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THE “FUNDAMENTAL THEOREM” FOR THE HIGHER
ALGEBRAIC K -THEORY OF STRONGLY \mathbb{Z} -GRADED RINGS

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ABSTRACT. The “fundamental theorem” for algebraic K -theory expresses the K -groups of a LAURENT polynomial ring $L[t, t^{-1}]$ as a direct sum of two copies of the K -groups of L (with a degree shift in one copy), and certain groups NK_q^\pm . It is shown here that a modified version of this result generalises to strongly \mathbb{Z} -graded rings; rather than the algebraic K -groups of L , the splitting involves groups related to the shift actions on the category of L -modules coming from the graded structure. (These actions are trivial in the classical case). The analogues of the groups NK_q^\pm are identified with the reduced K -theory of homotopy nilpotent twisted endomorphisms, and appropriate versions of MAYER-VIETORIS and localisation sequences are established.

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1 INTRODUCTION

1.1 THE FUNDAMENTAL THEOREM FOR THE ALGEBRAIC K -THEORY OF RINGS

The “fundamental theorem”, also known as the BASS-HELLER-SWAN formula, expresses the algebraic K -groups of a LAURENT polynomial ring $R_0[t, t^{-1}]$ as a direct sum

$$K_q(R_0[t, t^{-1}]) \cong K_q R_0 \oplus K_{q-1}(R_0) \oplus NK_q^+(R_0) \oplus NK_q^-(R_0). \quad (1.1)$$

It will be shown that this result largely depends on the structure of $R_0[t, t^{-1}]$ as a \mathbb{Z} -graded ring, and that a similar splitting can be established in much greater generality.

By way of analogy we think of any \mathbb{Z} -graded ring $R = \bigoplus_{k \in \mathbb{Z}} R_k$ as a substitute for a LAURENT polynomial ring, and consider the subrings $R_{\leq 0} = \bigoplus_{k \leq 0} R_k$ and $R_{\geq 0} = \bigoplus_{k \geq 0} R_k$ as substitutes for the polynomial rings $R_0[t^{-1}]$ and $R_0[t]$. If the ring R is *strongly graded* in the sense that $R_k R_\ell = R_{k+\ell}$ for all integers k and ℓ , the analogy is appropriate, and a surprising number of results known for (LAURENT) polynomial rings can be proved to hold for the more general setting, *e.g.*, the characterisation of finite domination *via* NOVIKOV homology [HS17], the splitting for the algebraic K -theory of the projective line [HM20], and the connection between finite domination and non-commutative localisation [Hüt]. This is not idle play: the class of strongly \mathbb{Z} -graded rings is much bigger than the class of LAURENT polynomial rings. One specific example of a strongly \mathbb{Z} -graded rings is $K[A, B, C, D]/(AB + CD = 1)$ where K is a field, $\deg(A) = \deg(C) = 1$ and $\deg(B) = \deg(D) = -1$. (The only units in this ring are the non-zero elements of K in degree 0.) A natural infinite family of examples is formed by the LEAVITT path algebras associated with row-finite directed graphs without sink satisfying a certain condition Y [NÖ20, Theorem 1.3]; this includes all LEAVITT path algebras associated with finite directed graphs without sink.

Back to the fundamental theorem, the graded structure of $R = \bigoplus_{k \in \mathbb{Z}} R_k$ induces “shift functors” $s_k: M \mapsto M \otimes_{R_0} R_k$ on categories of R_0 -modules; in case of a *strongly* graded ring, these functors preserve projectivity and satisfy the relation $s_k \circ s_\ell \cong s_{k+\ell}$. The key point is to measure how non-trivial the resulting \mathbb{Z} -action on algebraic K -groups is, which is done by considering the kernel A_q and cokernel B_q of the *shift difference map*

$$\mathrm{sd}_* = \mathrm{id} - s_{-1}: K_q R_0 \rightarrow K_q R_0 .$$

It turns out that the groups A_{q-1} and B_q play the role of $K_{q-1} R_0$ and $K_q R_0$ in the classical formulation (1.1) of the fundamental theorem. More precisely, $K_q R$ will be shown to be an extension of A_{q-1} by a direct sum of B_q with two appropriately defined nil terms (Theorem 2.7); the nil terms are identified as the reduced algebraic K -theory of categories of homotopy nilpotent twisted endomorphisms (Theorem 12.1).

1.2 RELATION WITH OTHER WORK

The “classical” fundamental theorem for the higher algebraic K -theory of rings has been proved by QUILLEN and GRAYSON [Gra76]. It has been extended from the K -theory of rings to the K -theory of schemes by THOMASON and TROBAUGH [TT90, Theorem 6.6], and to the algebraic K -theory of spaces [HKV⁺01] by KLEIN, VOGELL, WALDHAUSEN, WILLIAMS and the author. A version for skew LAURENT polynomial rings has been discussed by YAO [Yao95].

More recently, the result has been established by LÜCK and STEIMLE for skew LAURENT extensions of additive categories [LS16], and by FONTES and OGLE [FO18] in the context of \mathbb{S} -algebras. Most of the recent accounts follow the pattern laid out in [HKV⁺01], which is in turn loosely based on [Gra76].

The present paper goes beyond previous generalisations inasmuch as it moves the focus away from the very special case of (skew) LAURENT polynomial rings to the essential information contained in the *graded* structure of the ring extension $R_0 \subset R$.

1.3 STRUCTURE OF THE PAPER

The paper is organised in a way that avoids forward references in proofs. §2 provides an overview of notation and main results. §3 discusses induced chain complexes. §4 introduces finite domination, an important finiteness condition for chain complexes. In §§5–6 the “projective line” and its K -theory are reviewed. §7 is devoted to an analysis of the “nil terms” ${}^R\mathrm{NK}_q^\pm$ in the fundamental theorem. §8 contains the proof that the “fundamental square” of the projective line (roughly speaking relating the K -groups of $R_{\leq 0}$, R and $R_{\geq 0}$ with those of the projective line) is homotopy cartesian, which leads to a proof of exactness of the MAYER-VIETORIS sequence in §9 and a proof of the fundamental theorem in §10. In §11 we establish a “localisation sequence” for algebraic K -theory, and finish the paper by identifying the nil terms as the reduced K -theory of categories of homotopy nilpotent twisted endomorphisms in §12.

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2 NOTATION AND MAIN RESULTS

NOTATION AND CONVENTIONS

The word “ring” will always refer to an associative unital ring, homomorphisms of rings respect the unit, and “modules” are understood to be unital and right, unless otherwise specified. Let $R = \bigoplus_{k \in \mathbb{Z}} R_k$ be a \mathbb{Z} -graded unital ring, so that $R_k R_\ell \subseteq R_{k+\ell}$ for all $k, \ell \in \mathbb{Z}$. (Here $R_k R_\ell$ is the set of finite sums of products xy with $x \in R_k$ and $y \in R_\ell$.) The component R_0 is a subring of R with the same unit element [Dad80, Proposition 1.4]. Two further subrings of note are

$$R_{\leq 0} = \bigoplus_{k \leq 0} R_k \quad \text{and} \quad R_{\geq 0} = \bigoplus_{k \geq 0} R_k .$$

There are ring inclusions

$$R_{\leq 0} \xleftarrow{i^-} R_0 \xrightarrow{i^+} R_{\geq 0} \quad \text{and} \quad R_{\leq 0} \xrightarrow{j^-} R \xleftarrow{j^+} R_{\geq 0} ,$$

and ring homomorphisms given by projection

$$R_{\leq 0} \xrightarrow{p^-} R_0 \xleftarrow{p^+} R_{\geq 0} ;$$

they satisfy the relations

$$j^- \circ i^- = j^+ \circ i^+ , \quad p^- \circ i^- = \text{id}_{R_0} \quad \text{and} \quad p^+ \circ i^+ = \text{id}_{R_0} .$$

These various maps are used to define induction functors for modules,

$$i_*^- : P \mapsto P \otimes_{R_0} R_{\leq 0}$$

and its relatives i_*^+ , j_*^\mp and p_*^\mp ; the resulting maps on algebraic K -groups are denoted by the same symbols as the functors.

STRONGLY \mathbb{Z} -GRADED RINGS

The \mathbb{Z} -graded ring R is called *strongly graded* if $R_k R_\ell = R_{k+\ell}$ for all $k, \ell \in \mathbb{Z}$, or equivalently, if $R_1 R_{-1} = R_0 = R_{-1} R_1$. This ensures that the ring multiplication yields R_0 -bimodule isomorphisms $R_k \otimes_{R_0} R_{-k} \cong R_0$ and, more generally, $R_k \otimes_{R_0} R_\ell \cong R_{k+\ell}$; consequently, each R_k is an invertible R_0 -bimodule, and hence a finitely generated projective (left and right) R_0 -module [HS17, Proposition 1.6]. Similarly, one verifies [HS17, Lemma 1.9] that

$$R_{\leq q} = \bigoplus_{k \leq q} R_k \quad \text{and} \quad R_{\geq -p} = \bigoplus_{k \geq -p} R_k$$

are finitely generated projective (left and right) modules over $R_{\leq 0}$ and $R_{\geq 0}$, respectively, for all $p, q \in \mathbb{Z}$.

THE GROUPS ${}^R\text{NK}_q^\pm(R_0)$

Let R be a \mathbb{Z} -graded ring.

DEFINITION 2.1. We define the groups ${}^R\text{NK}_q^\mp(R_0)$ by

$${}^R\text{NK}_q^-(R_0) = \text{coker} (i_*^- : K_q(R_0) \rightarrow K_q(R_{\leq 0}))$$

and

$${}^R\text{NK}_q^+(R_0) = \text{coker} (i_*^+ : K_q(R_0) \rightarrow K_q(R_{\geq 0})) .$$

Thus we have a split short exact sequence

$$0 \longrightarrow K_q(R_0) \begin{array}{c} \xrightarrow{i_*^+} \\ \xleftarrow{p_*^+} \end{array} K_q(R_{\geq 0}) \longrightarrow {}^R\text{NK}_q^+(R_0) \longrightarrow 0$$

resulting in isomorphisms

$$\begin{aligned} K_q(R_{\geq 0}) &\cong K_q(R_0) \oplus {}^R\text{NK}_q^+(R_0) \\ \text{and} \quad {}^R\text{NK}_q^+ &\cong \ker (p_*^+ : K_q(R_{\geq 0}) \rightarrow K_q(R_0)) . \end{aligned} \tag{2.2}$$

Of course these remarks hold for ${}^R\text{NK}_q^-(R_0)$ and $K_q(R_{\leq 0})$ *mutatis mutandis*.

SHIFTS AND SHIFT DIFFERENCES

The \mathbb{Z} -graded over-ring R of the ring R_0 defines endofunctors of the category of R_0 -modules, the *shift functors*, by

$$s_j : P \mapsto P \otimes_{R_0} R_j \quad (j \in \mathbb{Z}) .$$

If R is *strongly* graded then R_j is an invertible R_0 -bimodule whence s_j defines an auto-equivalence of the category of finitely generated projective R_0 -modules, with resulting isomorphisms on algebraic K -groups

$$s_j : K_q R_0 \rightarrow K_q R_0 .$$

DEFINITION 2.3. Suppose that R is a strongly \mathbb{Z} -graded ring. The q th *shift difference map of R_0 relative to R* is defined as

$$sd_* = \text{id} - s_{-1} : K_q R_0 \rightarrow K_q R_0 . \tag{2.4}$$

The kernel of $sd_* : K_q R_0 \rightarrow K_q R_0$ is called the q th *shift kernel of R_0 relative to R* , and is denoted by the symbol ${}^R_{\text{ker}} K_q R_0$. The cokernel of this map is called the q th *shift cokernel of R_0 relative to R* , and is denoted by the symbol ${}^R_{\text{coker}} K_q R_0$.

It might be worth pointing out that we could just as well use the shift functor s_1 in place of s_{-1} ; in view of the R_0 -bimodule isomorphism $R_{-1} \otimes_{R_0} R_1 \cong R_0$, this would lead to the same description of the shift kernel, and to an isomorphic description of the shift cokernel. In any case, the exact sequence

$$0 \rightarrow {}^R_{\text{ker}} K_q R_0 \rightarrow K_q R_0 \xrightarrow{sd_*} K_q R_0 \rightarrow {}^R_{\text{coker}} K_q R_0 \rightarrow 0$$

determines the q th shift kernel and q th shift cokernel up to canonical isomorphism.

REMARK 2.5. If there is an R_0 -bimodule isomorphism $R_{-1} \cong R_0$ then sd_* is the zero map whence ${}^R_{\text{ker}} K_q R_0 = {}^R_{\text{coker}} K_q R_0 = K_q R_0$. This happens, for example, in case R is a LAURENT polynomial ring $R = R_0[t, t^{-1}]$ with a central indeterminate t .

THE FIRST NEGATIVE K -GROUP

Let $R = \bigoplus_{k \in \mathbb{Z}} R_k$ be a \mathbb{Z} -graded ring.

DEFINITION 2.6. We define the first negative K -group of R_0 relative to R as

$${}^R K_{-1}(R_0) = \text{coker}(j_*^- + j_*^+) ,$$

where the ring inclusions j^- and j^+ induce the map

$$j_*^- + j_*^+ : K_0(R_{\leq 0}) \oplus K_0(R_{\geq 0}) \rightarrow K_0(R) .$$

In the case of a LAURENT polynomial ring $R_0 \subseteq R_0[t, t^{-1}]$ this recovers the group $K_{-1}(R_0) = {}^{R_0[t, t^{-1}]} K_{-1}(R_0)$ as defined by BASS.

THE FUNDAMENTAL THEOREM

The fundamental theorem expresses $K_q(R_0[t, t^{-1}])$ as a direct sum of groups $K_q(R_0)$, $K_{q-1}(R_0)$, $NK_q^+(R_0)$ and $NK_q^-(R_0)$. In the more general context of $R_0 \subseteq R$ with R strongly \mathbb{Z} -graded, the fundamental result reads as follows:

THEOREM 2.7 (The fundamental theorem for the algebraic K -theory of strongly \mathbb{Z} -graded rings). *Let R be a strongly \mathbb{Z} -graded ring. There are short exact sequences of ABELIAN groups*

$$\begin{aligned}
 0 \rightarrow {}^R\text{NK}_q^-(R_0) \oplus {}_{\text{coker}}^R K_q(R_0) \oplus {}^R\text{NK}_q^+(R_0) \\
 \rightarrow K_q(R) \rightarrow {}_{\text{ker}}^R K_{q-1}(R_0) \rightarrow 0 \quad (\text{for } q > 0)
 \end{aligned}
 \tag{2.7a}$$

and

$$\begin{aligned}
 0 \rightarrow {}^R\text{NK}_0^-(R_0) \oplus {}_{\text{coker}}^R K_0(R_0) \oplus {}^R\text{NK}_0^+(R_0) \\
 \rightarrow K_0(R) \rightarrow {}^R K_{-1}(R_0) \rightarrow 0 .
 \end{aligned}
 \tag{2.7b}$$

THE MAYER-VIETORIS SEQUENCE

Let R be a strongly \mathbb{Z} -graded ring. By analogy with algebraic geometry, one can consider a “projective line” which is obtained by “gluing $\text{spec } R_{\leq 0}$ and $\text{spec } R_{\geq 0}$ along their intersection $\text{spec } R$ ”; more precisely, one can define the analogue of the category of quasi-coherent sheaves on the projective line as the category of certain diagrams of modules. The projective line \mathbb{P}^1 in this sense was introduced and its K -theory computed by MONTGOMERY and the author [HM20]; the relevant parts of the theory will be surveyed in §5 below.

