



**QUEEN'S  
UNIVERSITY  
BELFAST**

## Triangular Objects and Systematic K-Theory

Hüttemann, T., & Zhang, Z. (2017). Triangular Objects and Systematic *K*-Theory. *Communications in Algebra*, 45(7), 2757-2774. <https://doi.org/10.1080/00927872.2016.1226870>

**Published in:**  
Communications in Algebra

**Document Version:**  
Peer reviewed version

**Queen's University Belfast - Research Portal:**  
[Link to publication record in Queen's University Belfast Research Portal](#)

### **Publisher rights**

© 2016 Taylor & Francis.

This is an Accepted Manuscript of an article published by Taylor & Francis in *Communications in Algebra* on 07 Oct 2016, available online: <http://www.tandfonline.com/10.1080/00927872.2016.1226870>

### **General rights**

Copyright for the publications made accessible via the Queen's University Belfast Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

### **Take down policy**

The Research Portal is Queen's institutional repository that provides access to Queen's research output. Every effort has been made to ensure that content in the Research Portal does not infringe any person's rights, or applicable UK laws. If you discover content in the Research Portal that you believe breaches copyright or violates any law, please contact [openaccess@qub.ac.uk](mailto:openaccess@qub.ac.uk).

### **Open Access**

This research has been made openly available by Queen's academics and its Open Research team. We would love to hear how access to this research benefits you. – Share your feedback with us: <http://go.qub.ac.uk/oa-feedback>

# TRIANGULAR OBJECTS AND SYSTEMATIC $K$ -THEORY

THOMAS HÜTTEMANN AND ZUHONG ZHANG

ABSTRACT. We investigate modules over “systematic” rings. Such rings are “almost graded” and have appeared under various names in the literature; they are special cases of the  $G$ -systems of GRZESZCZUK. We analyse their  $K$ -theory in the presence of conditions on the support, and explain how this generalises and unifies calculations of graded and filtered  $K$ -theory scattered in the literature. Our treatment makes systematic use of the formalism of idempotent completion and a theory of triangular objects in additive categories, leading to elementary and transparent proofs throughout.

## CONTENTS

Introduction	2
Acknowledgements	2
<b>Part 1. General theory of lower triangular categories</b>	<b>2</b>
1. The idempotent completion of an additive category	2
2. Exact structures	4
3. Lower triangular categories	4
4. Subobject and quotient object functors	5
5. Algebraic $K$ -theory	8
6. Generalisations	8
7. The application template	9
<b>Part 2. Systematic algebra</b>	<b>10</b>
8. Systematic rings and modules	10
9. Systematically free and projective modules	11
10. Strongly systematic rings and modules	12
<b>Part 3. Algebraic <math>K</math>-theory</b>	<b>14</b>
11. Systematic $K$ -theory	14
· Systematic $K$ -theory	14
12. Systematic $K$ -theory of rings with positive support	14
· Ordering the subgroup	14
· Ordering the quotient	18
13. Equivariant $K$ -theory of affine toric schemes	19
References	20

---

*Date:* April 13, 2016.

*2010 Mathematics Subject Classification.* Primary 19D50; Secondary 18E05.

## INTRODUCTION

The aim of this note is to provide a unified treatment of several results in algebraic  $K$ -theory, comparing “graded” (*resp.*, “filtered”)  $K$ -theory of graded (*resp.*, filtered) rings with the usual algebraic  $K$ -theory of the subring in degree (*resp.*, filtration degree) 0. A typical case is the following result:

**Theorem** (QUILLEN [Qui73, p. 107, Proposition]). *Let  $B$  a positively graded ring (that is, a  $\mathbb{Z}$ -graded ring with  $B_k = \{0\}$  for  $k < 0$ ). Then there is a  $\mathbb{Z}[x, x^{-1}]$ -linear isomorphism*

$$\mathbb{Z}[x, x^{-1}] \otimes_{\mathbb{Z}} K_n(B_0) \cong K_n^{\text{gr}}(B), \quad x^n \otimes P \mapsto P \otimes_{B_0} {}^{(n)}B,$$

with  $K_n^{\text{gr}}(B)$  denoting the algebraic  $K$ -theory of the category of finitely generated  $\mathbb{Z}$ -graded projective  $B$ -modules, and  ${}^{(n)}B$  denoting the graded module with  ${}^{(n)}B_k = B_{k-n}$ .

The proof given in *loc.cit.* is short but subtle, involving certain non-canonical isomorphisms between various modules; it is quite surprising that, after passing to suitable quotients, all constructions become functorial and hence induce maps on QUILLEN  $K$ -groups. Similar complications can be found, in more explicit form, in [HH13] and [Hüt13].

In this paper we propose an alternative approach which, while still based on the additivity theorem (or a version of “characteristic filtrations”), is more explicit and transparent. In Part 1 we develop an axiomatic setup for applying the additivity theorem to triangular objects in additive categories. In Part 2 we introduce systematic rings and modules, a notion that generalises and unifies both graded and filtered algebra at once. In Part 3 we study the algebraic  $K$ -theory of systematic rings; our computations specialise to various known calculations of graded and filtered  $K$ -theory scattered in the literature. We end the paper with remarks on the algebraic  $K$ -theory of affine toric schemes.

## ACKNOWLEDGEMENTS

Work on this paper began during a research visit of the first author to Beijing Institute of Technology in Summer 2014. Their hospitality and financial support is gratefully acknowledged.

**Part 1. General theory of lower triangular categories**

## 1. THE IDEMPOTENT COMPLETION OF AN ADDITIVE CATEGORY

Let  $\mathcal{A}$  be an additive category. The *idempotent completion*, or KAROUBI *envelope*, of  $\mathcal{A}$  is the additive category  $\text{Idem } \mathcal{A}$  defined as follows: Objects are pairs  $(A, p)$  with  $p: A \longrightarrow A$  an idempotent morphism in  $\mathcal{A}$  (that is, objects are “projections” in  $\mathcal{A}$ ); a morphism  $f: (A, p) \longrightarrow (B, q)$  is a morphism  $f: A \longrightarrow B$  in  $\mathcal{A}$  such that  $qfp = f$  (or, equivalently,  $fp = f = qf$ ); in particular,  $\text{Idem } \mathcal{A}((A, p), (A', p'))$  is a subset of  $\mathcal{A}(A, A')$ . Identity morphisms are given by  $\text{id}_{(A, p)} = p$ , and composition of morphisms is inherited from composition in  $\mathcal{A}$ .

The functor  $A \mapsto (A, \text{id}_A)$  and  $f \mapsto f$  is an embedding of  $\mathcal{A}$  as a full subcategory of  $\text{Idem } \mathcal{A}$ . It is an equivalence if and only if all idempotents

in  $\mathcal{A}$  split (*i.e.*, if and only if for every idempotent morphism  $p: A \longrightarrow A$  there exist morphisms  $r: A \longrightarrow A'$  and  $s: A' \longrightarrow A$  such that  $r \circ s = \text{id}_{A'}$  and  $p = s \circ r$ ).

A standard example is the category  $\mathcal{A}$  with objects the based finitely generated free  $R$ -modules  $R^n$ ,  $n \geq 0$ , and morphisms the  $R$ -linear maps, for some fixed unital ring  $R$ . The category  $\mathcal{A}$  is equivalent to the category of all finitely generated free  $R$ -modules, and its idempotent completion  $\text{Idem } \mathcal{A}$  is equivalent to the category of finitely generated projective  $R$ -modules.

A functor  $\Phi: \mathcal{A} \longrightarrow \mathcal{B}$  between additive categories induces a functor

$$\text{Idem } \Phi = \hat{\Phi}: \text{Idem } \mathcal{A} \longrightarrow \text{Idem } \mathcal{B}, \quad (F, p) \mapsto (\Phi(F), \Phi(p));$$

for a morphism  $f: (F, p) \longrightarrow (G, q)$  we have

$$\hat{\Phi}(f) = \Phi(f): (\Phi(F), \Phi(p)) \longrightarrow (\Phi(G), \Phi(q)) \in \text{Idem } \mathcal{B},$$

as  $f = qfp$  implies  $\Phi(f) = \Phi(q)\Phi(f)\Phi(p)$  by functoriality.

**Lemma 1.1.** *If  $\Phi: \mathcal{A} \longrightarrow \mathcal{B}$  is additive then so is the induced functor  $\hat{\Phi}: \text{Idem } \mathcal{A} \longrightarrow \text{Idem } \mathcal{B}$ .*

*Proof.* By definition of additive functor, the functor  $\Phi$  yields group homomorphisms  $\mathcal{A}(A, A') \longrightarrow \mathcal{B}(\Phi(A), \Phi(A'))$ . Hence so does  $\hat{\Phi}$  as its effect on underlying morphisms is that of  $\Phi$ ; that is,  $\hat{\Phi}$  is additive.  $\square$

Suppose that  $\Phi, \Psi: \mathcal{A} \longrightarrow \mathcal{B}$  are additive functors, and that  $\tau: \Phi \overset{\cdot}{\longrightarrow} \Psi$  is a natural transformation. Then  $\hat{\tau}$ , defined by

$$\hat{\tau}_{(A,p)} = \Psi(p) \circ \tau_A: (\Phi(A), \Phi(p)) \longrightarrow (\Psi(A), \Psi(p)),$$

is a natural transformation of functors  $\hat{\Phi} \overset{\cdot}{\longrightarrow} \hat{\Psi}$ . Indeed, we have

$$\begin{aligned} \Psi(p) \circ \hat{\tau}_{(A,p)} \circ \Phi(p) &= \Psi(p) \circ (\Psi(p) \circ \tau_A) \circ \Phi(p) && \text{(definition of } \hat{\tau}) \\ &= \Psi(p) \circ (\tau_A \circ \Phi(p)) \circ \Phi(p) && \text{(naturality of } \tau) \\ &= \Psi(p) \circ (\tau_A \circ \Phi(p)) && \text{(\Phi(p) idempotent)} \\ &= \Psi(p) \circ (\Psi(p) \circ \tau_A) && \text{(naturality of } \tau) \\ &= \Psi(p) \circ \tau_A && \text{(\Psi(p) idempotent)} \\ &= \hat{\tau}_{(A,p)} && \text{(definition of } \hat{\tau}) \end{aligned}$$

so that  $\hat{\tau}_{(A,p)}$  is a morphism in  $\text{Idem } \mathcal{B}$ . To verify naturality, fix a morphism  $a: (A, p) \longrightarrow (A', p')$  in  $\text{Idem } \mathcal{A}$ . Then we compute

$$\begin{aligned} \hat{\tau}_{(A',p')} \circ \hat{\Phi}(a) &= \Psi(p') \circ \tau_{A'} \circ \Phi(a) && \text{(definition of } \hat{\tau}) \\ &= \Psi(p') \circ \Psi(a) \circ \tau_A && \text{(naturality of } \tau) \\ &= \Psi(a) \circ \Psi(p) \circ \tau_A && \text{(as } p'a = a = ap) \\ &= \hat{\Psi}(a) \circ \hat{\tau}_{(A,p)} && \text{(definition of } \hat{\tau}). \end{aligned}$$

