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Concentration of Measure: Non-Asymptotic Analysis for Uplink MU-MIMO

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***Abstract*—This paper considers uplink multiple-user multiple-input multiple-output (MU-MIMO) systems, in which multiple single-antenna users transmit signals to a multiple-antenna base station (BS) simultaneously. The maximum-ratio-combining (MRC) detection scheme is applied at the BS. Our main focus is on the non-asymptotic concentration of measure analysis for the instantaneous rate. Firstly, the tail probability of the instantaneous rate is derived, from which, a trade-off function is proposed and optimized. Our solution determines the coefficient included in the tail probability and reveals a trade-off between the tail probability and the offset of the instantaneous rate from its mean value. Subsequently, based on the tail probability, a narrow interval that the instantaneous rate falls within with high probability is provided. We show that this narrow interval shrinks with the number of BS antennas. Finally, we use our non-asymptotic results to theoretically characterize the outage probability.**

***Index Terms*—Concentration of measure, massive MIMO, MIMO, non-asymptotic, tail probability.**

I. Introduction

During the past decades, the fundamental metrics of wireless communications, such as the ergodic capacity [1] and the outage probability [2] have been comprehensively analyzed, with the derived results being inherently complicated. Recently, a series of works have analyzed large-scale MIMO networks with the aid of random matrix theory [3] or the law of large numbers [4]. However, in the former case, the final results are complicated and provide limited, if any, insights. In the latter case, the final results are indeed simpler, yet, they have focused, to date, on the ergodic rate [5].

We now point out that the ergodic rate may overestimate or underestimate the instantaneous performance, even in the small-to-moderate number of antennas regime [6]. This is why some existing works have focused on making the wireless channels less random using, for example, the massive MIMO [4] or reflective intelligent surfaces (RIS) [7] technologies. Thus, investigating the impact of the randomness of channels on the instantaneous performance with a simple method is indeed important and can facilitate the inherently challenging outage probability analysis.

In this context, a non-asymptotic analysis was presented in [8], [9]. This theoretical framework aims at providing the statistical properties of the random performance metrics for an arbitrary number of antenna elements [8]. Henceforth, "non-asymptotic" indicates that we will not be looking into limiting results (e.g., those asymptotic results obtained under the law of large numbers), but rather into capturing the dependence of the statistical properties of the instantaneous performance metric on its dimensional parameters (e.g., the number of antennas at the BS) and other system parameters (e.g., the instantaneous signal-to-noise ratio (SNR)). One of the fundamental tools for the non-asymptotic analysis is the concentration of measure (CoM) [10], due to the following reasons [11]: firstly, it provides an achievable confidence level, under which, the instantaneous performance metrics fluctuate within a narrow region; secondly, this method illuminates explicitly and more precisely the interactions between different parameters, even in the high-dimensional regime. In this context, the CoM has been used in [6] and [12] for the non-asymptotic analysis of the channel capacity.

Motivated by the above discussion, a MU-MIMO system is considered, in which, multiple single-antenna users transmit signals simultaneously to a multiple-antenna BS. A linear MRC detector is deployed at the BS. The main contributions of this paper are as follows:

1) A CoM-based non-asymptotic analysis of the instantaneous rate is presented. To the best of our knowledge, this paper is the first to pursue an non-asymptotic analysis for uplink MU-MIMO networks with the linear detection scheme at the BS.
2) The tail probability of the instantaneous rate is presented. From this result, the achievable confidence level (the tail probability) is derived and a trade-off function is proposed and optimized, which is used to control the trade-off between the achievable confidence level and the offset of the instantaneous rate from its mean value.
3) This paper also provides a narrow interval that the instantaneous rate falls within with high probability. The analytical and simulation results reveal that when the number of BS antennas increases, this narrow interval shrinks while the instantaneous rate will stabilize.
4) The application of the non-asymptotic results on the outage probability analysis is presented in detail.

II. System Model

This paper considers a simple uplink transmission process, in which K_u single-antenna users transmit signals simultaneously to an N_b-antenna BS. To facilitate our analysis, we consider the coherent case hereafter, i.e., the BS has perfect channel state information (CSI) of all users.

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Firstly, all users transmit signals to the BS and the complex received signal vector $\mathbf{y} \in \mathcal{C}^{N_b \times 1}$ at the BS is

$$\mathbf{y} = \sum_{k=1}^{K_u} \mathbf{h}_k x_k + \mathbf{n} = \mathbf{H}\mathbf{x} + \mathbf{n}, \tag{1}$$

where $\mathbf{h}_k \in \mathcal{C}^{N_b \times 1}$ denotes the uplink channel vector from the k-th user to the BS, which is circularly-symmetric complex Gaussian distributed, denoted by $\mathcal{CN}(\mathbf{0}, \gamma_k^2 \mathbf{I}_{N_b})$, in which γ_k^2 represents the large-scale fading coefficient. The matrix $\mathbf{H} = [\mathbf{h}_1, \ldots, \mathbf{h}_{K_u}] \in \mathcal{C}^{N_b \times K_u}$ denotes the channel matrix from all users to the BS, which includes independent columns. The scalar x_k is the symbol transmitted by the k-th user with the distribution $\mathcal{CN}(0, P_k)$ and $\mathbf{x} = [x_1, \ldots, x_{K_u}]^T$ includes independent elements, where the superscript $(\cdot)^T$ denotes the transpose operator. The vector $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I}_{N_b})$ is the additive white Gaussian noise (AWGN) at the BS.