THEOREM 2.8 (MAYER-VIETORIS sequence). *Let R be a strongly \mathbb{Z} -graded ring. There is a long exact sequence of algebraic K -groups*

$$\begin{array}{ccccccc}
 & & & & \dots\dots & \xrightarrow{\gamma} & K_{q+1}(R) \\
 \xrightarrow{\delta} & K_q(\mathbb{P}^1) & \xrightarrow{\beta} & K_q(R_{\leq 0}) \oplus K_q(R_{\geq 0}) & & \xrightarrow{\gamma} & K_q(R) \\
 \xrightarrow{\delta} & K_{q-1}(\mathbb{P}^1) & \xrightarrow{\beta} & K_{q-1}(R_{\leq 0}) \oplus K_{q-1}(R_{\geq 0}) & & \xrightarrow{\gamma} & K_{q-1}(R) \\
 & & & \vdots & & & \\
 \xrightarrow{\delta} & K_0(\mathbb{P}^1) & \xrightarrow{\beta} & K_0(R_{\leq 0}) \oplus K_0(R_{\geq 0}) & & \xrightarrow{\gamma} & K_0(R) \\
 & & & \rightarrow & & \xrightarrow{\gamma} & {}^R K_{-1}(R_0) \rightarrow 0 .
 \end{array}$$

HOMOTOPY NILPOTENT TWISTED ENDOMORPHISMS AND THE LOCALISATION SEQUENCE

Let R be a strongly \mathbb{Z} -graded ring. We define ${}^R\mathbf{Nil}^+(R_0)$ to be the category of pairs (Z, ζ) , with Z an R_0 -finitely dominated bounded chain complexes of

projective R_0 -modules, and $\zeta: Z \otimes_{R_0} R_1 \rightarrow Z \otimes_{R_0} R_0$ a homotopy nilpotent twisted endomorphism, cf. §7. Let ${}^R\mathbf{Nil}_q^+(R_0)$ denote the q th reduced algebraic K -group of ${}^R\mathbf{Nil}^+(R_0)$ (with respect to the quasi-isomorphisms as weak equivalences and the levelwise split monomorphisms as cofibrations); that is, ${}^R\mathbf{Nil}_q^+(R_0)$ is the kernel of the homomorphism

$$K_q({}^R\mathbf{Nil}^+(R_0)) \rightarrow K_q(R_0), \quad (Z, \zeta) \mapsto Z.$$

THEOREM 2.9. *Let R be a strongly \mathbb{Z} -graded ring. There are isomorphisms ${}^R\mathbf{NK}_{q+1}^-(R_0) \cong {}^R\mathbf{Nil}_q^+(R_0)$, and the groups $K_q({}^R\mathbf{Nil}^+(R_0))$ fit into a long exact sequence*

$$\begin{aligned} \dots \rightarrow K_{q+1}R \rightarrow K_q({}^R\mathbf{Nil}^+(R_0)) \xrightarrow{\phi} K_qR_{\geq 0} \rightarrow K_qR \\ \rightarrow \dots \rightarrow K_0({}^R\mathbf{Nil}^+(R_0)) \xrightarrow{\phi} K_0R_{\geq 0} \rightarrow K_0R \end{aligned}$$

with ϕ induced by the functor sending (Z, ζ) to the $R_{\geq 0}$ -module complex Z with $R_{\geq 0}$ acting through ζ .

There is a symmetric version involving a category ${}^R\mathbf{Nil}^-(R_0)$, using R_{-1} in place of R_1 , and a corresponding group ${}^R\mathbf{Nil}_q^-(R_0)$. The complete version of the result is formulated and proved in Theorems 11.1 and 12.1 below.

RIGHT REGULAR RINGS

A ring is called *right regular* if every finitely generated right module has a finite resolution by finitely generated projective right modules. A right regular ring is automatically right NOETHERIAN [Bas68, p. 122]. A graded ring is called *right graded regular* if every finitely generated graded right module has a finite resolution by finitely generated projective graded right modules, with degree-preserving differentials. A graded ring which is right regular as a ring is also right graded regular.

If R is a strongly \mathbb{Z} -graded ring, the category of R_0 -modules (*resp.*, of finitely generated R_0 -modules, *resp.*, of finitely generated projective R_0 -modules) is equivalent *via* the functor $P \mapsto P \otimes_{R_0} R$ to the category of graded R -modules (*resp.*, finitely generated graded R -modules, *resp.*, finitely generated projective graded R -modules), see DADE [Dad80, Theorem 2.8]. Thus the strongly graded ring R is right graded regular if and only if the ring R_0 is right regular.

THEOREM 2.10. *Suppose that R is a strongly \mathbb{Z} -graded ring. Suppose that R is right graded regular (or equivalently, that R_0 is right regular). Then the groups ${}^R\mathbf{NK}_q^\pm(R_0)$ are trivial for all $q \geq 0$, and there are short exact sequences*

$$0 \rightarrow {}^R_{\text{coker}}K_q(R_0) \rightarrow K_q(R) \rightarrow {}^R_{\text{ker}}K_{q-1}(R_0) \rightarrow 0 \quad (q > 0)$$

and

$$0 \rightarrow {}^R_{\text{coker}}K_0(R_0) \rightarrow K_0(R) \rightarrow {}^R K_{-1}(R_0) \rightarrow 0.$$

Proof. The $R_{\geq 0}$ -module R_0 has finite Tor-dimension since there is a resolution by finitely generated projective $R_{\geq 0}$ -modules [HM20, Lemma I.2.2]

$$0 \rightarrow R_{\geq 1} \xrightarrow{\subset} R_{\geq 0} \rightarrow R_0 \rightarrow 0 .$$

The R_0 -module $R_{\geq 0}$ is projective, by strong grading, and hence has finite Tor-dimension. A result of QUILLEN [Qui73, §6, Theorem 7] (for $A = B = R_{\geq 0}$ and $F_p A = \bigoplus_{k=0}^p R_k$) asserts that the ring inclusion $R_0 \subset R_{\geq 0}$ induces an isomorphism on K -groups; it follows that ${}^R\text{NK}_q^+(R_0)$ is trivial for all $q \geq 0$. A symmetric argument settles the case of ${}^R\text{NK}_q^-(R_0)$. The fundamental theorem 2.7 thus yields the desired short exact sequences. \square

3 INDUCED MODULES AND CHAIN COMPLEXES

THE GROTHENDIECK GROUP OF A RING

Let L be a rings. The group $K_0(L)$ is, by definition, the GROTHENDIECK group of the category $\mathbf{P}(L)$ of finitely generated projective L -modules; we denote the element corresponding to the module P by the symbol $[P]$. The cokernel of the map $K_0(\mathbb{Z}) \rightarrow K_0(L)$ induced by the induction functor $M \mapsto M \otimes_{\mathbb{Z}} L$ is the reduced GROTHENDIECK group $K_0(\mathbb{Z} \downarrow L)$ of L , more usually denoted by the symbol $\tilde{K}_0(L)$. We write the element corresponding to the module P by $\alpha([P])$. The following equivalences are well known:

$$[P] = [Q] \quad \text{in } K_0(L) \quad \Leftrightarrow \quad \exists k \geq 0: P \oplus L^k \cong Q \oplus L^k \quad (3.1a)$$

and

$$\alpha([P]) = \alpha([Q]) \quad \text{in } K_0(\mathbb{Z} \downarrow L) \quad \Leftrightarrow \quad \exists k, \ell \geq 0: P \oplus L^k \cong Q \oplus L^\ell \quad (3.1b)$$

In particular, $\alpha([P]) = 0$ if and only if P is stably free.

We will repeatedly make use of the fact that $K_0(L)$ can be described in other ways, for example using the machinery of WALDHAUSEN K -theory applied to the category $\text{Ch}^b \mathbf{P}(L)$ of bounded complexes of finitely generated projective L -modules (with quasi-isomorphisms as weak equivalences, and levelwise split monomorphisms as cofibrations). This description is such that a chain complex C in $\text{Ch}^b \mathbf{P}(L)$ gives rise to the element $[C] = \chi(C) = \sum_k (-1)^k [C_k]$ of $K_0(L)$. If C is contractible then $[C] = 0$.

INDUCED AND STABLY INDUCED MODULES

Let $f: L \rightarrow S$ be a ring homomorphism, with induced map

$$f_*: K_0(L) \rightarrow K_0(S) , \quad [P] \mapsto [P \otimes_L S] .$$

We will use the notation f_* also for the induction functor $- \otimes_L S$ so that $f_*([P]) = [f_*(P)]$.

DEFINITION 3.2. Let Q be a finitely generated projective S -module.

- (1) We say that Q is *induced from L* if there exists a finitely generated projective L -module P such that $f_*(P) \cong Q$.
- (2) The module Q is called *stably induced from L* if there exist a number $q \geq 0$ and a finitely generated projective L -module P such that $f_*(P) \cong Q \oplus S^q$.

Algebraic K -theory provides an obstruction to modules being stably induced, the obstruction group being $K_0(L \downarrow S) = \text{coker}(f_*)$. It comes equipped with a canonical map

$$\alpha: K_0(S) \rightarrow K_0(L \downarrow S) . \tag{3.3}$$

PROPOSITION 3.4. *Let Q be a finitely generated projective S -module. The following are equivalent:*

- (1) *The module Q is stably induced from L .*
- (2) *The element $\alpha([Q])$ of $K_0(L \downarrow S)$ is trivial.*

For example, if Q is a stably free S -module then Q is certainly stably induced from L so that $\alpha([Q]) = 0$ in $K_0(L \downarrow S)$

Proof of Proposition 3.4. If $Q \oplus S^s = P \otimes_L S$ then $[Q] = f_*([P]) - f_*([L^s])$ lies in the image of f_* whence $\alpha([Q]) = 0 \in K_0(L \downarrow S)$.

Conversely, suppose that $\alpha([Q]) = 0$. Then there exists $a \in K_0(L)$ such that $[Q] = f_*(a)$ in $K_0(S)$. We can find a finitely generated projective L -module M and a number $r \geq 0$ such that $a = [M] - [L^r]$ in $K_0(L)$. By applying f_* and re-arranging we find the equality

$$[Q \oplus S^r] = [Q] + [S^r] = f_*(a) + f_*([L^r]) = f_*([M]) \in K_0(S) .$$

This in turn implies that there exists $k \geq 0$ with

$$Q \oplus S^r \oplus S^k \cong f_*(M) \oplus S^k = f_*(M \oplus L^k) .$$

This shows Q to be stably induced. □

STABILISATION OF CHAIN COMPLEXES

As a matter of notation, we write $D(k, M)$ for the chain complex concentrated in degrees k and $k - 1$ with non-trivial entries M and differential the identity map of M .

DEFINITION 3.5. Let D be a chain complex of S -modules. We say that the chain complex D' is a *stabilisation* of D if $D' = D \oplus \bigoplus_k D(k, F_k)$ for finitely many finitely generated free S -modules F_k .

If D' is a stabilisation of D then there are mutually homotopy inverse chain homotopy equivalences $s: D \rightarrow D'$ (the inclusion) and $r: D' \rightarrow D$ (the projection) such that $rs = \text{id}_D$, and such that $\text{coker}(s) = \text{ker}(r) = \bigoplus_k D(k, F_k)$ is a contractible bounded complex with finitely generated free chain, boundary and cycle modules.

INDUCED AND STABLY INDUCED CHAIN COMPLEXES. THE CHAIN COMPLEX LIFTING PROBLEM

Let $f: L \rightarrow S$ be a ring homomorphism.

DEFINITION 3.6. Let D be a bounded complex of finitely generated projective S -modules.

- (1) We say that D is *induced from L* if there exists a bounded complex of finitely generated projective L -module C such that $f_*(C) \cong D$.
- (2) The complex D is called *stably induced from L* if there exist a bounded complex C of finitely generated projective L -modules such that $f_*(C)$ is isomorphic to a stabilisation of D .

Again, let D be a bounded complex of finitely generated projective S -modules; suppose that all chain modules D_k are induced from L . The *chain complex lifting problem* is to decide whether D is homotopy equivalent to a complex induced from L . We will address two variations of the theme below and show that

- if D is acyclic, then there exists an *acyclic* bounded complex C of finitely generated projective L -modules such that $f_*(C)$ is isomorphic to a stabilisation of D ;
- if f satisfies a certain strong flatness condition, then D is stably induced from L .

THE CHAIN COMPLEX LIFTING PROBLEM FOR ACYCLIC COMPLEXES

THEOREM 3.7. *Every acyclic bounded complex of stably induced modules is stably induced from an acyclic bounded complex. — More precisely, let D be an acyclic bounded chain complex of finitely generated projective S -modules concentrated in chain levels 0 to n such that each chain module D_k is stably induced from L . Then there exist a stabilisation D' of D and an acyclic bounded complex C' of finitely generated projective L -modules, both concentrated in chain levels 0 to n , such that $f_*(C')$ is isomorphic to D' .*

Proof. As D is acyclic there are finitely generated projective S -modules E_k , for $1 \leq k \leq n$, such that

$$D \cong \bigoplus_{k=1}^n D(k, E_k). \quad (3.8)$$

By construction we have $E_1 = D_0$ and consequently $\alpha([E_1]) = 0$, since D_0 is stably induced from L by hypothesis. By iteration, assuming that $\alpha([E_{k-1}]) = 0$ is known, we infer from $D_{k-1} = E_{k-1} \oplus E_k$ that $\alpha([E_k]) = 0$ as well.

Thus for $1 \leq k \leq n$ we can choose numbers $j_k \geq 0$ and finitely generated projective L -modules P_k such that

$$E_k \oplus S^{j_k} \cong f_*(P_k) .$$

We define chain complexes

$$D' = \bigoplus_{k=1}^n D(k, E_k \oplus S^{j_k}) \quad \text{and} \quad C' = \bigoplus_{k=1}^n D(k, P_k) ;$$

thus C' and D' are acyclic complexes concentrated in chain levels 0 to n . The computation

$$D' = \bigoplus_{k=1}^n D(k, E_k \oplus S^{j_k}) = \bigoplus_{k=1}^n \left(D(k, E_k) \oplus D(k, S^{j_k}) \right) \stackrel{(3.8)}{\cong} D \oplus \bigoplus_{k=1}^n D(k, S^{j_k})$$

confirms D' as a stabilisation of D . Finally, by construction we have isomorphisms of chain complexes

$$f_*(C') = \bigoplus_{k=1}^n D(k, f_*(P_k)) \cong \bigoplus_{k=1}^n D(k, E_k \oplus S^{j_k}) = D'$$

as required. □

THE CHAIN COMPLEX LIFTING PROBLEM FOR WELL-BEHAVED RING HOMOMORPHISMS

As before, let $f: L \rightarrow S$ be a ring homomorphism. We consider S as an L - L -bimodule, with L acting *via* f . To solve the chain complex lifting problem, we will assume that the following condition is satisfied:

The L - L -bimodule S is a filtered colimit of L - L -sub-bimodules S_j which are finitely generated projective right L -modules such that $S_j \otimes_L S = S$. (3.9)

In particular, S is a flat right L -module in this case.

EXAMPLE 3.10. The ring inclusion $f: L = R_{\leq 0} \xrightarrow{\subseteq} R = S$ with R a strongly \mathbb{Z} -graded ring satisfies condition (3.9) since $R = \bigcup_{k \geq 0} R_{\leq k}$ with $R_{\leq k}$ a finitely generated projective (left and right) $R_{\leq 0}$ -module [HM20, Lemma I.2.2] such that $R_{\leq k} \otimes_{R_{\leq 0}} R = R$ [HM20, Lemma I.2.4].

PROPOSITION 3.11. *Let D be a chain complex consisting of finitely generated projective S -modules concentrated in chain levels 0 to n . Suppose that each chain module D_k is induced from L so that $D_k \cong f_*(C'_k)$ for a finitely generated projective L -module C'_k . Suppose further that the ring homomorphism f satisfies condition (3.9). Then there exists a chain complex C of finitely generated projective L -modules, concentrated in chain levels 0 to n , such that $f_*(C) \cong D$.*

Proof. We denote the differentials of D by $d_k: D_k \rightarrow D_{k-1}$. Set $C_k = \{0\}$ for $k < 0$ and for $k > n$, and choose $C_n = C'_n$.

Since C_n is finitely generated, the composite map

$$C_n \rightarrow C_n \otimes_L S \cong D_n \xrightarrow{d_n} D_{n-1} = f_*(C'_{n-1}) \underset{(3.9)}{=} \operatorname{colim}_j C'_{n-1} \otimes_L S_j$$

factors through some stage j_n of the colimit system, resulting in a map

$$\partial_n: C_n \rightarrow C'_{n-1} \otimes_L S_{j_n} =: C_{n-1}$$

with target a finitely generated projective L -module, by our hypotheses on f , such that $f_*(\partial_n) \cong d_n$.

Since C_{n-1} is finitely generated, the composite map

$$C_{n-1} \rightarrow C_{n-1} \otimes_L S \cong D_{n-1} \xrightarrow{d_{n-1}} D_{n-2} = f_*(C'_{n-2}) \underset{(3.9)}{=} \operatorname{colim}_j C'_{n-2} \otimes_L S_j$$

factors through some stage j of the colimit system, resulting in a map

$$\tilde{\partial}_{n-1}: C_{n-1} \rightarrow C'_{n-2} \otimes_L S_j .$$

By replacing j with a larger index j_{n-1} (where “larger” refers to the filtered properties of the indexing category), we may assume that the composite with ∂_n is the zero map. In other words, we have a map

$$\tilde{\partial}_{n-1}: C_{n-1} \rightarrow C'_{n-2} \otimes_L S_{j_{n-1}} =: C_{n-2}$$

with target a finitely generated projective L -module, by our hypotheses on f , such that $f_*(\partial_n) \cong d_n$ and $\partial_{n-1} \circ \partial_n = 0$.

The process is repeated iteratively, until we have constructed C_0 and ∂_1 . □

PROPOSITION 3.12. *Let D be a chain complex consisting of finitely generated projective S -modules concentrated in chain levels $0 \leq k \leq n$, such that each chain module D_k is stably induced from L . Suppose that the ring homomorphism f satisfied condition (3.9). Then there exist a stabilisation D' of D and a chain complex C' of finitely generated projective L -modules, both concentrated in chain levels $0 \leq k \leq n + 1$, such that $f_*(C') \cong D'$.*

Proof. For $0 \leq k \leq n$ choose a number $s_k \geq 0$ such that $D_k \oplus S^{s_k}$ is induced from L , and choose a finitely generated projective L -module C_k with $f_*(C_k) \cong D_k \oplus S^{s_k}$. Set $D' = D \oplus \bigoplus_k D(k+1, S^{s_k})$. By construction, D' is concentrated in chain levels 0 to $n + 1$, and each chain module D' is induced from L (as it is the direct sum of a module induced from L with a finitely generated free module). Hence Proposition 3.11 yields a bounded complex C' of finitely generated projective L -modules, concentrated in chain levels 0 to $n + 1$, such that $f_*(C') \cong D'$. □

4 FINITE DOMINATION

Let C be a (possibly unbounded) chain complex of K -modules, for some unital ring K . We say that C is K -finitely dominated, or of type (FP) over K , if C is homotopy equivalent to a bounded complex of finitely generated projective K -modules.