**Lemma 1.2.** *If  $\tau$  is a natural isomorphism then  $\hat{\tau}$  is a natural isomorphism as well. Equivalent additive categories thus have equivalent idempotent completions.*

*Proof.* As  $\tau^{-1}$  is a natural transformation  $\Psi \xrightarrow{\cdot} \Phi$  the construction above yields a natural transformation  $\widehat{\tau^{-1}}: \widehat{\Psi} \xrightarrow{\cdot} \widehat{\Phi}$ . We claim that this is the inverse of  $\widehat{\tau}$ . Indeed, for an object  $(A, p) \in \text{Idem } \mathcal{A}$  we calculate

$$\begin{aligned} (\widehat{\tau^{-1}} \circ \widehat{\tau})_{(A,p)} &= \widehat{\tau^{-1}}_{(A,p)} \circ \widehat{\tau}_{(A,p)} \\ &= (\Phi(p) \circ \tau_A^{-1}) \circ (\Psi(p) \circ \tau_A) \quad (\text{definition of } \widehat{\tau} \text{ and } \widehat{\tau^{-1}}) \\ &= \Phi(p) \circ \tau_A^{-1} \circ \tau_A \circ \Psi(p) \quad (\text{naturality of } \tau) \\ &= \Phi(p) \quad (\text{as } p \circ p = p) \\ &= \text{id}_{\widehat{\Phi}(A,p)} \end{aligned}$$

so that  $\widehat{\tau^{-1}} \circ \widehat{\tau} = \text{id}_{\widehat{\Phi}}$ . A similar calculation shows  $\widehat{\tau} \circ \widehat{\tau^{-1}} = \text{id}_{\widehat{\Psi}}$  as well.  $\square$

We will from now on drop the decoration “ $\widehat{\phantom{x}}$ ” and let  $\Phi$  denote both the original additive functor and the induced functor  $\widehat{\Phi}$  discussed above. — We will make use of the following fact, which can be verified by explicit calculation:

**Lemma 1.3.** *Idempotent completion is compatible with filtered colimits:*

$$\text{Idem} \left( \varinjlim \mathcal{A}[S] \right) = \varinjlim \left( \text{Idem } \mathcal{A}[S] \right) ,$$

where  $S$  varies over a directed poset (or, more generally, a small filtered category) and  $S \mapsto \mathcal{A}[S]$  is a system of additive categories and additive functors.  $\square$

## 2. EXACT STRUCTURES

In this note we consider all additive categories as exact categories with the split exact structure, that is, by declaring a sequence to be exact if and only if it is split exact. By definition, a sequence  $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$  is split exact if there exists an isomorphism  $\chi: B \longrightarrow A \oplus C$  resulting in a commutative ladder diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \text{id}_A \downarrow & & \chi \downarrow & & \text{id}_C \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\text{incl}} & A \oplus C & \xrightarrow{\text{proj}} & C & \longrightarrow & 0 \end{array} .$$

## 3. LOWER TRIANGULAR CATEGORIES

Let  $\mathcal{A}$  be an additive category, and let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be full additive subcategories. We define the lower triangular category  $\mathfrak{LT}(\mathcal{A}_1, \mathcal{A}_2)$  to be the category which has objects

$$F = F_1 \oplus F_2 , \quad F_j \in \mathcal{A}_j$$

(the direct sum decomposition being part of the data), and has morphisms the lower triangular matrices

$$f = \begin{pmatrix} f_{11} & \\ f_{21} & f_{22} \end{pmatrix} : F_1 \oplus F_2 = F \longrightarrow G = G_1 \oplus G_2$$

with  $f_{ij} \in \mathcal{A}(F_j, G_i)$ . The category  $\mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$  comes with a faithful (but not full) forgetful functor to  $\mathcal{A}$ , for which we do not introduce special notation. Perhaps more importantly,  $\mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$  is itself an additive category; the direct sum of  $F_1 \oplus F_2$  and  $G_1 \oplus G_2$  in  $\mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$  is described by the following sum system:

$$F_1 \oplus F_2 \begin{array}{c} \xleftarrow{\begin{pmatrix} \text{id} & \\ 0 & \text{id} \end{pmatrix}} \\ \xrightarrow{\begin{pmatrix} \text{id} & 0 \\ 0 & \text{id} \end{pmatrix}} \end{array} (F_1 \oplus G_1) \oplus (F_2 \oplus G_2) \begin{array}{c} \xleftarrow{\begin{pmatrix} 0 & \text{id} \\ 0 & 0 \end{pmatrix}} \\ \xrightarrow{\begin{pmatrix} 0 & \text{id} \\ 0 & 0 \end{pmatrix}} \end{array} G_1 \oplus G_2$$

For future reference we record a rather trivial calculation:

**Lemma 3.1.** *Suppose that the morphism*

$$\begin{pmatrix} p_{11} & \\ p_{21} & p_{22} \end{pmatrix} : (A_1 \oplus A_2) \longrightarrow (A_1 \oplus A_2)$$

*in  $\mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$  is idempotent. Then we have equalities*

$$p_{11}^2 = p_{11} \ , \quad (3.2a)$$

$$p_{22}^2 = p_{22} \ , \quad (3.2b)$$

$$p_{21}p_{11} + p_{22}p_{21} = p_{21} \ ; \quad (3.2c)$$

*the latter implies, by multiplication with  $p_{11}$  from the right and re-arranging,*

$$p_{22}p_{21}p_{11} = 0 \ . \quad (3.2d)$$

□

There are embeddings of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  as full subcategories of  $\mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$ , given by sending an object  $A$  to  $A \oplus 0$  and  $0 \oplus A$ , respectively. These functors yield full embeddings (often suppressed from the notation in the following)

$$\epsilon_1 : \text{Idem } \mathcal{A}_1 \longrightarrow \text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2) \ , \quad (A_1, p_{11}) \mapsto \left( A_1 \oplus 0, \begin{pmatrix} p_{11} & \\ 0 & 0 \end{pmatrix} \right) \quad (3.3a)$$

and

$$\epsilon_2 : \text{Idem } \mathcal{A}_2 \longrightarrow \text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2) \ , \quad (A_2, p_{22}) \mapsto \left( 0 \oplus A_2, \begin{pmatrix} 0 & \\ 0 & p_{22} \end{pmatrix} \right) \ . \quad (3.3b)$$

#### 4. SUBJECT AND QUOTIENT OBJECT FUNCTORS

We keep the notation from the previous section. There are functors

$$S : \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2) \longrightarrow \mathcal{A}_2 \ , \quad A_1 \oplus A_2 \mapsto A_2 \ , \quad \begin{pmatrix} f_{11} & \\ f_{21} & f_{22} \end{pmatrix} \mapsto f_{22}$$

and

$$Q : \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2) \longrightarrow \mathcal{A}_1 \ , \quad A_1 \oplus A_2 \mapsto A_1 \ , \quad \begin{pmatrix} f_{11} & \\ f_{21} & f_{22} \end{pmatrix} \mapsto f_{11}$$

which induce functors on idempotent completions

$$S: (A_1 \oplus A_2, \begin{pmatrix} \alpha_{11} & \\ \alpha_{21} & \alpha_{22} \end{pmatrix}) \mapsto (A_2, \alpha_{22})$$

and

$$Q: (A_1 \oplus A_2, \begin{pmatrix} \alpha_{11} & \\ \alpha_{21} & \alpha_{22} \end{pmatrix}) \mapsto (A_1, \alpha_{11}) .$$

**Lemma 4.1.** *The functors*

$$S: \text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2) \longrightarrow \text{Idem } \mathcal{A}_2$$

and

$$Q: \text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2) \longrightarrow \text{Idem } \mathcal{A}_1$$

defined above are exact (that is, map split short exact sequences to split short exact sequences).

*Proof.* The functors  $S: \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2) \longrightarrow \mathcal{A}_2$  and  $Q: \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2) \longrightarrow \mathcal{A}_1$  are additive, hence so are the induced functors after idempotent completion, by Lemma 1.1. This is equivalent to the assertion under consideration.  $\square$

**Lemma 4.2.** *There is a short exact sequence of functors*

$$0 \longrightarrow S \longrightarrow \text{id}_{\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)} \longrightarrow Q \longrightarrow 0 .$$

More precisely,

- (1) for every object  $A = (A_1 \oplus A_2, \begin{pmatrix} p_{11} & \\ p_{21} & p_{22} \end{pmatrix})$  of  $\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$  there is a sequence in  $\text{Idem } \mathcal{A}$

$$0 \longrightarrow S(A) \xrightarrow{\begin{pmatrix} 0 \\ p_{22} \end{pmatrix} \sigma} A \xrightarrow{\begin{pmatrix} p_{11} & 0 \end{pmatrix} \pi} Q(A) \longrightarrow 0 , \quad (4.3)$$

and this sequence is split exact in the following way: There exists a morphism  $\rho: A \longrightarrow S(A)$  in  $\text{Idem } \mathcal{A}$  such that the induced map

$$\begin{pmatrix} \pi \\ \rho \end{pmatrix} : A \longrightarrow Q(A) \oplus S(A)$$

is an isomorphism in  $\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$ ;

- (2) the sequence (4.3) is natural in  $A$  with respect to morphisms in  $\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$ .