Then, the BS will detect x_k from the received signal vector $\mathbf{y}$ given in (1) with the MRC detection scheme, and the corresponding detecting vector $\mathbf{a}_k \in \mathcal{C}^{1 \times N_b}$ is

$$\mathbf{a}_k = \frac{\mathbf{h}_k^\dagger}{\|\mathbf{h}_k\|_2}, \quad k = 1, \ldots, K_u, \tag{2}$$

where the symbol $\|\cdot\|_2$ is the 2-norm of a vector and the superscript $(\cdot)^\dagger$ denotes the conjugate transpose. Multiplying $\mathbf{a}_k$ on both sides of (1), it is easy to see that the signal-to-interference-plus-noise ratio (SINR) for the k-th user under the power scaling law $P_k = \frac{E_k}{N_b}$, for all $k = 1, \ldots, K_u$, is

$$\begin{aligned} \mathrm{SINR}_k &= \frac{E_k}{N_b} \|\mathbf{h}_k\|_2^2 \left(\sum_{\substack{m=1 \\ m \neq k}}^{K_u} \frac{E_m}{N_b} \left| \frac{\mathbf{h}_k^\dagger}{\|\mathbf{h}_k\|_2} \mathbf{h}_m \right|^2 + \sigma^2 \right)^{-1} \\ &= \frac{\frac{E_k}{N_b} \|\mathbf{h}_k\|_2^2}{\sum\limits_{\substack{m=1 \\ m \neq k}}^{K_u} \frac{E_m}{N_b} |\tilde{h}_m|^2 + \sigma^2} = \frac{\frac{\gamma_k^2 E_k}{\sigma^2 N_b} \tilde{\mathbf{h}}_k^\dagger \tilde{\mathbf{h}}_k}{\frac{1}{N_b} \tilde{\mathbf{h}}_{I/k}^\dagger \mathbf{D}_{I/k} \tilde{\mathbf{h}}_{I/k} + 1}, \end{aligned} \tag{3}$$

where E_k is a fixed value and $\tilde{h}_m \triangleq \frac{\mathbf{h}_k^\dagger}{\|\mathbf{h}_k\|_2} \mathbf{h}_m$, $m \neq k$. By following the same methodology as in [5, Appendix A], we can show that the complex random variable $\tilde{h}_m$ is distributed as $\tilde{h}_m \sim \mathcal{CN}(0, \gamma_m^2)$ and it is independent of $\mathbf{h}_k$. The random vector $\tilde{\mathbf{h}}_k = \frac{1}{\gamma_k} \mathbf{h}_k$ is distributed as $\tilde{\mathbf{h}}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{N_b})$ and the random vector $\tilde{\mathbf{h}}_{I/k} \in \mathcal{C}^{(K_u - 1) \times 1}$ is defined as

$$\tilde{\mathbf{h}}_{I/k} \triangleq \left(\tfrac{1}{\gamma_1} \tilde{h}_1, \ \ldots, \ \tfrac{1}{\gamma_{k-1}} \tilde{h}_{k-1}, \ \tfrac{1}{\gamma_{k+1}} \tilde{h}_{k+1}, \ \ldots, \ \tfrac{1}{\gamma_{K_u}} \tilde{h}_{K_u} \right)^T,$$

whose distribution is $\tilde{\mathbf{h}}_{I/k} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{K_u - 1})$. The matrix $\mathbf{D}_{I/k} \in \mathcal{C}^{K_u - 1}$ is a diagonal matrix with the i-th diagonal element as $\frac{\gamma_i^2 E_i}{\sigma^2}$, for $i = 1, \ldots, k-1, k+1, \ldots, K_u$.

So, based on (3), the instantaneous rate (in nats/s)[1] of the k-th user under the power scaling law $\{P_k = \frac{E_k}{N_b}\}_{k=1}^{K_u}$ is

$$C_k = \ln\left(1 + \mathrm{SINR}_k\right), \tag{4}$$

which will be used to analyze the later tail probability (the achievable confidence level) in Section III-A.

Now, define the new random vector $\mathbf{g}_k$ and the block diagonal matrix $\mathbf{D}_t$, which are given by

$$\mathbf{g}_k \triangleq \begin{pmatrix} \tilde{\mathbf{h}}_k^T & \tilde{\mathbf{h}}_{I/k}^T \end{pmatrix}^T \in \mathcal{C}^{(N_b + K_u - 1) \times 1},$$

$$\mathbf{D}_t \triangleq \begin{pmatrix} \mathbf{D}_k & \mathbf{0}_{N_b \times (K_u - 1)} \\ \mathbf{0}_{(K_u - 1) \times N_b} & \mathbf{D}_{I/k} \end{pmatrix},$$

[1]For the ease of exposition, we assume that the bandwidth is equal to unity such that the expression in (4) and thereof represents the rate.

where $\mathbf{D}_k = \frac{\gamma_k^2 E_k}{\sigma^2} \mathbf{I}_{N_b}$. The spectral radius [13] of the diagonal matrix $\mathbf{D}_{I/k}$ and $\mathbf{D}_t$ are respectively given by

$$\rho_{\mathbf{D}_{I/k}} = \max_{j=1,\ldots k-1, k+1, \ldots, K_u} \frac{\gamma_j^2 E_j}{\sigma^2}, \tag{5}$$

$$\rho_{\mathbf{D}_t} = \max_{k=1,\ldots,K_u} \frac{\gamma_k^2 E_k}{\sigma^2}. \tag{6}$$

Then, (4) can be rewritten as

$$C_k = \underbrace{\ln\left(1 + \frac{1}{N_b} \mathbf{g}_k^\dagger \mathbf{D}_t \mathbf{g}_k\right)}_{C_{k,1}} - \underbrace{\ln\left(1 + \frac{1}{N_b} \tilde{\mathbf{h}}_{I/k}^\dagger \mathbf{D}_{I/k} \tilde{\mathbf{h}}_{I/k}\right)}_{C_{k,2}}, \tag{7}$$

which will be used to calculate the later non-asymptotic results in Section III-B.

In both (4) and (7), it is obvious that the instantaneous rate C_k depends on the random channels $\tilde{\mathbf{h}}_k$ and $\tilde{\mathbf{h}}_{I/k}$, whose randomness is uncontrollable. In the following, the CoM-based non-asymptotic analysis will be given to show explicitly how much the randomness of channels and the number of BS antennas affect the instantaneous rate.