Finite domination satisfies the “2-of-3 property” with respect to short exact sequences of chain complexes. We include a proof for completeness.

LEMMA 4.1. *Let C , D and E be bounded below complexes of projective K -modules, and suppose that there is a short exact sequence*

$$0 \rightarrow C \xrightarrow{f} D \xrightarrow{g} E \rightarrow 0 . \tag{4.2}$$

If any two of the complexes are K -finitely dominated then so is the third.

Proof. The given sequence gives rise to a short exact sequence of bounded complexes of projective K -modules

$$0 \rightarrow D \rightarrow \text{cyl}(g) \xrightarrow{h} \text{cone}(g) \rightarrow 0 ,$$

together with homotopy equivalences $E \rightarrow \text{cyl}(g)$ and $q: C[1] \rightarrow \text{cone}(g)$. By iteration, there is a short exact sequence

$$0 \rightarrow \text{cyl}(g) \rightarrow \text{cyl}(h) \rightarrow \text{cone}(h) \rightarrow 0 ,$$

together with homotopy equivalences $C[1] \rightarrow \text{cyl}(h)$ and $D[1] \rightarrow \text{cone}(h)$. As finite domination is invariant under suspension and homotopy equivalences, it suffices to prove that *if C and D are K -finitely dominated so is E .*

Since we are dealing with bounded below complexes of projective modules, the canonical map $\text{cone}(f) \rightarrow E$ is a homotopy equivalence. As C and D are K -finitely dominated there exist bounded complexes C' and D' of finitely generated projective K -modules and chain homotopy equivalences $\alpha: C \rightarrow C'$ and $\beta: D \rightarrow D'$. Choose a homotopy inverse α' of α , and let $h: \text{id}_C \simeq \alpha'\alpha$ be a homotopy. The chain map

$$\begin{pmatrix} \alpha & \\ \beta fh & \beta \end{pmatrix} : \text{cone}(f) \rightarrow \text{cone}(\beta f \alpha')$$

is a quasi-isomorphism (since α and β are) and hence a homotopy equivalence; its target is a bounded complex of finitely generated projective K -modules. Thus $E \simeq \text{cone}(f) \simeq \text{cone}(\beta f \alpha')$ shows that E is K -finitely dominated. \square

5 THE PROJECTIVE LINE ASSOCIATED WITH A STRONGLY \mathbb{Z} -GRADED RING

The projective line associated with a strongly \mathbb{Z} -graded ring has been introduced by MONTGOMERY and the author [HM20]. We recall definitions and K -theoretical results which will take a central place when establishing the fundamental theorem.

FROM NOW ON WE WILL ASSUME THROUGHOUT THAT $R = \bigoplus_{k \in \mathbb{Z}} R_k$ IS A STRONGLY \mathbb{Z} -GRADED RING UNLESS OTHER HYPOTHESES ARE SPECIFIED.

DEFINITION 5.1 (Sheaves and vector bundles on the projective line [HM20, Definitions II.1.1 and II.2.1]). A *quasi-coherent sheaf on \mathbb{P}^1* , or just *sheaf* for short, is a diagram

$$\mathcal{Y} = \left(Y^- \xrightarrow{v^-} Y^0 \xleftarrow{v^+} Y^+ \right) \tag{5.2}$$

where Y^- , Y^0 and Y^+ are modules over $R_{\leq 0}$, R and $R_{\geq 0}$, respectively, with an $R_{\leq 0}$ -linear homomorphisms v^- and an $R_{\geq 0}$ -linear homomorphism v^+ , such that the diagram of the adjoint R -linear maps

$$Y^- \otimes_{R_{\leq 0}} R \xrightarrow[\cong]{v^-} Y^0 \xleftarrow[\cong]{v^+} Y^+ \otimes_{R_{\geq 0}} R \tag{5.3}$$

consists of isomorphisms. This latter condition will be referred to as the *sheaf condition*. A *morphism $f = (f^-, f^0, f^+): \mathcal{Y} \rightarrow \mathcal{Z}$* between sheaves is a commutative diagram of the form

$$\begin{array}{ccccc} \mathcal{Y} & & Y^- & \xrightarrow{v^-} & Y^0 & \xleftarrow{v^+} & Y^+ \\ \downarrow & & \downarrow f^- & & \downarrow f^0 & & \downarrow f^+ \\ \mathcal{Z} & & Z^- & \xrightarrow{\zeta^-} & Z^0 & \xleftarrow{\zeta^+} & Z^+ \end{array}$$

with f^- , f^0 and f^+ homomorphisms of modules over $R_{\leq 0}$, R and $R_{\geq 0}$, respectively.

We call the sheaf \mathcal{Y} a *vector bundle* if its constituent modules are finitely generated projective modules over their respective ground rings. The category of vector bundles (and all morphisms of sheaves between them) is denoted by $\text{Vect}(\mathbb{P}^1)$.

Specific examples of vector bundles are the *twisting sheaves*

$$\mathcal{O}(k, \ell) = \left(R_{\leq k} \xrightarrow{\subseteq} R \xleftarrow{\supseteq} R_{\geq -\ell} \right)$$

where k and ℓ are integers. (Note that the diagrams $\mathcal{O}(k, \ell)$ may not be sheaves if the \mathbb{Z} -graded ring R fails to be *strongly* graded.) — Taking tensor product with twisting sheaves defines an interesting operation on the category of sheaves on \mathbb{P}^1 :

DEFINITION 5.4 (Twisting [HM20, Definition II.2.4]). Let \mathcal{Y} be a sheaf, and let $k, \ell \in \mathbb{Z}$. We define the (k, ℓ) th *twist of \mathcal{Y}* , denoted $\mathcal{Y}(k, \ell)$, to be the sheaf

$$\mathcal{Y}(k, \ell) = \left(Y^- \otimes_{R_{\leq 0}} R_{\leq k} \rightarrow Y^0 \otimes_R R \leftarrow Y^+ \otimes_{R_{\geq 0}} R_{\geq -\ell} \right),$$

with structure maps induced by those of \mathcal{Y} and the inclusion maps.

DEFINITION 5.5. Let $k, \ell \in \mathbb{Z}$. The (k, ℓ) th canonical sheaf functor $\Psi_{k, \ell}$ is defined by

$$\Psi_{k, \ell}: \mathbf{P}(R_0) \rightarrow \text{Vect}(\mathbb{P}^1), \quad P \mapsto P \otimes \mathcal{O}(k, \ell)$$

where $\mathbf{P}(R_0)$ is the category of finitely generated projective R_0 -modules, and the symbol $P \otimes \mathcal{O}(k, \ell)$ denotes the sheaf

$$P \otimes_{R_0} R_{\leq k} \rightarrow P \otimes_{R_0} R \leftarrow P \otimes_{R_0} R_{\geq -\ell}.$$

DEFINITION 5.6. The sheaf cohomology modules of the sheaf \mathcal{Y} of (5.2) are defined by the exact sequence

$$0 \rightarrow H^0\mathcal{Y} \rightarrow Y^- \oplus Y^+ \xrightarrow{v^- - v^+} Y^0 \rightarrow H^1\mathcal{Y} \rightarrow 0,$$

that is, $H^0\mathcal{Y} = \ker(v^- - v^+)$ and $H^1\mathcal{Y} = \text{coker}(v^- - v^+)$. We will also use the notation $\Gamma\mathcal{Y}$ for $H^0\mathcal{Y}$ and speak of *global sections* of \mathcal{Y} .

The R_0 -modules $H^0\mathcal{Y}$ and $H^1\mathcal{Y}$ depend functorially on \mathcal{Y} . Considering \mathcal{Y} as a diagram of R_0 -modules we have isomorphisms $H^q\mathcal{Y} = \varprojlim^q \mathcal{Y}$ for $q = 0, 1$.

One can explicitly compute the sheaf cohomology of twisting sheaves by direct inspection [HM20, Proposition II.3.4]. Similarly, one can show:

PROPOSITION 5.7. For $P \in \mathbf{P}(R_0)$ and $k, \ell \in \mathbb{Z}$ there is an isomorphism

$$\Gamma\Psi_{k, \ell}P \cong \bigoplus_{j=-\ell}^k s_j P$$

where $s_j P = P \otimes_{R_0} R_j$. The isomorphism is natural in P so that there are isomorphisms of functors

$$\begin{aligned} \Gamma \circ \Psi_{k, \ell} &\cong 0 \quad \text{if } k + \ell < 0, \\ \Gamma \circ \Psi_{0, 0} &\cong \text{id}, \\ \Gamma \circ \Psi_{k, -k} &\cong s_k. \end{aligned}$$

Moreover, if $k + \ell \geq -1$ then $H^1 \circ \Psi_{k, \ell}\mathcal{Y} = 0$. □

6 THE ALGEBRAIC K -THEORY OF THE PROJECTIVE LINE

It is technically more convenient to pass from the category $\text{Vect}(\mathbb{P}^1)$ of vector bundles to a certain subcategory on which the global sections functor Γ is exact:

DEFINITION 6.1. The category $\text{Vect}(\mathbb{P}^1)_0$ is the full subcategory of $\text{Vect}(\mathbb{P}^1)$ consisting of those objects \mathcal{Y} satisfying

$$H^1\mathcal{Y}(k, \ell) = 0 \quad \text{for all } k, \ell \in \mathbb{Z} \text{ with } k + \ell \geq 0.$$

We let $\text{Ch}^b \text{Vect}(\mathbb{P}^1)$ denote the category of bounded chain complexes of vector bundles; similarly, we denote by $\text{Ch}^b \text{Vect}(\mathbb{P}^1)_0$ the category of bounded chain complexes in the category $\text{Vect}(\mathbb{P}^1)_0$. A map f of vector bundles will be called an *h-equivalence*, or a *quasi-isomorphism*, if $f^?$ is a quasi-isomorphism of chain complexes of modules for each decoration $? \in \{-, 0, +\}$. As *h-equivalences* are defined homologically, they satisfy the saturation and extension axioms. All categories mentioned have a cylinder functor given by the usual mapping cylinder construction which satisfies the cylinder axiom.

DEFINITION 6.2 ([HM20, Definition III.1.1]). The *K*-theory space of the projective line is defined to be

$$K(\mathbb{P}^1) = \Omega |h\mathcal{S}\bullet\text{Ch}^b \text{Vect}(\mathbb{P}^1)| ,$$

where “*h*” stands for the category of *h*-equivalences.

LEMMA 6.3 ([HM20, Corollary III.1.4]). *The inclusion $\text{Vect}(\mathbb{P}^1)_0 \subseteq \text{Vect}(\mathbb{P}^1)$ induces a homotopy equivalence*

$$h\mathcal{S}\bullet\text{Ch}^b \text{Vect}(\mathbb{P}^1)_0 \xrightarrow{\cong} h\mathcal{S}\bullet\text{Ch}^b \text{Vect}(\mathbb{P}^1) ,$$

and hence a homotopy equivalence $\Omega |h\mathcal{S}\bullet\text{Ch}^b \text{Vect}(\mathbb{P}^1)_0| \xrightarrow{\cong} K(\mathbb{P}^1)$. □

For $k + \ell \geq -1$ the functor $\Psi_{k,\ell}: \text{Ch}^b \mathbf{P}(R_0) \rightarrow \text{Vect}(\mathbb{P}^1)_0$ is an exact functor between WALDHAUSEN categories (with quasi-isomorphisms and *h*-equivalences as weak equivalences, and cofibrations the monomorphisms with cokernel an object of the category under consideration).

THEOREM 6.4 ([HM20, Theorem III.5.1]). *Suppose that $R = \bigoplus_{k \in \mathbb{Z}} R_k$ is a strongly \mathbb{Z} -graded ring. There is a homotopy equivalence of *K*-theory spaces*

$$K(R_0) \times K(R_0) \rightarrow K(\mathbb{P}^1)$$

induced by the functor

$$\begin{aligned} \Psi_{-1,0} + \Psi_{0,0}: \text{Ch}^b \mathbf{P}(R_0) \times \text{Ch}^b \mathbf{P}(R_0) &\rightarrow \text{Ch}^b \text{Vect}(\mathbb{P}^1)_0 , \\ (C, D) &\mapsto \Psi_{-1,0}(C) \oplus \Psi_{0,0}(D) . \end{aligned} \quad \square$$

7 THE NIL TERMS

THE CATEGORY OF TWISTED ENDOMORPHISMS

Let M be an R_0 -module. A (positive) *twisted endomorphism* of M is an R_0 -linear map

$$\alpha: M \otimes_{R_0} R_1 \rightarrow M \otimes_{R_0} R_0 .$$

The collection of such pairs (M, α) , for various modules M and their twisted endomorphisms, forms a category $\text{Tw}^+ \text{End}(R_0)$; a morphism $f: (M, \alpha) \rightarrow (N, \beta)$ is an R_0 -linear map $f: M \rightarrow N$ with $\beta \circ (f \otimes \text{id}_{R_1}) = (f \otimes \text{id}_{R_0}) \circ \alpha$. — The category $\text{Tw}^- \text{End}(R_0)$ of (negative) twisted endomorphisms of the form $M \otimes_{R_0} R_{-1} \rightarrow M \otimes_{R_0} R_0$ is defined analogously.

In case of a strongly \mathbb{Z} -graded ring R we have the equality $R_{n+1} = R_n R_1$, or (what is the same) the isomorphism $R_{n+1} \cong R_n \otimes_{R_0} R_1$. Given an object (M, α) of $\text{Tw}^+ \text{End}(R_0)$, we can thus recursively define the n th iteration of α to be the map

$$\alpha^{(n)}: M \otimes_{R_0} R_n \rightarrow M \otimes_{R_0} R_0 \quad (n \geq 0)$$

determined by

$$\alpha^{(0)} = \text{id} \quad \text{and} \quad \alpha^{(1)} = \alpha,$$

with $\alpha^{(n+1)}$ being the composition

$$\begin{aligned} M \otimes_{R_0} R_{n+1} &\cong M \otimes_{R_0} R_n \otimes_{R_0} R_1 \xrightarrow{\alpha^{(n)} \otimes \text{id}_{R_1}} M \otimes_{R_0} R_0 \otimes_{R_0} R_1 \\ &\cong M \otimes_{R_0} R_1 \xrightarrow{\alpha} M \otimes_{R_0} R_0. \end{aligned}$$

We say that the twisted endomorphism α of M is *nilpotent* if $\alpha^{(n)} = 0$ for $n \gg 0$.

Every $R_{\geq 0}$ -module M determines an object (M, μ) of $\text{Tw}^+ \text{End}(R_0)$, with M considered as an R_0 -module by restriction of scalars, and μ defined by $m \otimes r \mapsto mr \otimes 1$ for $m \in M$ and $r \in R_1$. There results a functor

$$\Phi = \Phi^+ : \text{Mod-}R_{\geq 0} \rightarrow \text{Tw}^+ \text{End}(R_0);$$

an analogous construction provides us with a functor

$$\Phi^- : \text{Mod-}R_{\leq 0} \rightarrow \text{Tw}^- \text{End}(R_0).$$

LEMMA 7.1. *The functors $\Phi = \Phi^+$ and Φ^- are isomorphisms of categories.*

Proof. This is just the theory of modules over tensor rings, since (thanks to the strong grading) $R_{\geq 0}$ is the tensor ring of R_1 over R_0 , and $R_{\leq 0}$ is the tensor ring of R_{-1} over R_0 . \square

As a matter of notation, for $(M, \alpha) \in \text{Tw}^+ \text{End}(R_0)$ we will denote the $R_{\geq 0}$ -module $\Phi^{-1}(M, \alpha)$ simply by the symbol M ; if we wish to stress the twisted endomorphism, we shall write M^α . The module structure of M^α and the maps $\alpha^{(n)}$ are related by the square diagram (7.2) below;

$$\begin{array}{ccc} M \otimes_{R_0} R_n & \xrightarrow{\alpha^{(n)}} & M \otimes_{R_0} R_0 \\ \downarrow \cong & & \downarrow \cong \\ M^\alpha \otimes_{R_{\geq 0}} R_{\geq n} & \longrightarrow & M^\alpha \otimes_{R_{\geq 0}} R_{\geq 0} \end{array} \tag{7.2}$$

the vertical maps, induced by inclusions, are isomorphisms of R_0 -modules ([HM20, Proposition I.2.12]), and the bottom horizontal map is the obvious one, mapping $x \otimes r \in M^\alpha \otimes_{R_{\geq 0}} R_{\geq n}$ to $x \otimes r \in M^\alpha \otimes_{R_{\geq 0}} R_{\geq 0}$. The diagram commutes.