*Proof.* Let  $A = (A_1 \oplus A_2, \begin{pmatrix} p_{11} & \\ p_{21} & p_{22} \end{pmatrix})$  be an object of  $\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$ . Then we have the sequence (4.3) of composable morphisms in  $\text{Idem } \mathcal{A}$ ; explicitly:

$$0 \longrightarrow (A_2, p_{22}) \xrightarrow{\begin{pmatrix} 0 \\ p_{22} \end{pmatrix} \sigma} \left( A_1 \oplus A_2, \begin{pmatrix} p_{11} & \\ p_{21} & p_{22} \end{pmatrix} \right) \xrightarrow{\begin{pmatrix} p_{11} & 0 \end{pmatrix} \pi} (A_1, p_{11}) \longrightarrow 0 .$$

Clearly  $\pi \circ \sigma = 0$ , and note that as objects of  $\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$  we have

$$(A_1, p_{11}) \oplus (A_2, p_{22}) = \left( A_1 \oplus A_2, \begin{pmatrix} p_{11} & \\ & p_{22} \end{pmatrix} \right) .$$

We define  $\rho: A \longrightarrow S(A)$  to be the morphism in  $\text{Idem } \mathcal{A}$

$$\rho = \begin{pmatrix} p_{22} p_{21} & p_{22} \end{pmatrix} : \left( A_1 \oplus A_2, \begin{pmatrix} p_{11} & \\ p_{21} & p_{22} \end{pmatrix} \right) \longrightarrow (A_2, p_{22}) ;$$

we claim that  $\rho$  and  $\pi$  yield a morphism in  $\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$

$$\begin{pmatrix} \pi \\ \rho \end{pmatrix} = \begin{pmatrix} p_{11} & & \\ p_{22}p_{21} & p_{22} & \end{pmatrix} : \\ \left( A_1 \oplus A_2, \begin{pmatrix} p_{11} & & \\ p_{21} & p_{22} & \end{pmatrix} \right) \longrightarrow \left( A_1 \oplus A_2, \begin{pmatrix} p_{11} & & \\ p_{21} & p_{22} & \end{pmatrix} \right).$$

Indeed, using the equations (3.2a) and (3.2b) we calculate

$$\begin{pmatrix} p_{11} & & \\ & p_{22} & \end{pmatrix} \cdot \begin{pmatrix} p_{11} & & \\ p_{22}p_{21} & p_{22} & \end{pmatrix} \cdot \begin{pmatrix} p_{11} & & \\ p_{21} & p_{22} & \end{pmatrix} = \begin{pmatrix} p_{11} & & \\ p_{22}p_{21}p_{11} + p_{22}p_{21} & p_{22} & \end{pmatrix},$$

which coincides with the map  $(\begin{smallmatrix} \pi \\ \rho \end{smallmatrix})$  by (3.2d) as required.

Next, we define

$$M = \begin{pmatrix} p_{11} & & \\ p_{21}p_{11} & p_{22} & \end{pmatrix} : \left( A_1 \oplus A_2, \begin{pmatrix} p_{11} & & \\ p_{21} & p_{22} & \end{pmatrix} \right) \longrightarrow \left( A_1 \oplus A_2, \begin{pmatrix} p_{11} & & \\ p_{21} & p_{22} & \end{pmatrix} \right);$$

this is a morphism in  $\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$  as

$$\begin{pmatrix} p_{11} & & \\ p_{21} & p_{22} & \end{pmatrix} \cdot \begin{pmatrix} p_{11} & & \\ p_{21}p_{11} & p_{22} & \end{pmatrix} \cdot \begin{pmatrix} p_{11} & & \\ p_{21} & p_{22} & \end{pmatrix} = \begin{pmatrix} p_{11} & & \\ p_{21}p_{11} + p_{22}p_{21}p_{11} & p_{22} & \end{pmatrix} = M,$$

using (3.2a), (3.2b) and (3.2d) again.

Finally, we calculate

$$M \cdot \begin{pmatrix} \pi \\ \rho \end{pmatrix} = \begin{pmatrix} p_{11} & & \\ p_{21}p_{11} & p_{22} & \end{pmatrix} \cdot \begin{pmatrix} p_{11} & & \\ p_{22}p_{21} & p_{22} & \end{pmatrix} = \begin{pmatrix} p_{11} & & \\ p_{21} & p_{22} & \end{pmatrix}$$

(using (3.2c) for the (2,1)-entry of the last matrix) which is the identity map of  $(A_1 \oplus A_2, \begin{pmatrix} p_{11} & & \\ p_{21} & p_{22} & \end{pmatrix})$ . Similarly,

$$\begin{pmatrix} \pi \\ \rho \end{pmatrix} \cdot M = \begin{pmatrix} p_{11} & & \\ p_{22}p_{21} & p_{22} & \end{pmatrix} \cdot \begin{pmatrix} p_{11} & & \\ p_{21}p_{11} & p_{22} & \end{pmatrix} = \begin{pmatrix} p_{11} & & \\ p_{21} & p_{22} & \end{pmatrix}$$

(using (3.2d) for the (2,1)-entry of the last matrix) which is the identity map of  $(A_1 \oplus A_2, \begin{pmatrix} p_{11} & & \\ p_{21} & p_{22} & \end{pmatrix})$ . This shows that  $(\begin{smallmatrix} \pi \\ \rho \end{smallmatrix})$  is an isomorphism in  $\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$  with inverse  $M$ , and finishes the proof of part (1).

It remains to verify naturality of the sequence (4.3) with respect to morphisms in  $\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$

$$\begin{pmatrix} f_{11} & & \\ f_{21} & f_{22} & \end{pmatrix} : \left( A_1 \oplus A_2, \begin{pmatrix} p_{11} & & \\ p_{21} & p_{22} & \end{pmatrix} \right) \longrightarrow \left( B_1 \oplus B_2, \begin{pmatrix} q_{11} & & \\ q_{21} & q_{22} & \end{pmatrix} \right).$$

Using the defining property of a morphism in the idempotent completion

$$\begin{pmatrix} f_{11} & & \\ f_{21} & f_{22} & \end{pmatrix} \cdot \begin{pmatrix} p_{11} & & \\ p_{21} & p_{22} & \end{pmatrix} = \begin{pmatrix} f_{11} & & \\ f_{21} & f_{22} & \end{pmatrix} = \begin{pmatrix} q_{11} & & \\ q_{21} & q_{22} & \end{pmatrix} \cdot \begin{pmatrix} f_{11} & & \\ f_{21} & f_{22} & \end{pmatrix},$$

it is a routine calculation to verify commutativity of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (A_2, p_{22}) & \xrightarrow{\begin{pmatrix} 0 \\ p_{22} \end{pmatrix}} & (A_1 \oplus A_2, p) & \xrightarrow{\begin{pmatrix} p_{11} & 0 \end{pmatrix}} & (A_1, p_{11}) & \longrightarrow & 0 \\ & & \downarrow f_{22} & & \downarrow \begin{pmatrix} f_{11} \\ f_{21} \end{pmatrix} & & \downarrow f_{11} & & \\ 0 & \longrightarrow & (B_2, q_{22}) & \xrightarrow{\begin{pmatrix} 0 \\ q_{22} \end{pmatrix}} & (B_1 \oplus B_2, q) & \xrightarrow{\begin{pmatrix} q_{11} & 0 \end{pmatrix}} & (B_1, q_{11}) & \longrightarrow & 0 \end{array}$$

which finishes the proof of part (2).  $\square$



**Remark 4.4.** The splitting map  $\rho$  from Lemma 4.2 (1) is *not* natural in  $A$ .

## 5. ALGEBRAIC $K$ -THEORY

**Lemma 5.1.** *There is an isomorphism of QUILLEN  $K$ -groups*

$$(Q_*, S_*): K_n(\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)) \xrightarrow{\cong} K_n(\text{Idem } \mathcal{A}_1) \oplus K_n(\text{Idem } \mathcal{A}_2) ,$$

given by sending  $A = (A_1 \oplus A_2, \begin{pmatrix} p_{11} & \\ & p_{22} \end{pmatrix})$  to  $Q(A) = (A_1, p_{11})$  and  $S(A) = (A_2, p_{22})$ . The inverse  $\epsilon_{1*} + \epsilon_{2*}$  is induced by the functor

$$\begin{aligned} \epsilon_1 + \epsilon_2: \text{Idem } \mathcal{A}_1 \times \text{Idem } \mathcal{A}_2 &\longrightarrow \text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2) , \\ ((A_1, p_{11}), (A_2, p_{22})) &\mapsto (A_1 \oplus A_2, \begin{pmatrix} p_{11} & \\ & p_{22} \end{pmatrix}) . \end{aligned}$$

*Proof.* Using the embeddings of categories (3.3a) and (3.3b), we can rephrase the conclusion of Lemma 4.2: There is a short exact sequence of endo-functors of  $\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)$  and natural transformations

$$0 \longrightarrow \epsilon_2 S \longrightarrow \text{id} \longrightarrow \epsilon_1 Q \longrightarrow 0 .$$

By QUILLEN's additivity theorem [Qui73, Corollary 1, p. 106] we have  $(\epsilon_2 S)_* + (\epsilon_1 Q)_* = \text{id}$ , and this sum factors as

$$\begin{aligned} K_n(\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)) &\xrightarrow{(Q_*, S_*)} K_n(\text{Idem } \mathcal{A}_1) \oplus K_n(\text{Idem } \mathcal{A}_2) \\ &\xrightarrow{\epsilon_{1*} + \epsilon_{2*}} K_n(\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2)) . \end{aligned}$$

On the other hand  $(Q_*, S_*) \circ (\epsilon_{1*} + \epsilon_{2*})$  is the identity map of the group  $K_n(\text{Idem } \mathcal{A}_1) \oplus K_n(\text{Idem } \mathcal{A}_2)$ . Hence  $(Q_*, S_*)$  is an isomorphism with inverse  $\epsilon_{1*} + \epsilon_{2*}$ .  $\square$

## 6. GENERALISATIONS

Let  $\mathcal{A}$  be an additive category as before, and let  $\mathcal{A}_q$ ,  $1 \leq q \leq r$ , be a finite collection of full additive subcategories. We define the lower triangular category  $\mathfrak{L}\mathfrak{T}(\mathcal{A}_q; 1 \leq q \leq r)$  to be the category which has objects

$$F = \bigoplus_{q=1}^r F_q , \quad F_j \in \mathcal{A}_j$$

(the direct sum decomposition being part of the data), and has morphisms the lower triangular matrices

$$f = \begin{pmatrix} f_{11} & & & \\ f_{21} & f_{22} & & \\ \vdots & \vdots & \ddots & \\ f_{r1} & f_{r2} & \cdots & f_{rr} \end{pmatrix} : \bigoplus_{q=1}^r F_q = F \longrightarrow G = \bigoplus_{q=1}^r G_q$$

with  $f_{ij} \in \mathcal{A}(F_j, G_i)$ . The category  $\mathfrak{L}\mathfrak{T}(\mathcal{A}_q; 1 \leq q \leq r)$  is an additive category, and we have  $\mathfrak{L}\mathfrak{T}(\mathcal{A}_1, \mathcal{A}_2) = \mathfrak{L}\mathfrak{T}(\mathcal{A}_q; 1 \leq q \leq 2)$ .