III. Performance Analysis

In this section, a detailed performance analysis of the instantaneous rate, which includes the tail probability (the achievable confidence level), non-asymptotic results (the confidence lower and upper bounds), and the application in outage probability, will be provided.

A. Tail Probability

We first introduce some key preliminary lemmas. From [8, Proposition 1] and [10, Theorem 5.6], the following lemma about the tail probability can be obtained.

Lemma 1: Consider the complex random vector $\mathbf{h} \in \mathcal{C}^{N \times 1}$ with the distribution $\mathbf{h} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_N)$ and a positive semi-definite deterministic matrix $\mathbf{R} \in \mathcal{C}^{N \times N}$ with the bounded spectral radius ρ_s. Consider any real valued function $f(x)$, where $x \in \mathbb{R}^+$ is a real valued positive variable, such that the function $g(x) = f(x^2)$ is a Lipschitz function with the corresponding Lipschitz norm $\|g\|_{\mathcal{L}}$. Then, for arbitrary $\delta > 0$, the following tail probability holds true,

$$\Pr\left\{\left| f\left(\mathbf{h}^\dagger \mathbf{R} \mathbf{h}\right) - \mathbb{E}\left\{ f\left(\mathbf{h}^\dagger \mathbf{R} \mathbf{h}\right)\right\}\right| > \delta \right\} \le 2\exp\left[-\frac{\delta^2}{\rho_s \|g\|_{\mathcal{L}}^2}\right]. \tag{8}$$

The second useful lemma is about the tail probability of the inter-user interference term in (3), which can be obtained by following the similar process as in [10, Example 2.12].

Lemma 2: For any $\varepsilon > 0$, the tail probability of the form $z_k = \frac{1}{N_b} \tilde{\mathbf{h}}_{I/k}^\dagger \mathbf{D}_{I/k} \tilde{\mathbf{h}}_{I/k}$, which appears in (3), is

$$\Pr\left\{ z_k - \mathbb{E}\{z_k\} > \frac{\rho_{\mathbf{D}_{I/k}}}{N_b} \varepsilon + \frac{1}{N_b} \sqrt{2\mathrm{tr}\left(\mathbf{D}_{I/k}^2\right)} \sqrt{\varepsilon} \right\} \le e^{-\varepsilon}, \tag{9}$$

where $\mathbb{E}\{z_k\} = \frac{1}{N_b} \mathrm{tr}\left(\mathbf{D}_{I/k}\right)$, the notations $\mathrm{tr}(\cdot)$ and $\mathbb{E}\{\cdot\}$ denote the trace and expectation operator, respectively.

Now, looking at the SINR_k in (3), let

$$s_k \triangleq \frac{\gamma_k^2 E_k}{\sigma^2 N_b} \left(\frac{1}{N_b} \tilde{\mathbf{h}}_{I/k}^\dagger \mathbf{D}_{I/k} \tilde{\mathbf{h}}_{I/k} + 1 \right)^{-1} \le \frac{\gamma_k^2 E_k}{\sigma^2 N_b}, \tag{10}$$

which is bounded under finite $\mathrm{SNR}_k = \frac{E_k}{\sigma^2}$. Then, from (3) and (4), the instantaneous rate can be rewritten as

$$C_k = \ln\left(1 + s_k \tilde{\mathbf{h}}_k^\dagger \tilde{\mathbf{h}}_k\right). \tag{11}$$

So, in terms of Lemma 1, we define the following function

$$f_{\mathrm{in}}(x) \triangleq \ln\left(1 + x\right), \quad x > 0. \tag{12}$$

We can see that $g_{\text{in}}(x) = f_{\text{in}}(x^2)$ is a Lipschitz function with the Lipschitz norm $\|g_{\text{in}}\|_{\mathcal{L}} = 1$. Under the condition of knowing $\tilde{\mathbf{h}}_{I/k}$, which is equivalent to knowing z_k defined in Lemma 2, we can set the matrix $\mathbf{R}$ in Lemma 1 as $\mathbf{R} = s_k \mathbf{I}_{N_b}$, so the corresponding bounded spectral radius is $\rho_s = s_k$. With this setting and Lemma 1, for arbitrary $\delta > 0$, the following conditional tail probability for the instantaneous rate C_k in (11) can be obtained:

$$\Pr\left\{ \left| C_k - \mathbb{E}\left\{C_k\right\} \right| > \delta \,\middle|\, z_k \right\} \leq 2\exp\left[-\frac{\delta^2}{\frac{\gamma_k^2 E_k}{\sigma_n^2 N_b}} \left(z_k + 1\right) \right]. \quad (13)$$

Then, by using (9) in Lemma 2 and (13), we have:

Theorem 1: For arbitrary $t > 0$, the tail probability for the instantaneous rate C_k is given by

$$\Pr\left\{ \left| C_k - \mathbb{E}\left\{C_k\right\} \right| > \sqrt{\frac{\gamma_k^2 E_k}{\sigma^2}} \sqrt{t} \right\} \leq \min\left\{1, \Pr_k^{\text{tail}}(t)\right\}, \quad (14)$$

where the term $\Pr_k^{\text{tail}}(t)$ represents the tail probability, i.e., the probability that the instantaneous rate C_k deviates from its mean value by a given level, given by

$$\Pr_k^{\text{tail}}(t) = 2\exp\left[-N_b t\right] + 2\Pr_{\text{inter}}^{\text{tail}}(t), \quad (15)$$

and $\Pr_{\text{inter}}^{\text{tail}}(t)$ is

$$\Pr_{\text{inter}}^{\text{tail}}(t) = \exp\left[-\left\{ \left(\text{tr}\left(\mathbf{D}_{I/k}\right) + N_b\right) t + \frac{3\text{tr}\left(\mathbf{D}_{I/k}^2\right) t^2}{2\left(\rho_{\mathbf{D}_{I/k}} t + 1\right)} \right\} \right]. \quad (16)$$