THE CHARACTERISTIC SEQUENCE OF A TWISTED ENDOMORPHISM

Let R be a strongly \mathbb{Z} -graded ring. Multiplication defines an R_0 - $R_{\geq 0}$ -bimodule isomorphism $\vartheta: R_1 \otimes_{R_0} R_{\geq 0} \rightarrow R_{\geq 1}$, cf. [HM20, Lemma 1.2.4]. Thus for any $R_{\geq 0}$ -module M there results a map of $R_{\geq 0}$ -modules

$$\chi_M = \chi: M \otimes_{R_0} R_{\geq 1} \rightarrow M \otimes_{R_0} R_{\geq 0} \tag{7.3}$$

defined as the composition

$$M \otimes_{R_0} R_{\geq 1} \xrightarrow[\cong]{\text{id} \otimes \vartheta^{-1}} M \otimes_{R_0} R_1 \otimes_{R_0} R_{\geq 0} \xrightarrow{a \otimes \text{id}} M \otimes_{R_0} R_{\geq 0}$$

with $a(m \otimes x) = mx$ the action of scalars on the module M .

PROPOSITION 7.4 (Characteristic sequence of a twisted endomorphism). *Let (M, α) be an object of $\text{Tw}^+\text{End}(R_0)$. There is a short exact sequence of $R_{\geq 0}$ -modules, natural in (M, α) ,*

$$0 \rightarrow M^\alpha \otimes_{R_0} R_{\geq 1} \xrightarrow{\chi_M} M^\alpha \otimes_{R_0} R_{\geq 0} \xrightarrow{\pi} M^\alpha \rightarrow 0, \tag{7.5}$$

with $\pi(m \otimes r) = mr$. The sequence is split exact as a sequence of R_0 -modules.

This generalises the usual characteristic sequence of a module equipped with an endomorphism [Bas68, Proposition XII.1.1].

Proof. The sequence (7.5) is nothing but the “canonical resolution” of [Hüt, Lemma 6.2] for the $R_{\geq 0}$ -module M^α associated with (M, α) . \square

CHAIN COMPLEXES OF TWISTED ENDOMORPHISMS

A bounded chain complex in the category $\text{Tw}^+\text{End}(R_0)$, that is, an object of the category $\text{Ch}^b \text{Tw}^+\text{End}(R_0)$, consists of a pair (C, α) where C is a bounded chain complex of R_0 -modules, and

$$\alpha: C \otimes_{R_0} R_1 \rightarrow C \otimes_{R_0} R_0$$

is a map of R_0 -module complexes.

DEFINITION 7.6. The mapping half-torus of $(C, \alpha) \in \text{Ch}^b \text{Tw}^+\text{End}(R_0)$ is the complex of $R_{\geq 0}$ -modules

$$\mathfrak{H}(C, \alpha) = \text{cone} \left(\chi_C: C \otimes_{R_0} R_{\geq 1} \rightarrow C \otimes_{R_0} R_{\geq 0} \right),$$

with $\chi_C = \chi$ defined in (7.3).

The mapping half-torus provides an exact additive functor defined on the category $\text{Tw}^+\text{End}(R_0)$ to the category of chain complexes of $R_{\geq 0}$ -modules. It comes equipped with a natural $R_{\geq 0}$ -linear map $\mathfrak{H}(C, \alpha) \rightarrow C^\alpha$ induced from the short exact sequence (7.5).

PROPOSITION 7.7 ([Hüt, Corollaries 6.5 and 6.8]). *Suppose that (C, α) is a bounded chain complex in $\text{Tw}^+\text{End}(R_0)$.*

- (1) *The map $\mathfrak{H}(C, \alpha) \rightarrow C^\alpha$ is a quasi-isomorphism.*
- (2) *If C is a complex of projective R_0 -modules, the map $\mathfrak{H}(C, \alpha) \rightarrow C^\alpha$ is a chain homotopy equivalence of R_0 -module complexes.*
- (3) *If C is an R_0 -finitely dominated complex of projective $R_{\geq 0}$ -modules, then $\mathfrak{H}(C, \alpha)$ is $R_{\geq 0}$ -finitely dominated. \square*

HOMOTOPY NILPOTENT TWISTED ENDOMORPHISMS

Let (C, α) be a chain complex in the category $\text{Tw}^+\text{End}(R_0)$. We say that α is *homotopy nilpotent* if the chain map

$$\alpha^{(n)} : C \otimes_{R_0} R_n \rightarrow C \otimes_{R_0} R_0 \tag{7.8}$$

is null homotopic for some (equivalently, all) sufficiently large $n \geq 0$. Using diagram (7.2), this amounts to saying that the map of R_0 -module complexes

$$C^\alpha \otimes_{R_{\geq 0}} R_{\geq n} \rightarrow C^\alpha \otimes_{R_{\geq 0}} R_{\geq 0} = C^\alpha$$

(induced by the inclusion map $R_{\geq n} \subseteq R_{\geq 0}$) is null homotopic for $n \gg 0$. Since $R_{\geq n}$ is an invertible $R_{\geq 0}$ -bimodule with inverse $R_{\geq -n}$, this in turn is equivalent to

$$C^\alpha = C^\alpha \otimes_{R_{\geq 0}} R_{\geq 0} \rightarrow C^\alpha \otimes_{R_{\geq 0}} R_{\geq -n} \tag{7.9}$$

being null homotopic for $n \gg 0$. Under suitable finiteness assumptions, this is equivalent to the stronger condition that the map (7.9) is null homotopic for $n \gg 0$ as a map of $R_{\geq 0}$ -module complexes:

LEMMA 7.10. *Let C be an $R_{\geq 0}$ -finitely dominated complex of $R_{\geq 0}$ -modules. The following statements are equivalent:*

- (1) *The induced complex $C \otimes_{R_{\geq 0}} R$ is acyclic.*
- (2) *For $n \gg 0$ the obvious map $\nu_{0,n} : C \otimes_{R_{\geq 0}} R_{\geq 0} \rightarrow C \otimes_{R_{\geq 0}} R_{\geq -n}$, considered as a map of $R_{\geq 0}$ -module chain complexes, is null homotopic.*
- (3) *For $n \gg 0$ the obvious map $\nu_{0,n} : C \otimes_{R_{\geq 0}} R_{\geq 0} \rightarrow C \otimes_{R_{\geq 0}} R_{\geq -n}$, considered as a map of R_0 -module chain complexes, is null homotopic.*

If these equivalent conditions hold, the complex C is R_0 -finitely dominated.

Proof. All three statements and the conclusion on finite domination are invariant under homotopy equivalence of complexes of $R_{\geq 0}$ -modules. As C is $R_{\geq 0}$ -finitely dominated we may, and will, assume from the outset that C is a bounded complex of finitely generated projective $R_{\geq 0}$ -modules.

We use the notation

$$\nu: C \otimes_{R_{\geq 0}} R_{\geq 0} \rightarrow C \otimes_{R_{\geq 0}} R$$

and

$$\nu_{k,n}: C \otimes_{R_{\geq 0}} R_{\geq -k} \rightarrow C \otimes_{R_{\geq 0}} R_{\geq -k-n}$$

for the obvious $R_{\geq 0}$ -linear maps sending $x \otimes r$ to $x \otimes r$.

(1) \Rightarrow (2): The map ν is certainly null homotopic as its target is contractible by hypothesis, and a choice of null homotopy yields a map

$$H: \text{cyl}(\text{id}_C) \rightarrow C \otimes_{R_{\geq 0}} R = \bigcup_{n \geq 0} C \otimes_{R_{\geq 0}} R_{\geq -n}$$

restricting to ν and 0, respectively, on the “ends” of the cylinder. Since the source is a bounded complex of finitely generated projective modules, the map H factors through a finite stage of the increasing union, resulting in a number $N \geq 0$ and a map

$$H': \text{cyl}(\text{id}_C) \rightarrow C \otimes_{R_{\geq 0}} R_{\geq -N} . \tag{7.11}$$

By construction, H' is a null homotopy of $\nu_{0,N}$, and hence $\nu_{0,n} \simeq 0$ for all $n \geq N$.

(2) \Rightarrow (3): This is a tautology.

(3) \Rightarrow (1): Let n be sufficiently large so that $\nu_{0,n} \simeq 0$ as a map of R_0 -module complexes. As $\nu_{k,n}$ is isomorphic to $\nu_{0,n} \otimes_{R_0} R_{-k}$, as a map of R_0 -modules ([HM20, Proposition I.2.12]), we conclude that $\nu_{k,n} \simeq 0$ for all $k \geq 0$. In particular, the maps $\nu_{k,n}$ induce the trivial map on homology. Since $C \otimes_{R_{\geq 0}} R = \text{colim}_{n \geq 0} C \otimes_{R_{\geq 0}} R_{\geq -n}$, and since homology commutes with filtered colimits, we conclude by cofinality that

$$H_*(C \otimes_{R_{\geq 0}} R) = \text{colim}_{n \geq 0} H_*(C \otimes_{R_{\geq 0}} R_{\geq -n}) = \text{colim}_{k \geq 0} H_*(C \otimes_{R_{\geq 0}} R_{\geq -kn}) ,$$

the last colimit taken with respect to the maps $H_*(\nu_{kn,n})$ which are trivial. It follows that the colimit is trivial, whence $C \otimes_{R_{\geq 0}} R$ is acyclic.

Suppose now that the equivalent conditions of the Lemma are satisfied. Going back to (7.11) and the notation used there, we know that the chain map

$$\nu_{0,N}: C \cong C \otimes_{R_{\geq 0}} R_{\geq 0} \rightarrow C \otimes_{R_{\geq 0}} R_{\geq -N}$$

is null homotopic. Hence its mapping cone is homotopy equivalent to the mapping cone $M = C \oplus \Sigma(C \otimes_{R_{\geq 0}} R_{\geq -N})$ of the zero map between the same complexes, which contains C as a direct summand. On the other hand, the map $\nu_{0,N}$ is injective (since C is assumed to consist of projective

$R_{\geq 0}$ -modules), hence its mapping cone is quasi-isomorphic to the cokernel K of $\nu_{0,N}$. The ℓ th chain module K_ℓ of K is isomorphic, as an $R_{\geq 0}$ -module, to $C_\ell \otimes_{R_{\geq 0}} (R_{\geq -N}/R_{\geq 0})$. Since C_ℓ is a direct summand of $R_{\geq 0}^{n_\ell}$, for suitable $n_\ell \geq 0$, the module K_ℓ is a direct summand of

$$\begin{aligned} R_{\geq 0}^{n_\ell} \otimes_{R_{\geq 0}} (R_{\geq -N}/R_{\geq 0}) &\cong \left(R_{\geq 0} \otimes_{R_{\geq 0}} (R_{\geq -N}/R_{\geq 0}) \right)^{n_\ell} \\ &\cong (R_{\geq -N}/R_{\geq 0})^{n_\ell} \cong \left(\bigoplus_{q=-N}^{-1} R_q \right)^{n_\ell}, \end{aligned}$$

the last isomorphism being R_0 -linear. Since R is strongly graded each R_q is a finitely generated projective R_0 -module so that K_ℓ is a finitely generated projective R_0 -module as well. Thus the (bounded) complex K is R_0 -finitely dominated. Since C consists of projective R_0 -modules so does M , and as M is quasi-isomorphic to K by the above arguments there is in fact an R_0 -linear homotopy equivalence $M \simeq K$. It follows that M is R_0 -finitely dominated, hence so is its direct summand C . — The finiteness result also follows from Theorem 8.1 of [Hüt]; in the notation used there the ring $R_*((t^{-1}))$ contains R so that $C \otimes_{R_{\geq 0}} R_*((t^{-1})) \cong C \otimes_{R_{\geq 0}} R \otimes_R R_*((t^{-1}))$ is acyclic as required. \square

From Lemma 7.10, and from the fact that the maps (7.8) and (7.9) are isomorphic as R_0 -linear maps, we conclude:

COROLLARY 7.12. *Let (C, α) be an object of $\text{Ch}^b \text{Tw}^+ \text{End}(R_0)$. Suppose that the associated $R_{\geq 0}$ -module complex C^α is $R_{\geq 0}$ -finitely dominated. The twisted endomorphism α is homotopy nilpotent if and only if $C^\alpha \otimes_{R_{\geq 0}} R$ is acyclic. \square*

For later use we record the following useful fact:

LEMMA 7.13. *Let (Z, ζ) be an object of $\text{Ch}^b \text{Tw}^+ \text{End}(R_0)$. Suppose that the associated $R_{\geq 0}$ -module complex Z^ζ is an $R_{\geq 0}$ -finitely dominated complex of projective $R_{\geq 0}$ -modules. Suppose further that $Z^\zeta \otimes_{R_{\geq 0}} R$ is acyclic. Then Z is an R_0 -finitely dominated bounded complex of projective R_0 -modules.*

Proof. Since R is strongly \mathbb{Z} -graded the complex Z consists of projective R_0 -modules. By hypothesis, Z^ζ is homotopy equivalent to a bounded complex C of finitely generated projective $R_{\geq 0}$ -modules. Since $Z^\zeta \otimes_{R_{\geq 0}} R$ is acyclic so is $C \otimes_{R_{\geq 0}} R$. This forces C , hence Z^ζ , to be R_0 -finitely dominated in view of Lemma 7.10. \square

THE NIL CATEGORY

We are now in a position to define the nil category, the category of homotopy nilpotent endomorphisms of chain complexes satisfying a suitable finiteness constraint. In fact there are two variants, corresponding to the two subrings $R_{\geq 0}$ and $R_{\leq 0}$ of R .

DEFINITION 7.14. The *positive nil category* ${}^R\mathbf{Nil}^+(R_0)$ of R_0 relative to R is the full subcategory of $\text{Ch}^b(\text{Tw}^+\text{End}(R_0))$ consisting of those chain complexes (C, α) such that

- (1) the underlying R_0 -module complex C consists of projective R_0 -modules, and is R_0 -finitely dominated;
- (2) the chain map $\alpha: C \otimes_{R_0} R_1 \rightarrow C \otimes_{R_0} R_0$ is homotopy nilpotent in the sense that $\alpha^{(n)} \simeq 0$ for $n \gg 0$.

The *negative nil category* ${}^R\mathbf{Nil}^-(R_0)$ of R_0 relative to R is defined analogously, using the category $\text{Tw}^-\text{End}(R_0)$ of negative twisted endomorphisms in place of $\text{Tw}^+\text{End}(R_0)$.

REMARK 7.15. Let (C, α) be an object of ${}^R\mathbf{Nil}^+(R_0)$. By Proposition 7.7, the $R_{\geq 0}$ -linear map $\mathfrak{H}(C, \alpha) \rightarrow C^\alpha$ is a quasi-isomorphism with $R_{\geq 0}$ -finitely dominated source. The same map is an R_0 -linear homotopy equivalence. Thus the second condition in the definition of ${}^R\mathbf{Nil}^+(R_0)$ is equivalent to the condition $\mathfrak{H}(C, \alpha) \otimes_{R_{\geq 0}} R \simeq 0$, by Corollary 7.12.

A morphism in the category ${}^R\mathbf{Nil}^+(R_0)$ is called a *cofibration* if it is injective and its cokernel consists of projective R_0 -modules. We say the morphism is a *weak equivalence*, or a *q-equivalence*, if it is a quasi-isomorphism of R_0 -module complexes.

LEMMA 7.16. *These definitions equip ${}^R\mathbf{Nil}^+(R_0)$ with the structure of a category with cofibrations and weak equivalences.*

Proof. This is mostly straightforward; the gluing lemma holds, for example, since it holds in the category of bounded complexes of projective R_0 -modules. What needs explicit verification is that the requisite pushouts exist within the category ${}^R\mathbf{Nil}^+(R_0)$. Using the functor Φ and its inverse $(A, \alpha) \mapsto A^\alpha$ we identify ${}^R\mathbf{Nil}^+(R_0)$ with a full subcategory of the category \mathcal{K} of bounded chain complexes of $R_{\geq 0}$ -modules. Let (A, α) , (B, β) and (C, γ) be objects in ${}^R\mathbf{Nil}^+(R_0)$, let $f: (A, \alpha) \rightarrow (B, \beta)$ be a cofibration, and let $g: (A, \alpha) \rightarrow (C, \gamma)$ be an arbitrary morphism in ${}^R\mathbf{Nil}^+(R_0)$. We can then form the pushout diagram

$$\begin{array}{ccc} A^\alpha & \xrightarrow{f} & B^\beta \\ \downarrow g & & \downarrow \\ C^\gamma & \xrightarrow{f'} & P^\pi \end{array}$$

in the category \mathcal{K} , and claim that (P, π) is an object of $\text{Ch}^b(\text{Tw}^+\text{End}(R_0))$. As f is a cofibration in ${}^R\mathbf{Nil}^+(R_0)$ it is an injective map, and the same is thus true for its pushout f' . Let K^κ denote the cokernel of f , associated with $(K, \kappa) \in \text{Ch}^b(\text{Tw}^+\text{End}(R_0))$. From general properties of pushout squares we

know that $\text{coker}(f') \cong K^\kappa$. Hence we obtain a commutative ladder diagram of short exact sequences of chain complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\alpha & \longrightarrow & B^\beta & \longrightarrow & K^\kappa & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & C^\gamma & \longrightarrow & P^\pi & \longrightarrow & K^\kappa & \longrightarrow & 0 \end{array}$$

Again, since f is a cofibration its cokernel K^κ consists of projective R_0 -modules. Thus both sequences are levelwise split short exact sequences of R_0 -module complexes, and P consists of projective R_0 -modules. By Lemma 4.1, applied to the top row, the complex K^κ is R_0 -finitely dominated, which implies, by applying Lemma 4.1 to the bottom row, that so is P^π .