**Proposition 6.1.** *Let  $\mathcal{A}$  be an additive category as before, and let  $\mathcal{A}_q$ ,  $1 \leq q \leq r$ , be a finite collection of full additive subcategories. There are exact (=additive) functors*

$$T_k : \text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_q; 1 \leq q \leq r) \longrightarrow \text{Idem } \mathcal{A}_k ,$$

$$\left( \bigoplus_{q=1}^r P_q, \begin{pmatrix} p_{11} & & & \\ p_{21} & p_{22} & & \\ \vdots & \vdots & \ddots & \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{pmatrix} \right) \mapsto (P_k, p_{kk}) ,$$

$$\begin{pmatrix} f_{11} & & & \\ f_{21} & f_{22} & & \\ \vdots & \vdots & \ddots & \\ f_{r1} & f_{r2} & \cdots & f_{rr} \end{pmatrix} \mapsto f_{kk} .$$

These functors induce an isomorphism on  $K$ -groups

$$(T_{1*}, T_{2*}, \dots, T_{r*}) : K_n(\text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_q; 1 \leq q \leq r)) \longrightarrow \bigoplus_{k=1}^r K_n(\text{Idem } \mathcal{A}_k) ;$$

the inverse isomorphisms are induced by the functor

$$\prod_{q=1}^r \text{Idem } \mathcal{A}_q \longrightarrow \text{Idem } \mathfrak{L}\mathfrak{T}(\mathcal{A}_q; 1 \leq q \leq r) ,$$

$$((P_q, p_{qq}))_{q=1}^r \mapsto \left( \bigoplus_{q=1}^r P_q, \begin{pmatrix} p_{11} & & & \\ & p_{22} & & \\ & & \ddots & \\ & & & p_{rr} \end{pmatrix} \right) .$$

*Proof.* In view of the obvious identification

$$\mathfrak{L}\mathfrak{T}(\mathcal{A}_q; 1 \leq q \leq r) = \mathfrak{L}\mathfrak{T}(\mathfrak{L}\mathfrak{T}(\mathcal{A}_q; 1 \leq q \leq r-1), \mathcal{A}_r) ,$$

and the analogous equality for idempotent completions, this follows from a straightforward induction on  $r$ . For  $r = 2$ , the Proposition has been verified in Lemmas 4.1, 4.2 and 5.1; for  $r \leq 1$  the Proposition is trivial. (Alternatively, consider the  $T_k$  as successive quotients of an admissible filtration of the identity functor, and apply Corollary 2, p. 107 of [Qui73].)  $\square$

## 7. THE APPLICATION TEMPLATE

For the reader's convenience we end this part with a basic "template" for applying the abstract machinery. The template has to be adjusted to the actual situation under consideration, as seen in our applications later in the paper.

Given an additive category  $\mathcal{A}$ , the task is to compute  $K_n(\text{Idem } \mathcal{A})$ . We proceed following these steps:

- (AT1) Filter the category  $\mathcal{A}$  by full subcategories  $\mathcal{A}[S]$ , where  $S$  varies over the directed poset of non-empty finite subsets of a partially ordered set  $G$  or, more generally, over any upwards directed sub-poset of the power-set of  $G$  with union  $G$ ; ordering is by inclusion. (The poset structure of  $G$  is irrelevant at this stage.) This yields a corresponding filtration  $(\text{Idem } \mathcal{A})[S] = \text{Idem } (\mathcal{A}[S])$  of  $\text{Idem } \mathcal{A}$ .
- (AT2) For  $S = \{s\}$  a one-element set, identify  $\mathcal{A}[s] = \mathcal{A}[S]$  with some other interesting additive category  $\mathcal{A}'[s]$ . This yields automatically an identification of  $\text{Idem } \mathcal{A}[s]$  with  $\text{Idem } \mathcal{A}'[s]$ , by Lemma 1.2.

- (AT3) For  $S = \{s_1, s_2, \dots, s_r\}$  with  $r \geq 2$  identify  $\mathcal{A}[S]$  with the category  $\mathfrak{L}\mathfrak{T}(\mathcal{A}[\{s_q\}]; 1 \leq q \leq r)$ . (Here we will make use of the order relation of  $G$  which will influence the indexing of the elements  $s_j$ .)
- (AT4) From Proposition 6.1, and from steps (AT2) and (AT3), we obtain isomorphisms  $\bigoplus_{s \in S} K_n(\text{Idem } \mathcal{A}'[s]) \xrightarrow{\cong} K_n(\text{Idem } \mathcal{A}[S])$  which are natural in  $S$  with respect to set inclusion.
- (AT5) As  $\text{Idem } \mathcal{A} = \bigcup_S \text{Idem } \mathcal{A}[S]$  by step (AT1), and as  $K$ -theory commutes with filtered unions [Qui73, p. 104], we obtain the isomorphism

$$K_n(\text{Idem } \mathcal{A}) \cong \bigoplus_{s \in G} K_n(\text{Idem } \mathcal{A}'[s]) .$$

If  $\mathcal{A}'[s] = \mathcal{A}'$  does not depend on  $s$  we obtain an isomorphism

$$K_n(\text{Idem } \mathcal{A}) \cong \bigoplus_{s \in G} K_n(\text{Idem } \mathcal{A}') \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} K_n(\text{Idem } \mathcal{A}') , \quad (7.1)$$

where  $\mathbb{Z}[G]$  denotes the free ABELIAN group with basis  $G$ .

## Part 2. Systematic algebra

### 8. SYSTEMATIC RINGS AND MODULES

Given a subset  $B$  of a ring  $R$  and a subset  $A$  of a right  $R$ -module, we let  $AB$  denote the set of finite sums of products  $ab$  with  $a \in A$  and  $b \in B$ . — Let  $G$  be a group, multiplicatively written. A (*unital*)  $G$ -systematic ring is a unital ring  $R$  together with a family  $(R_g)_{g \in G}$  of additive subgroups of  $(R, +)$  such that

- (SR1)  $R = \sum_{g \in G} R_g$  (that is, the subgroups  $R_g$  generate  $R$  as an ABELIAN group),
- (SR2)  $R_g R_h \subseteq R_{gh}$  for all  $g, h \in G$ ,
- (SR3)  $1 \in R_1$ .

The first two conditions define what is called a  $G$ -system by GRZESZCZUK [Grz85]. The last condition  $1 \in R_1$  is redundant for finite  $G$ , see [Grz85, Theorem 1]. — The unital  $G$ -systematic rings are precisely the homomorphic images of unital  $G$ -graded rings. Indeed, if  $\pi: R' \rightarrow R$  is a surjective ring homomorphism with  $R'$  a  $G$ -graded ring, then setting  $R_g = \pi(R'_g)$  makes  $R$  into a  $G$ -systematic ring. Conversely, if  $R$  is  $G$ -systematic define  $R'_g = \{g\} \times R_g$  and  $R' = \bigoplus_{g \in G} R'_g$ ; this is a  $G$ -graded ring with multiplication determined by  $(g, a) \cdot (h, b) = (gh, ab)$ , and the obvious map  $\pi: (g, a) \mapsto a$  is a surjective ring homomorphism. Note that  $\ker \pi$  need not be a graded ideal.

A filtered ring  $R$  equipped with an increasing or decreasing filtration  $(F^k R)_{k \in \mathbb{Z}}$  can be considered as a  $\mathbb{Z}$ -systematic ring by setting  $R_k = F^k R$ , provided that  $R = \bigcup_k F^k R$  and  $1 \in F^0 R$ .

Given a  $G$ -systematic ring  $R$ , a  $G$ -systematic  $R$ -module is a unital right  $R$ -module  $M$  together with a family  $(M_g)_{g \in G}$  of additive subgroups of  $(M, +)$  such that

- (SM1)  $M = \sum_{g \in G} M_g$  (that is, the subgroups  $M_g$  generate  $M$  as an ABELIAN group),

(SM2)  $M_g R_h \subseteq M_{gh}$  for all  $g, h \in G$ .

A homomorphism  $f: M \longrightarrow N$  of  $G$ -systematic modules is an  $R$ -linear map such that  $f(M_g) \subseteq N_g$  for all  $g \in G$ . Direct sums are given by the prescription

$$\left( \sum_{g \in G} M_g \right) \oplus \left( \sum_{g \in G} N_g \right) = \sum_{g \in G} (M_g \oplus N_g) .$$

This defines the additive category  $\mathbf{Syst}_G\text{-}R$  of  $G$ -systematic  $R$ -modules. — Every  $R$ -module  $M$  can be considered as a  $G$ -systematic  $R$ -module when equipped with the trivial systematic structure  $M_g = M$ . This defines a functor from the category of  $R$ -modules to the category of systematic  $R$ -modules which is right adjoint to the functor which forgets the systematic structure.

