with a position (z_1, z_2) obeying (18) in a jittered partition with a double-triangular partition, the overall gain is

$$g = \sum_{z_1 \in \frac{1}{m} \{0, 2, \dots, 2\lfloor \frac{m-2}{2} \rfloor\}} \sum_{z_2 \in \frac{1}{m} \{0, \dots, m-1\}} \mathbf{1}_{(z_1, z_2) \in T} [c_0 m^{-6} + c_1 z_1 m^{-5} - c_2 z_2 m^{-5}].$$

As $0 \leq z_1, z_2 \leq 1$, we have $[c_0 m^{-6} + c_1 z_1 m^{-5} - c_2 z_2 m^{-5}] \leq c_3 m^{-5}$ for all m (where we have put $c_3 = c_0 + c_1$), so

$$g \leq c_3 m^{-5} (2\lfloor \frac{m-2}{2} \rfloor + 1)m \leq c_3 m^{-3}.$$

To show that this is the correct rate of convergence note that the rectangle

$$R = \{(z_1, z_2) \in [0, 1]^2 : z_1 \geq \frac{1}{2}, z_2 \leq \frac{c_1}{4c_2}\}$$

is contained in T for all m , and that $\mathbf{z} \in R$ implies $[c_0 m^{-6} + c_1 z_1 m^{-5} - c_2 z_2 m^{-5}] \geq \frac{c_1}{4} m^{-5}$, so

$$g \geq \frac{c_1}{4} m^{-5} \sum_{z_1 \in \frac{1}{m} \{0, 2, \dots, 2\lfloor \frac{m-2}{2} \rfloor\}} \sum_{z_2 \in \frac{1}{m} \{0, \dots, m-1\}} \mathbf{1}_{(z_1, z_2) \in R} \geq c_4 m^{-3}$$

for some constant $c_4 > 0$. Putting things together, we see that the gain g behaves like $m^{-3} = N^{-3/2}$ as $m \rightarrow \infty$. Note that symmetry considerations with respect to the main diagonal would allow to modify almost double as many rectangles, but this will of course not change the asymptotic order of the gain. Moreover, for small m , i.e. $m < 150$, the gain is not as big due to effects of the absolute constants. Concluding, exploiting the local modifications suggested in this paper to their limit yields an improvement of jittered sampling with N points in the order of $N^{-3/2}$.

Table 2 shows the result of a second numerical experiment for $N = 10^2$. We generated 10000 instances of jittered samples, and compare them to different modified point sets based on a particular jittered sampling set; i.e. we compare \mathcal{P}_{jit} to \mathcal{P}_{01} as defined above as well as to $\mathcal{P}_{z_2=0}$ which is a partition in which we modify all pairs of rectangles in the top row and to \mathcal{P}_{all} in which all eligible rectangles are replaced by triangles. Note that in the present case (18) is satisfied for every \mathbf{z} in

$$\left\{ \mathbf{z} \in [0, 1]^2 : z_2 < \frac{1}{45} + \frac{1}{3}z_1 \right\}.$$

$\mathbb{E}\mathcal{L}_2(\cdot)$	\mathcal{P}_{jit}	\mathcal{P}_{01}	$\mathcal{P}_{z_2=0}$	\mathcal{P}_{all}
$N = 10^2$	0.00016366	0.000163152	0.000162172	0.00016101

TABLE 2. Comparison of expected \mathcal{L}_2 -discrepancy of classical jittered sampling with point sets stemming from different modifications. The values in the table give the empirical mean of the \mathcal{L}_2 -discrepancy of 10000 individual samples. We calculated the discrepancy of individual samples with Warnock’s formula.

4. CONCLUDING REMARKS AND OPEN PROBLEMS

In this final section we collect various open problems for future research.

- (1) **\mathcal{L}_p -discrepancy.** It is of course natural to ask whether our result also holds for the expected \mathcal{L}_p -discrepancy. The advantage of the case $p = 2$ is that the contributions of the individual sampling points to the mean discrepancy behave additively due to Proposition 3. This is not the case for $p \neq 2$, and already the case $p = 4$, for which an analogue [14, Proposition 2] of Proposition 3 is known, does not have this simple structure. The proof of an extension of Theorem 1 to $p \neq 2$ appears therefore to require a substantially new ingredient. We recall that the main tool utilised in the proof of the Strong Partition Principle [14, Theorem 1], is an inequality due to Hoeffding. However, it appears that generalisations of Hoeffding’s result based on the theory of majorisations [15] do not suffice to extend our Theorem 1.
- (2) **Star discrepancy.** Naturally, we are not only interested in the expected \mathcal{L}_p -discrepancy, but also in a related result for the star discrepancy. However, at the moment even proving a Strong Partition Principle for the star discrepancy seems out of reach.
- (3) **Generalized \mathcal{L}_p -discrepancy.** We recall that the main idea of Hickernell’s generalization is to not only consider the ordinary \mathcal{L}_p -discrepancy of a point set, but also the discrepancies of all projections to lower dimensional faces of the unit cube. In the two-dimensional case this means that we also need to include the discrepancy of the projections of the point set to the x - and the y -axis. Direct calculations seem to indicate that our construction does not improve jittered sampling with respect to Hickernell’s notion. We leave it as an open question whether there is another construction that can actually improve the discrepancy of jittered sampling with respect to Hickernell’s notion.
- (4) **Asymptotic gain.** Finally, our result is of theoretical interest as it shows the existence of stratified samples which improve classical jittered sampling. But our gain concerns only lower order terms of the expected discrepancy. It would be very interesting to