Application of the exact half-torus functor yields another commutative ladder diagram of short exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{H}(A, \alpha) & \longrightarrow & \mathfrak{H}(B, \beta) & \longrightarrow & \mathfrak{H}(K, \kappa) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathfrak{H}(C, \gamma) & \longrightarrow & \mathfrak{H}(P, \pi) & \longrightarrow & \mathfrak{H}(K, \kappa) & \longrightarrow & 0 \end{array}$$

consisting of bounded complexes of finitely generated projective $R_{\geq 0}$ -modules; in particular, both short exact sequences are levelwise split. By Remark 7.15 the chain complexes $\mathfrak{H}(A, \alpha)$, $\mathfrak{H}(B, \beta)$ and $\mathfrak{H}(C, \gamma)$ are $R_{\geq 0}$ -finitely dominated. Hence $\mathfrak{H}(K, \kappa)$ and $\mathfrak{H}(P, \pi)$ are $R_{\geq 0}$ -finitely dominated as well, by two applications of Lemma 4.1.

The bottom row yields a short exact sequence

$$0 \rightarrow \mathfrak{H}(C, \gamma) \otimes_{R_{\geq 0}} R \rightarrow \mathfrak{H}(P, \pi) \otimes_{R_{\geq 0}} R \rightarrow \mathfrak{H}(K, \kappa) \otimes_{R_{\geq 0}} R \rightarrow 0 .$$

The first and third entry are acyclic, by Corollary 7.12, hence so is the middle entry. Applying the Corollary again leads us to conclude that (P, π) is an object of ${}^R\mathbf{Nil}^+(R_0)$. \square

For L a unital ring write $\mathbf{FD}(L)$ for the category of L -finitely dominated bounded complexes of projective L -modules. It is well-known, and can be verified without difficulty using WALDHAUSEN’s approximation theorem, that the inclusion $\text{Ch}^b \mathbf{P}(L) \xrightarrow{\simeq} \mathbf{FD}(L)$ induces a homotopy equivalence on K -theory spaces

$$K(L) = \Omega|q\mathcal{S}\cdot\text{Ch}^b \mathbf{P}(L)| \xrightarrow{\simeq} \Omega|q\mathcal{S}\cdot\mathbf{FD}(L)| , \tag{7.17}$$

where “ q ” stands for quasi-isomorphisms as usual, and the cofibrations are the injective maps with levelwise projective cokernel. Hence the forgetful functors

$$o^\mp : {}^R\mathbf{Nil}^\mp(R_0) \rightarrow \mathbf{FD}(R_0) , \quad (Z, \zeta) \mapsto Z$$

yield group homomorphisms

$$o_*^- : K_q({}^R\mathbf{Nil}^-(R_0)) \rightarrow K_q(R_0) \quad \text{and} \quad o_*^+ : K_q({}^R\mathbf{Nil}^+(R_0)) \rightarrow K_q(R_0) . \tag{7.18}$$

With the lemma and notation in place, we can now define

$${}^R\mathbf{Nil}_q^-(R_0) = \ker \left(o_*^- : K_q({}^R\mathbf{Nil}^-(R_0)) \rightarrow K_q(R_0) \right) \tag{7.19a}$$

and

$${}^R\mathbf{Nil}_q^+(R_0) = \ker \left(o_*^+ : K_q({}^R\mathbf{Nil}^+(R_0)) \rightarrow K_q(R_0) \right) . \tag{7.19b}$$

We will establish in Theorem 12.1 that there are isomorphisms

$${}^R\mathbf{NK}_{q+1}^-(R_0) \cong {}^R\mathbf{Nil}_q^+(R_0) \quad \text{and} \quad {}^R\mathbf{NK}_{q+1}^+(R_0) \cong {}^R\mathbf{Nil}_q^-(R_0) ,$$

identifying the groups ${}^R\mathbf{NK}_{q+1}^\mp(R_0)$ with the q th reduced algebraic K -group of the category ${}^R\mathbf{Nil}^\pm(R_0)$.

8 THE FUNDAMENTAL SQUARE

8.1 THE FUNDAMENTAL SQUARE OF THE PROJECTIVE LINE

Let $f = (f^-, f^0, f^+)$ be a morphism $\mathcal{Y} \rightarrow \mathcal{Z}$ in the category $\text{Ch}^b \text{Vect}(\mathbb{P}^1)$ of bounded chain complexes of vector bundles on the projective line associated with a strongly \mathbb{Z} -graded ring R . We say that f is an h^γ -equivalence, for the decoration $\gamma \in \{-, 0, +\}$, if the component f^γ is a quasi-isomorphism of chain complexes. Together with the previous notion of cofibrations, *viz.*, levelwise split injections in each component, this equips $\text{Ch}^b \text{Vect}(\mathbb{P}^1)$ with three new structures of a category with cofibrations and weak equivalences. Note that an h^- -equivalence is automatically an h^0 -equivalence as well since in the commutative square

$$\begin{array}{ccc} Y^- \otimes_{R_{\leq 0}} R & \xrightarrow{v_\#^-} & Y^0 \\ f^- \otimes R \downarrow & & \downarrow f^0 \\ Z^- \otimes_{R_{\leq 0}} R & \xrightarrow{\zeta_\#^-} & Z^0 \end{array}$$

the horizontal maps are isomorphisms (sheaf condition), and the left vertical map is a quasi-isomorphism because R is a flat left $R_{\leq 0}$ -module thanks to the strong grading. — Similarly, every h^+ -equivalence is an h^0 -equivalence.

Thus the identity functor yields a commutative square of K -theory spaces

$$\begin{array}{ccc}
 h\mathcal{S}_\bullet \text{Ch}^b \text{Vect}(\mathbb{P}^1) & \longrightarrow & h_+\mathcal{S}_\bullet \text{Ch}^b \text{Vect}(\mathbb{P}^1) \\
 \downarrow & & \downarrow \\
 h_-\mathcal{S}_\bullet \text{Ch}^b \text{Vect}(\mathbb{P}^1) & \longrightarrow & h_0\mathcal{S}_\bullet \text{Ch}^b \text{Vect}(\mathbb{P}^1)
 \end{array} \tag{8.1}$$

which we call the *fundamental square* of the projective line. Associated with it, by taking vertical homotopy fibres, is the map

$$\alpha: h\mathcal{S}_\bullet \text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_-} \rightarrow h_+\mathcal{S}_\bullet \text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_0} \tag{8.2}$$

where the notation is as in WALDHAUSEN’s fibration theorem [Wal85, Theorem 1.6.4]. Explicitly, $\text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_-}$ is the full subcategory of $\text{Ch}^b \text{Vect}(\mathbb{P}^1)$ consisting of those objects \mathcal{Y} for which $\mathcal{Y} \rightarrow 0$ is an h_- -equivalence, that is, which satisfy $Y^- \simeq *$ and $Y^0 \simeq *$, and $\text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_0}$ is the full subcategory of $\text{Ch}^b \text{Vect}(\mathbb{P}^1)$ consisting of those objects \mathcal{Y} for which $\mathcal{Y} \rightarrow 0$ is an h_0 -equivalence, that is, which satisfy $Y^0 \simeq *$.

THE FIBRES OF THE FUNDAMENTAL SQUARE

The (homotopy) fibres of the maps in the fundamental square can be identified explicitly: they are homotopy equivalent to K -theory spaces of categories of homotopy nilpotent twisted endomorphisms. We will use this identification presently to conclude that the fundamental square is homotopy cartesian.

THEOREM 8.3 (Fibres of the fundamental square). *The functor*

$$\begin{aligned}
 F: \text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_0} &\rightarrow {}^R\mathbf{Nil}^+(R_0) , \\
 \mathcal{Y} = (Y^- \rightarrow Y^0 \leftarrow Y^+) &\mapsto \Phi(Y^+) ,
 \end{aligned}$$

defined on the category of bounded chain complexes of vector bundles \mathcal{Y} with $Y^0 \simeq *$, induces a homotopy equivalence

$$h_+\mathcal{S}_\bullet \text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_0} \xrightarrow{\sim} q\mathcal{S}_\bullet {}^R\mathbf{Nil}^+(R_0) , \tag{8.3a}$$

and, by restriction, a homotopy equivalence

$$h\mathcal{S}_\bullet \text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_-} \xrightarrow{\sim} q\mathcal{S}_\bullet {}^R\mathbf{Nil}^+(R_0) \tag{8.3b}$$

where the letter “ q ” stands for weak equivalences in the category ${}^R\mathbf{Nil}^+(R_0)$.

By symmetry, there are analogous homotopy equivalences

$$h_-\mathcal{S}_\bullet \text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_0} \xrightarrow{\sim} q\mathcal{S}_\bullet {}^R\mathbf{Nil}^-(R_0) , \tag{8.4a}$$

and, by restriction,

$$h\mathcal{S}_\bullet \text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_+} \xrightarrow{\sim} q\mathcal{S}_\bullet {}^R\mathbf{Nil}^-(R_0) . \tag{8.4b}$$

Proof. The functor F is well defined. Indeed, the component Y^+ of \mathcal{Y} is a bounded complex of finitely generated projective $R_{\geq 0}$ -modules, hence Y^+ consists of projective R_0 -modules. The hypothesis $Y^+ \otimes_{R_{\geq 0}} R \cong Y^0 \simeq *$ implies that Y^+ is R_0 -finitely dominated by Lemma 7.13. Moreover, the associated twisted endomorphism

$$Y^+ \otimes_{R_0} R_1 \rightarrow Y^+ \otimes_{R_0} R_0$$

is homotopy nilpotent by Corollary 7.12.

We will employ WALDHAUSEN’s approximation theorem [Wal85, Theorem 1.6.7], combined with the standard observation that the map required by property (App 2) does not have to be a cofibration (since it can be replaced by one in the presence of cylinder functors). — We can in fact treat cases (8.3a) and (8.3b) at the same time as the proofs are almost identical.

To start with, a morphism $a = (a^-, a^0, a^+)$ in $\text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_0}$ is an h_+ -equivalence, by definition, if and only if $F(a) = a^+$ is a quasi-isomorphism. A morphism $b = (b^-, b^0, b^+)$ in $\text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_-}$ is an h -equivalence if and only if $F(b) = b^+$ is a quasi-isomorphism since b^- and b^0 are maps between acyclic complexes, hence are quasi-isomorphisms in any case. In other words, the functor F satisfies property (App 1) in both cases under consideration.

Let $(Z, \zeta) \in {}^R\mathbf{Nil}^+(R_0)$ and $Y^+ \in \text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_0}$, and let $f: \Phi(Y^+) \rightarrow (Z, \zeta)$ be a morphism in ${}^R\mathbf{Nil}^+(R_0)$. We can equivalently consider f as an $R_{\geq 0}$ -linear map of chain complexes $Y^+ \rightarrow Z^\zeta$. By Remark 7.15, the chain complex Z^ζ is quasi-isomorphic to the $R_{\geq 0}$ -finitely dominated complex $\mathfrak{H}(Z, \zeta)$ so that there exists a bounded complex \bar{E} of finitely generated projective $R_{\geq 0}$ -modules and a quasi-isomorphism $e: \bar{E} \rightarrow Z^\zeta$. We can lift f up to homotopy to an $R_{\geq 0}$ -linear map $g: Y^+ \rightarrow \bar{E}$ so that eg is homotopic to f . A choice of homotopy $f \simeq eg$ determines a map $h': \text{cyl}(g) \rightarrow Z^\zeta$ such that the composite $Y^+ \xrightarrow{i_0} \text{cyl}(g) \xrightarrow{h'} Z^\zeta$ coincides with f , and such that the composite $E \xrightarrow{i_1} \text{cyl}(g) \xrightarrow{h'} Z^\zeta$ coincides with e . In particular, h' is a quasi-isomorphism of $R_{\geq 0}$ -module complexes. (Here i_0 and i_1 are the front and back inclusion into the mapping cylinder.) We can now form a complex X^+ by attaching to $\text{cyl}(g)$ a direct sum of contractible complexes of the type $D(\ell, M)$, with M a finitely generated projective $R_{\geq 0}$ -module, so that all chain modules of X^+ except possibly one are finitely generated free modules. We let a^+ denote the composition of i_0 with the inclusion $\text{cyl}(g) \rightarrow X^+$, and let h^+ denote the composition of the projection $X^+ \rightarrow \text{cyl}(g)$ with h' . We have constructed a commutative diagram

$$\begin{array}{ccc} Y^+ & \xrightarrow{a^+} & X^+ \\ & \searrow f & \downarrow \simeq h^+ \\ & & Z^\zeta \end{array}$$

where X^+ is a bounded complex of finitely generated projective $R_{\geq 0}$ -modules, and h^+ is a quasi-isomorphism.

We will now extend X^+ to a vector bundle $\mathcal{X} = (X^-, X^0, X^+)$ and a^+ to a map of vector bundles $a = (a^-, a^0, a^+): \mathcal{Y} \rightarrow \mathcal{X}$. This may involve enlarging X^+ further, by taking the direct sum with contractible complexes of the form $D(k, R_{\geq 0}^k)$ and composing a^+ and h^+ with the obvious inclusion and projection maps. We shall keep the notation X^+ and h^+ even for the modified data.

Let $X^0 = X^+ \otimes_{R_{\geq 0}} R$. Recall that Z^ζ is quasi-isomorphic to X^+ , via the map h^+ , and to the half-torus $\mathfrak{H}(Z, \zeta)$, by Remark 7.15. Both X^+ and $\mathfrak{H}(Z, \zeta)$ are bounded complexes of finitely generated projective $R_{\geq 0}$ -modules, so these complexes are homotopy equivalent. It follows that $X^0 \simeq \mathfrak{H}(Z, \zeta) \otimes_{R_{\geq 0}} R$ is contractible, by Remark 7.15 again; this implies that $[X^0] = \sum_k (-1)^k [X_k^0] = 0 \in K_0(R)$. As all chain modules of X^0 are free with the possible exception of a single module, we conclude that X^0 consists of stably free modules, and hence consists of modules which are stably induced from $R_{\leq 0}$. We now appeal to Theorem 3.7: we can modify X^+ by taking direct sum with finitely many contractible complexes of the form $D(k, R_{\geq 0}^j)$, thereby modifying X^0 in an analogous manner, so that X^0 is isomorphic to $X^- \otimes_{R_{\leq 0}} R$ for a bounded acyclic complex X^- of finitely generated projective $R_{\leq 0}$ -modules.

We have thus constructed a vector bundle

$$\mathcal{X} = (X^- \rightarrow X^0 \leftarrow X^+)$$

together with a quasi-isomorphism $h^+: X^+ \rightarrow Z^\zeta$ and a map $a^+: Y^+ \rightarrow X^+$ such that $h^+ \circ a^+ = f$. The map a^+ induces a compatible map $a^0 = \xi_\#^+ \circ (a^+ \otimes \text{id}) \circ (\xi_\#^+)^{-1}$, see the diagram in Fig. 1. For each chain level ℓ the composite

$$\begin{array}{ccccc} Y^- & \xrightarrow{v^-} & Y^0 & \xleftarrow[\cong]{\xi_\#^+} & Y^+ \otimes_{R_{\geq 0}} R \\ & & \downarrow a^0 & & \downarrow a^+ \otimes \text{id} \\ X^- & \xrightarrow{\xi^-} & X^0 & \xleftarrow[\cong]{\xi_\#^+} & X^+ \otimes_{R_{\geq 0}} R \end{array}$$

Figure 1: Diagram used in proof of Theorem 8.3

map

$$Y_\ell^- \xrightarrow{v^-} Y_\ell^0 \xrightarrow{a^0} X_\ell^0 \cong X_\ell^- \otimes_{R_{\leq 0}} R = \bigcup_{q \geq 0} X_\ell^- \otimes_{R_{\leq 0}} R_{\leq q}$$

factors through some term $X_\ell^- \otimes_{R_{\leq 0}} R_{\leq q}$ so that, by choosing $q \gg 0$, the map $a^0 \circ v^-$ factorises as

$$Y^- \xrightarrow{a^-} X^- \otimes_{R_{\leq 0}} R_{\leq q} \rightarrow X^0,$$

the second map in this composition given by $x \otimes r \mapsto \xi^-(x) \cdot r$. That is, there exists $q \geq 0$ and $a^- : Y^- \rightarrow X^- \otimes_{R_{\leq 0}} R_{\leq q}$ such that $a = (a^-, a^0, a^+) : \mathcal{Y} \rightarrow \mathcal{X}(q, 0)$ is a map of vector bundles, and such that $h^+ \circ F(a) = f$. Since X^- and X^0 are contractible by construction, $\mathcal{X}(q, 0)$ is an object of $\text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h-} \subseteq \text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_0}$.