For  $M$  a  $G$ -systematic module and  $a$  an element of  $G$  we use the symbol  $(^a)M$  to denote the  $a$ -shift of  $M$ ; this is the  $G$ -systematic module which is  $M$  as an  $R$ -module, with systematic structure given by  $(^a)M_g = M_{a^{-1}g}$ . Clearly  $(^b)(^a)M = (^{ba})M$  so that shifting defines a left  $G$ -action on the category  $\mathbf{Syst}_G\text{-}R$ .

## 9. SYSTEMATICALLY FREE AND PROJECTIVE MODULES

Let  $R$  be a  $G$ -systematic ring. We are interested in the category  $\mathcal{P}_G$  of finitely generated systematically projective  $R$ -modules, which are direct summands of direct sums of modules of the form  $(^g)R$ . In other words,  $\mathcal{P}_G$  is the idempotent completion of the additive category  $\mathcal{F}_G$  of finitely generated systematically free modules, which has objects all finite direct sums with summands of the form  $(^g)R$ . We will in fact work with *systematically free based modules* throughout, that is, free modules equipped with a choice of preferred basis elements. (Morphisms are not required to respect basis elements.) The preferred generator of  $(^g)R$  is the unit element  $1 \in (^g)R_g$ .

Given a set  $S \subseteq G$  we let  $\mathcal{F}_G[S]$  denote the full additive subcategory of  $\mathbf{Syst}_G\text{-}R$  with objects the  $G$ -systematically free based modules of the form

$$\bigoplus_{s \in S} (^s)R^{m_s}$$

for integers  $m_s \geq 0$ , of which only finitely many are allowed to be non-zero. This is the additive category of *finitely generated systematically free based modules with generators having degrees in  $S$* . We denote the idempotent completion  $\text{Idem } \mathcal{F}_G[S]$  by  $\mathcal{P}_G[S]$ , and call  $\mathcal{P}_G[S]$  the category of *finitely generated systematically projective modules with generators having degrees in  $S$* .

Given sets  $S \subseteq T \subseteq G$  we have inclusions of full subcategories  $\mathcal{F}_G[S] \subseteq \mathcal{F}_G[T]$  and  $\mathcal{P}_G[S] \subseteq \mathcal{P}_G[T]$ , resulting in systems of additive categories indexed by the power set of  $G$  (ordered by inclusion). In view of Lemma 1.3 we observe

$$\mathcal{F}_G = \bigcup_S \mathcal{F}_G[S] \quad \text{and} \quad \mathcal{P}_G = \text{Idem } \mathcal{F}_G = \bigcup_S \mathcal{P}_G[S] \quad (9.1)$$

whenever we let  $S$  vary over the full power set of  $G$ , or some upwards directed sub-poset covering all of  $G$ .

## 10. STRONGLY SYSTEMATIC RINGS AND MODULES

**Definition 10.1.** We call a  $G$ -systematic ring  $K$  (resp., a  $G$ -systematic  $K$ -module  $M$ ) *strongly systematic* if for all  $g_1, g_2 \in G$  we have an equality  $K_{g_1}K_{g_2} = K_{g_1g_2}$  (resp.,  $M_{g_1}K_{g_2} = M_{g_1g_2}$ ).

Strongly systematic rings are exactly the homomorphic images of strongly graded rings, and coincide with “CLIFFORD systems” in the sense of DADE [Dad70], and with “almost strongly graded rings” satisfying  $1 \in K_1$  in the terminology of NĂSTĂSESCU and VAN OYSTAEYEN [NvO82, §I.8]. — Clearly a strongly  $G$ -systematic ring  $K$  satisfies  $1 \in K_{g^{-1}}K_g$  for all  $g \in G$ . Conversely, suppose that  $K$  is a  $G$ -systematic ring with  $1 \in K_{h^{-1}}K_h$  for all  $h \in G$ . Then

$$K_gK_h \subseteq K_{gh} \subseteq K_{gh}(K_{h^{-1}}K_h) \subseteq K_gK_h$$

so that  $K_gK_h = K_{gh}$  for all  $g, h \in G$ : The ring  $K$  is strongly systematic.

**Lemma 10.2.** Let  $Q$  be a group, let  $K = \sum_{g \in Q} K_g$  be a  $Q$ -systematic ring, and let  $M = \sum_{g \in Q} M_g$  be a  $Q$ -systematic  $K$ -module. If  $K$  is strongly systematic then so is  $M$ .

*Proof.* For strongly systematic  $K$ , since  $1 \in K_1$  we have

$$M_gK_h \subseteq M_{gh} = M_{gh}K_1 = M_{gh}K_{h^{-1}}K_h \subseteq M_gK_h$$

for all  $g, h \in Q$ . That is, we have  $M_gK_h = M_{gh}$  as required.  $\square$

**Lemma 10.3** (cf. NĂSTĂSESCU and VAN OYSTAEYEN [NvO82, I.8.2]). Let  $K$  be a  $Q$ -systematic ring, and let  $a \in Q$  be such that  $1 \in K_aK_{a^{-1}}$ . Then the right  $K_1$ -module  $K_a$  is finitely generated projective.

*Proof.* By hypothesis there is a finite sum decomposition  $1 = \sum_j \alpha_j \beta_j$  with  $\alpha_j \in K_a$  and  $\beta_j \in K_{a^{-1}}$ . Define  $K_1$ -linear maps

$$\rho_j: K_a \longrightarrow K_1, \quad r \mapsto \beta_j r,$$

and observe that  $\sum_j \alpha_j \rho_j = \text{id}_{K_a}$  as

$$\sum_j \alpha_j \rho_j(r) = \sum_j \alpha_j \beta_j r = 1 \cdot r = r$$

so that the collection of the  $\alpha_j$  and  $\rho_j$  form a dual basis of  $K_a$ . Consequently  $K_a$  is finitely generated projective as a right  $K_1$ -module, see BOURBAKI [Bou98, §II.2.6, Proposition 12].  $\square$

Let  $K$  be a  $Q$ -systematic ring, and let  $L$  be a right  $K_1$ -module. We consider the tensor product  $L \otimes_{K_1} K$  as a  $Q$ -systematic  $K$ -module with  $(L \otimes_{K_1} K)_q$  the subgroup generated by the primitive tensors  $\ell \otimes k$  with  $\ell \in L$  and  $k \in K_q$ . In other words,  $(L \otimes_{K_1} K)_q$  is the image of map  $\omega: L \otimes_{K_1} K_q \longrightarrow L \otimes_{K_1} K$ . The map  $\omega$  is injective in case  $L$  is a projective (or, more generally, flat)  $K_1$ -module, in which case we will tacitly identify  $L \otimes_{K_1} K_q$  with  $(L \otimes_{K_1} K)_q$ .

**Lemma 10.4.** Let  $K$  be a  $Q$ -systematic ring, and let  $a \in Q$  be such that  $1 \in K_{a^{-1}}K_a$ . The natural map

$$\nu_{(a)K}: K_{a^{-1}} \otimes_{K_1} K \longrightarrow {}^{(a)}K, \quad s \otimes r \mapsto sr$$

is an isomorphism of  $Q$ -systematic  $K$ -modules.

*Proof.* By hypothesis there is a finite sum decomposition  $1 = \sum_j \alpha_j \beta_j$  with  $\alpha_j \in K_{a-1}$  and  $\beta_j \in K_a$ . Define

$$\tau: {}^{(a)}K \longrightarrow K_{a-1} \otimes_{K_1} K, x \mapsto \sum_j \alpha_j \otimes (\beta_j x).$$

This map is clearly  $K$ -linear, and it is systematic as for  $x \in {}^{(a)}K_g = K_{a-1}g$  we have  $\beta_j x \in K_a K_{a-1}g \subseteq K_g$ . Now  $\nu_{(a)K} \circ \tau(x) = \sum_j \alpha_j \beta_j x = x$ , and

$$\tau \circ \nu_{(a)K}(s \otimes r) = \tau(sr) = \sum_j \alpha_j \otimes (\beta_j sr).$$

But  $s \in K_{a-1}$  so that  $\beta_j s \in K_1$ ; consequently,  $\alpha_j \otimes (\beta_j sr) = (\alpha_j \beta_j s) \otimes r$ , with  $\alpha_j \beta_j \in K_1$ , so that

$$\tau \circ \nu_{(a)K}(s \otimes r) = \sum_j \alpha_j \beta_j \cdot s \otimes r = s \otimes r.$$

We have shown that both compositions  $\tau \circ \nu_{(a)K}$  and  $\nu_{(a)K} \circ \tau$  are identity maps, as required.  $\square$

**Proposition 10.5.** *Let  $Q$  be a group, and let  $K$  be a strongly  $Q$ -systematic ring. Then the category  $\mathcal{P}_Q$  of finitely generated  $Q$ -systematically projective  $K$ -modules is equivalent to the category of finitely generated projective  $K_1$ -modules via the functor  $\rho$  that sends a module  $M = \sum_{q \in Q} M_q$  to its component  $M_1$ . The inverse equivalence is given by the functor  $\tau$  sending  $L$  to the  $Q$ -systematic module  $L \otimes_{K_1} K = \sum_{q \in Q} L \otimes_{K_1} K_q$ .*

*Proof.* As  $K$  is strongly systematic, each of its components  $K_a$  is a finitely generated projective  $K_1$ -module by Lemma 10.3. It follows that  $\rho: M \mapsto M_1$  maps finitely generated systematically projective  $K$ -modules to finitely generated projective  $K_1$ -modules.

If  $L$  is a finitely generated free  $K_1$ -module then  $L \otimes_{K_1} K$  is (isomorphic to) a finite direct sum of copies of  $K$ , that is,  $L \otimes_{K_1} K$  is a finitely generated systematically free module. It follows that  $\tau: L \mapsto L \otimes_{K_1} K$  maps finitely generated projective  $K_1$ -modules to finitely generated systematically projective  $K$ -modules.

For a projective  $K_1$ -module  $L$  we have  $(L \otimes_{K_1} K)_1 = L \otimes_{K_1} K_1 \cong L$  (natural in  $L$ ) so that  $\rho \circ \tau \cong \text{id}$ .