Proof: See Appendix A. ■

Remark 1: In Theorem 1, the offset of the instantaneous rate C_k from its mean value is

$$\Theta_k(t) = \sqrt{\frac{\gamma_k^2 E_k}{\sigma^2}} \sqrt{t}. \quad (17)$$

The tail probability is $\Pr_k^{\text{tail}}(t)$, which means that the achievable confidence level, i.e., the least reliable probability that the instantaneous rate C_k deviates from its mean value not exceeding the given level $\Theta_k(t)$, is

$$\Pr_k^{\text{con}}(t) = \max\left\{0, 1 - \Pr_k^{\text{tail}}(t)\right\}. \quad (18)$$

From Theorem 1, it is obvious that lower $\Theta_k(t)$ indicates higher concentration of the instantaneous rate C_k around its mean value, while larger $\Pr_k^{\text{con}}(t)$ implies higher reliability. From (17) and (18), the coefficient t is very important since it inherently affects the offset $\Theta_k(t)$ and the achievable confidence level $\Pr_k^{\text{con}}(t)$. Ideally, it is better to have lower offset $\Theta_k(t)$ along with higher confidence level $\Pr_k^{\text{con}}(t)$. Unfortunately, higher values of t lead to higher offset and higher confidence level and vice versa. Thus, there is a trade-off between the offset and the achievable confidence level. To measure this trade-off, a useful function is now proposed:

$$\mathcal{T}_k(t) \triangleq \frac{\Pr_k^{\text{con}}(t)}{\Theta_k(t)}, \quad (19)$$

which indicates the achievable confidence level for per unit of the offset.

We next propose an optimization problem, that is, for arbitrary small $\delta > 0$, we choose $t \in [\delta, +\infty)$ in Theorem 1 to maximize $\mathcal{T}_k(t)$ as follows:

$$\arg\max_{t \geq \delta} \mathcal{T}_k(t). \quad (20)$$

Since it is very difficult to find the optimal solution t of (20) directly, we provide its suboptimal solution by maximizing the lower bound of $\mathcal{T}_k(t)$, i.e., we seek to determine $t \in [\delta, +\infty)$ by solving the following optimization problem

$$\arg\max_{t \geq \delta} \frac{\max\left\{0, 1 - 4\exp\left[-N_b t\right]\right\}}{\Theta_k(t)}, \quad (21)$$

where we have used (18) and the fact that

$$\Pr_{\text{inter}}^{\text{tail}}(t) < \exp\left[-N_b t\right]. \quad (22)$$

Corollary 1: For an arbitrary small $\delta > 0$, the optimal solution of (21) (or the suboptimal solution of (20)) is as:

$$t^* = \frac{-1}{N_b}\left\{ W_{-1}\left(\frac{-1}{8} e^{-1/2}\right) + \frac{1}{2} \right\}, \quad (23)$$

where $W_n(x)$ denotes the Lambert W function [14].

Proof: In (21), when $\delta \leq t < \frac{\ln 4}{N_b}$, we have $1 - 4\exp\left[-N_b t\right] < 0$. This means that the lower bound of the achievable confidence level (the numerator in (21)) is equal to zero. Hence, in this case, from a physical perspective, the optimal solution seeks to make the offset value $\Theta_k(t)$ in (17) approach to zero, i.e., the optimal solution is $t = \delta$. On the other hand, in the region $t \geq \frac{\ln 4}{N_b}$, (21) is equivalent to

$$\arg\max_{t \geq \frac{\ln 4}{N_b}} f(t), \quad (24)$$

where $f(t)$ is defined as

$$f(t) \triangleq \frac{1 - 4\exp\left[-N_b t\right]}{\sqrt{t}}. \quad (25)$$

The first derivative of $f(t)$ is

$$\frac{\partial f(t)}{\partial t} = \frac{1}{2t^{\frac{3}{2}}}\left[(8N_b t + 4)e^{-N_b t} - 1\right] = 0. \quad (26)$$

In the region $t \geq \frac{\ln 4}{N_b}$, the solution of (26) is t^* given in (23). When $t \geq t^*$, we have $\frac{\partial f(t)}{\partial t} \leq 0$, whilst when $\frac{\ln 4}{N_b} \leq t < t^*$, we have $\frac{\partial f(t)}{\partial t} > 0$. This proves the existence and uniqueness of the solution of (24) and the exact solution is t^* in the region $t \in [\frac{\ln 4}{N_b}, +\infty)$. Finally, we choose the optimal $t \in \{\delta, t^*\}$ to maximize $\mathcal{T}_k(t)$ in (19). It is easy to show that $\mathcal{T}_k(t^*) > \mathcal{T}_k(\delta) = 0$, which implies that the optimal $t^{\text{opt}} = t^*$. This completes the proof. ■

Remark 2: From (22) and the simulation results later, we have that the suboptimal solution in Corollary 1 will be more accurate with lower $\text{SNR}_k = \frac{E_k}{\sigma^2}$, and it is actually tight enough in the low to medium $\{\text{SNR}_k\}_{k=1}^{K_u}$. Meanwhile, the gap between the suboptimal solution in (23) and the exact optimal solution will be eliminated for an increasing number of BS antennas N_b. Also, from (23), the suboptimal solution t^* decreases with respect to N_b in the order $\mathcal{O}\left(\frac{1}{N_b}\right)$, where the notation $\mathcal{O}(\cdot)$ denotes the big $\mathcal{O}$ operation.