know whether jittered sampling has the optimal order of magnitude or whether there are stratified point sets with $N = m^d$ points with an asymptotic gain over classical jittered sampling.

REFERENCES

- [1] C. Aistleitner, Covering numbers, dyadic chaining and discrepancy. *J. Complexity* 27 (2011), 531–540.
- [2] W.W.L. Chen, M.M.Skrikanov, Explicit constructions in the classical mean squares problem in irregularity of point distribution. *J. Reine Angew. Math.*, 545 (2002), 67–95.
- [3] H. Davenport, Note on irregularities of distribution. *Mathematika* 3 (1956), 131–135.
- [4] B. Doerr, A lower bound for the discrepancy of a random point set. *J. Complexity* 30 (2014), 16–20.
- [5] B. Doerr, A sharp discrepancy bound for jittered sampling, arXiv:2103.15712.
- [6] J. Dick, F. Pillichshammer, *Digital Nets and Sequences*, Cambridge Univ. Press, Cambridge, 2010.
- [7] J. Dick, F. Pillichshammer, Explicit constructions of point sets and sequences with low discrepancy, Kritzer, Peter (ed.) et al., *Uniform distribution and quasi-Monte Carlo methods. Discrepancy, integration and applications. Radon Series on Computational and Applied Mathematics* 15, 63–86 (2014).
- [8] J. Dick and F. Pillichshammer, Optimal \mathcal{L}_2 -discrepancy bounds for higher order digital sequences over the finite field \mathbb{F}_2 , *Acta Arith.* 162, No. 1 (2014), 65–99.
- [9] M. Gnewuch, H. Pasing, C. Weiss, A generalized Faulhaber inequality, improved bracketing covers and applications to discrepancy, arXiv:2010.11479.
- [10] S. Heinrich, E. Novak, G. Wasilkowski and H. Wozniakowski, The inverse of the star-discrepancy depends linearly on the dimension. *Acta Arith.* 96 (2001), no. 3, 279–302.
- [11] F.J. Hickernell, The mean square discrepancy of randomized nets, *ACM Trans. Modeling Comput. Simul.* 6, no 4 (1996), 274–296.
- [12] F.J. Hickernell, A generalized discrepancy and quadrature error bound, *Math. Comp.* 67, no 221 (1998), 299–322.
- [13] W. Hoeffding, On the distribution of the number of successes in independent trials, *Ann. Math. Statist.* 27 (1956), 713–721.
- [14] M. Kiderlen, F. Pausinger, Discrepancy of stratified samples from partitions of the unit cube, *Monatsh Math* (2021). <https://doi.org/10.1007/s00605-021-01538-4>
- [15] A. W. Marshall, I. Olkin, B. C. Arnold. *Inequalities: theory of majorization and its applications*. Springer Series in Statistics. Springer, New York, second edition, 2011.
- [16] J. Matoušek, On the L_2 -Discrepancy of Anchored Boxes, *J. Complexity* 14 (1998), 527–556.
- [17] K. F. Roth, On irregularities of distribution. *Mathematika* 1 (1954), 73–79.
- [18] R. Y. Rubinstein, *Simulation and the Monte Carlo Method*, Wiley, New York, 1981.
- [19] F. Pausinger, S. Steinerberger, On the discrepancy of jittered sampling, *J. Complexity* 33 (2016), 199–216.
- [20] F. Pausinger, M. Rachh, S. Steinerberger, Optimal jittered sampling for two points in the unit square, *Statist. Probab. Lett.* 132 (2018), 55–61.
- [21] S. K. Thompson, *Sampling*, 3rd edn, Wiley, New York, 2012.

AARHUS UNIVERSITY, AARHUS, DENMARK
Email address: `kiderlen@math.au.dk`

QUEEN'S UNIVERSITY BELFAST, BELFAST, UNITED KINGDOM.
Email address: `f.pausinger@qub.ac.uk`