Define a natural transformation  $\nu: \tau \circ \rho \xrightarrow{\cdot} \text{id}$  by

$$\nu_M: M_1 \otimes_{K_1} K \longrightarrow M, \quad m \otimes k \mapsto mk.$$

This is a map of systematic modules: the image of  $(M_1 \otimes_{K_1} K)_q = M_1 \otimes_{K_1} K_q$  is contained in  $M_q$ . As  $K$  is strongly systematic we have  $1 \in K_1 = K_{a-1}K_a$  for every  $a \in Q$ . Hence for  $M = {}^{(a)}K$  a systematically free module on one generator the map  $\nu_M = \nu_{(a)K}$  is an isomorphism by Lemma 10.4. By allowing direct sums we see that  $\nu_M$  is an isomorphism for (finitely generated) systematically free modules; further allowing retracts of free modules yields the statement of the proposition.  $\square$

**Remark 10.6.** (1) The proof shows that we also obtain an equivalence between the category of  $Q$ -systematically projective  $K$ -modules and the category of projective  $K_1$ -modules (without any finite generation

hypothesis). However, the theorem does not extend to all modules, nor to all finitely generated modules. For example, consider  $K = \mathbb{Z}[1/2]$  as a  $\mathbb{Z}$ -systematic ring with systematic structure defined by  $K_k = \{2^k \cdot x \mid x \in \mathbb{Z}\}$  so that  $K_0 = \mathbb{Z}$ . Then  $L = \mathbb{Z}/2$  is a non-trivial  $K_0$ -module, but  $\tau(L) = L \otimes_{K_0} K = (\mathbb{Z}/2) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] = 0$  so that  $\rho \circ \tau(L) = (L \otimes_{K_0} K)_0 \not\cong L$ .

- (2) The functor  $\rho$  is additive and preserves injections, but not surjections. For example, with  $K = \mathbb{Z}[1/2]$  as above, multiplication by 2 is a surjective (in fact, bijective) self-map of  $K$ ; application of  $\rho$  results in the self-map of  $\mathbb{Z}$  given by multiplication by 2, which is not onto.

### Part 3. Algebraic $K$ -theory

#### 11. SYSTEMATIC $K$ -THEORY

**Systematic  $K$ -theory.** The  $G$ -systematic  $K$ -theory of  $R$  is the algebraic  $K$ -theory of the additive category  $\mathcal{P}_G$  of finitely generated  $G$ -systematically projective  $R$ -modules with respect to the split exact structure. We write  $K_n^{G\text{-syst}}(R) = K_n(\mathcal{P}_G)$ . As  $G$  acts on the left of the category  $\mathcal{P}_G$  by shifting, the groups  $K_n^{G\text{-syst}}(R)$  acquire a left  $\mathbb{Z}[G]$ -module structure described by

$$(g, P) \mapsto {}^{(g)}P \quad \text{for } g \in G .$$

#### 12. SYSTEMATIC $K$ -THEORY OF RINGS WITH POSITIVE SUPPORT

Suppose now we are given an extension of multiplicative groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1 . \quad (12.1)$$

Our goal is to analyse the  $G$ -systematic  $K$ -theory of a  $G$ -systematic ring  $R$  in terms of the  $N$ -systematic or  $H$ -systematic  $K$ -theory of certain subrings of  $R$ , under the assumption that  $R$  has “positive support” in a suitable sense. This requires, among other things, to have a partial order on either  $N$  or  $H$ .

**Ordering the subgroup.** Suppose that the extension (12.1) is split so that  $G = N \rtimes H$  is a semi-direct product with respect to some group homomorphism  $\theta: H \longrightarrow \text{Aut}(N)$ . We simply write  $hn$  for  $\theta(h)(n)$ , so that the group multiplication takes the form  $(n, h)(n', h') = (n \cdot hn', hh')$ , and inverses are computed as  $(n, h)^{-1} = (h^{-1}n^{-1}, h^{-1})$ . Given a  $G$ -systematic module  $M$  we define

$$M^H = \sum_{h \in H} M_{(1, h)} .$$

Applied to  $M = R$  this provides us with the  $H$ -systematic ring  $R^H$ , and in general  $M^H$  is an  $H$ -systematic  $R^H$ -module by restriction of scalars.

**Lemma 12.2.** *Let  $s \in N$  be a fixed element. The category  $\mathcal{F}_H$  of finitely generated  $H$ -systematically free  $R^H$ -modules is equivalent to the category  $\mathcal{F}_{N \rtimes H}[s \rtimes H]$ . The equivalence takes the module  ${}^{(h)}R^H$  to the module  ${}^{(s, h)}R$ .*

*Proof.* Finitely generated free based modules are entirely determined by their generators. Hence there is a bijection of objects determined by the assignment

$${}^{(h)}R^H \mapsto {}^{(s,h)}R .$$

Morphisms  ${}^{(s,h_1)}R \longrightarrow {}^{(s,h_2)}R$  in  $\mathcal{F}_{N \rtimes H}[s \rtimes H]$  are in bijective correspondence with elements of

$${}^{(s,h_2)}R_{(s,h_1)} = R_{(s,h_2)^{-1}(s,h_1)} = R_{(1,h_2^{-1}h_1)} ,$$

as morphisms of free modules on one generator are determined by the image of the generator. On the other hand, morphisms  ${}^{(h_1)}R^H \longrightarrow {}^{(h_2)}R^H$  in  $\mathcal{F}_H$  are in bijective correspondence with elements of

$${}^{(h_2)}R_{h_1}^H = R_{h_2^{-1}h_1}^H = R_{(1,h_2^{-1}h_1)} ,$$

which is the same set. Composition of morphisms corresponds to multiplication of elements in both categories. This shows that the set of morphisms  ${}^{(s,h_1)}R \longrightarrow {}^{(s,h_2)}R$  and the set of morphisms  ${}^{(h_1)}R^H \longrightarrow {}^{(h_2)}R^H$  are the same. The claim follows by considering finite direct sums of modules (which amounts to considering matrices instead of single ring elements).  $\square$

**Theorem 12.3.** *Let  $G = N \rtimes H$  be a semi-direct product as before. Suppose that  $N$  is equipped with a translation-invariant partial order (so that  $n \geq n'$  implies  $anb \geq an'b$  for all  $a, b, n, n' \in N$ ) which is also invariant under the action of  $H$  (so that  $n \geq n'$  implies  $hn \geq hn'$  for all  $n, n' \in N$  and  $h \in H$ ). Let  $R = \sum_{(n,h) \in N \rtimes H} R_{(n,h)}$  be an  $(N \rtimes H)$ -systematic ring with support in  $N^+ \rtimes H$ , where  $N^+ = \{n \in N \mid n \geq 1\}$  is the positive cone of  $N$  as usual; that is, suppose that  $R_{(n,h)} = \{0\}$  whenever  $n \notin N^+$ . There are canonical  $\mathbb{Z}[N \rtimes H]$ -module isomorphisms*

$$\mathbb{Z}[N \rtimes H] \otimes_{\mathbb{Z}[H]} K_n^{H\text{-syst}}(R^H) \cong K_n^{(N \rtimes H)\text{-syst}}(R) ,$$

described by the formula  $(s, h) \otimes {}^{(h'')}R^H \mapsto {}^{(s, hh'')}R$ .

*Proof. Step (AT1).* The category  $\mathcal{F}_G$  is the filtered union of the categories  $\mathcal{F}_G[S \rtimes H]$ , where  $S$  ranges over the directed poset  $\mathbb{P}(N)^{\text{fin}}$  of non-empty finite subsets of  $N$  (ordered by inclusion). Consequently,  $\mathcal{P}_G$  is the filtered union of the categories  $\mathcal{P}_G[S \rtimes H]$  by Lemma 1.3.

**Step (AT2).** It has been observed in Lemma 12.2 that  $\mathcal{F}_G[s \rtimes H]$  is equivalent to the category  $\mathcal{F}_H$ . By Lemma 1.2, the categories  $\mathcal{P}_G[s \rtimes H]$  and  $\mathcal{P}_H$  are consequently equivalent.

**Steps (AT3) and (AT4).** Let  $S$  be a finite non-empty subset of  $N$ . We write  $S$  in the form  $S = \{s_1, s_2, \dots, s_r\}$  where  $s_i > s_j$  implies  $i < j$  (that is, we choose a linear extension of the poset  $S$ ). We claim that the categories  $\mathcal{F}_G[S \rtimes H]$  and  $\mathfrak{L}\mathfrak{T}(\mathcal{F}_G[s_q \rtimes H]; 1 \leq q \leq r)$  are equivalent. Every object  $P$  of  $\mathcal{F}_G[S \rtimes H]$  is, by definition, a direct sum of the form  $P = P_1 \oplus P_2 \oplus \dots \oplus P_r$ , where

$$P_j = \bigoplus_{k=1}^{n_j} {}^{(s_j, h_{jk})}R$$

with certain integers  $n_j \geq 0$  and element  $h_{jk} \in H$ . That is,  $P$  is a collection of objects  $P_j$  of  $\mathcal{F}_G[s_j \rtimes H]$ . A morphism  $f$  from  $P = P_1 \oplus P_2 \oplus \dots \oplus P_r$  to  $Q = Q_1 \oplus Q_2 \oplus \dots \oplus Q_r$  is a matrix  $(f_{ij})$  with  $f_{ij}: P_j \longrightarrow Q_i$  a  $G$ -systematic



map. To prove the desired equivalence of categories it is enough to verify that this matrix is necessarily a lower triangular matrix.