B. Non-Asymptotic Results

Based on Theorem 1, the non-asymptotic results, i.e., the lower and upper bound that the instantaneous rate C_k of (7) fluctuates within, are presented as follows:

Corollary 2: With the probability exceeding $\Pr_k^{\text{con}}(t)$, which has been given in (18), the instantaneous rate C_k will fall within the region

$$\bar{C}_k^{\text{lower}} - \sqrt{\frac{\gamma_k^2 E_k}{\sigma^2}} \sqrt{t} < C_k < \bar{C}_k^{\text{upper}} + \sqrt{\frac{\gamma_k^2 E_k}{\sigma^2}} \sqrt{t}, \quad (27)$$

where the terms $\bar{C}_k^{\text{lower}}$ and $\bar{C}_k^{\text{upper}}$ denote the lower and upper bound of the mean value $\mathbb{E}\{C_k\}$ respectively and are as

$$\bar{C}_k^{\text{lower}} = \bar{C}_k - \frac{1}{4}\frac{\rho_{\mathbf{D}_t}}{N_b}, \quad (28)$$

$$\bar{C}_k^{\text{upper}} = \bar{C}_k + \frac{1}{4}\frac{\rho_{\mathbf{D}_{I/k}}}{N_b}. \quad (29)$$

In (28) and (29), the term $\bar{C}_k$ is given by

$$\bar{C}_k = \ln\left(1 + \gamma_k^2 E_k \left(\sigma^2 + \frac{1}{N_b}\sum_{\substack{m=1\\m\neq k}}^{K_u} \gamma_m^2 E_m\right)^{-1}\right). \quad (30)$$

Proof: See Appendix B. ■

Remark 3: From Corollary 2, the gap between the lower and the upper bound of the instantaneous rate is

$$\Delta_k^{\text{gap}} = \frac{1}{4}\frac{\rho_{\mathbf{D}_t}}{N_b} + \frac{1}{4}\frac{\rho_{\mathbf{D}_{I/k}}}{N_b} + 2\sqrt{\frac{\gamma_k^2 E_k}{\sigma^2}}\sqrt{t}. \quad (31)$$

Substituting the suboptimal t^* in (23) of Corollary 1 into (31), it is obvious that the gap Δ_k^{gap} in (31) reduces as $\mathcal{O}\left(\frac{1}{\sqrt{N_b}}\right)$ and approaches to 0 when the number of BS antennas grows large, while the instantaneous rate also tends to stabilize to the constant rate $\bar{C}_k$ given by (30).

Meanwhile, substituting the suboptimal solution t^* given by (23) into (17) and (18), the achievable confidence level approaches to

$$\Pr_k^{\text{con}}(t) \xrightarrow{N_b\to\infty} 1 - 4\exp\left\{W_{-1}\left(\frac{-1}{8}e^{-1/2}\right) + \frac{1}{2}\right\} = 0.8736, \quad (32)$$

while the offset value $\Theta_k(t)$ approaches to

$$\Theta_k(t) \xrightarrow{N_b\to\infty} 0. \quad (33)$$

All the results in (31)–(33) imply that in the massive MIMO regime, the instantaneous rate C_k tends to stabilize with the least reliable probability 0.8736.

C. Application in Outage Probability Analysis

The outage probability is defined as:

$$\Pr_k^{\text{out}} = \Pr\left\{C_k < R^*\right\}, \quad (34)$$

where R^* denotes the target data rate and C_k is the instantaneous rate for the k-th user given in (4) and (7). In terms of the tail probability obtained in Theorem 1, let

$$R^* = \begin{cases} \mathbb{E}\{C_k\} - \Theta_k(t), & R^* < \mathbb{E}\{C_k\} \\ \mathbb{E}\{C_k\} + \Theta_k(t), & R^* \geq \mathbb{E}\{C_k\} \end{cases}, \quad (35)$$

in which, the offset $\Theta_k(t)$ given in (17) is a function of t. Thus, given R^* and $\mathbb{E}\{C_k\}$, (35) can be obtained by adjusting the value t, and under the two conditions, the corresponding positive coefficient t is as,

$$t_{\text{out}} = \left(\mathbb{E}\{C_k\} - R^*\right)^2 \frac{\sigma^2}{\gamma_k^2 E_k}. \quad (36)$$

Then, the outage probability $\Pr_k^{\text{out}}$ in (34) can be rewritten as

$$\Pr_k^{\text{out}} = \begin{cases} \Pr\left\{C_k - \mathbb{E}\{C_k\} < -\Theta_k(t_{\text{out}})\right\}, & R^* < \mathbb{E}\{C_k\} \\ 1 - \Pr\left\{C_k - \mathbb{E}\{C_k\} \geq \Theta_k(t_{\text{out}})\right\}, & R^* \geq \mathbb{E}\{C_k\} \end{cases}. \quad (37)$$

Now, combine (37) and the result obtained in Theorem 1, the following corollary can be obtained.

Corollary 3: For a given target rate R^*, the outage probability $\Pr_k^{\text{out}}$ for the k-th user given in (34) is as follows:[2]

i) If $R^* < \mathbb{E}\{C_k\}$, then we have

$$\Pr_k^{\text{out}} \leq \min\left\{1, \frac{\Pr_k^{\text{tail}}(t_{\text{out}})}{2}\right\} \leq \min\left\{1, \frac{\Pr_k^{\text{tail}}\left(t_{\text{out}}^{\text{low}}\right)}{2}\right\}, \quad (38)$$

where the coefficient t_{out} and the tail probability $\Pr_k^{\text{tail}}(\cdot)$ have been given in (36) and (15), respectively. The coefficient $t_{\text{out}}^{\text{low}}$ denotes the lower bound of t_{out} under this situation, which is given by

$$t_{\text{out}}^{\text{low}} = \left(\max\left\{R^*, \bar{C}_k^{\text{lower}}\right\} - R^*\right)^2 \frac{\sigma^2}{\gamma_k^2 E_k}, \quad (39)$$

where $\bar{C}_k^{\text{lower}}$ has been given in (28).

ii) If $R^* \geq \mathbb{E}\{C_k\}$, then we have

$$\Pr_k^{\text{out}} \geq \max\left\{0, 1 - \frac{\Pr_k^{\text{tail}}(t_{\text{out}})}{2}\right\} \geq \max\left\{0, 1 - \frac{\Pr_k^{\text{tail}}\left(t_{\text{out}}^{\text{upper}}\right)}{2}\right\}, \quad (40)$$

where $t_{\text{out}}^{\text{upper}}$ denotes the upper bound of t_{out} in this scenario, which is given by

$$t_{\text{out}}^{\text{upper}} = \left(\min\left\{R^*, \bar{C}_k^{\text{upper}}\right\} - R^*\right)^2 \frac{\sigma^2}{\gamma_k^2 E_k}, \quad (41)$$

where $\bar{C}_k^{\text{upper}}$ has been given in (29).