This proves that F satisfies property (App 2) in addition to (App 1), and the approximation theorem applies. \square

COROLLARY 8.5. *The map α of (8.2) is a homotopy equivalence. Thus, the fundamental square (8.1) is homotopy cartesian.*

Proof. The previous Theorem asserts that in the chain of maps

$$h\mathcal{S}\bullet\text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h-} \xrightarrow{\alpha} h_+\mathcal{S}\bullet\text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h_0} \xrightarrow{F} q\mathcal{S}\bullet{}^R\mathbf{Nil}^+(\mathbb{R}_0)$$

both F and $F \circ \alpha$ are homotopy equivalences. It follows that α is a homotopy equivalence as well. Since the fundamental square consists of connected spaces it is homotopy cartesian. \square

AUXILIARY CATEGORIES

We let \mathbb{C}_+ denote the category of bounded chain complexes of “vector bundles on $\text{spec}(R_{\geq 0})$ ”, that is, diagrams of the form

$$Y^0 \xleftarrow{v^+} Y^+$$

with Y^0 and Y^+ being bounded complexes of finitely generated projective over R and $R_{\geq 0}$, respectively, subject to the condition that the adjoint map $Y^0 \leftarrow Y^+ \otimes_{R_{\geq 0}} R$ be an isomorphism. A *morphism* $g = (g^0, g^+)$ from $Y^0 \xleftarrow{v^+} Y^+$ to $Z^0 \xleftarrow{\zeta^+} Z^+$ consists of an R -linear map $g^0 : Y^0 \rightarrow Z^0$ and an $R_{\geq 0}$ -linear map $g^+ : Y^+ \rightarrow Z^+$ such that $\zeta^+ \circ g^+ = g^0 \circ v^+$.

By \mathbb{D}_+ we denote the full subcategory of \mathbb{C}_+ consisting of objects $Y^0 \xleftarrow{v^+} Y^+$ such that all chain modules Y_k^0 are stably induced from $R_{\leq 0}$, that is, such that

$$[Y_k^0] \in \text{im} (K_0(R_{\leq 0}) \rightarrow K_0(R)) ,$$

or equivalently, such that $\alpha([Y_k^0]) = 0$ in $K_0(R_{\leq 0} \downarrow R)$ for all k .

LEMMA 8.6. *Every object of \mathbb{D}_+ stably extends to an object of $\text{Ch}^b \text{Vect}(\mathbb{P}^1)$. That is, given an object $Y^0 \xleftarrow{v^+} Y^+$ of \mathbb{D}_+ there exist an object*

$$Z^- \xrightarrow{\zeta^-} Z^0 \xleftarrow{\zeta^+} Z^+$$

of $\text{Ch}^b \text{Vect}(\mathbb{P}^1)$ and finitely many numbers $j_k \geq 0$ together with a commutative diagram

$$\begin{array}{ccc}
 Y^0 \oplus \bigoplus_k D(k, R^{j_k}) & \xleftarrow{v^+ \oplus \text{inc}} & Y^+ \oplus \bigoplus_k D(k, R_{\geq 0}^{j_k}) \\
 \downarrow = & & \downarrow = \\
 Z^0 & \xleftarrow{\zeta^+} & Z^+ ,
 \end{array}$$

where “inc” denotes the obvious inclusion map based on the ring inclusion $R \supseteq R_{\geq 0}$.

Proof. We apply Proposition 3.12 to the ring inclusion $f: R_{\leq 0} \rightarrow R$ and the chain complex $D = Y^0$. We obtain a stabilisation

$$D' = Y^0 \oplus \bigoplus_k D(k, R^{j_k})$$

of $D = Y^0$ and a bounded complex of finitely generated projective $R_{\leq 0}$ -modules $Z^- = C'$ such that there is an isomorphism $i: Z^- \otimes_{R_{\leq 0}} R \rightarrow D'$. Write Z^0 in place of D' , let $\zeta^-: Z^- \rightarrow Z^0$ be the $R_{\leq 0}$ -linear map adjoint to i , and define

$$Z^+ = Y^+ \oplus \bigoplus_k D(k, R_{\geq 0}^{j_k}) ;$$

with $\zeta^+ = v^+ \oplus \text{inc}$ the data satisfies all the requirements of the Lemma. \square

A morphism $g = (g^0, g^+)$ in \mathbb{C}_+ is called a *cofibration* if both g^0 and g^+ are levelwise injections such that $\text{coker}(g)$ is an object of \mathbb{C}_+ . We call g an *h_+ -equivalence* if g^+ is a quasi-isomorphism (and hence so is $g^0 \cong g^+ \otimes_{R_{\geq 0}} R$).

LEMMA 8.7. *With these definitions, both \mathbb{C}_+ and \mathbb{D}_+ are categories with cofibrations and weak equivalences.*

Proof. This is clear for \mathbb{C}_+ since this category is equivalent to the category of bounded chain complexes of finitely generated projective $R_{\geq 0}$ -modules, via the functors

$$(Y^0 \leftarrow Y^+) \mapsto Y^+ \quad \text{and} \quad M^+ \mapsto (M^+ \otimes_{R_{\geq 0}} R \leftarrow M^+) .$$

It remains to observe that the cokernel of a cofibration g in \mathbb{D}_+ is automatically an object of \mathbb{D}_+ since, given $g_k^0: Y_k^0 \rightarrow Z_k^0$, we have the equality

$$\alpha([\text{coker } g_k^0]) = \alpha([Z_k^0]) - \alpha([Y_k^0]) = 0$$

in the group $K_0(R_{\leq 0} \downarrow R)$. \square

We make the analogous symmetric definitions for \mathbb{C}_- and \mathbb{D}_- . The category \mathbb{C}_0 is defined to be the category of bounded chain complexes of finitely generated projective R -modules, with \mathbb{D}_0 the full subcategory of those objects Y^0 such that all chain modules Y_k^0 are stably induced from both $R_{\leq 0}$ and $R_{\geq 0}$, that is, such that

$$[Y_k^0] \in \text{im}(K_0 R_{\leq 0} \rightarrow K_0 R) \cap \text{im}(K_0 R_{\geq 0} \rightarrow K_0 R) \quad \text{for all } k .$$

Lemmas 8.6 and 8.7 are valid *mutatis mutandis* for these categories.

LEMMA 8.8. *The forgetful functor*

$$(Y^0 \leftarrow Y^+) \mapsto Y^+$$

induces isomorphisms on algebraic K -groups

$$K_q \mathbb{D}_+ \xrightarrow{\cong} K_q R_{\geq 0} \quad (q > 0)$$

*and an injection $K_0 \mathbb{D}_+ \xrightarrow{\subseteq} K_0 R_{\geq 0}$. — These statements hold *mutatis mutandis* for the analogous maps $K_q \mathbb{D}_- \rightarrow K_q R_{\leq 0}$ and $K_q \mathbb{D}_0 \rightarrow K_q R$.*

Proof. As remarked before, the functor $\mathcal{Y} \mapsto Y^+$ establishes an equivalence of \mathbb{C}_+ with the category of bounded chain complexes of finitely generated projective $R_{\geq 0}$ -modules, and \mathbb{D}_+ corresponds to the subcategory of complexes Y^+ such that $Y^+ \otimes_{R_{\geq 0}} R$ consists of modules which are stably induced from $R_{\leq 0}$. Let \mathbb{D}' and \mathbb{C}' denote the full subcategories of complexes concentrated in chain level 0. Then the map of K -groups in question can be computed using QUILLEN's Q -construction, applied to the inclusion of exact categories $\mathbb{D}' \subseteq \mathbb{C}'$. The result is now an immediate consequence of GRAYSON cofinality [Gra79, Theorem 1.1] since \mathbb{D}' is closed under extension in \mathbb{C}' ; indeed, an object $\mathcal{Y} \in \mathbb{C}'$ is in \mathbb{D}' if and only if $\alpha([Y^0]) = 0 \in K_0(R_{\leq 0} \downarrow R)$, and K -theory is additive on short exact sequences. To see that the former category is cofinal in the latter it suffices to observe that every finitely generated projective module can be complemented to a finitely generated free one, which is automatically stably induced. \square

THE CORNERS OF THE FUNDAMENTAL SQUARE

We can now identify the corners of the fundamental square as the algebraic K -theory spaces of the categories $\mathbb{D}_-, \mathbb{D}_0$ and \mathbb{D}_+ .

LEMMA 8.9. *The forgetful functor $\Phi^+ : \text{Ch}^b \text{Vect}(\mathbb{P}^1) \rightarrow \mathbb{D}_+$, given by*

$$\mathcal{Y} = (Y^- \rightarrow Y^0 \leftarrow Y^+) \mapsto \Phi^+(\mathcal{Y}) = (Y^0 \leftarrow Y^+) ,$$

induces a homotopy equivalence on \mathcal{S}_\bullet -constructions with respect to h_+ -equivalences:

$$h_+ \mathcal{S}_\bullet \text{Ch}^b \text{Vect}(\mathbb{P}^1) \xrightarrow{\cong} h_+ \mathcal{S}_\bullet \mathbb{D}_+$$

*The statement holds *mutatis mutandis* for \mathbb{D}_0 and \mathbb{D}_- as well.*

Proof. By definition of h_+ -equivalences the forgetful functor respects and detects weak equivalences. Let

$$\mathcal{X} = (X^- \xrightarrow{\xi^-} X^0 \xleftarrow{\xi^+} X^+)$$

be an object of $\text{Ch}^b \text{Vect}(\mathbb{P}^1)$, and let

$$\overleftarrow{\mathcal{Y}} = (Y^0 \xleftarrow{v^+} Y^+)$$

be an object of \mathbb{D}_+ . Given a morphism

$$g = (g^0, g^+): \Phi^+(\mathcal{X}) \rightarrow \overleftarrow{\mathcal{Y}}$$

we construct the object $\mathcal{Z} \in \text{Ch}^b \text{Vect}(\mathbb{P}^1)$ associated with $\overleftarrow{\mathcal{Y}}$ as described in Lemma 8.6; denote the composition of g with the inclusion $\overleftarrow{\mathcal{Y}} \rightarrow \Phi^+(\mathcal{Z})$ by $h = (h^0, h^+)$. The composite map

$$X^- \xrightarrow{\xi^-} X^0 \xrightarrow{h^0} Z^0$$

factors as

$$X^- \xrightarrow{h^-} Z^- \otimes_{R_{\leq 0}} R_{\leq q} \xrightarrow{\cong} Z^- \otimes_{R_{\leq 0}} R \cong Z^0$$

for sufficiently large $q \geq 0$; this is because X^- is a bounded complex of finitely generated $R_{\leq 0}$ -modules, and because

$$Z^0 \cong Z^- \otimes_{R_{\leq 0}} R = Z^- \otimes_{R_{\leq 0}} \bigcup_{k \geq 0} R_{\leq k} = \bigcup_{k \geq 0} Z^- \otimes_{R_{\leq 0}} R_k .$$

This results in a map

$$h = (h^-, h^0, h^+): \mathcal{X} \rightarrow \mathcal{Z}(q, 0)$$

in $\text{Ch}^b \text{Vect}(\mathbb{P}^1)$ (we identify the canonically isomorphic complexes $\Phi^+(\mathcal{Z}) = Z^+$ and $\Phi^+(\mathcal{Z}(q, 0)) = Z^+ \otimes_{R_{\geq 0}} R$ here). The projection map $p: \Phi^+(\mathcal{Z}(q, 0)) \rightarrow \overleftarrow{\mathcal{Y}}$ is an h_+ -equivalence, and satisfies the condition $p \circ \Phi^+(h) = g$. By WALDHAUSEN’s approximation theorem [Wal85, Theorem 1.6.7], Φ^+ induces a homotopy equivalence on \mathcal{S}_\bullet -constructions. \square

9 ESTABLISHING THE MAYER-VIETORIS SEQUENCE

We will now establish the MAYER-VIETORIS sequence of Theorem 2.8. As before let R be a strongly \mathbb{Z} -graded ring. The induction functors

$$j_*^- = (- \otimes_{R_{\leq 0}} R) \quad \text{and} \quad j_*^+ = (- \otimes_{R_{\geq 0}} R)$$

give rise to maps $\gamma = j_*^- - j_*^+ : K_q(R_{\leq 0}) \oplus K_q(R_{\geq 0}) \rightarrow K_q(R)$, for $q \geq 0$.

In view of Lemma 8.9, the homotopy cartesian fundamental square of the projective line (8.1) yields a MAYER-VIETORIS sequence of algebraic K -groups

$$\dots \xrightarrow{\gamma} K_{q+1}\mathbb{D}_0 \xrightarrow{\delta} K_q\mathbb{P}^1 \xrightarrow{\beta=(\beta_-, \beta_+)} K_q\mathbb{D}_- \oplus K_q\mathbb{D}_+ \xrightarrow{\gamma=\gamma_- - \gamma_+} K_q\mathbb{D}_0 \xrightarrow{\delta} \dots,$$

for $q \geq 0$, ending with a surjective homomorphism $K_0\mathbb{D}_- \oplus K_0\mathbb{D}_+ \rightarrow K_0\mathbb{D}_0$. By Lemma 8.8 this sequence coincides, for $q > 0$, with the one of Theorem 2.8, so we only need to look at its tail end. Using Lemma 8.8 again we construct the following commutative diagram:

$$\begin{array}{ccccccccc} K_1(R) & \rightarrow & K_0(\mathbb{P}^1) & \rightarrow & K_0(R_{\leq 0}) \oplus K_0(R_{\geq 0}) & \xrightarrow{\bar{\gamma}} & K_0(R) & \rightarrow & {}^R K_{-1}(R_0) & \rightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ = & & = & & \subseteq & & \subseteq & & & & \\ K_1(\mathbb{D}_0) & \rightarrow & K_0(\mathbb{P}^1) & \xrightarrow{\beta} & K_0(\mathbb{D}_-) \oplus K_0(\mathbb{D}_+) & \xrightarrow{\gamma} & K_0(\mathbb{D}_0) & \longrightarrow & 0 & & \end{array}$$

We know that the bottom row is exact, and argue that this remains true for the top row. — Replacing the target of β by a larger group does not change $\ker(\beta)$, so the top row is exact at $K_0(\mathbb{P}^1)$. It is exact at $K_0(R)$ and ${}^R K_{-1}(R_0)$ by definition of the latter group. Thus it remains to verify exactness at the third entry.

So let $(x, y) \in \ker \bar{\gamma}$ be given. We can find finitely generated projective modules P and Q over $R_{\leq 0}$ and $R_{\geq 0}$, respectively, and numbers $p, q \geq 0$, such that

$$x = [P] - [R_{\leq 0}^p] \in K_0(R_{\leq 0}) \quad \text{and} \quad y = [Q] - [R_{\geq 0}^q] \in K_0(R_{\geq 0}).$$

The condition $\bar{\gamma}(x, y) = 0$ translates into the equality

$$[P \otimes_{R_{\leq 0}} R] - [R^p] = [Q \otimes_{R_{\geq 0}} R] - [R^q]$$

in $K_0(R)$. Consequently, there exists $s \geq 0$ such that

$$(P \otimes_{R_{\leq 0}} R) \oplus R^q \oplus R^s \cong (Q \otimes_{R_{\geq 0}} R) \oplus R^p \oplus R^s.$$

As the right-hand module is induced from $R_{\geq 0}$ this shows that $P \otimes_{R_{\leq 0}} R$ is stably induced from $R_{\geq 0}$ so that the diagram

$$P \rightarrow P \otimes_{R_{\leq 0}} R$$

defines an object of \mathbb{D}_- . Since $R_{\leq 0}^p \rightarrow R^p$ is an object of \mathbb{D}_- as well we conclude that $x \in K_0\mathbb{D}_-$. By a symmetric argument we can show $y \in K_0\mathbb{D}_+$. As $\gamma(x, y) = 0$ (the fourth vertical map is injective), and as the bottom row is exact, we know that $(x, y) = \beta(z)$ for some $z \in K_0(\mathbb{P}^1)$, which shows exactness of the top row. This completes the proof of Theorem 2.8. \square

10 PROOF OF THE FUNDAMENTAL THEOREM

THE K -THEORY OF THE PROJECTIVE LINE REVISITED

Recall from Theorem 6.4 that there are isomorphisms of K -groups

$$\tilde{\alpha} = (\Psi_{-1,0} \ \Psi_{0,0}): K_q R_0 \oplus K_q R_0 \rightarrow K_q \mathbb{P}^1, \\ ([P], [Q]) \mapsto [\Psi_{-1,0}(P)] + [\Psi_{0,0}(Q)],$$

where $\Psi_{k,\ell}$ stands for the canonical sheaf functors (Definition 5.5). By precomposing $\tilde{\alpha}$ with the invertible map $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, we obtain modified isomorphisms

$$\alpha = \begin{pmatrix} -\Psi_{-1,0} + \Psi_{0,0} & \Psi_{0,0} \end{pmatrix}: K_q R_0 \oplus K_q R_0 \rightarrow K_q \mathbb{P}^1, \\ ([P], [Q]) \mapsto -[\Psi_{-1,0}(P)] + [\Psi_{0,0}(P)] + [\Psi_{0,0}(Q)]. \tag{10.1}$$

(This map cannot be confused with the map α from (3.3).) As we have a cylinder functor at our disposal, the additivity theorem implies that we can model the minus sign by taking suitable “homotopy cofibres” of maps of functors. Concretely, the isomorphisms α are induced by the functor

$$A: \text{Ch}^b \mathbf{P}(R_0) \times \text{Ch}^b \mathbf{P}(R_0) \rightarrow \text{Vect}(\mathbb{P}^1)_0, \\ (C, D) \mapsto \text{cone}(\Psi_{-1,0}(C) \rightarrow \Psi_{0,0}(C)) \oplus \Psi_{0,0}(D) \tag{10.2}$$

and the ensuing homotopy equivalence of K -theory spaces

$$\text{cone}(\Psi_{-1,0} \rightarrow \Psi_{0,0}) + \Psi_{0,0}: K(R_0) \times K(R_0) \rightarrow K(\mathbb{P}^1),$$

where “+” refers to the H -space structure given by direct sum. Alternatively, we can use the functor

$$A': \text{Ch}^b \mathbf{P}(R_0) \times \text{Ch}^b \mathbf{P}(R_0) \rightarrow \text{Vect}(\mathbb{P}^1)_0, \\ (C, D) \mapsto \Sigma \Psi_{-1,0}(C) \oplus \Psi_{0,0}(C) \oplus \Psi_{0,0}(D);$$

the maps induced by A and A' are homotopic, by the additivity theorem.