We thus want to show that for  $i < j$  we must have  $f_{ij} = 0$ . Indeed, it is enough to show that every systematic homomorphism  ${}^{(s_j, h_{jk})}R \longrightarrow {}^{(s_i, h_{il})}R$  is necessarily trivial. For the latter is determined by the image of the generator  $1 \in {}^{(s_j, h_{jk})}R_{(s_j, h_{jk})}$  in the set

$$\begin{aligned} {}^{(s_i, h_{il})}R_{(s_j, h_{jk})} &= R_{(s_i, h_{il})^{-1} \cdot (s_j, h_{jk})} = R_{(h_{il}^{-1} s_i^{-1}, h_{il}^{-1}) \cdot (s_j, h_{jk})} \\ &= R_{(h_{il}^{-1} s_i^{-1} \cdot h_{il}^{-1} s_j, h_{il}^{-1} h_{jk})} = R_{(h_{il}^{-1} (s_i^{-1} s_j), h_{il}^{-1} h_{jk})} . \end{aligned}$$

As  $j > i$  we cannot have  $s_j \geq s_i$ , by choice of linear extension, and as the order is invariant under the action of  $H$  by hypothesis,  $h_{il}^{-1} s_j \not\geq h_{il}^{-1} s_i$  and thus

$$h_{il}^{-1} (s_i^{-1} s_j) = h_{il}^{-1} s_i^{-1} \cdot h_{il}^{-1} s_j = (h_{il}^{-1} s_i)^{-1} \cdot h_{il}^{-1} s_j \not\geq 1 .$$

This means that  $h_{il}^{-1} (s_i^{-1} s_j) \notin N^+$  whence  ${}^{(s_i, h_{il})}R_{(s_j, h_{jk})}$  must be the trivial group, by our hypothesis on the support of  $R$ . In other words, the generator of the source must map to 0, and so the homomorphism under investigation is trivial.

By Proposition 6.1 the inclusion functors  $\mathcal{P}_G[s_j \times H] \longrightarrow \mathcal{P}_G[S \times H]$  induce isomorphisms  $\bigoplus_{s \in S} K_n(\mathcal{P}_G[s \times H]) \xrightarrow{\cong} K_n(\mathcal{P}_G[S \times H])$ . From the previous step we conclude that we have isomorphisms

$$\mathbb{Z}[S] \otimes_{\mathbb{Z}} K_n^{H\text{-syst}}(R^H) \cong \bigoplus_{s \in S} K_n^{H\text{-syst}}(R^H) \xrightarrow{\cong} K_n(\mathcal{P}_G[S \times H]) \quad (12.4)$$

(where  $\mathbb{Z}[S]$  denotes the free ABELIAN group on  $S$ ) which are determined by the assignments  $s \otimes {}^{(h)}R^H \mapsto {}^{(s, h)}R$ . These isomorphisms are natural with respect to set inclusions  $S \subseteq T$ .

**Step (AT5).** Upon passing to the colimit over  $S \in \mathbb{P}(N)^{\text{fin}}$  in (12.4) we obtain an isomorphism

$$\omega: \mathbb{Z}[N] \otimes_{\mathbb{Z}} K_n^{H\text{-syst}}(R^H) \xrightarrow{\cong} K_n^{(N \rtimes H)\text{-syst}}(R) , \quad s \otimes {}^{(h)}R^H \mapsto {}^{(s, h)}R$$

of ABELIAN groups.

We have maps of left  $\mathbb{Z}[N]$ -modules

$$\mathbb{Z}[N] \otimes_{\mathbb{Z}} K_n^{H\text{-syst}}(R^H) \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} \mathbb{Z}[N \rtimes H] \otimes_{\mathbb{Z}[H]} K_n^{H\text{-syst}}(R^H)$$

described by the formulæ

$$\alpha(s \otimes P) = (s, 1) \otimes P \quad \text{and} \quad \beta((s, h) \otimes P) = s \otimes {}^{(h)}P ;$$

that is,  $\alpha$  is induced by the inclusion of rings  $\mathbb{Z}[N] \longrightarrow \mathbb{Z}[N \rtimes H]$ . To show that  $\beta$  is well defined, it suffices to note that the assignment  $((s, h), P) \mapsto s \otimes {}^{(h)}P$  is  $\mathbb{Z}[H]$ -balanced: Indeed, we have

$$((s, h) \cdot (1, h'), P) = ((s, hh'), P) \mapsto s \otimes {}^{(hh')}P$$

and

$$((s, h), {}^{(h')}P) \mapsto s \otimes {}^{(h)(h')}P = s \otimes {}^{(hh')}P .$$

As  $(1)P = P$  we have  $\beta \circ \alpha = \text{id}$ . But

$$\begin{aligned} \alpha \circ \beta((s, h) \otimes P) &= \alpha(s \otimes ({}^h P)) = (s, 1) \otimes ({}^h P) \\ &= (s, 1) \cdot (1, h) \otimes P = (s, h) \otimes P \end{aligned}$$

so that  $\beta$  is in fact an isomorphism.

The composite map

$$\omega \circ \beta: \mathbb{Z}[N \rtimes H] \otimes_{\mathbb{Z}[H]} K_n^{H\text{-syst}}(R^H) \longrightarrow K_n^{(N \rtimes H)\text{-syst}}(R)$$

is described by the formula

$$\omega \circ \beta: (s, h) \otimes ({}^{h''} R^H) \mapsto (s, hh'') R,$$

as can be seen readily by tracing the definitions. It is bijective by what we have proved before; it is in fact an isomorphism of left  $\mathbb{Z}[N \rtimes H]$ -modules as, on the one hand,

$$\omega \circ \beta((s', h') \cdot (s, h) \otimes ({}^{h''} R^H)) = \omega \circ \beta((s' \cdot h' s, h' h) \otimes ({}^{h''} R^H)) = (s' \cdot h' s, h' h h'') R$$

while, on the other hand,

$$(s', h') \omega \circ \beta((s, h) \otimes ({}^{h''} R^H)) = (s', h')(s, hh'') R = (s' \cdot h' s, h' h h'') R. \quad \square$$

The theorem can be applied in rather more common situations. For example, a  $G$ -graded ring  $R = \bigoplus_{g \in G} R_g$  may be considered as a  $G$ -systematic ring. The category  $\mathcal{F}_G$  is then equivalent to the category of finitely generated  $G$ -graded free  $R$ -modules so that Theorem 12.3 gives a description of the graded  $K$ -theory of  $R$ . This unifies various calculations of graded  $K$ -theory in the literature, for example those by QUILLEN [Qui73, p. 107, Proposition] (the case  $N = \mathbb{Z}$  and  $H = \{1\}$ ), HAZRAT and HÜTTEMANN [HH13, Theorem 1] (the case of a direct product  $\mathbb{Z} \times H$ ) and HÜTTEMANN [Hüt13] (the case  $N = \mathbb{Z}^n$  and  $H = \{1\}$ , and  $R$  having support in a polyhedral pointed cone). — It should be pointed out explicitly that the category of *all* systematic modules over a graded ring can contain non-graded modules, but that a *systematically projective* module over a graded ring is automatically graded.

The theorem also applies to a positively filtered ring  $R$ , that is, a ring equipped with an ascending chain of additive subgroups  $F^k R$ ,  $k \in \mathbb{Z}$ , such that  $F^{-1} R = \{0\}$ ,  $1 \in F^0 R$ ,  $F^k R \cdot F^\ell R \subseteq F^{k+\ell} R$ , and  $\bigcup_k F^k R = R$ . Setting  $R_k = F^k R$  we obtain a  $\mathbb{Z}$ -systematic ring; the category  $\mathcal{F}_{\mathbb{Z}}$  is now equivalent to the category of finitely generated filt-free based  $R$ -modules in the sense of NĂSTĂSESCU and VAN OYSTAEYEN [NvO82, §D IV]. Application of Theorem 12.3 to the trivial group extension  $N = G = \mathbb{Z}$  and  $H = \{1\}$  then says that the filtered  $K$ -theory of  $R$  is isomorphic to the tensor product of  $\mathbb{Z}[x, x^{-1}] = \mathbb{Z}[\mathbb{Z}]$  with the (usual)  $K$ -theory of  $F^0 R$ .

A positively filtered ring as above determines an associated  $\mathbb{Z}$ -graded ring with support in  $\mathbb{N}$ . Applying Theorem 12.3 to both the filtered ring and the associated graded ring, and noting that the “degree 0”-pieces are the same, we immediately get the following:

**Corollary 12.5.** *Let  $R$  be a positively filtered ring, and let  $B$  denote the associated  $\mathbb{Z}$ -graded ring  $\bigoplus_{k \geq 0} F^k R / F^{k-1} R$ . The filtered  $K$ -theory of  $R$  and the  $\mathbb{Z}$ -graded  $K$ -theory of  $B$  are both isomorphic to  $\mathbb{Z}[x, x^{-1}] \otimes_{\mathbb{Z}} K_n(F^0 R)$  as left  $\mathbb{Z}[x, x^{-1}]$ -modules; the action of the indeterminate  $x$  corresponds to shifting of filtered and graded modules, respectively.  $\square$*

**Ordering the quotient.** Given a (possibly non-split) group extension of multiplicative groups

$$1 \longrightarrow N \xrightarrow{\subseteq} G \xrightarrow{\pi} H \longrightarrow 1$$

and a translation-invariant partial order “ $\geq$ ” on  $H$ , we write  $H^+$  for the positive cone  $\{h \in H \mid h \geq 1\}$  of  $H$  and define  $G^+ = \pi^{-1}(H^+)$ . Let  $R = \sum_{g \in G} R_g$  be a  $G$ -systematic ring, and let  $R_N = \sum_{n \in N} R_n$  be the  $N$ -systematic subring determined by  $N$ .

**Theorem 12.6.** *Suppose the  $G$ -systematic ring  $R$  has support in  $G^+$  in the sense that  $R_g = \{0\}$  whenever  $g \notin G^+$ . Any choice of set-theoretic section  $\sigma: H \longrightarrow G$  of  $\pi$  determines isomorphisms of ABELian groups*

$$\Psi: \bigoplus_H K_q^{N\text{-syst}}(R_N) \xrightarrow{\cong} K_q^{G\text{-syst}}(R); \quad (12.7)$$

the restriction of this isomorphism to the  $h$ -summand is induced by the functor determined by  ${}^{(n)}R_N \mapsto {}^{(\sigma(h) \cdot n)}R$ .