Remark 4: Corollary 3 and the simulations later on indicate that under a lower level of target data rate, the user terminals can transmit signals successfully with high probability. Under these circumstances, it is reasonable to analyze the largest outage probability, as given in (38). However, with a higher level of the target data rate, it is extremely difficult for the users to transmit signals successfully. Thus, in this scenario, it is more practical to analyze the smallest outage probability, as given in (40).

IV. Numerical Results

Our numerical results are carried out based on the following setting unless otherwise stated: the number of users $K_u = 10$ and the transmit power $E_1 = \ldots = E_{K_u} = E_p = 30$dBm; the noise variance $\sigma^2 = 1$ and the channel gain parameters $\gamma_1^2 = \ldots = \gamma_{K_u}^2 = 1$.

A. Optimal Solution t

Firstly, Fig. 1 shows the variation of the coefficient t, which appears in the tail probability of Theorem 1 and the non-asymptotic results of Corollary 2. The optimal t is determined by solving the optimization problem (20) and has been calculated in Corollary 1. In this figure, the "optimal-simu" results are computed by using the built-in "fminbnd" function in MATLAB. From Fig. 1, the suboptimal solution obtained in Corollary 1 is tighter with lower transmit power E_p at users' side and the gap between the optimal and suboptimal solution will be eliminated with a massive number of BS antennas. Furthermore, the optimal and suboptimal t decreases with the number of BS antennas N_b until to 0.

B. Tail Probability and Non-asymptotic Results

The tail probability (achievable confidence level) and non-asymptotic results of the instantaneous rate given in Theorem 1 and Corollary 2 are illustrated in Fig. 2. The related coefficient t has been analyzed in Fig. 1. Specifically, in Fig. 2, the "optimal" results are numerically computed with the optimal solution t obtained by the built-in "fminbnd" function in MATLAB, while the "suboptimal" results are computed by using the suboptimal solution t in Corollary 1. This figure shows that the upper and lower bounds obtained from the optimal and suboptimal t respectively are almost identical.

Firstly, this figure confirms the correctness of our achievable confidence level and non-asymptotic results. As expected, with the reliable probability satisfying (18), the instantaneous rate can fall within a narrow region between their

[2]As the term $\Pr_k^{\text{tail}}(t_{\text{out}})$ in both (38) and (40) is a function of $\mathbb{E}\{C_k\}$, the direct calculation of the value $\mathbb{E}\{C_k\}$ is difficult and complicated. So, we use its lower and upper bound provided in the non-asymptotic analysis.

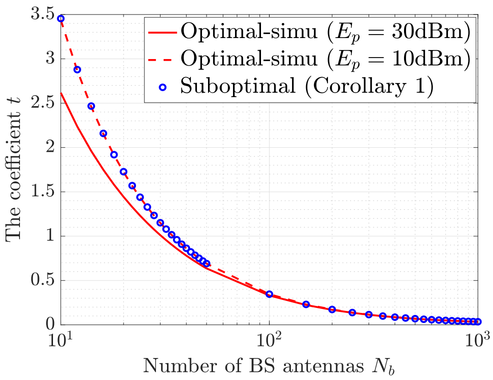

Fig. 1: The solution t of the optimization problem (20), which is given in Corollary 1, versus N_b.

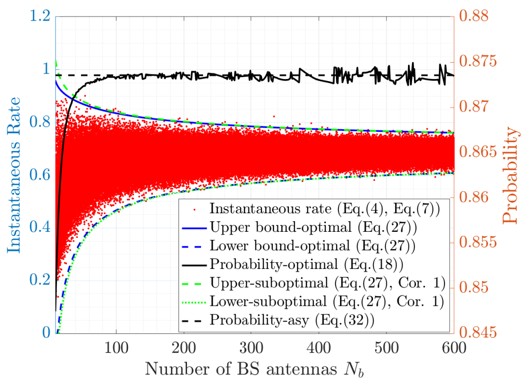

Fig. 2: The achievable confidence level in (18), the analytical confidence bounds in Corollary 2 and 500 Monte Carlo realizations of the instantaneous rate versus N_b.

corresponding lower and upper bound, which are given by (27). Furthermore, with the increase of the number of BS antennas, the achievable confidence level will increase to the corresponding asymptotic result, which is equal to 0.8736. Also, the gap between its lower and upper bound reduces with the number of BS antennas. One of the reasons is that the coefficient t decreases as shown in Fig. 1 when the number of BS antennas is large enough. This demonstrates that the instantaneous rate will tend to stabilize and concentrate on a certain value under the reliable probability exceeding 0.8736.

C. Outage Probability

Fig. 3 simulates all results in the outage probability anaysis in Corollary 3. Here, the "Upper-Eq.(38-1)" and "Upper-Eq.(38-2)" results represent the first and second upper bound respectively obtained in (38), while the "Lower-Eq.(40-1)" and "Lower-Eq.(40-2)" results are from the first and second lower bound obtained in (40). The "Mean" represents the mean value $\mathbb{E}\{C_k\}$ of the instantaneous rate. This figure reveals that in a lower transmission target rate level (e.g. $[0, 0.4]$ in this figure), the largest outage probability is close to 0, i.e, the outage events barely occur in this situation, whilst under a higher target rate (e.g. [0.85, 1]), the smallest outage probability almost approaches to 1, i.e., the outage events nearly always happen. This implies that the non-asymptotic results (lower and upper bound) of the outage probability are extremely simple and useful for acquisition of insights.