It will be convenient to have an explicit homotopy inverse for α . We record the following fact:

LEMMA 10.3. *The functor*

$$\Xi: \text{Vect}(\mathbb{P}^1)_0 \rightarrow \text{Ch}^b \mathbf{P}(R_0) \times \text{Ch}^b \mathbf{P}(R_0), \\ \mathcal{Y} \mapsto \begin{pmatrix} \Sigma \Gamma \mathcal{Y}(1, 0) \oplus \Gamma \mathcal{Y} \oplus \Gamma \mathcal{Y}(1, -1) \\ \Sigma \Gamma \mathcal{Y}(1, -1) \oplus \Gamma \mathcal{Y}(1, 0) \end{pmatrix}$$

induces a homotopy inverse on the level of K -theory spaces; here Γ is the “global sections” functor from Definition 5.6.

Proof. The functor A is known to induce a homotopy equivalence. Hence in view of the above remarks on A' it is enough to show that the map induced by $\Xi \circ A'$ is homotopic to the identity. Recalling the natural isomorphisms

$$\Gamma\Psi_{k,\ell}P \cong \bigoplus_{j=-\ell}^k s_j P \otimes_{R_0} R_j \quad \text{and} \quad (\Psi_{k,\ell}P)(a, b) \cong \Psi_{k+a,\ell+b}P ,$$

from Proposition 5.7, where $s_j P = P \otimes_{R_0} R_j$ is the j th shift of P , one calculates readily that the first component of $\Xi \circ A'(C, D)$ is

$$\begin{aligned} & \Sigma^2\Psi_{0,0}(C) \oplus \Sigma\Gamma\Psi_{1,0}(C) \oplus \Sigma\Gamma\Psi_{1,0}(D) \\ & \oplus \Sigma\Gamma\Psi_{-1,0}(C) \oplus \Gamma\Psi_{0,0}(C) \oplus \Gamma Y_{0,0}(D) \\ & \oplus \Sigma\Gamma\Psi_{0,-1}(C) \oplus \Gamma\Psi_{1,-1}(C) \oplus \Gamma\Psi_{1,-1}(D) \\ \cong & \Sigma^2 C \oplus \Sigma C \oplus \Sigma s_1 C \oplus \Sigma D \oplus \Sigma s_1 D \\ & \oplus C \oplus D \\ & \oplus s_1 C \oplus s_1 D , \end{aligned}$$

while the second component is

$$\begin{aligned} & \Sigma^2\Psi_{0,-1}(C) \oplus \Sigma\Gamma\Psi_{1,-1}(C) \oplus \Sigma\Gamma\Psi_{1,-1}(D) \\ & \oplus \Sigma\Gamma\Psi_{0,0}(C) \oplus \Gamma\Psi_{1,0}(C) \oplus \Gamma Y_{1,0}(D) \\ \cong & \Sigma s_1 C \oplus \Sigma s_1 D \\ & \oplus \Sigma C \oplus C \oplus s_1 C \oplus D \oplus s_1 D . \end{aligned}$$

Recalling that suspension represents the H -space structure inverse on K -theory spaces, this shows that the composition $\Xi \circ A'$ induces a map that is homotopic to the identity as required. \square

THE MODIFIED MAYER-VIETORIS SEQUENCE

On a much more elementary level, we have automorphisms

$$\eta = \begin{pmatrix} \text{id} & -i_*^- p_*^+ \\ 0 & \text{id} \end{pmatrix} : K_q(R_{\leq 0}) \oplus K_q(R_{\geq 0}) \rightarrow K_q(R_{\leq 0}) \oplus K_q(R_{\geq 0}) \quad (q \geq 0)$$

with inverse given by

$$\eta^{-1} = \begin{pmatrix} \text{id} & i_*^- p_*^+ \\ 0 & \text{id} \end{pmatrix} : K_q(R_{\leq 0}) \oplus K_q(R_{\geq 0}) \rightarrow K_q(R_{\leq 0}) \oplus K_q(R_{\geq 0}) .$$

Both maps are induced by functors, with the minus sign modelled by suspension (making implicit use of the additivity theorem and the presence of a cylinder functor again). Explicitly, η is induced by the functor

$$\begin{aligned} \text{Ch}^b \mathbf{P}(R_{\leq 0}) \times \text{Ch}^b \mathbf{P}(R_{\geq 0}) & \rightarrow \text{Ch}^b \mathbf{P}(R_{\leq 0}) \times \text{Ch}^b \mathbf{P}(R_{\geq 0}) , \\ (C, D) & \mapsto (C \oplus \Sigma i_*^- p_*^+ D, D) . \end{aligned}$$

We use the map η and its inverse to construct a modified MAYER-VIETORIS sequence

$$\begin{aligned} \dots &\xrightarrow{\gamma\eta^{-1}} K_{q+1}R \xrightarrow{\alpha^{-1}\delta} K_qR_0 \oplus K_qR_0 \\ &\xrightarrow{\eta\beta\alpha} K_qR_{\leq 0} \oplus K_qR_{\geq 0} \xrightarrow{\gamma\eta^{-1}} K_qR \xrightarrow{\alpha^{-1}\delta} \dots \end{aligned} \tag{10.4}$$

ending with the exact sequence

$$K_0R_0 \oplus K_0R_0 \xrightarrow{\eta\beta\alpha} K_0R_{\leq 0} \oplus K_0R_{\geq 0} \xrightarrow{\gamma\eta^{-1}} K_0R \rightarrow {}^R K_{-1}R_0 \rightarrow 0 .$$

Note that any exact sequence of the form $A \xrightarrow{f} B \xrightarrow{g} C$ gives rise to a short exact sequence

$$0 \rightarrow A/\ker(f) \rightarrow B \rightarrow \operatorname{im}(g) \rightarrow 0 ;$$

in this way, sequence (10.4) can be split into short exact sequences

$$\begin{aligned} 0 \rightarrow (K_qR_{\leq 0} \oplus K_qR_{\geq 0})/\ker(\gamma\eta^{-1}) &\xrightarrow{\gamma\eta^{-1}} K_qR \\ &\xrightarrow{\alpha^{-1}\delta} \operatorname{im}(\alpha^{-1}\delta) \rightarrow 0 \quad (q > 0) \end{aligned}$$

and

$$0 \rightarrow (K_0R_{\leq 0} \oplus K_0R_{\geq 0})/\ker(\gamma\eta^{-1}) \xrightarrow{\gamma\eta^{-1}} K_0R \rightarrow {}^R K_{-1}R_0 \rightarrow 0 .$$

From exactness of the modified MAYER-VIETORIS sequence (10.4) again we have the equalities $\ker(\gamma\eta^{-1}) = \operatorname{im}(\eta\beta\alpha)$ and $\operatorname{im}(\alpha^{-1}\delta) = \ker(\eta\beta\alpha)$, so we obtain short exact sequences

$$0 \rightarrow \operatorname{coker}(\eta\beta\alpha) \rightarrow K_qR \rightarrow \ker(\eta\beta\alpha) \rightarrow 0 \quad (q > 0) \tag{10.5}$$

and

$$0 \rightarrow \operatorname{coker}(\eta\beta\alpha) \rightarrow K_0R \rightarrow {}^R K_{-1}R_0 \rightarrow 0 . \tag{10.6}$$

THE MAP $\eta\beta\alpha$

On the level of categories the effect of the map $\beta\alpha$ is to send $(P, Q) \in \operatorname{Ch}^b \mathbf{P}(R_0) \times \operatorname{Ch}^b \mathbf{P}(R_0)$ to $(C, D) \in \operatorname{Ch}^b \mathbf{P}(R_{\leq 0}) \times \operatorname{Ch}^b \mathbf{P}(R_{\geq 0})$ where

$$\begin{aligned} C &= \operatorname{cone}(P \otimes_{R_0} R_{\leq -1} \xrightarrow{\subseteq} P \otimes_{R_0} R_{\leq 0}) \oplus (Q \otimes_{R_0} R_{\leq 0}) \\ &= i_*^- \operatorname{cone}(s_{-1}P \rightarrow P) \oplus i_*^- Q \end{aligned}$$

and

$$\begin{aligned} D &= \operatorname{cone}(P \otimes_{R_0} R_{\geq 0} \xrightarrow{\cong} P \otimes_{R_0} R_{\geq 0}) \oplus (Q \otimes_{R_0} R_{\geq 0}) \\ &\simeq Q \otimes_{R_0} R_{\geq 0} = i_*^+ Q ; \end{aligned}$$

here s_{-1} denotes the shift functor

$$s_{-1} : Q \mapsto Q \otimes_{R_0} R_{-1}$$

which maps the category of finitely generated projective R_0 -modules to itself. — To determine the effect of the map η on (C, D) , recall first that weakly equivalent functors induce homotopic maps on K -theory spaces. So we can, for example, replace D by $i_*^+ Q$. The first component of $\eta\beta\alpha(P, Q)$ then is, up to homotopy,

$$i_*^- \operatorname{cone}(s_{-1}P \rightarrow P) \oplus i_*^- Q \oplus \Sigma i_*^- p_*^+ i_*^+ Q .$$

Since $p_*^+ i_*^+ Q = Q$, and since suspension represents a homotopy inverse for direct sum [Wal85, Proposition 1.6.2], we can thus model the effect of $\eta\beta\alpha$ by the functor

$$(P, Q) \mapsto (i_*^- \operatorname{cone}(s_{-1}P \rightarrow P), i_*^+ Q) .$$

On the level of K -groups, this means that the effect of $\eta\beta\alpha$ is described by the diagonal matrix

$$\eta\beta\alpha = \begin{pmatrix} i_*^-(\operatorname{id} - s_{-1}) & 0 \\ 0 & i_*^+ \end{pmatrix} : K_q R_0 \oplus K_q R_0 \rightarrow K_q R_{\leq 0} \oplus K_q R_{\geq 0} . \quad (10.7)$$

THE KERNEL OF $\eta\beta\alpha$

In view of (10.7), $\ker(\eta\beta\alpha) = \ker(i_*^-(s_{-1} - \operatorname{id})) \oplus \ker(i_*^-)$. Now the maps i_*^\mp are split injective (with left inverse p_*^\mp), thus

$$\ker(\eta\beta\alpha) \cong \ker(\operatorname{sd}_*) \oplus \{0\} = {}^R_{\ker} K_q R_0 , \quad (10.8)$$

where sd_* denotes the shift difference map $\operatorname{sd}_* = \operatorname{id} - s_{-1} : K_q R_0 \rightarrow K_q R_0$ from (2.4) with kernel ${}^R_{\ker} K_q R_0$ (Definition 2.3).

THE COKERNEL OF $\eta\beta\alpha$

From the specific representation of $\eta\beta\alpha$ in (10.7) we read off that

$$\operatorname{coker}(\eta\beta\alpha) = \operatorname{coker}(i_*^- \circ \operatorname{sd}_*) \oplus \operatorname{coker}(i_*^+) ,$$

the second summand being nothing but ${}^R \operatorname{NK}_q^+ R_0$ by definition. To identify the first summand we compose $i_*^- \circ \operatorname{sd}_*$ with the splitting isomorphism (2.2):

$$K_q(R_0) \xrightarrow{\operatorname{sd}_*} K_q(R_0) \xrightarrow{i_*^-} K_q(R_{\leq 0}) \xrightarrow[\cong]{S=(c, p_*^-)} {}^R \operatorname{NK}_q^- \oplus K_q(R_0)$$

(Here $c : K_q(R_{\leq 0}) \rightarrow {}^R \operatorname{NK}_q^-$ is the canonical projection onto the cokernel of i_*^- .) Thus $\operatorname{coker}(i_*^- \circ \operatorname{sd}_*) \cong \operatorname{coker}(S \circ i_*^- \circ \operatorname{sd}_*)$. Since $c \circ i_*^- = 0$, and since $p_*^- \circ i_*^-$ is the identity, we conclude that this group is isomorphic to the cokernel of the composition

$$K_q R_0 \xrightarrow{\operatorname{sd}_*} K_q R_0 \xrightarrow{\subseteq} {}^R \operatorname{NK}_q^- R_0 \oplus K_q R_0 ,$$

which is ${}^R\mathbf{NK}_q^- \oplus {}_{\text{coker}}{}^R K_q R_-$. In total, this results in an isomorphism

$$\text{coker}(\eta\beta\alpha) \cong {}^R\mathbf{NK}_q^- R_0 \oplus {}_{\text{coker}}{}^R K_q R_0 \oplus {}^R\mathbf{NK}_q^+ R_0 . \tag{10.9}$$

Putting the identifications (10.8) and (10.9) of the kernel and cokernel of $\eta\beta\alpha$ into the sequences (10.5) and (10.6) gives precisely the advertised Fundamental Theorem 2.7. \square

11 THE LOCALISATION SEQUENCE

In this section we will establish the long exact “localisation” sequence of Theorem 2.9. There are in fact two “mirror-symmetric” versions, which we state in full for completeness.

THEOREM 11.1 (“Localisation sequence”). *Let R be a strongly \mathbb{Z} -graded ring. There are long exact sequences of algebraic K -groups*

$$\begin{aligned} \dots \rightarrow K_{q+1}R \rightarrow K_q({}^R\mathbf{Nil}^+(R_0)) \xrightarrow{\phi} K_q R_{\geq 0} \rightarrow K_q R \\ \rightarrow \dots \rightarrow K_0({}^R\mathbf{Nil}^+(R_0)) \xrightarrow{\phi} K_0 R_{\geq 0} \rightarrow K_0 R \end{aligned} \tag{11.1a}$$

and

$$\begin{aligned} \dots \rightarrow K_{q+1}R \rightarrow K_q({}^R\mathbf{Nil}^-(R_0)) \xrightarrow{\phi} K_q R_{\leq 0} \rightarrow K_q R \\ \rightarrow \dots \rightarrow K_q({}^R\mathbf{Nil}^-(R_0)) \xrightarrow{\phi} K_0 R_{\leq 0} \rightarrow K_0 R \end{aligned} \tag{11.1b}$$

with ϕ induced by the forgetful functor $(Z, \zeta) \mapsto Z$ on the category ${}^R\mathbf{Nil}^\pm(R_0)$.