*Proof.* The proof is similar to that of Theorem 12.3; we give a short account of the relevant arguments. — Let  $g, g' \in G$  be such that  $g^{-1}g' \notin G^+$ , i.e., such that  $\pi(g) \not\leq \pi(g')$ . Then there are no non-trivial homomorphisms  $\eta: {}^{(g')}R \longrightarrow {}^{(g)}R$ . Indeed,  $\eta$  is determined by the image of the generator  $1 \in {}^{(g')}R_{g'}$ , that is, by the element  $\eta(1) \in {}^{(g)}R_{g'} = R_{g^{-1}g'}$ ; but  $R_{g^{-1}g'} = \{0\}$  by our condition on the support of  $R$ .

Now let  $S = \{s_1, s_2, \dots, s_r\} \subseteq H$  be a finite subset; we may assume, by renumbering if necessary, that  $s_i > s_j$  implies  $i < j$ . It follows from the previous paragraph that we can identify  $\mathcal{F}_G[\pi^{-1}S]$  with the lower triangular category  $\mathfrak{L}\mathfrak{T}(\mathcal{F}_G[\pi^{-1}(s_q)]; 1 \leq q \leq r)$ ; from Proposition 6.1 we infer that the inclusion functors induce isomorphisms

$$\bigoplus_{s \in S} K_q(\mathcal{P}_G[\pi^{-1}(s)]) \longrightarrow K_q(\mathcal{P}_G[\pi^{-1}S]) \quad (12.8)$$

which are natural in  $S$ .

Now observe that the category  $\mathcal{F}_N$  of finitely generated  $N$ -graded free  $R_N$ -modules is equivalent to  $\mathcal{F}_G[\pi^{-1}(s)]$ , via the functor that takes  ${}^{(n)}R_N \in \mathcal{F}_N$  to the  $G$ -graded free  $R$ -module  ${}^{(\sigma(s)n)}R$ . This is a bijection on objects as  $\pi^{-1}(s) = \sigma(s)N$ . Morphisms  ${}^{(n_1)}R_N \longrightarrow {}^{(n_2)}R_N$  are in bijective correspondence with elements of  ${}^{(n_2)}R_N{}_{n_1} = (R_N)_{n_2^{-1}n_1}$ , and morphisms  ${}^{(\sigma(s)n_1)}R \longrightarrow {}^{(\sigma(s)n_2)}R$  are in bijective correspondence with elements of  ${}^{(\sigma(s)n_2)}R_{\sigma(s)n_1} = R_{n_2^{-1}n_1}$ , which is the same set. From Lemma 1.2 we conclude that  $\mathcal{P}_G[\pi^{-1}(s)]$  is equivalent to the category  $\mathcal{P}_N$  of finitely generated  $N$ -graded projective  $R_N$ -modules.

Combining this with the isomorphism (12.8), and passing to the limit with respect to  $S$ , we obtain an isomorphism of  $K$ -groups as stated in the Theorem.  $\square$

**Remark 12.9.** Define a group  $N \tilde{\times}_\sigma H$  which has  $N \times H$  as underlying set, has neutral element  $(\sigma(1)^{-1}, 1)$ , and has multiplication law

$$(n_1, h_1) \cdot (n_2, h_2) = (\sigma(h_1 h_2)^{-1} \sigma(h_1) \sigma(h_2) n_1^{\sigma(h_2)}, h_1 h_2)$$

where  $a^b = b^{-1} a b$ . The map  $\mu(n, h) = \sigma(h) n$  is a group isomorphism  $\mu: N \tilde{\times}_\sigma H \longrightarrow G$  with inverse  $\alpha(g) = (\sigma(\pi(g))^{-1} g, \pi(g))$ . The left-hand side of (12.7) has an  $N \tilde{\times}_\sigma H$ -module structure described by saying that the action of  $(n_1, h_1)$  sends the module  ${}^{(n_2)} R_N$  in the  $h_2$ -summand to the module  ${}^{(\bar{n})} R_N$  in the  $h_1 h_2$ -summand, where  $\bar{n} = \sigma(h_1 h_2)^{-1} \sigma(h_1) \sigma(h_2) n_1^{\sigma(h_2)}$ . The isomorphism then becomes a  $\mu$ -semilinear map in the sense that  $\Psi((n, h) \cdot x) = \mu(n, h) \cdot \Psi(x)$ .

Theorem 12.6 and Proposition 10.5 (applied to  $Q = N$  and  $K = R_N$ ) yield:

**Corollary 12.10.** *Suppose the  $G$ -systematic ring  $R$  has support in  $G^+$ . Suppose further that the  $N$ -systematic ring  $R_N$  is strongly systematic. Then there are isomorphisms of  $K$ -groups*

$$\bigoplus_H K_n(R_1) \cong K_n^{G\text{-gr}}(R) . \quad \square$$

### 13. EQUIVARIANT $K$ -THEORY OF AFFINE TORIC SCHEMES

Corollary 12.10 generalises a result on graded  $K$ -theory obtained by AU, HUANG and WALKER [AHW09, Theorem 1]. In their set-up,  $G$  is an ABELIAN group,  $A \subseteq G$  a sub-monoid,  $R = B[A]$  the monoid ring over a commutative ground ring  $B$ , and  $N$  is the group of invertible elements in  $A$ . The partial order on  $H = G/N$  is defined by

$$h' \geq h \quad \Leftrightarrow \quad g' g^{-1} \in A ,$$

where  $g, g' \in G$  are representatives of the cosets  $h = gN$  and  $h' = g'N$ . (That is,  $A = G^+$  in the notation of Theorem 12.6.) Corollary 12.10 then reduces to the cited result

$$\mathbb{Z}[G/N] \otimes_{\mathbb{Z}} K_n(B) \cong \bigoplus_{G/N} K_n(B) \cong K_n^{G\text{-gr}}(B[A]) .$$

As explained in *loc.cit.*, the result can be further specialised and translated into the language of affine toric schemes. We will look at a slight generalisation. So suppose in addition to the above that  $G \cong \mathbb{Z}^r$  is an  $r$ -dimensional lattice, and that  $A = \sigma^\vee \cap G$  for some rational polyhedral  $r$ -dimensional cone  $\sigma^\vee \subseteq G \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$ . Then  $R$  is the coordinate ring of the (possibly singular) affine toric  $B$ -scheme  $X = \text{Spec} R$ .

Suppose further that  $R'$  is a commutative  $G$ -graded  $R$ -algebra, that is, a commutative  $G$ -graded ring equipped with a degree-preserving ring homomorphism  $\nu: R \longrightarrow R'$ . Geometrically this corresponds to a morphism of affine schemes

$$X' = \text{Spec} R' \longrightarrow \text{Spec} R = X ;$$

due to the presence of  $G$ -gradings, the  $r$ -dimensional algebraic torus  $T = \text{Spec}B[G]$  acts on both source and target, and the morphism is  $T$ -equivariant.

Note now that since  $R_N$  is strongly graded so is  $R'_N$ ; indeed, for any  $n, n' \in G$  we have  $R'_n \otimes R'_{n'} \supseteq R'_n \otimes \nu(R_{n'})$ , and the restriction

$$R'_n \otimes \nu(R_{n'}) \longrightarrow R'_{nn'}$$

of the multiplication map in  $R'$  is surjective since any module over a strongly graded ring is itself strongly graded (Lemma 10.2 or [Dad80, Theorem 2.8], applied to the right  $R_N$ -module  $R'_N$ ). As  $T$ -equivariant vector bundles on  $\text{Spec}R'$  correspond to finitely generated  $G$ -graded projective  $R'$ -modules, Corollary 12.10 yields the following generalisation of [AHW09, Theorem 4]:

**Theorem 13.1.** *If in the situation set out above the  $G$ -graded  $R$ -algebra  $R'$  has support in  $G^+$ , there are isomorphisms of ABELian groups*

$$K_n^T(X') \cong \bigoplus_{G/N} K_n(R'_1) \cong \mathbb{Z}[G/N] \otimes_{\mathbb{Z}} K_n(R'_1) . \quad \square$$

#### REFERENCES

- [AHW09] Suanne Au, Mu-wan Huang, and Mark E. Walker. The equivariant  $K$ -theory of toric varieties. *J. Pure Appl. Algebra*, 213(5):840–845, 2009.
- [Bou98] Nicolas Bourbaki. *Algebra I. Chapters 1–3*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.
- [Dad70] E. C. Dade. Compounding Clifford’s theory. *Ann. of Math. (2)*, 91:236–290, 1970.
- [Dad80] Everett C. Dade. Group-graded rings and modules. *Math. Z.*, 174(3):241–262, 1980.
- [Grz85] P. Grzeszczuk. On  $G$ -systems and  $G$ -graded rings. *Proc. Amer. Math. Soc.*, 95(3):348–352, 1985.
- [HH13] Roozbeh Hazrat and Thomas Hüttemann. On Quillen’s calculation of graded  $K$ -theory. *J. Homotopy Relat. Struct.*, 8(2):231–238, 2013.
- [Hüt13] Thomas Hüttemann. A note on the graded  $K$ -theory of certain graded rings. *Comm. Algebra*, 41(8):2991–2995, 2013.
- [NvO82] C. Năstăsescu and F. van Oystaeyen. *Graded ring theory*, volume 28 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1982.
- [Qui73] Daniel Quillen. Higher algebraic  $K$ -theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

THOMAS HÜTTEMANN, QUEEN’S UNIVERSITY BELFAST, SCHOOL OF MATHEMATICS AND PHYSICS, PURE MATHEMATICS RESEARCH CENTRE, BELFAST BT7 1NN, UK

*E-mail address:* [t.huettemann@qub.ac.uk](mailto:t.huettemann@qub.ac.uk)

*URL:* [http://www.qub.ac.uk/puremaths/Staff/Thomas\\_Huettemann/](http://www.qub.ac.uk/puremaths/Staff/Thomas_Huettemann/)

ZHANG ZUHONG, BEIJING INSTITUTE OF TECHNOLOGY, SCHOOL OF MATHEMATICS, 5 SOUTH ZHONGGUANCUN STREET, HAIDIAN DISTRICT, 100081 BEIJING, CHINA

*E-mail address:* [zuhong@hotmail.com](mailto:zuhong@hotmail.com)