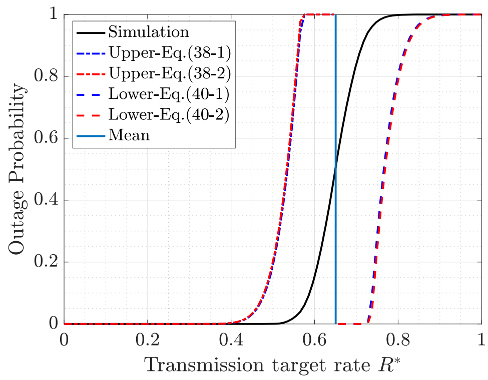

Fig. 3: The outage probability in Corollary 3 vesus the transmission rate R^* with $N_b = 100$.

V. Conclusions

Different from the previous existing works, we provided a CoM-based non-asymptotic analysis of the instantaneous rate for the uplink MU-MIMO system. Specifically, the achievable confidence level (tail probability) was first provided for a moderate-to-large number of the BS antennas. Following this mathematical formulation, a trade-off function was proposed and optimized, and the corresponding optimal solution was calculated. Then, based on the confidence level, a narrow interval that the instantaneous rate falls within with high probability was also presented. Finally, our non-asymptotic results were applied on the outage probability analysis and valuable insights were obtained.

Appendix A
Proof of Theorem 1

We first define the random variable

$$x_k \triangleq C_k - \mathbb{E}\{C_k\}, \tag{42}$$

and the function

$$p(z_k) \triangleq \exp\left[-\delta^2 \left(\frac{\gamma_k^2 E_k}{\sigma^2 N_b}\right)^{-1} (z_k+1)\right], \tag{43}$$

where the instantaneous rate C_k has been given in (4), and z_k has been defined in Lemma 2. Then, we have

$$\Pr\{|x_k| > \delta\} = \mathbb{E}_{z_k}\left\{\Pr\left\{|x_k| > \delta \middle| z_k\right\}\right\} \triangleq \text{①}. \tag{44}$$

Let $f_{Z_k}(z_k)$ denote the PDF of the random variable z_k. For arbitrary $c > 0$, we have

$$\begin{aligned} \text{①} = & \int_0^c \Pr\left\{|x_k| > \delta \middle| z_k\right\} f_{Z_k}(z_k) dz_k \\ & + \int_c^{+\infty} \Pr\left\{|x_k| > \delta \middle| z_k\right\} f_{Z_k}(z_k) dz_k. \end{aligned} \tag{45}$$

Then, substituting the result obtained in (13) into (45) yields,

$$\begin{aligned} \text{①} & \le 2\int_0^c p(z_k) f_{Z_k}(z_k) dz_k + 2\int_c^{+\infty} p(z_k) f_{Z_k}(z_k) dz_k \\ & \overset{(a)}{\le} 2p(0)\int_0^c f_{Z_k}(z_k) dz_k + 2p(c)\int_c^{+\infty} f_{Z_k}(z_k) dz_k \\ & \overset{(b)}{\le} 2p(0) + 2p(c)\Pr\{z_k > c\}, \end{aligned} \tag{46}$$

where the inequality (a) is obtained by using the non-increasing property of the function $p(z_k)$ given in (43) with respect to z_k. The inequality (b) is due to

$$\int_0^c f_{Z_k}(z_k) dz_k \le 1.$$

We also define

$$a_1 \triangleq \frac{\delta^2\sigma^2\rho_{\mathbf{D}_{I/k}}}{\gamma_k^2 E_k}+1, \qquad a_2 \triangleq \frac{\delta^2\sigma^2}{\gamma_k^2 E_k}\sqrt{2\mathrm{tr}\left(\mathbf{D}_{I/k}^2\right)},$$

$$a_3 \triangleq \frac{\delta^2\sigma^2 \mathrm{tr}\left(\mathbf{D}_{I/k}\right)}{\gamma_k^2 E_k}+\frac{\delta^2\sigma^2 N_b}{\gamma_k^2 E_k}.$$

By setting the coefficient c in (46) as

$$c=\frac{\rho_{\mathbf{D}_{I/k}}}{N_b}\varepsilon+\frac{1}{N_b}\sqrt{2\mathrm{tr}\left(\mathbf{D}_{I/k}^2\right)}\sqrt{\varepsilon}+\frac{1}{N_b}\mathrm{tr}\left(\mathbf{D}_{I/k}\right), \tag{47}$$

and using Lemma 2, the last term in (46) is upper bounded

$$2p(c)\Pr\{z_k>c\}\leq 2\exp\left[-\left(a_3-\frac{a_2^2}{4a_1}+a_1\left(\sqrt{\varepsilon}+\frac{a_2}{2a_1}\right)^2\right)\right]. \tag{48}$$

By setting

$$\varepsilon=\frac{a_2^2}{4a_1^2}=\frac{\sigma^4\mathrm{tr}\left(\mathbf{D}_{I/k}^2\right)\delta^4}{2\left(\sigma^2\rho_{\mathbf{D}_{I/k}}\delta^2+\gamma_k^2E_k\right)^2},$$

(48) can be further written as

$$2p(c)\Pr\{z_k>c\}\leq 2\exp\left[-\left\{\frac{\left[\sigma^2\mathrm{tr}\left(\mathbf{D}_{I/k}\right)+\sigma^2N_b\right]\delta^2}{\gamma_k^2E_k}\right.\right.$$
$$\left.\left.+\frac{3}{2}\sigma^4\mathrm{tr}\left(\mathbf{D}_{I/k}^2\right)\delta^4\left(\gamma_k^2E_k\left(\sigma^2\rho_{\mathbf{D}_{I/k}}\delta^2+\gamma_k^2E_k\right)\right)^{-1}\right\}\right]. \tag{49}$$

Now, substituting (49) and the value $p(0)$ obtained in (43) into (46), and further into (44), and at the same time setting $t=\frac{\sigma^2\delta^2}{\gamma_k^2E_k}$, the final result in Theorem 1 can be obtained.