Proof. We will show that the sequence (11.1a) is exact; the argument for sequence (11.1b) is similar. — We apply WALDHAUSEN’s fibration theorem to the right-hand vertical map in the fundamental square (8.1). In view of Lemma 8.9 this results in exact sequences of K -groups

$$\dots \rightarrow K_{q+1}\mathbb{D}_+ \rightarrow K_{q+1}\mathbb{D}_0 \rightarrow \pi_q\Omega|h_+\text{Vect}(\mathbb{P}^1)^{h_0}| \xrightarrow{k'} K_q\mathbb{D}_+ \rightarrow K_q\mathbb{D}_0$$

for $q \geq 0$. In view of Lemma 8.8 and Theorem 8.3, this proves exactness of the sequence (11.1a) in the $q \geq 1$ range down to the term $K_1 R_{\geq 0}$. Lemma 8.8 also states that in the commutative diagram in Fig. 2 the two vertical maps labelled f and g are injective. As the bottom row is exact, the top row is automatically exact as well except possibly at $K_0 R_{\geq 0}$. Let $x \in K_0 R_{\geq 0}$ be an element with $j_*^+(x) = 0$. We can write $x = [P] - [Q]$, for finitely generated projective $R_{\geq 0}$ -modules P and Q . The condition $j_*^+[P] - j_*^+[Q] = j_*^+(x) = 0 \in K_0 R$, that is, $[P \otimes_{R_{\geq 0}} R] = [Q \otimes_{R_{\geq 0}} R] \in K_0 R$, means that there exists a number $\ell \geq 0$ together with an isomorphism

$$j_*^+(P) \oplus R^\ell = (P \otimes_{R_{\geq 0}} R) \oplus R^\ell \cong (Q \otimes_{R_{\geq 0}} R) \oplus R^\ell = j_*^+(Q) \oplus R^\ell .$$

$$\begin{array}{ccccccc}
 K_1 R_{\geq 0} & \longrightarrow & K_1 R & \longrightarrow & \pi_0 \Omega |h_+ \text{Vect}(\mathbb{P}^1)^{h_0}| & \xrightarrow{k} & K_0 R_{\geq 0} & \xrightarrow{j_*^+} & K_0 R \\
 \uparrow = & & \uparrow = & & \uparrow = & & \uparrow \subseteq f & & \uparrow \subseteq g \\
 K_1 \mathbb{D}_+ & \longrightarrow & K_1 \mathbb{D}_0 & \longrightarrow & \pi_0 \Omega |h_+ \text{Vect}(\mathbb{P}^1)^{h_0}| & \xrightarrow{k'} & K_0 \mathbb{D}_+ & \xrightarrow{j'_*} & K_0 \mathbb{D}_0
 \end{array}$$

Figure 2: Diagram used to establish the tail end of the localisation sequence

Let P' be a complement of P so that $P' \oplus P$ is a finitely generated free $R_{\geq 0}$ -module. Then $j_*^+(P' \oplus P) \oplus R^\ell$ is a finitely generated free R -module, and

$$j_*^+(P' \oplus P) \oplus R^\ell = j_*^+(P') \oplus j_*^+(P) \oplus R^\ell \cong j_*^+(P') \oplus j_*^+(Q) \oplus R^\ell = j_*^+(P' \oplus Q) \oplus R^\ell$$

so that $j_*^+(P' \oplus Q) \oplus R^\ell$ is a finitely generated free R module as well. Consequently, both $j_*^+(P' \oplus P)$ and $j_*^+(P' \oplus Q)$ stably extend to $R_{\leq 0}$; thus the two diagrams

$$p = \left(j_*^+(P' \oplus P) \leftarrow P' \oplus P \right) \quad \text{and} \quad q = \left(j_*^+(P' \oplus Q) \leftarrow P' \oplus Q \right)$$

are objects of \mathbb{D}_+ . Let us consider the element $z = [p] - [q] \in K_0 \mathbb{D}_+$. We calculate

$$f(z) = f([p] - [q]) = [P' \oplus P] - [P' \oplus Q] = [P] - [Q] = x$$

(see Lemma 8.8 for the effect of f). As $x \in \ker(j_*^+)$, and as g is an injection, this implies $z \in \ker(j'_*)$, and by exactness of the lower horizontal sequence we infer that $z = k'(y)$ for some $y \in \pi_0 \Omega |h_+ \text{Vect}(\mathbb{P}^1)^{h_0}|$. Then $k(y) = fk'(y) = f(z) = x$ so that $x \in \text{im } k$. This proves the top row to be exact at $K_0 R_{\geq 0}$, and establishes together with Theorem 8.3 the tail end of the long exact sequence.

It remains to identify the map ϕ in the sequence (11.1a). To this end, let $\overline{\mathbf{FD}}(R_{\geq 0})$ denote the category of bounded complexes of $R_{\geq 0}$ -modules which are quasi-isomorphic to a bounded complex of finitely generated projective $R_{\geq 0}$ -modules, and consist of projective R_0 -modules. The inclusion functor

$$\text{Ch}^b \mathbf{P}(R_{\geq 0}) \rightarrow \overline{\mathbf{FD}}(R_{\geq 0})$$

is exact and yields isomorphisms on algebraic K -groups (with respect to weak equivalences the quasi-isomorphisms, and cofibrations the monomorphisms with levelwise R_0 -projective cokernel) as can be checked with the help of WALDHAUSEN's approximation theorem. This inclusion is the right hand vertical map in the square diagram of Fig. 3. The top horizontal arrow maps the complex

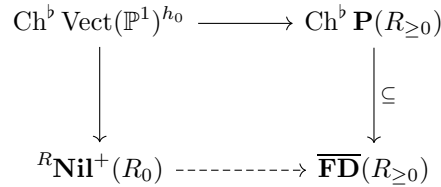


Figure 3: Diagram used in proof of localisation sequence

of vector bundles \mathcal{Z} to its component Z^+ . The left-hand vertical map is the one discussed in Theorem 8.3, identifying the group $\pi_q \Omega |h_+ \mathrm{Vect}(\mathbb{P}^1)^{h_0}|$ with $K_q({}^R\mathbf{Nil}^+(R_0))$; it is given by the assignment

$$\mathcal{Z} = \left(Z^- \rightarrow Z^0 \xleftarrow{\zeta^+} Z^+ \right) \mapsto (Z^+, \zeta)$$

with ζ the composition $Z^+ \otimes_{R_0} R_1 \cong Z^+ \otimes_{R_{\geq 0}} R_{\geq 1} \rightarrow Z^+ \otimes_{R_{\geq 0}} R_{\geq 0} \cong Z^+ \otimes_{R_0} R_0$. Note that Z^+ consists of projective R_0 -modules since R is *strongly* \mathbb{Z} -graded, that Z^+ is R_0 -finitely dominated by Lemma 7.13, and that ζ is homotopy nilpotent by Corollary 7.12. Using the functor $(Z, \zeta) \mapsto Z^\zeta$ in the lower horizontal position renders the square diagram commutative; the complex Z^ζ is indeed an object of $\overline{\mathbf{FD}}(R_{\geq 0})$ by Remark 7.15. The induced map on algebraic K -groups is the map ϕ occurring in the sequence (11.1a). — The proof is complete. \square

12 THE K -THEORY OF HOMOTOPY NILPOTENT TWISTED ENDOMORPHISMS

The groups ${}^R\mathrm{NK}_0^\mp R_0$ can be interpreted as obstruction groups for finitely generated projective modules over $R_{\leq 0}$ or $R_{\geq 0}$, respectively, to be stably induced from R_0 (Proposition 3.4). For $q > 0$ the groups ${}^R\mathrm{NK}_q^\pm R_0$ are isomorphic to the (reduced) algebraic K -groups of the category of homotopy nilpotent twisted endomorphisms, as will be shown in this section. This will also complete the proof of Theorem 2.9.

THEOREM 12.1 (${}^R\mathrm{NK}_q^\mp$ is K -theory of homotopy nilpotent twisted endomorphisms). *For $q > 0$ there are natural isomorphisms of ABELian groups*

$${}^R\mathrm{NK}_q^+(R_0) \xrightarrow{\cong} {}^R\mathrm{Nil}_{q-1}^-(R_0) = \ker \left(o_*^- : K_{q-1}({}^R\mathbf{Nil}^-(R_0)) \rightarrow K_{q-1}(R_0) \right) \quad (12.1a)$$

and

$${}^R\mathrm{NK}_q^-(R_0) \xrightarrow{\cong} {}^R\mathrm{Nil}_{q-1}^+(R_0) = \ker \left(o_*^+ : K_{q-1}({}^R\mathbf{Nil}^+(R_0)) \rightarrow K_{q-1}(R_0) \right) \quad (12.1b)$$

with the groups ${}^R\mathrm{Nil}_{q-1}^\mp(R_0)$ from (7.19a) and (7.19b), and maps o_*^\mp as introduced in (7.18) induced by the forgetful functors $o^\mp : (Z, \zeta) \mapsto Z$.

Proof. Upon taking the homotopy fibre of the top horizontal map in the fundamental square (8.1) we obtain a long exact sequence of K -groups containing the snippet

$$K_q\mathbb{P}^1 \rightarrow K_q\mathbb{D}_+ \xrightarrow{\nabla} K_{q-1}\text{Vect}(\mathbb{P}^1)^{h+} \xrightarrow{\iota} K_{q-1}\mathbb{P}^1 \rightarrow K_{q-1}\mathbb{D}_+ ,$$

with ι induced by the inclusion functor $\text{Ch}^b \text{Vect}(\mathbb{P}^1)^{h+} \rightarrow \text{Ch}^b \text{Vect}(\mathbb{P}^1)$; the symbol ∇ stands for a connecting homomorphism in the long exact sequence of homotopy groups associated with the fibration. — Using the isomorphism $\alpha: K_qR_0 \oplus K_qR_0 \rightarrow K_q\mathbb{P}^1$ from (10.1), and with $K_k(R_{\geq 0})$ in place of $K_k\mathbb{D}_+$ (Lemma 8.8), we obtain the exact sequence

$$\begin{aligned} K_qR_0 \oplus K_qR_0 &\xrightarrow{\epsilon} K_qR_{\leq 0} \xrightarrow{\nabla} K_{q-1}\text{Vect}(\mathbb{P}^1)^{h+} \\ &\xrightarrow{\alpha^{-1}\iota} K_{q-1}R_0 \oplus K_{q-1}R_0 \xrightarrow{\epsilon} K_{q-1}R_{\leq 0} . \end{aligned}$$

The map ϵ is the difference of maps induced by the functors

$$(P, Q) \mapsto (P \otimes_{R_0} R_{\geq 0}) \oplus (Q \otimes_{R_0} R_{\geq 0}) \quad \text{and} \quad (P, Q) \mapsto P \otimes_{R_0} R_{\geq 0} ,$$

thus ϵ is the usual split injection induced from $i^+: Q \mapsto Q \otimes_{R_0} R_{\geq 0}$ on the second summand, and is the zero map on the first. Hence the image of the map $\alpha^{-1}\iota$, which equals the kernel of ϵ , is $K_{q-1}R_0 \oplus \{0\}$, and there results another exact sequence

$$0 \rightarrow \{0\} \oplus K_qR_0 \xrightarrow{i_*^+} K_qR_{\geq 0} \xrightarrow{\nabla} K_{q-1}\text{Vect}(\mathbb{P}^1)^{h+} \xrightarrow{\text{pr}_1\alpha^{-1}\iota} K_{q-1}R_0 \rightarrow 0 .$$

By Theorem 8.3 we have an isomorphism

$$\omega: K_{q-1}\text{Vect}(\mathbb{P}^1)^{h+} \xrightarrow{\cong} K_{q-1}{}^R\text{Nil}^-(R_0) .$$

Claim: The composition $\text{pr}_1\alpha^{-1}\iota\omega^{-1}$ is induced
by the forgetful functor $o^-: (Y, v) \mapsto Y$. (12.2)

Assuming the claim for the moment, we obtain an exact sequence

$$K_qR_0 \xrightarrow{i_*^+} K_qR_{\geq 0} \xrightarrow{\omega\nabla} K_{q-1}{}^R\text{Nil}^-(R_0) \xrightarrow{o^-} K_{q-1}R_0 ,$$

giving the isomorphism

$${}^R\text{NK}_q^+ = \text{coker } i_*^+ = K_qR_{\geq 0} / \ker(\omega\nabla) \cong \text{im}(\omega\nabla) = \ker(o^-) = {}^R\text{Nil}_{q-1}^-R_0 .$$

This establishes (12.1a). The isomorphism of (12.1b) is verified using a symmetric argument, employing the isomorphism

$$\begin{aligned} K_qR_0 \oplus K_qR_0 &\rightarrow K_q\mathbb{P}^1 , \\ ([P], [Q]) &\mapsto [\Psi_{0,0}(P)] - [\Psi_{0,-1}(P)] + [\Psi_{0,0}(Q)] , \end{aligned} \tag{12.3}$$

in place of α .

It remains to verify Claim (12.2). By Lemma 10.2 the isomorphism α^{-1} is induced by the functor

$$\begin{aligned} \Xi: \text{Vect}(\mathbb{P}^1)_0 &\rightarrow \text{Ch}^b \mathbf{P}(R_0) \times \text{Ch}^b \mathbf{P}(R_0), \\ \mathcal{Y} &\mapsto \begin{pmatrix} \Sigma\Gamma\mathcal{Y}(1,0) \oplus \Gamma\mathcal{Y} \oplus \Gamma\mathcal{Y}(1,-1) \\ \Sigma\Gamma\mathcal{Y}(1,-1) \oplus \Gamma\mathcal{Y}(1,0) \end{pmatrix} \end{aligned}$$

Now consider the diagram in Fig. 4. It is not a commutative diagram of functors, but on the level of K -theory spaces it is homotopy commutative. Since all vertical arrows marked as inclusion maps induce identity maps on K -groups, and since ω induces an isomorphism on K -groups, this establishes Claim (12.2). — The vertical arrows a and b result in homotopy equivalences on K -theory

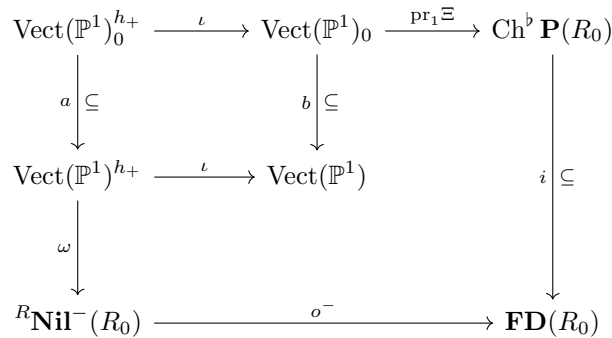


Figure 4: Diagram used to establish Claim (12.2)

spaces with respect to h -equivalences; this is analogous to Lemma 6.3, and the arguments of Lemma III.1.3 and Corollary III.1.4 of [HM20] carry over verbatim. For the arrow i see (7.17). As to the alleged homotopy commutativity, recall that for $\mathcal{Z} \in \text{Vect}(\mathbb{P}^1)_0$ we have $\lim^1_{\leftarrow} \mathcal{Z}(k, \ell) = H^1 \mathcal{Z}(k, \ell) = 0$ when $k + \ell \geq 0$, by definition of the category $\text{Vect}(\mathbb{P}^1)_0$; consequently, the map $\Gamma \mathcal{Z}(k, \ell) \rightarrow \text{holim} \mathcal{Z}(k, \ell)$ is a quasi-isomorphism when $k + \ell \geq 0$, where “holim” stands for the homotopy limit or homotopy pullback construction, which is the dual of the double mapping cylinder. Furthermore, for $\mathcal{Z} \in \text{Vect}(\mathbb{P}^1)_0^{h+}$ we have $\mathcal{Z}(k, \ell)^+ = Z^+ \otimes_{R_{\geq 0}} R_{\geq -\ell} \simeq *$ and $\mathcal{Z}(k, \ell)^0 = Z^0 \otimes_R R \simeq *$, which for $k + \ell \geq 0$ implies

$$\begin{aligned} \Gamma \mathcal{Z}(k, \ell) &\simeq \text{holim} (Z^- \otimes_{R_{\leq 0}} R_{\leq k} \rightarrow Z^0 \otimes_R R \leftarrow Z^+ \otimes_{R_{\geq 0}} R_{\geq -\ell}) \\ &\simeq \text{holim} (Z^- \otimes_{R_{\leq 0}} R_{\leq k} \rightarrow * \leftarrow *) \simeq Z^- \otimes_{R_{\leq 0}} R_{\leq k}. \end{aligned}$$

Thus the composition $i \text{pr}_1 \Xi \iota$, that is, the functor that sends $\mathcal{Y} \in \text{Vect}(\mathbb{P}^1)_0^{h+}$ to

$$\Sigma\Gamma\mathcal{Y}(1,0) \oplus \Gamma\mathcal{Y} \oplus \Gamma\mathcal{Y}(1,-1),$$

is weakly equivalent to the functor that sends \mathcal{Y} to

$$(\Sigma Y^- \otimes_{R_{\leq 0}} R_{\leq 1}) \oplus Y^- \oplus (Y^- \otimes_{R_{\leq 0}} R_{\leq 1}) ;$$

by the additivity theorem, the induced map on K -theory spaces is homotopic to the map induced by $\mathcal{Y} \mapsto Y^-$ since suspension models an inverse for the H -space structure. But the formula $\mathcal{Y} \mapsto Y^-$ describes the effect of $o^- \circ \omega$ as well. This finishes the proof. \square

REMARK 12.4. The groups $K_q({}^R\mathbf{Nil}^+(R_0))$ can also be obtained as QUILLEN K -groups of the exact category of finitely generated projective R_0 -modules P equipped with nilpotent twisted endomorphisms $P \otimes_{R_0} R_1 \rightarrow P$, as is implied by the work of LÜCK and STEIMLE [LS16]. In their notation, \mathcal{A} is the category of finitely generated projective R_0 -modules, and Φ is the endofunctor $- \otimes_{R_0} R_1$ of \mathcal{A} ; for \mathcal{A}^\times one substitutes the category of projective R_0 -modules. All relevant formulas and arguments of *loc. cit.* §§8.2–3 and from the beginning of the proof of their Theorem 8.2 carry over without further changes, establishing the claim. (It should be noted that the category $\mathcal{A}_\Phi[t, t^{-1}]$ appearing in other parts of [LS16] does not correspond to the category of all finitely generated projective R -modules but to modules induced from R_0 only.)

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