Appendix B
Proof of Corollary 2

Before the proof, a useful lemma that will be used is presented firstly, which is as follows:

Lemma 3: [10, Theorem 5.5] Under the same assumption as given in Lemma 1, for all $\lambda\in\mathbb{R}$, we have:

$$\ln\mathbb{E}\left\{e^{\lambda\left(f\left(\mathbf{h}^\dagger\mathbf{R}\mathbf{h}\right)-\mathbb{E}\left\{f\left(\mathbf{h}^\dagger\mathbf{R}\mathbf{h}\right)\right\}\right)}\right\}\leq\frac{\lambda^2}{4}\rho_s\|g\|_{\mathcal{L}}^2. \tag{50}$$

Now, with the result (14) in Theorem 1, with the probability exceeding $\mathrm{Pr}_k^{\mathrm{con}}$ given in (18), we have

$$\mathbb{E}\{C_k\}-\sqrt{\frac{\gamma_k^2E_k}{\sigma^2}}\sqrt{t}<C_k<\mathbb{E}\{C_k\}+\sqrt{\frac{\gamma_k^2E_k}{\sigma^2}}\sqrt{t}. \tag{51}$$

From (51), we only need to calculate the term $\mathbb{E}\{C_k\}$. Recall the rate C_k in (7), we have

$$\mathbb{E}\{C_k\}=\mathbb{E}\{C_{k,1}\}-\mathbb{E}\{C_{k,2}\}. \tag{52}$$

Thus, we need to calculate the term $\mathbb{E}\left\{C_{k,1}\right\}$ and $\mathbb{E}\left\{C_{k,2}\right\}$ in (52), respectively. Looking at $C_{k,1}$ and $C_{k,2}$, which have been given in (7), for a finite $\mathrm{SNR}_k=\frac{E_k}{\sigma^2}$, for all $k=1,\ldots,K_u$, both the spectral radiuses $\rho_{\mathbf{D}_{I/k}}$ and $\rho_{\mathbf{D}_t}$ of the diagonal matrix $\mathbf{D}_{I/k}$ and $\mathbf{D}_t$, which are given in (5) and (6) respectively, are bounded.

Then, according to Lemma 3, define the function

$$f_e(x)=\ln(1+x), \quad x>0.$$

So, the function $g_e(x)=f_e(x^2)$ is a Lipschitz function with the Lipschitz norm $\|g_e\|_{\mathcal{L}}=1$. Regarding the term $C_{k,1}$, in Lemma 3, let $\lambda=1$, the vector $\mathbf{h}=\mathbf{g}_k$, and the matrix $\mathbf{R}=\frac{1}{N_b}\mathbf{D}_t$, which indicates that the spectral radius is $\rho_s=\frac{\rho_{\mathbf{D}_t}}{N_b}$. Then, it is easy to obtain that

$$\ln\left(\mathbb{E}\left\{e^{C_{k,1}}\right\}\right)-\frac{1}{4}\frac{\rho_{\mathbf{D}_t}}{N_b}$$
$$\overset{(\tilde{a})}{\leq}\mathbb{E}\{C_{k,1}\}=\mathbb{E}\left\{\ln\left(e^{C_{k,1}}\right)\right\}\overset{(a)}{\leq}\ln\left(\mathbb{E}\left\{e^{C_{k,1}}\right\}\right). \tag{53}$$

Similarly, in Lemma 3, setting $\mathbf{h}=\tilde{\mathbf{h}}_{I/k}$ and $\mathbf{R}=\frac{1}{N_b}\mathbf{D}_{I/k}$, which means the spectral radius is $\rho_s=\frac{\rho_{\mathbf{D}_{I/k}}}{N_b}$, we have,

$$\ln\left(\mathbb{E}\left\{e^{C_{k,2}}\right\}\right)-\frac{1}{4}\frac{\rho_{\mathbf{D}_{I/k}}}{N_b}$$
$$\overset{(\tilde{b})}{\leq}\mathbb{E}\{C_{k,2}\}=\mathbb{E}\left\{\ln\left(e^{C_{k,2}}\right)\right\}\overset{(b)}{\leq}\ln\left(\mathbb{E}\left\{e^{C_{k,2}}\right\}\right). \tag{54}$$

In (53) and (54), the inequalities $(\tilde{a})$ and $(\tilde{b})$ are from Lemma 3 while the inequalities (a) and (b) are obtained by using the Jensen's inequality. Substituting (53) and (54) into (52) yields

$$\ln\left(\mathbb{E}\left\{e^{C_{k,1}}\right\}\right)-\ln\left(\mathbb{E}\left\{e^{C_{k,2}}\right\}\right)-\frac{1}{4}\frac{\rho_{\mathbf{D}_t}}{N_b}\leq\mathbb{E}\{C_k\}$$
$$\leq\ln\left(\mathbb{E}\left\{e^{C_{k,1}}\right\}\right)-\ln\left(\mathbb{E}\left\{e^{C_{k,2}}\right\}\right)+\frac{1}{4}\frac{\rho_{\mathbf{D}_{I/k}}}{N_b}, \tag{55}$$

where the terms $\ln\left(\mathbb{E}\left\{e^{C_{k,i}}\right\}\right)$, for $i=1,2$, are as:

$$\ln\left(\mathbb{E}\left\{e^{C_{k,1}}\right\}\right)=\ln\left(1+\frac{\gamma_k^2E_k}{\sigma^2}+\frac{1}{N_b}\sum_{\substack{m=1\\m\neq k}}^{K_u}\frac{\gamma_m^2E_m}{\sigma^2}\right), \tag{56}$$

$$\ln\left(\mathbb{E}\left\{e^{C_{k,2}}\right\}\right)=\ln\left(1+\frac{1}{N_b}\sum_{\substack{m=1\\m\neq k}}^{K_u}\frac{\gamma_m^2E_m}{\sigma^2}\right). \tag{57}$$

Substituting (56) and (57) into (55), the bounds of the term $\mathbb{E}\{C_k\}$ in (51) can be obtained and likewise Corollary 2 is also obtained.

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